The viewshed problem: a theoretical analysis and a new algorithm for finding the viewshed of a given point on a triangulated terrain

Haluk ELİŞ

Department of Industrial Engineering, Bilkent University, 06800, Ankara, Turkey

Barbaros TANSEL*

Department of Industrial Engineering, Bilkent University, 06800, Ankara, Turkey

Osman OĞUZ†

Department of Industrial Engineering, Bilkent University, 06800, Ankara, Turkey

ABSTRACT

We give a comprehensive theoretical treatment for calculating the viewshed of a given point, present an analytical solution to the viewshed problem and a new algorithm for finding the viewshed on a triangulated terrain. We implement our algorithm on a real terrain. Some algorithms make use of the horizon information of the terrain to calculate viewshed. The vertices of the horizon of the terrain are projected onto the supporting plane of the triangle of interest to find the visible region on the triangle. We show that this approach is erroneous. We offer an alternative model in which we project relevant triangles of the terrain onto the triangle of interest. It is shown that the invisible region on a given triangle caused by another triangle is characterized by a nonlinear system of equations, for which a closed-form solution is given. Our analytical treatment results in a new exact algorithm which can be used in studies such as optimal placement of guards on terrains where exact viewsheds of guards are needed.

* Posthumously

† Corresponding author: ooguz@bilkent.edu.tr

Keywords: viewshed analysis on terrains; projection of a triangle; a polynomial-time viewshed algorithm; an application of polyhedral theory

1. Introduction

Viewshed calculations on terrains have important applications in such diverse areas as military (siting defense instruments against enemy intrusion) (Ray *et al.* 1992), archeology (Lake et al. 1998), forest protection from fires (Goodchild and Lee 1989, Pompa-García *et al.* 2010) and computer graphics (Bittner and Wonka 2003). Some problems related to viewshed analysis are posed and solution algorithms are presented in Lee (1991). Nagy (1994) discusses several issues regarding terrain visibility, such as visibility graphs and visibility invariants. There are factors, such as weather, vegetation, time of day, etc., that might affect visibility (see, for example, Marcos 2007 and Magoč *et al.* 2010).

Consider a region ‘S’ in R^3, which represents the surface of a geographical terrain. The digital representation of S is known as a “Digital Terrain Model (DTM)” (Li *et al.* 2005). A DTM is, in general terms, an approximation of the real surface obtained by sampling points from the terrain with x, y, and z coordinate information and assigning a function among those points. There are two mainly used approximations of terrains: Regular square grids (RSG) and triangulated irregular networks (TIN) (De Floriani and Magillo 2003).

Our goal, in this paper, is to give an analysis of the viewshed of a given point ‘V’ on a triangulated terrain (see Figure 1) and to present a new algorithm for calculating the viewshed of V. Line-of-sight (LOS) between two points V and Y on S is described as the straight line segment between those points and if LOS between V and Y is not blocked by the terrain then two points are said to be mutually visible (intervisible) (see Figure 2). The viewshed of V is defined to be those portions of the terrain visible from V. The formal definitions of visibility and viewshed for TINs are given in section 2. We discuss the previous work on viewshed calculations in section 3. In section 4, we give an analysis of viewshed and present our algorithm with an implementation on a real terrain.

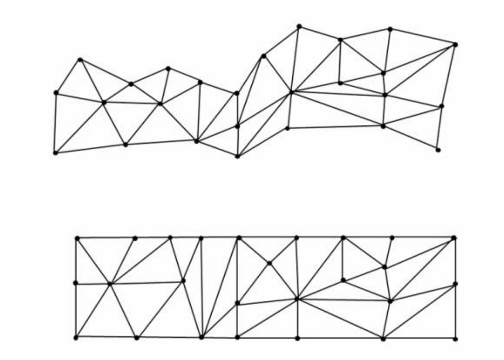


Figure 1. Triangulated irregular network (3 and 2 dimensional views)

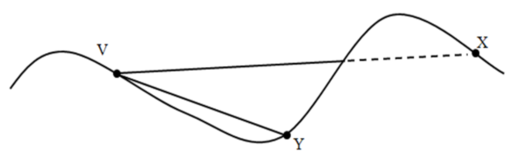


Figure 2. Cross-section of a terrain surface. V and Y are visible but V and X are not.

2. Visibility on TINs

Nagy (1994) posed the question "Are surfaces tangent to a line-of-sight visible?" since, as he pointed out, "... computer implementation requires unambiguous specifications." The result in this section provides a definite answer to this question and lays the theoretical background for the analysis of the viewshed problem.

