New linear and positive semidefinite programming based approximation hierarchies for polynomial optimisation

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April 18, 2014

Abstract

In this paper we consider the set of polynomials which are nonnegative over a subset of the nonnegative orthant, where this subset is described by homogeneous polynomial inequalities. The study of such a set of polynomials is motivated by copositivity and the fact that any bounded polynomial optimisation problem can be reformulated into a conic optimisation problem over such a set. The main work of this paper is to introduce a new hierarchy of linear inner approximations for such a set. This hierarchy can then be improved through the use of positive semidefiniteness. Advantages to these approaches are discussed and some examples are presented.

Keywords real algebraic geometry; copositive optimisation; approximation hierarchy; conic optimisation; non-negative polynomials; polynomial optimisation

Mathematics Subject Classification (2010) 90C05; 90C25; 12D15; 14P10

1 Introduction

Polynomials optimisation problems are problems where we want to optimise a polynomial objective function over a feasible set defined by a set of polynomial inequalities (we call such a feasible set a semialgebraic set). Several NP-hard problems can be formulated in this form. We provide a short list of some examples of such problems: (i) optimising a homogeneous quadratic polynomial over the non-negative orthant, which is equivalent to testing matrix copositivity [10, 25]; (ii) 0-1 linear optimisation problems [35, 36, 40]; (iii)

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quadratic optimisation problems including for example the Quadratic assignment problem, the Graph partitioning problem and the MAX-CUT problem [17, 24, 30, 31, 29, 34].

Polynomial optimisation problems attract big attention in theoretical and applied mathematics. Real algebraic geometry and semialgebraic geometry are subfields in algebra strongly related with polynomial optimisation problems. Since these problems are in general very difficult, it is a natural choice to look for tractable relaxations. These relaxations are often based on some variant of a “*positivestellensatz*” for given semialgebraic set. Many researchers have proposed hierarchies of such relaxations which are based on moment and sums-of-squares approximations of the original problem, and give semidefinite optimisation problems. Lasserre [20] proposed a hierarchy of semidefinite optimisation problems which under certain conditions converge with their optimal values to the optimal value of the original polynomial optimisation problem, see also [37]. We suggest the reader to consider also [22] for a beautiful and comprehensive overview of the results from this area.

Copositive optimisation, where the main focus is on symmetric matrices that give non-negative quadratic forms over the non-negative orthant, is another very active area of research. It can be considered as a special subfield in polynomial optimisation but has recently been considered as a field in its own right, since several NP-hard problems like the Stability number problem, the Quadratic assignment problem and the Graph partitioning problem [9, 30, 31, 27] get very simple conic optimisation structure, see also [28]. The copositive optimisation approach was extended beyond polynomial optimisation problems by Burer [8] and Dickinson, Eichfelder and Povh [16, 12] to quadratic optimisation problems over general closed sets, hence the question of how to provide good approximation hierarchies for copositive optimisation problems are getting even more relevant. Many of the authors who approached NP-hard problems by copositive optimisation used approximation hierarchies based on moments and sums-of-squares to provide new lower or upper bounds for the optimal values of the original problems. Parrilo, de Klerk and Pasechnik [26, 9] introduced two monotonic hierarchies of cones that approximate the cone of copositive matrices from the inside. One hierarchy consists of cones described by linear constraints and the other contains cones described by positive semidefinite constraints, see also [4] for an alternative description of these cones.

Motivated by the Stability number problem, Peña et al. [27] used a similar sums-of-squares idea to propose a new hierarchy sandwiched between the hierarchies from [26] and [9]. Dong [14] used tensors to describe the duals of the linear approximations from [26, 9] and the semidefinite approximations from [27]. This paper was actually part of the original inspiration for our research. Bundfuss and Dür [6, 7] used a very nice idea of simplicial decompositions to construct outer and inner approximation hierarchy of cones described by linear inequalities for the cone of copositive matrices. We point out that a comprehensive overview of the recent results in the area of copositive optimisation can be
found in the following survey papers [3, 8, 15] and in the recent dissertation [10].

1.1 Contribution

Motivated by the tensor idea from [14] we propose a hierarchy of approximations which provides lower bounds for any polynomial optimisation problem with a bounded feasible set. These lower bounds then converge to the optimal value of the original problem as the level of the hierarchy tends to infinity.

If there are a finite number of polynomials in the original problem, then these lower bounds would come from linear optimisation problems. This thus inherits the advantages of linear optimisation, for example the ability to solve large problems relatively quickly and the ability to use warm-starts (which we will show in Subsection 5.1 could help when moving from one level of the hierarchy to another).

It is also possible to add positive semidefinite constraints in order to improve our approximations (see Subsection 5.2). The new lower bounds would then come from positive semidefinite optimisation problems. The disadvantage of this is that we lose some of the advantages connect to linear optimisation. The advantage of this approach is that positive semidefinite approximations can often be a lot more accurate than linear approximations. This approach can also be compared to standard sum-of-squares approximations. The advantage of our approach over these, is that in the traditional sum-of-squares approaches, the order of the positive semidefinite constraints goes rapidly as the level of the hierarchy increases and it is this large order of the PSD constraints which can be a severe problem for solvers. In comparison, the order of the positive semidefinite constraints in our problem does not grow with the level of the hierarchy (although the number of constraints does grow).

1.2 Notation

In this subsection we will introduce some notation that will be used throughout this paper.

For a strictly positive integer $m$ and a nonnegative integer $t$ we define the following, where we shall exclude the $m$ from the notation when it is equal to one:

$$
\mathbb{R}^m := \text{The set of real vectors of order } m;$$

$$\mathbb{R}_+^m := \text{The set of non-negative real vectors of order } m;$$

$$\mathbb{N}^m := \text{The set of non-negative integer vectors of order } m;$$

$$\mathbb{N}_t^m := \{\alpha \in \mathbb{N}^m \mid e^T \alpha = t\}.$$

where $e \in \mathbb{R}^m$ is the all-ones vector. Also, for $i \in \{1, \ldots, m\}$, we define $e_i \in \mathbb{R}^m$ to be the unit vector with $i$-th component equal to one and all other components equal to
zero. For \( e_i \) and \( e_k \), the value of \( m \) will be apparent from the context. For \( x \in \mathbb{R}^m \) we refer to its \( i \)-th component via either \( (x)_i \) or \( x_i \). We define inner product of \( x, y \in \mathbb{R}^m \) as 
\[
\langle x, y \rangle := \sum_{i=1}^{m} x_i y_i
\]
and consider the standard Euclidean norm \( \|x\|_2 = \sqrt{\langle x, x \rangle} \).
We note that \(|N^m_n| = (m + t - 1)! / (t! (m - 1)!)
and define
\[
\mathbb{R}^{N^m_n} := \text{The set of real vectors of order } |N^m_n|, \text{ indexed by elements in } N^m_n.
\]

The definitions of the inner product and Euclidean norm are then naturally extended for this.
For \( x \in \mathbb{R}^n \), \( \alpha \in \mathbb{N}^n \) and \( t \in \mathbb{N} \), we define \( x^\alpha \in \mathbb{R}^n \) and \( x^t \in \mathbb{R}^{N^n_n} \) as follows (where \( 0^0 := 1 \)):
\[
x^\alpha := \prod_{i=1}^{n} x_i^{\alpha_i}, \quad x^t := (x^\alpha)_{\alpha \in N^n_n}.
\]

**Example 1.1.** For \( x^T = \begin{pmatrix} 3 & 5 \end{pmatrix} \) and \( \alpha^T = \begin{pmatrix} 2 & 0 \end{pmatrix} \), we have \( x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} = 9 \),
\[
N^2_2 = \left\{ \begin{pmatrix} 2 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \right\}, \quad x^2 = \begin{pmatrix} x^{[2,0]^T} \\ x^{[0,2]^T} \\ x^{[1,1]^T} \end{pmatrix} = \begin{pmatrix} 9 \\ 25 \\ 15 \end{pmatrix}.
\]

We let \( \text{deg}(f) \) denote the degree of a polynomial \( f \) and let \( \mathbb{R}_t[x] \) denote the set of homogeneous polynomials of degree \( t \) with real coefficients acting on \( \mathbb{R}^n \). Note that for any \( f \in \mathbb{R}_t[x] \) there exists a unique \( f \in \mathbb{R}^{N^n_n} \) such that \( f(x) = \langle f, x^t \rangle = \sum_{\alpha \in N^n_n} (f)_{\alpha} x^\alpha \).
Using this fact, from now on we shall freely interchange between a function \( f \in \mathbb{R}_t[x] \) and a vector \( f \in \mathbb{R}^{N^n_n} \).

For a Euclidean space \( Z \) and a set \( A \subseteq Z \), we denote the closure of \( A \) by \( \text{cl} A \) and denote the interior of \( A \) by \( \text{int} A \). We then say that \( A \) is closed if \( A = \text{cl} A \). We further define the dual of \( A \), the conic hull of \( A \) and the convex hull of \( A \) as follows respectively:
\[
A^* := \{ x \in Z \mid \langle x, y \rangle \geq 0 \text{ for all } y \in A \},
\]
\[
\text{conic} A := \left\{ \sum_{i=1}^{m} \theta_i z_i \mid m \in \mathbb{N}, \theta \in \mathbb{R}^m_+, z_i \in A \text{ for all } i \right\},
\]
\[
\text{conv} A := \left\{ \sum_{i=1}^{m} \theta_i z_i \mid m \in \mathbb{N}, \theta \in \mathbb{R}^m_+, e^T \theta = 1, z_i \in A \text{ for all } i \right\}.
\]

We then say that \( A \) is convex if \( A = \text{conv} A \). Lastly, we say that \( A \) is a cone if for all \( x \in A \) and all \( \alpha \in \mathbb{R}^+ \) we have \( \alpha x \in A \).

**Remark 1.2.** By “conic” we are refering to the conic hull in Euclidean space. This is not
the same as considering the so called polynomial cone (see Subsection 7.2).

Remark 1.3. Note that being a cone is not equivalent to having \( \mathcal{A} = \text{conic} \mathcal{A} \), since the latter should also be convex.

Remark 1.4. Also note that if \( \mathcal{A} = \emptyset \) then \( \text{conic} \mathcal{A} = \{0\} \).

We finish this subsection by recalling the following well known results on dual sets. We shall assume that the reader is familiar with these results throughout the paper.

**Theorem 1.5** ([2]). For a Euclidean space \( \mathcal{Z} \) and sets \( \mathcal{A}, \mathcal{B} \subseteq \mathcal{Z} \) we have

\[
\text{int}(\mathcal{A}^{*}) = \{ y \in \mathcal{Z} \mid \langle x, y \rangle > 0 \text{ for all } x \in \mathcal{A} \},
\]

\[
\mathcal{A}^{*} = (\text{cl} \mathcal{A})^{*}, \quad \mathcal{A}^{**} = \text{cl conic} \mathcal{A}, \quad \mathcal{A} \subseteq \mathcal{B} \Rightarrow \mathcal{A}^{*} \supseteq \mathcal{B}^{*}.
\]

**1.3 Cones we are considering**

In this paper we shall consider \( d, n \in \mathbb{N} \setminus \{0\} \), along with \( \{Y_{j} \mid j \in \mathbb{N}\} \) being a set of closed convex cones such that \( Y_{j} \subseteq \mathbb{R}^{n_{j}} \) for all \( j \) and \( Y_{0} = \mathbb{R}^{+} \). We then let

\[
Y := \{ x \in \mathbb{R}^{n} \mid x^{j} \in Y_{j} \text{ for all } j \in \mathbb{N} \},
\]

\[
G := \{ f \in \mathbb{R}^{N_{d}} \mid f(x) \geq 0 \text{ for all } x \in Y \}.
\] (1)

The main focus of this paper will be on the set \( G \), in particular providing a hierarchy of inner approximations for it. One motivation for the study of this set is provided by copositivity, which shall be discussed in Subsection 7.1. Another motivation is provided by polynomial optimisation, which shall be discussed in the following section. Before considering this, we shall first look at some basic well-known properties of these sets.

**Theorem 1.6.** Both \( Y \) and \( G \) are closed cones, with \( G \) also being convex and full-dimensional. Furthermore, we have

\[
\text{int} G = \{ f \in \mathbb{R}^{N_{d}} \mid f(x) > 0 \text{ for all } x \in Y \setminus \{0\} \},
\]

\[
G^{*} = \text{conv} \{ x^{d} \mid x \in Y \}.
\]

**Proof.** It is trivial to see that both \( Y \) and \( G \) are cones, with \( G \) also being convex.

We now recall the well-known result (see e.g. [5]) that for Euclidean spaces \( \mathcal{X}, \mathcal{Z} \), a closed set \( \mathcal{A} \subseteq \mathcal{Z} \) and a continuous function \( g: \mathcal{X} \to \mathcal{Z} \), we have that the set \( \{ x \in \mathcal{X} \mid g(x) \in \mathcal{A} \} \) is closed. This implies that for all \( j \in \mathbb{N} \) we have that the set \( \{ x \in \mathbb{R}^{n} \mid x^{j} \in Y_{j} \} \) is closed. Combining this with the well-known result that the intersection of (infinitely many) closed sets is closed implies that \( Y \) is closed.

Using similar arguments it can also be shown that the set \( G \) is closed.

We shall now prove the characterisation of \( \text{int} G \). It can then trivially be seen that this is a nonempty set and thus \( G \) is full-dimensional.
We let $\mathcal{M} = \{x^d \mid x \in \mathcal{Y}, \|x\|_2 = 1\}$ and note that we trivially have $\mathcal{G} = \mathcal{M}^*$ and $\mathcal{M}$ does not contain the origin. Therefore

$$\text{int} \mathcal{G} = \{f \in \mathbb{R}^n \mid \langle f, y \rangle > 0 \text{ for all } y \in \mathcal{M}\} = \{f \in \mathbb{R}^n \mid f(x) > 0 \text{ for all } x \in \mathcal{Y} : \|x\|_2 = 1\} = \{f \in \mathbb{R}^n \mid f(x) > 0 \text{ for all } x \in \mathcal{Y} \setminus \{0\}\}.$$ 

Finally we shall prove the characterisation of $\mathcal{G}^*$. Using $\mathcal{M}$ given above we note that $\mathcal{G}^* = \mathcal{M}^{**} = \text{cl conic } \mathcal{M}$. Furthermore, it can be seen that $\mathcal{M}$ is a compact set whose convex hull does not contain the origin. By [18, Proposition 1.4.7] this then implies that the conic hull of $\mathcal{M}$ is closed. Noting that the conic hull of $\mathcal{M}$ is given by $\text{conv} \{x^d \mid x \in \mathcal{Y}\}$, this completes the proof.

**Remark 1.7.** Note that for $d = 1$ we have that $\mathcal{G}^*$ is the convex hull of $\mathcal{Y}$.

## 2 Polynomial optimisation

In this section we recall some basic results, definitions and techniques in connection to (polynomial) optimisation.

For an optimisation problem $\mathcal{O}$, we shall let $\text{feas}(\mathcal{O})$ be its feasible set, $\text{opt}(\mathcal{O})$ be its set of optimal solutions and $\text{val}(\mathcal{O})$ be its optimal value.