Let $S \in R^3$ be the surface of interest to be represented as a TIN. Let $P = \{p_1,...,p_n\}$ be the set of 'n' points sampled from S, $\overline{p_i} \in R^2$ be the projection of point p_i onto x-y plane, and $\overline{P} = \{ \overline{p_1},\ldots,\overline{p_n} \}$ be the set of points that are projected. The triangulation of $\overline{P}$ is defined as the maximal planar subdivision whose vertex set is $\overline{P}$ (De Berg *et al.* 1997). Let T* be the resulting triangulation and E* be the set of edges formed after the triangulation, i.e.

$E^*=\{ (\bar{p}_i,\bar{p}_j) : i,j \in \{1,\ldots,n\}, i\neq j\}$. Then, those points p_i and p_j in P for which $(\bar{p}_i, \bar{p}_j)$ is an edge in E* are connected with a straight line to obtain a TIN representation T of S. Note that both T and T* are compact and connected point sets, and in addition, T* is convex. Since T is compact we may restrict T to the nonnegative orthant without loss of generality. Note that $T^* \subseteq R^2$ is the projection of $T \subseteq R^3$, i.e. $T^*= \{(x,y):(x,y,z) \in T$ for a unique $z\geq 0\}$. We may also view T* as a subset of R^3 by taking T* to be the set of points (x,y,0) such that $(x,y,z) \in T$ for $z\geq 0$.

We assume that visibility is a symmetric concept, that is, if A is visible from B then B is visible from A. We define the visibility function v(.,.) to be a two-valued function that assigns value 1 or 0 to each pair of points A, B on T depending on whether the points are intervisible or not;

$$v: R^3 \times R^3 \rightarrow \{0,1\}, \quad \text{with } v(A,B) = v(B,A) = \begin{cases} 1, & \text{if A and B are intervisible} \\ 0, & \text{otherwise} \end{cases}$$

For a point $A \in R^3$, let x(A), y(A) and z(A) denote the x, y and z coordinates of point A respectively. Let A and B be points on T for which an intervisibility query is to be made, and A* and B* be the projections of A and B on x-y plane. We assume that K*L*M* and P*Q*R* are triangles in T* which A* and B* lie in respectively, and that vertices K*, L*, M*, P*, Q*, and R* are the projections of K, L, M, P, Q, and R respectively. We note that A and B must lie in triangles KLM and PQR. Since a point in a triangle can be written as a convex combination of the vertices of the triangle, there exist unique $\lambda_1, \lambda_2, \lambda_3$ and $\mu_1, \mu_2, \mu_3 \in R$ such that

$$A^*=\lambda_1 K^*+ \lambda_2 L^*+ \lambda_3 M^*,\ \lambda_1+ \lambda_2+ \lambda_3=1,\ \lambda_i\geq 0,\ i=1,2,3 \text{ and}$$

$$B^*= \mu_1 P^*+ \mu_2 Q^*+ \mu_3 R^*,\ \mu_1+ \mu_2+ \mu_3=1,\ \mu_i\geq 0,\ i=1,2,3.$$

The elevations of A and B can be found as the convex combination of the elevations of the vertices K, L, M, and P, Q, R respectively, using λ_1, λ_2, λ_3 and μ_1, μ_2, μ_3;

$$z(A)= \lambda_1 z(K)+ \lambda_2 z(L)+ \lambda_3 z(M) \text{ and } z(B)= \mu_1 z(P) + \mu_2 z(Q) + \mu_3 (R)$$

Let $Y \in R^3$ be such that $Y^* \in T^*$. Let $\tau(Y)$ be the point where the line through Y and Y* intersects T, in words $\tau(Y)$ is the projection of Y on T. Note that for a point Y on T, Y= $\tau(Y)$.

Definition 2.1. Let A, B ∈ T. Then $v(A,B)=0$ if and only if $\exists\, \lambda' \in (0,1)$ such that $\lambda' z(A)+(1-\lambda')z(B) < z(\tau(\lambda' A+(1-\lambda')B))$.

Definition 2.1. says that if the height of the open line segment (LOS) between points A and B is strictly less than that of the terrain at some point between A and B, then A and B are not intervisible. The mutual visibility of two points is generally given as a definition in the visibility literature. We establish it as a theorem using definition 2.1 and lemma 2.1.

Lemma 2.1. Let A, B ∈ T. If $v(A,B)=1$ then it must be true that $\lambda z(A)+(1-\lambda)z(B) \geq z(\tau(\lambda A+(1-\lambda)B))\ \forall\, \lambda \in [0,1]$ (^). The inequality in (^) can not be strict $\forall\ \lambda \in (0,1)$.