Consider a general polynomial optimisation problem with a compact feasible set:

$$\begin{align*}
\min_{x \in \mathbb{R}^{n-1}} & \quad \tilde{g}_1(x) \\
\text{s.t.} & \quad \tilde{f}_i(x) \geq 0 \quad \forall i \in \mathcal{I} \quad (P2)
\end{align*}$$

Note that equality constraints can be represented as 2 inequality constraints, so for simplicity we will not explicitly consider these types of constraints. Also note that we have let the dimension of the variables in this problem be equal to $(n-1)$.

We let $d = \deg(\tilde{g}_1)$ and $d_i = \deg(\tilde{f}_i)$ for all $i$. Without loss of generality we shall assume that $d, d_i > 0$ for all $i$.

As the feasible set is compact, again without loss of generality, we shall assume that for all feasible $x$ we have $x \in \mathbb{R}^{n-1}_+$ and $1 - e^T x \geq 0$. We shall further assume that this latter bounding inequality is explicitly given in the problem.

For a polynomial $\tilde{f}_i : \mathbb{R}^{n-1} \to \mathbb{R}$ we let the polynomial $f_i : \mathbb{R}^n \to \mathbb{R}$ be its homogenisation. By this we mean that $f_i$ is homogeneous polynomial of degree $d_i$ such that $\tilde{f}_i(x) = f_i \left((x^T 1)^T\right)$ for all $x \in \mathbb{R}^{n-1}$. For example, if $n = 3$ and $\tilde{f}_1(x) = x_1^2 + 4x_2 + 3$, then we would have $f_1(x) = x_1^2 + 4x_2x_3 + 3x_3^2$. As another example, the polynomial “$1 - e^T x$” corresponding to the bounding inequality would become “$2x_n - e^T x$".
Similarly, for \( \tilde{g}_1 : \mathbb{R}^{n-1} \to \mathbb{R} \) we let the polynomial \( g_1 : \mathbb{R}^n \to \mathbb{R} \) be its homogenisation. Letting \( g_2(x) = x_n^d \), we then have that (P2) is equivalent to the following problem:

\[
\begin{align*}
\min_{x} & \quad g_1(x) \\
\text{s.t.} & \quad f_i(x) \geq 0 \quad \forall i \in \mathcal{I} \\
& \quad g_2(x) = 1 \\
& \quad x \in \mathbb{R}_+^n,
\end{align*}
\]

where

**Assumption 2.1.** All the polynomials are homogeneous of degree greater than zero.

**Assumption 2.2.** We have \( d = \deg(g_1) = \deg(g_2) \).

**Assumption 2.3.** \( g_2(x) > 0 \) for all \( x \in \mathbb{R}_+^n \setminus \{0\} \) such that \( f_i(x) \geq 0 \) for all \( i \in \mathcal{I} \).

**Assumption 2.4.** We have that \( \text{val} \ (P3) \neq -\infty \) and if \( \text{feas} \ (P3) \neq \emptyset \) then \( \text{opt} \ (P3) \neq \emptyset \).

To see that Assumption 2.3 holds, we first note that we trivially have \( g_2(x) \geq 0 \) for all \( x \in \mathbb{R}_+^n \). Furthermore, if \( g_2(x) = 0 \) for some \( x \in \mathbb{R}_+^n \) with \( f_i(x) \geq 0 \) for all \( i \in \mathcal{I} \), then the polynomial \( 2x_n - e^T x \) coming from the bounding inequality would imply that \( -e^T x \geq 0 \), which in turn implies that \( x = 0 \).

Assumption 2.4 comes from the fact that the feasible set is compact and the objective function is a polynomial. This assumption is in fact implied by Assumptions 2.1 to 2.3, however for clarity we have stated it explicitly.

Note that the new problem (P3) only has one more variable than the original problem (P2).

We now let \( \mathcal{Y}_0 = \mathbb{R}_+ \) and for all \( j \in \mathbb{N} \setminus \{0\} \) we let \( \mathcal{Y}_j \subseteq \mathbb{R}^{N'}_+ \) such that

\[
\mathcal{Y}_j = \{ y \in \mathbb{R}^{N'}_+ | \langle f_i, y \rangle \geq 0 \text{ for all } i \in \mathcal{I} \text{ such that } \deg(f) = j \}.
\]

We note the following properties for these sets (for proof see e.g. [33]):

- The \( \mathcal{Y}_j \)'s are closed convex cones, as the intersection of (infinitely many) closed convex cones is itself a closed convex cone.

- If we had a finite number of polynomials in the original problem then the \( \mathcal{Y}_j \)'s would all be polyhedral cones.

- For all \( j \in \mathbb{N} \setminus \{0\} \) we have \( \mathcal{Y}_j^* = \text{cl conic} \{ f_i | i \in \mathcal{I}, \deg(f_i) = j \} \subseteq \mathbb{R}^{N'}_+ \).
Now letting $\mathcal{Y}$ and $\mathcal{G}$ be as given in Subsection 1.3, we have that problem (P3) can be relaxed to the following:

$$\begin{align*}
\min_{x,v} & \quad \langle g_1, v \rangle \\
\text{s.t.} & \quad \langle g_2, v \rangle = 1 \\
& \quad v = x^d \in \mathcal{G}^* \\
& \quad x \in \mathbb{R}^n.
\end{align*}$$

(P5)

This can then be further relaxed to

$$\begin{align*}
\min_{v \in \mathbb{R}^n} & \quad \langle g_1, v \rangle \\
\text{s.t.} & \quad \langle g_2, v \rangle = 1 \\
& \quad v \in \mathcal{G}^*.
\end{align*}$$

(P6)

The standard dual problem to (P6) is then

$$\max_{\lambda \in \mathbb{R}} \quad \lambda$$

$$\text{s.t.} \quad g_1 - \lambda g_2 \in \mathcal{G}.$$ 

(D6)

The following theorem shows the equivalence of the optimisation problems (P3), (P6) and (D6).

**Theorem 2.5.** We consider problems (P3), (P6) and (D6), along with Assumptions 2.1 to 2.4 holding and the $\mathcal{Y}_i$’s given in (4). Then we have:

i. $\text{val} (P3) = \text{val} (P6) = \text{val} (D6)$.

ii. $\text{feas} (P6) = \text{conv}\{x^d \mid x \in \text{feas} (P3)\}$.

iii. $\text{opt} (P6) = \text{conv}\{x^d \mid x \in \text{opt} (P3)\}$.

iv. For $\lambda \in \mathbb{R}$, we have that $\lambda$ is a strictly feasible point of (D6) (i.e. $g_1 - \lambda g_2 \in \text{int} \mathcal{G}$) if and only if $\lambda < \text{val} (P3)$.

**Proof.** We split this proof up into proving the individual statements, although not in the order given in the theorem:
iv. For \( \lambda \in \mathbb{R} \) we have

\[
g_1 - \lambda g_2 \in \text{int}(G) \quad \Leftrightarrow \quad g_1(x) - \lambda g_2(x) > 0 \text{ for all } x \in \mathcal{Y} \setminus \{0\}
\]

(from Theorem 1.6)

\[
\Leftrightarrow \quad g_1(x) - \lambda g_2(x) > 0 \text{ for all } x \in \mathcal{Y} : g_2(x) > 0
\]

(from Assumption 2.3)

\[
\Leftrightarrow \quad g_1(x) > \lambda \text{ for all } x \in \mathcal{Y} : g_2(x) = 1
\]

(from Assumptions 2.1 and 2.2)

\[
\Leftrightarrow \quad \text{val}(P3) > \lambda \quad \text{(from Assumption 2.4)}.
\]

i. As (D6) is the dual of (P6), which is a relaxation of (P5), which is in turn a relaxation of (P3), we have

\[
\text{val}(P3) \geq \text{val}(P6) \geq \text{val}(D6).
\]

Now, considering statement iv, we have \( \text{val}(D6) \geq \text{val}(P3) \), which then proves the required result.

ii. It is trivial to see that \( \text{feas}(P6) \supseteq \text{conv}\{x^d \mid x \in \text{feas}(P3)\} \). We shall now prove the opposite inclusion relation.

We consider an arbitrary \( v \in \text{feas}(P6) \). From Theorem 1.6 and Carathéodory’s theorem, there exists \( \hat{x}_1, \ldots, \hat{x}_m \in \mathcal{Y} \setminus \{0\} \) such that \( v = \sum_{i=1}^{m} \hat{x}_i^d \). From Assumption 2.3 we have \( g_2(\hat{x}_i) > 0 \) for all \( i \). For all \( i = 1, \ldots, m \) we now let \( \theta_i = g_2(\hat{x}_i) > 0 \) and \( x_i = \theta_i^{-1/d} \hat{x}_i \in \mathcal{Y} \). We then have the following, which implies the required result:

\[
g_2(x_i) = \theta_i^{-1} g_2(\hat{x}_i) = 1 \quad \text{for all } i = 1, \ldots, m,
\]

\[
v = \sum_{i=1}^{m} \theta_i x_i^d, \quad 1 = \langle g_2, v \rangle = \sum_{i=1}^{m} \theta_i.
\]

iii. This follows directly from i and ii. \( \square \)

The fact that any bounded polynomial problem can be reformulated into this form means that it has a huge number of potential applications. This includes problems with binary variables as we have that \( x \in \{0, 1\} \) if and only if \( x(x - 1) = 0 \). Since many polynomial optimisation problems with binary variables are NP-hard (e.g. the Quadratic assignment problem [34]), the approach from this paper can be used to handle very hard problems.
2.1 Example

As an example, we apply this to the following optimisation problem, which is represented in Fig. 1a. Note that the optimum is attained at the points \( A = 0 \) and \( C = \begin{pmatrix} 1/2 & 1/2 \end{pmatrix}^T \), and the optimal value is equal to zero.

\[
\begin{align*}
\min_{\mathbf{x}} \quad & x_1 - x_2 \\
\text{s.t.} \quad & 2x_1^2 - x_2 \geq 0 \\
& 1 - 2x_1 \geq 0 \\
& \mathbf{x} \in \mathbb{R}^2_+.
\end{align*}
\] (7)

This is then equivalent to the following problem:

\[
\begin{align*}
\min_{\mathbf{x}} \quad & x_1 - x_2 \\
\text{s.t.} \quad & 2x_1^2 - x_2x_3 \geq 0 \\
& x_3 - 2x_1 \geq 0 \\
& x_3 - x_1 - x_2 \geq 0 \\
& x_3 = 1 \\
& \mathbf{x} \in \mathbb{R}^3_+.
\end{align*}
\]

For this we have

\[
\begin{align*}
n &= 3, & d &= 1, & g_1(\mathbf{x}) &= x_1 - x_2, & g_2(\mathbf{x}) &= x_3, \\
f_1(\mathbf{x}) &= 2x_1^2 - x_2x_3, & f_2(\mathbf{x}) &= x_3 - 2x_1, & f_3(\mathbf{x}) &= x_3 - x_1 - x_2, & \mathcal{I} &= \{1, 2, 3\}.
\end{align*}
\]

In terms of problems (P6) and (D6), this corresponds to

\[
\begin{align*}
g_1 &= \begin{pmatrix} 1 & -1 & 0 \end{pmatrix}^T, & g_2 &= \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T, \\
\mathcal{Y}_0 &= \mathbb{R}_+, & \mathcal{Y}_1 &= \{ \mathbf{y} \in \mathbb{R}^{N_1} \mid 2(\mathbf{y})_{e_1} \leq (\mathbf{y})_{e_3}, \quad (\mathbf{y})_{e_1} + (\mathbf{y})_{e_2} \leq (\mathbf{y})_{e_3} \}, \\
\mathcal{Y}_2 &= \{ \mathbf{y} \in \mathbb{R}^{N_2} \mid (\mathbf{y})_{e_2 + e_3} \leq 2(\mathbf{y})_{e_2} \}, & \mathcal{Y}_i &= \mathbb{R}^{N_i}, & \text{for all } i \geq 3, \\
\mathcal{Y} &= \{ \mathbf{x} \in \mathbb{R}^3_+ \mid 2x_1 \leq x_3, \quad x_1 + x_2 \leq x_3, \quad x_2x_3 \leq 2x_1^2 \}.
\end{align*}
\]
Feasible set for Problem (7).

The combination of both gray areas represents the cone $G^*$ from (8), intersected by hyperplane $(v)_{e_3} = 1$. The inner set represents feasible set for the original problem.

Figure 1: Graphical represents for the example in Subsection 2.1

$$G^* = \text{conv}(\mathcal{Y}) = \left\{ \begin{pmatrix} (v)_{e_1} \\ (v)_{e_2} \\ (v)_{e_3} \end{pmatrix} \middle| 0 \leq (v)_{e_2} \leq (v)_{e_1} \leq \frac{1}{2}(v)_{e_3} \right\}$$

$$= \text{conic} \left\{ \begin{pmatrix} 0 \\ 1 \\ 1 \\ 2 \\ 2 \end{pmatrix} , \begin{pmatrix} 1 \\ 0 \\ 1 \\ 2 \end{pmatrix} , \begin{pmatrix} 0 \\ 1 \\ -2 \\ 0 \\ 1 \end{pmatrix} \right\} ,$$

$$G = \left\{ \begin{pmatrix} (f)_{e_1} \\ (f)_{e_2} \\ (f)_{e_3} \end{pmatrix} \middle| (f)_{e_3} \geq 0, \ (f)_{e_1} + 2(f)_{e_3} \geq 0, \ (f)_{e_1} + (f)_{e_2} + 2(f)_{e_3} \geq 0 \right\}$$

$$= \text{conic} \left\{ \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 1 \end{pmatrix} \right\} ,$$

Therefore Problem (7) can be relaxed to a linear optimisation problem of the form (P5), which is given below, along with its dual (D6). The set of feasible solutions for the reformulated problem is visualised in Fig. 1b.
\[
\min_{\nu, u \in \mathbb{R}^3} \left\{ \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \nu \right\} \begin{pmatrix} v = \nu_1 \\ 0 \\ 2 \end{pmatrix} + \nu_2 \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + \nu_3 \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \quad \nu \in \mathbb{R}_+^n, \quad \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, v = 1 \right\}
\]

\[
= \min_{\nu \in \mathbb{R}_+^3} \{ \nu_2 | \nu_1 + 2\nu_2 + 2\nu_3 = 1, \quad \nu \in \mathbb{R}_+^n \} = 0, \quad \max_{\lambda \in \mathbb{R}} \{ \lambda | -\lambda \geq 0, \quad 1 - 2\lambda \geq 0, \quad -2\lambda \geq 0 \} = 0.
\]

Furthermore, we have
\[
\text{opt (9)} = \text{conv} \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0.5 \\ 0.5 \\ 1 \end{pmatrix} \right\}.
\]

3 Approximation hierarchy for $\mathcal{G}$ and $\mathcal{G}^*$

In this section we shall introduce a hierarchy of inner approximations for $\mathcal{G}$, which is based on the following positivstellensatz:

Theorem 3.1. [13, Theorems 3.5 and 4.1] Let \( \{f_0\} \cup \{f_i | i \in \mathcal{I} \} \subseteq \bigcup_{i \in \mathcal{I}} \mathbb{R}_i[x] \) such that \( 1 \in \{f_i | i \in \mathcal{I} \} \) and \( f_0(x) > 0 \) for all \( x \in \mathbb{R}_+^n \setminus \{0\} : f_i(x) \geq 0 \forall i \in \mathcal{I} \). Then for some \( r \in \mathbb{N} \) there exists a subset \( \mathcal{J} \subseteq \mathcal{I} \) of finite cardinality and a set \( \{g_j | j \in \mathcal{J} \} \subseteq \bigcup_{i \in \mathcal{I}} \mathbb{R}_i[x], \) with all of their coefficients non-negative, such that \( (e^T x)^r f_0(x) = \sum_{j \in \mathcal{J}} f_j(x) g_j(x) \).

We now define the following set for \( r \in \mathbb{N} \):

\[
\mathcal{G}_r := \left\{ h \in \mathbb{R}_N^q \middle| \exists f_p \in \mathcal{Y}_{r+d-e^T p}^* \text{ for all } p \in \mathbb{N}_i, i = 0, \ldots, r + d \right\}
\]

\[
\begin{align*}
\text{such that} \quad (e^T x)^r h(x) = \sum_{i=0}^{r+d} \sum_{p \in \mathbb{N}_i} x^p f_p(x)
\end{align*}
\]

We then have the following theorem on how this set provides a hierarchy of inner approximations for $\mathcal{G}$:

Theorem 3.2. Consider $\mathcal{G}$ and $\mathcal{G}_r$ given in (1) and (10) respectively. For all \( r \in \mathbb{N} \) we
have
\[ \mathcal{G}_r \subseteq \mathcal{G}, \]  
\[ \mathcal{G}_r \subseteq \mathcal{G}_{r+1}, \]  
\[ \text{int} \mathcal{G} \subseteq \bigcup_{s \in \mathbb{N}} \mathcal{G}_s. \]  
(11) \[ (12) \]  
(13)

Proof. (11) follows directly from the definitions, whilst (13) follows trivially from Theorems 1.6 and 3.1. We are now left to prove (12).

We consider an arbitrary \( \mathbf{h} \in \mathcal{G}_r \) and \( \mathbf{f}_p \in \mathcal{Y}_{r+d-\mathbf{e}^T \mathbf{p}} \) for all \( \mathbf{p} \in \mathbb{N}_i^n, i = 0, \ldots, r + d \) such that \( (\mathbf{e}^T \mathbf{x})^r \mathbf{h}(\mathbf{x}) = \sum_{i=0}^{r+d} \sum_{\mathbf{p} \in \mathbb{N}_i^n} \mathbf{x}^\mathbf{p} \mathbf{f}_\mathbf{p}(\mathbf{x}) \). We now define the following

\[ \hat{f}_0(\mathbf{x}) := 0 \quad \text{and} \quad \hat{f}_m(\mathbf{x}) := \sum_{\substack{k=1, \ldots, n: \ m_k > 0 \quad m \in \mathbb{N}_j^n \quad j = 1, \ldots, r + d + 1.}} \]

As \( \mathcal{Y}_i^r \) is a convex cone for all \( i \), we get that \( \hat{f}_m \in \mathcal{Y}_{r+d+1-\mathbf{e}^T \mathbf{m}} \) for all \( \mathbf{m} \). We also have the following, which completes the proof:

\[ (\mathbf{e}^T \mathbf{x})^{r+1} \mathbf{h}(\mathbf{x}) = (\mathbf{e}^T \mathbf{x}) \sum_{i=0}^{r+d} \sum_{\mathbf{p} \in \mathbb{N}_i^n} \mathbf{x}^\mathbf{p} \mathbf{f}_\mathbf{p}(\mathbf{x}) = \sum_{i=1}^{r+d+1} \sum_{\mathbf{m} \in \mathbb{N}_i^n} \mathbf{x}^\mathbf{m} \hat{f}_\mathbf{m}(\mathbf{x}). \]

Special cases of this hierarchy have in fact previously been used to provide two approximation hierarchies for the copositive cone, which shall be discussed in Subsection 7.1.

For all \( r \in \mathbb{N} \), we clearly have that \( \mathcal{G}_r \) is a convex cone. We also have \( \left( \mathcal{Y}_d \cup \mathbb{R}^{\mathbb{N}_i^n}_+ \right) \subseteq \mathcal{G}_r \), which implies that it is also full-dimensional. In general it is not pointed and it is an open question whether it is in general closed.

It is relatively simple but long winded to give an explicit description for this set, along with that of its dual. In order to make this paper more concise, we have decided to omit this description.

4 Approximating problems (P6) and (D6)

We now consider how this hierarchy can be used in connection to the polynomial optimisation reformulation in Section 2.

Naturally, we can take problems (P6) and (D6), and substitute in \( \mathcal{G}_r \) for \( \mathcal{G} \), which
gives us the following:

\[
\begin{align*}
\min_{v \in \mathbb{R}^n} \langle g_1, v \rangle \\
\text{s.t.} \quad \langle g_2, v \rangle &= 1 \\
\quad v \in \mathcal{G}_r^* 
\end{align*}
\]  
(P14_r)

\[
\begin{align*}
\max_{\lambda \in \mathbb{R}} \lambda \\
\text{s.t.} \quad g_1 - \lambda g_2 \in \mathcal{G}_r 
\end{align*}
\]  
(D14_r)

We now present the following theorem in connection to this:

**Theorem 4.1.** For \( r \in \mathbb{N} \), we consider problems (P3), (P14_r) and (D14_r), along with Assumptions 2.1 to 2.3 holding and the \( Y_i \)'s given in (4). Then we have:

i. \( \text{val} (D14_r) \leq \text{val} (P14_r) \) for all \( r \in \mathbb{N} \).

ii. \( \text{val} (P14_r) \leq \text{val} (P14_{r+1}) \leq \text{val} (P3) \) for all \( r \in \mathbb{N} \).

iii. \( \text{val} (D14_r) \leq \text{val} (D14_{r+1}) \leq \text{val} (P3) \) for all \( r \in \mathbb{N} \).

iv. \( \lim_{r \to \infty} \text{val} (D14_r) = \lim_{r \to \infty} \text{val} (P14_r) = \text{val} (P3) \).

**Proof.** We shall prove each point in turn:

i. This comes from the fact that (P14_r) is the dual problem to (D14_r).

ii, iii. These follow directly from Theorems 2.5 and 3.2.

iv. We first recall from Theorem 2.5 that \( \text{val} (P3) \neq -\infty \). We now split the remainder of this proof into two cases:

i. \( \text{val} (P3) = \infty \):

From Theorem 2.5, for an arbitrary \( \lambda \in \mathbb{R} \) we have \( g_1 - \lambda g_2 \in \text{int} \mathcal{G} \), and thus, by Theorem 3.2, there exists \( r \in \mathbb{N} \) such that \( g_1 - \lambda g_2 \in \mathcal{G}_r \). We therefore have \( \lambda \leq \text{val} (D14_r) \leq \text{val} (P3) \). The result now follows from statements i and ii.

ii. \( \text{val} (P3) < \infty \):

We consider an arbitrary \( \varepsilon > 0 \) and \( \lambda = \text{val} (P3) - \varepsilon \). From Theorem 2.5 we have \( g_1 - \lambda g_2 \in \text{int} \mathcal{G} \), and thus, by Theorem 3.2, there exists \( r \in \mathbb{N} \) such that \( g_1 - \lambda g_2 \in \mathcal{G}_r \). We therefore have \( \text{val} (P3) - \varepsilon = \lambda \leq \text{val} (D14_r) \leq \text{val} (P3) \). The result now follows from statements i and ii.

5 Advantages to this approach

In this section we shall look at some of the theoretical advantages to this new approach.
5.1 Warm-starts

If there were a finite number of polynomials in problem (P3), then the \( Y_i \)'s and \( Y_i^* \)'s would all be polyhedral cones, and so the approximations would simply be linear optimisation problems \[39\]. We could then use the simplex algorithm to solve these problems. One advantage of the simplex algorithm is that it makes good use of warm-starts. If a function \((e^T x) h(x)\) can be written in the required non-negative form, then so can the function \((e^T x)^{r+} h(x)\) (see the proof of (12) in Theorem 3.2). Using this, we could take an optimal solution from a previous iteration and provide a warm-start for the next iteration.

5.2 Adding redundant constraints

Suppose that for all \( i \in \mathbb{N} \) we have a closed convex cone \( \hat{Y}_i \subseteq \mathbb{R}^{n_i} \) such that

\[
\{ x^i \mid x \in Y \} \subseteq \hat{Y}_i \subseteq Y_i.
\]

Then for all \( r \in \mathbb{N} \) we have the following where \( G \) and \( G_r \) correspond to the original \( Y_i \)'s, and \( \hat{G} \) and \( \hat{G}_r \) correspond to the new \( \hat{Y}_i \)'s:

\[
G = \hat{G} \supseteq \hat{G}_r \supseteq G_r.
\]

In other words, the smaller cones would provide an approximation of the same set which is at least as good, if not better. These smaller cones can be created by adding redundant constraints. One example of a redundant constraints that we could add for the problem (P3) are "\( f_i(x) f_j(x) \geq 0 \)" where \( i, j \in I \). Adding such constraints would increase the size of the problem, so this would have to be balanced with the improved accuracy.

Another example of a redundant inequality to add is positive semidefiniteness. A symmetric \( m \times m \) matrix \( A \) is defined to be positive semidefinite (PSD) if \( u^T A u \geq 0 \) for all \( u \in \mathbb{R}^m \). The set of PSD matrices is a proper cone (i.e. closed, convex, pointed and full-dimensional) which has been widely studied, including optimising over it \[1\]. We now define the linear functional \( A \), taking vectors from \( \mathbb{R}^{n_2} \) to symmetric \( n \times n \) matrices such that for all \( z \in \mathbb{R}^{n_2} \) and \( i, j \in \{1, \ldots, n\} \) we have \((Az)_{ij} = (z)_{e_i+e_j}\).

For any \( u, x \in \mathbb{R}^n \) we have

\[
u^T (Ax^2) u = \sum_{i,j=1}^n (x^2)_{e_i+e_j} u_i u_j = \sum_{i,j=1}^n x_i x_j u_i u_j = (x^T u)^2 \geq 0\]

Therefore, we may add the redundant constraint of \( Az \) being PSD to \( Y_2 \). Examples of this technique being applied are provided in Example 6.3 and Subsection 7.1.

The disadvantage of this approach is that we no longer have linear optimisation problems, but instead positive semidefinite optimisation problems. This means that we are...
no longer able to use warm-starts and solving positive semidefinite optimisation problems generally takes a lot longer than linear optimisation problems.

The advantage of this approach is that positive semidefinite approximations can often be a lot more accurate than linear approximations (see [21]). This approach can also be compared to standard sum-of-squares approximations. The advantage of our approach over these is that in the traditional sum-of-squares approaches the order of the PSD constraints goes rapidly as the level of the hierarchy increases (normally growing of the order of \(n^r\)). It is this large order of the PSD constraints which can be a severe problem for solvers. In comparison, our approach only has PSD constraints of order \(n\) (although the number of these constraints does grow rapidly).

### 6 Examples

In this section we shall consider a few illustrative examples:

**Example 6.1.** We return to the example from Subsection 2.1. We are considering:

\[
\begin{align*}
n &= 3, \quad d = 1, \quad g_1(x) = x_1 - x_2, \quad g_2(x) = x_3, \\
f_1(x) &= 2x_1^2 - x_2x_3, \quad f_2(x) = x_3 - 2x_1, \quad f_3(x) = x_3 - x_1 - x_2, \quad I = \{1, 2, 3\}, \\
Y_0^r &= \mathbb{R}_+, \quad Y_1^r = \text{conic}\{f_2, f_3\}, \quad Y_2^r = \text{conic}\{f_1\}, \\
Y_r^* &= \{0\} \text{ for all } i \geq 3, \quad G = \text{conic}\{f_2, g_1, h\}, \text{ where } h(x) = x_2.
\end{align*}
\]

For all \(r\) we have \((Y_0^r \cup \mathbb{R}_+^{N}) \subseteq G_r \subseteq G\). From this we see that in this case \(G_r = G\) if and only if \(g_1 \in G_r\). In fact, from the following it can be seen that \(g_1 \in G_2\), and thus \(G_r = G\) for all \(r \geq 2\).

\[
(e^T x)^2 g_1(x) = (\frac{1}{2} x_1 + 3 x_2 + 2 x_3) f_1(x) + \left(\frac{5}{2} x_1 x_2 + x_1 x_3 + 2 x_2 x_3\right) f_2(x) + x_2^2 f_3(x).
\]

By computing the approximations explicitly we get that the optimal values for the zeroth, first and second level approximations are \(-1\), \(-\frac{1}{6}\) and 0 respectively. The exact descriptions for \(G_r^*\) are visualised in Fig. 2.

**Example 6.2 (Immediate convergence).** For \(0 \leq \varepsilon < 1\) we consider

\[
\begin{align*}
n &= d = 2, \quad \mathcal{Y}_0 = \mathbb{R}_+, \quad \mathcal{Y}_i = \mathbb{R}_+^{N_i} \text{ for all } i \in \mathbb{N} \setminus \{0, 2\}, \\
\mathcal{Y}_2 &= \left\{ \begin{pmatrix} (y)_{2e_1} \\ (y)_{2e_2} \\ (y)_{e_1+e_2} \end{pmatrix} \begin{array}{c}
(2 - \varepsilon)(y)_{2e_1} \geq \varepsilon(y)_{2e_1}, \\
(2 - \varepsilon)(y)_{2e_2} \geq \varepsilon(y)_{2e_2}, \\
(y)_{e_1+e_2} = 0
\end{array} = \text{conic}\left\{ \begin{pmatrix} 2 - \varepsilon \\ \varepsilon \\ 0 \end{pmatrix}, \begin{pmatrix} \varepsilon \\ 0 \\ 0 \end{pmatrix} \right\} \right. \\
&= \text{conic}\left\{ \begin{pmatrix} 2 - \varepsilon \\ \varepsilon \\ 0 \end{pmatrix}, \begin{pmatrix} \varepsilon \\ 0 \\ 0 \end{pmatrix} \right\}
\end{align*}
\]

By computing the approximations explicitly we get that the optimal values for the zeroth, first and second level approximations are \(-1\), \(-\frac{1}{6}\) and 0 respectively. The exact descriptions for \(G_r^*\) are visualised in Fig. 2.
Figure 2: The cones $\mathcal{G}_0^* \supseteq \mathcal{G}_1^* \supseteq \mathcal{G}_2^* = \mathcal{G}^* = \text{conv}\{\mathcal{Y}\}$ from Example 6.1 intersected with the hyperplane $(v)_{e_3} = 1$.

We then have

$$\mathcal{Y} = \left\{ x \in \mathbb{R}_+^2 \mid \begin{array}{l}(2 - \varepsilon)x_1^2 \geq \varepsilon x_2^2, \\ (2 - \varepsilon)x_2^2 \geq \varepsilon x_1^2, \\ x_1x_2 = 0 \end{array} \right\} = \begin{cases} \text{conic}\{e_1\} \cup \text{conic}\{e_2\} & \text{if } \varepsilon = 0, \\ \{0\} & \text{if } 0 < \varepsilon < 1, \end{cases}$$

$$\mathcal{G}^* = \begin{cases} \text{conic} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} & \text{if } \varepsilon = 0, \\ \{0\} & \text{if } 0 < \varepsilon < 1. \end{cases}$$

For $r = 0$ it can be shown that

$$\mathcal{G}_0^* = \text{conic} \begin{pmatrix} 2 - \varepsilon & \varepsilon \\ \varepsilon & 0 \end{pmatrix} \cup \begin{pmatrix} 2 - \varepsilon & \varepsilon \\ \varepsilon & 0 \end{pmatrix}.$$

If we consider $\varepsilon = 0$, then this implies that $\mathcal{G}_r^* = \mathcal{G}^*$ for all $r \in \mathbb{N}$.

Alternatively, if we consider $0 < \varepsilon < 1$, then we have $\mathcal{G}_0^* \neq \mathcal{G}^*$, but it can be shown that $\mathcal{G}_r^* = \mathcal{G}^*$ for all $r \in \mathbb{N}$ with $r \geq 1$.

Note that this result is independent of how small $\varepsilon$ is (provided that it remains positive). We thus see that, in general, a small change in one of the $\mathcal{Y}_i$’s may have a large effect on the $\mathcal{G}_i^*$’s. Furthermore, whenever $\mathcal{G}^* = \{0\}$, we get that $\mathcal{G}^* = \mathcal{G}_r^*$ for some finite $r \in \mathbb{N}$, otherwise the limit would not converge to $\{0\}$. 
Example 6.3 (Infinite convergence and positive semidefinite constraints). We consider
\[ n = d = 2, \quad \mathcal{Y}_0 = \mathbb{R}_+, \quad \mathcal{Y}_i = \mathbb{R}^{N_2} \text{ for all } i \in \mathbb{N} \setminus \{0, 2\}, \]
\[ \mathcal{Y}_2 = \left\{ y \in \mathbb{R}^{N_2} \mid 2(y)_{2e_2} = 2(y)_{e_1 + e_2} \geq (y)_{2e_1} \right\}. \]

We then have
\[ \mathcal{Y} = \{ x \in \mathbb{R}^2_+ \mid 2x_1^2 = 2x_1x_2 \geq x_1^2 \} = \text{conic} \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}, \]
\[ G^* = \left\{ \begin{pmatrix} (v)_{2e_1} \\ (v)_{2e_2} \\ (v)_{e_1 + e_2} \end{pmatrix} \left| (v)_{2e_1} = (v)_{2e_2} = (v)_{e_1 + e_2} \right\} = \text{conic} \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix} \right\}. \]

It can be shown that for this we have
\[ G_r^* = \text{conic} \left\{ \begin{pmatrix} 1 - 2^{-r} \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 + 2^{-r} \\ 1 \\ 1 \end{pmatrix} \right\}. \]

As we can clearly see, this does tend towards the required set, however \( G_r^* \neq G^* \) for all \( r \in \mathbb{N} \).

If we consider continuing this example by adding the redundant positive semidefinite constraint from Subsection 5.2 to \( \mathcal{Y}_2 \) then it can be shown that
\[ \mathcal{Y}_2 = \left\{ y \in \mathbb{R}^{N_2} \mid 2(y)_{2e_2} = 2(y)_{e_1 + e_2} \geq (y)_{2e_1}, \begin{pmatrix} (y)_{2e_1} \\ (y)_{e_1 + e_2} \end{pmatrix} \text{ is PSD} \right\} \]
\[ = \left\{ y \in \mathbb{R}^{N_2} \mid 2(y)_{2e_1} \geq 2(y)_{2e_2} = 2(y)_{e_1 + e_2} \geq (y)_{2e_1} \geq 0 \right\}, \]
\[ G_r^* = \text{conic} \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 + 2^{-r} \\ 1 \\ 1 \end{pmatrix} \right\}. \]

Thus, although adding the positive semidefinite constraint does improve the approximations, in this case we still have \( G_r^* \neq G^* \) for all \( r \in \mathbb{N} \).

However, in some cases adding the redundant positive semidefinite constraint will give us \( G_r^* = G^* \) when there was originally a gap for all \( r \). An example of this happening will be given in Subsection 7.1.
7 Relation to previous work

In this section we will compare this new hierarchy to some previous hierarchies. We assume that the reader already has some knowledge of these previous hierarchies.

7.1 Copositivity

In this subsection we shall look at how special cases of our hierarchy have previously been used to approximate the cone of copositive matrices.

A symmetric matrix $A$ is defined to be copositive if $x^T A x \geq 0$ for all non-negative vectors $x$. We then denote the cone of $n \times n$ symmetric copositive matrices by $\mathcal{COP}_n$. For surveys on copositivity, we recommend the reader to read [8, 10, 15]. If we now consider $\mathcal{G}$ for $d = 2$, $\mathcal{Y}_0 = \mathbb{R}_+$ and $\mathcal{Y}_i \subseteq \mathbb{R}^{n_i}$ for all $i \in \mathbb{N} \setminus \{0\}$, then we get that $\mathcal{G}$ is equivalent to $\mathcal{COP}_n$. Our hierarchy, $\mathcal{G}_r$, then provides an inner approximation hierarchy to the cone of copositive matrices.

If we consider $d = 2$, $\mathcal{Y}_0 = \mathbb{R}_+$ and $\mathcal{Y}_i \subseteq \mathbb{R}^{n_i}$ for all $i \in \mathbb{N} \setminus \{0\}$, then we get that $\mathcal{G}_r$ is equivalently to the hierarchy which was introduced in [9] and is denoted by $\mathcal{C}_n$. We have that $\mathcal{C}_n$ is always a polyhedral cone, but for $n \geq 2$ we have that $\mathcal{COP}_n$ is not a polyhedral cone, and thus $\mathcal{COP}_n \neq \mathcal{C}_n$ for all $r \in \mathbb{N}$.

If we add the PSD constraint to $\mathcal{Y}_2$, then $\mathcal{G}$ remains equivalent to $\mathcal{COP}_n$, and $\mathcal{G}_r$ is equivalent to the hierarchy which was introduced in [27] and is denoted by $\mathcal{Q}_n$. For $n \leq 4$ it was shown that $\mathcal{Q}_n^0 = \mathcal{COP}_n$. Therefore adding the positive semidefinite constraint can make the approximations exact when there was originally a gap.

Another inner approximation hierarchy to the cone of copositive matrices is provided by the Parrilo cones [26], and for $r = 1$, we have that $\mathcal{Q}_n^1$ is equal to the Parrilo-1 cone [27]. In [11] it was found that for $n \geq 5$, this cone is not invariant under scalings of the variables. Therefore, in general, $\mathcal{G}_r$ is not invariant under scalings of the variables. These scalings are equivalent to replacing $e$ in the definition of $\mathcal{G}_r$ from (10) with an alternative strictly positive vector. It was also shown in [11] that $\mathcal{Q}_n^r \neq \mathcal{COP}_n$ for all $n \geq 5$.

The duals to $\mathcal{C}_n$ and $\mathcal{Q}_n^r$ were previously studied in [14], and in fact the work in that paper provided part of the initial inspiration to our paper.

7.2 Krivine-Stengle Positivstellensatz

This was proven in [19] and then rediscovered in [38]. See also [23, Section 3.6.1] for an introduction to this result.

In this a set of nonhomogeneous polynomials $\{f_i \mid i \in \mathcal{I}\}$ was considered, along with the set $\mathcal{W} = \{x \in \mathbb{R}^n \mid f_i(x) \geq 0 \text{ for all } i \in \mathcal{I}\}$. The set $\mathcal{M}$ was then defined to be the smallest set such that $\{f_i \mid i \in \mathcal{I}\} \subseteq \mathcal{M}$, and if $f, g \in \mathcal{M}$ and $\lambda \in \mathbb{R}_+$, then we have $(f + g) \in \mathcal{M}$ and $fg \in \mathcal{M}$ and $\lambda f \in \mathcal{M}$. This is referred to as the polynomial cone of
\{f_i \mid i \in I\}. This should not be confused with the conic hulls in Euclidean space that we considered in our paper.

It was shown that if we have a polynomial \( h \) such that \( h(x) > 0 \) for all \( x \in W \) then there exists \( f, g \in \mathcal{M} \) such that \( pf = g + 1 \).

By letting \( \mathcal{M}_r \) be the set of polynomials in \( \mathcal{M} \) of degree less than or equal to \( r \), from this we get that if we have a polynomial \( h \) such that \( h(x) > 0 \) for all \( x \in W \) then there exists \( r \in \mathbb{N} \) and \( f, g \in \mathcal{M}_r \) such that \( pf = g + 1 \). For a fixed \( r \), we have that \( \mathcal{M}_r \) is a polyhedral cone, and thus checking whether there exists \( f, g \in \mathcal{M}_r \) such that \( pf = g + 1 \) is a linear optimisation problem.

Another way of writing our new hierarchy would be to consider a set of homogeneous polynomials \( \{f_i \mid i \in I\} \) and to let \( \tilde{\mathcal{M}} \) be the smallest set such that \( \{1\} \cup \{f_i \mid i \in I\} \subseteq \tilde{\mathcal{M}} \), and if \( f, g \in \tilde{\mathcal{M}} \) and \( \lambda \in \mathbb{R}_+ \) and \( i \in \{1, \ldots, n\} \), then we have \( (f+g) \in \tilde{\mathcal{M}} \) and \( x_i f(x) \in \tilde{\mathcal{M}} \) and \( \lambda f \in \tilde{\mathcal{M}} \). Then letting \( \tilde{\mathcal{M}}_r \) be the set of polynomials in \( \tilde{\mathcal{M}} \) of degree less than or equal to \( r \), our hierarchy is equivalent to saying that if we have a polynomial \( h \) such that \( h(x) > 0 \) for all \( x \in Y \setminus \{0\} \) then there exists \( r \in \mathbb{N} \) such that \( (e^Tx)^rh(x) \in \tilde{\mathcal{M}}_r \).

Written in this form, some similarities between our hierarchy and that of Krivine-Stengle are apparent. In particular, for a fixed polynomial \( h \) and a fixed \( r \), checking the existence for both is a linear optimisation problem. However our hierarchy has the following two advantages over that by Krivine and Stengle:

- For a fixed \( r \), we multiply the polynomial that we are considering with a fixed polynomial \( (e^T x)^r \), rather than a variable polynomial. This allows us to have \( h \) as a variable whilst still having a linear optimisation problem.

- For \( \tilde{\mathcal{M}} \), we only consider \( x_i f(x) \), rather than \( fg \). This reduces the computational size of the hierarchy.

### 7.3 Putinar-Vasilescu Positivstellensatz

The Putinar-Vasilescu Positivstellensatz is the following. This is based on sum-of-squares, a good introduction to which is provided by [23].

**Theorem 7.1** ([32, Theorem 1]). Consider homogeneous polynomials of even degree \( f_1, \ldots, f_m \), and the set \( W = \{x \in \mathbb{R}^n \mid f_i(x) \geq 0 \text{ for all } i = 1, \ldots, m\} \). If we have a polynomial \( h \) such that \( h(x) > 0 \) for all \( x \in W \setminus \{0\} \) then there exists an \( r \in \mathbb{N} \) and sums-of-squares polynomials \( g_0, \ldots, g_m \) such that \( (x^T x)^r h(x) = g_0(x) + \sum_{i=1}^m f_i(x) g_i(x) \).

We can consider applying this to our case by noting that we have \( x \in \mathbb{R}^n_+ \) if and only if there exists \( z \in \mathbb{R}^n \) such that \( x_i = z_i^2 \) for all \( i \) (written as \( x = z \circ z \)). Using this observation, it can be shown that in our case the Putinar-Vasilescu Positivstellensatz is equivalent to the following:
Theorem 7.2 ([13, Theorem 3.2]). Consider homogeneous polynomials $f_1, \ldots, f_m$, and the set $\mathcal{Y} = \{ x \in \mathbb{R}_+^n \mid f_i(x) \geq 0 \text{ for all } i = 1, \ldots, m \}$. If we have a polynomial $h$ such that $h(x) > 0$ for all $x \in \mathcal{Y} \setminus \{0\}$ then $(e^T x)_r h(x) = g_0(x) + \sum_{i=1}^{m} f_i(x) g_i(x)$ for some $r \in \mathbb{N}$ and $g_i(z \circ z)$ being sum-of-squares (equivalently $g_i(x) = \sum_{j=1}^{\infty} \sum_{\alpha \in \mathbb{N}_0^n} x^\alpha \tilde{g}_{i,\alpha}(x)$, where $\tilde{g}_{i,\alpha}$ are sum-of-squares).

Therefore our hierarchy can be seen as taking this, but restricting to $\tilde{g}_{i,\alpha}$ to be of degree zero. As stated in Subsection 5.2, the disadvantages of this approach in comparison with ours are:

- You are required to perform positive semidefinite optimisation, which prevents you from using warm-starts.
- The order of the individual positive semidefinite constraints can grow very large, which means that the hierarchy quickly becomes computationally intractable.

An area of further research would be to consider what happens when the degree of $\tilde{g}_{i,\alpha}$ is restricted to low but nonzero values. This would allow for the introduction of positive semidefinite constraints of low order.

Our hierarchy could have been described in a similar form to this, without the use of the $\mathcal{Y}_i$’s, however we have avoided this as we wished to allow for the possibility of adding other types of constraints, which would be equivalent to infinitely many polynomial constraints. For example, if we have a cone $K \subseteq \mathbb{R}^{N_r}$ which is the projection of a spectrahedron, then we could consider constraints of the form $f(x) \geq 0$ for all $f \in K$ by intersecting $\mathcal{Y}_i$ with $K^*$. Our approximations would then similarly be the projections of spectrahedrons, and thus conic optimisation problems over these approximations would be positive semidefinite optimisation problems.

8 Conclusions

In this paper we showed how a large class of polynomial problems (including all those with a bounded feasible set) can be reformulated as a polynomial optimisation problem with specific structure. We were then able to reformulate this new problem as a conic optimisation problem with one constraint. All of the difficulty of the original problem was contained in the cone that we were optimising over. For this conic optimisation problem, there was strong duality with its dual. We then introduced a hierarchy of inner approximations to the cone in the dual problem. This also provided a hierarchy of outer approximations to the cone in the primal problem. As the level of the hierarchies tended towards infinity, these approximations tended towards the cones that we were interested in. These hierarchies then provided lower bounds to our original problems, which similarly
tended towards the actual solution as the level of the hierarchy tended towards infinity. Finally we suggested advantages of our approach, along with considering some examples.

Future work in connection with this would be to implement the approximations and see how they perform in practice.

Acknowledgements

This research was started whilst the first author was at the Johann Bernoulli Institute for Mathematics and Computer Science, University of Groningen, the Netherlands, and the authors would like to thank this institute for the support that it provided.

The second author wishes to thank to Slovenian research agency for support via program P1-0383 and project L74119 and to Creative Core FISNM-3330-13-500033 ‘Simulations’ project funded by the European Union.

References