Proof (by contradiction). Suppose A and B are mutually visible. Definition 2.1. implies that $\nexists\, \lambda' \in (0,1)$ such that $\lambda' z(A)+(1-\lambda')z(B) < z(\tau(\lambda' A+(1-\lambda')B))$. Thus, the inequality must be ≥ $\forall\, \lambda \in (0,1)$. Suppose, to the contrary, that we use > in (^) $\forall\, \lambda \in (0,1)$. Let λ_n and x_n be two sequences such that $\lambda_n = 1 - (\frac{1}{n+1})$ and $x_n = \lambda_n A+(1-\lambda_n)B$. Obviously, $\lambda_n \to 1$ and this implies $x_n \to A$. Then (^) becomes $\lambda_n z(A)+(1-\lambda_n)z(B) > z(\tau(\lambda_n A+(1-\lambda_n)B)) = z(\tau(x_n)),\ \forall\, n \in N$. But $\lambda_n \to 1$ implies $z(A) > z(A)$, and the second assertion follows. The first assertion follows from the preceding result and the fact that λ=1 and λ=0 satisfy (^) ■

Theorem 2.1. Let A, B $\in$ T. Then, v(A,B)=1 if and only if

$$\lambda z(A)+(1-\lambda)z(B) \geq z(\tau(\lambda A+(1-\lambda)B)) \; \forall \, \lambda \in [0,1].$$

Theorem 2.1. concludes that if the LOS between A and B is tangent to the surface of T then A and B must be intervisible. Let V be a point on T. We denote the viewshed of V by *VS*(V), and define *VS*(V) to be the subset of T visible from V, i.e. $VS(V)=\{B \in T: v(V,B)=1\}$. Next section discusses the previous work on viewshed calculations.

3. Previous work

Goodchild and Lee (1989), and Lee (1991) consider a triangle, as a whole, visible or not. When all three edges of a triangle are visible the triangle is visible, otherwise not visible, and thus giving an approximation to the problem. Ben-Moshe et al (2008) present two approximation algorithms for computing *VS*(V). The survey papers by De Floriani and Magillo (1994) and De Floriani and Magillo (2003) discuss exact algorithms for calculating viewsheds. An algorithm computes the lower envelopes of a set of triangles and has a worst case complexity of $O(n^2 \log n)$. Viewshed problem on TINs can be considered as a special case of the hidden surface removal problem. Hidden surface removal problems involve the computation of visible parts of objects in the scene on a viewplane vertical to yz (or xy) plane when viewed from a point located at infinity and several hidden-surface removal algorithms exist (Mc Kenna (1987), Reif and Sen (1995), Katz *et al.* (1992), De Berg and Gray (2010),). However, hidden surface removal algorithms are mainly of theoretical interest and their specific implementation on TINs remains problematic (Nagy (1994), Dercole (2003), De Floriani and Magillo (2003)).

Viewshed algorithm by De Floriani and Magillo (1997) make use of the horizon information of the terrain. Horizon of a terrain with respect to a given viewpoint is described as the farthest set of points seen by the viewpoint. Several algorithms for horizon computation exist (see Edelsbrunner et al. (1989), Stewart (1998), Dercole (2003)). De Floriani and Magillo (1997) compute a star-shaped polygon around the viewpoint and find the "current" horizon of the polygon. Then, the vertices of the current horizon are projected onto the triangle of interest, which is not in the polygon yet, to find the visible portion of the triangle. In section 4.2., we show that this approach is erroneous and present an alternative model. It is known that viewshed algorithms take $\Omega(n^2)$ time (Cole and Sharir).

Riggs and Dean (2007) point out that there is disagreement in viewsheds computed by different softwares, and disagreement between the real (on the field) viewshed and the viewshed computed by commercial softwares. They conclude that there is a need for improved viewshed algorithms.

4. Analysis

4.1. Preliminaries

Let T and T* be as defined in section 2, $V=(V_1,V_2,V_3)$ be the viewpoint on T whose viewshed is to be calculated and KLM be a triangle on T. We assume T is in the nonnegative orthant. We want to find the region on KLM which is invisible to V. The complement of this region gives the visible region and the union of visible regions over all triangles gives *VS*(V).

Let PQR be a triangle that might affect the visibility of (i.e. cast a shadow on) KLM when viewed from V. We call KLM the "Target Triangle" (TT) and PQR the "Blocking Triangle Candidate" (BTC). Let $\bar{P}, \bar{Q}, \bar{R}, \bar{K}, \bar{L}, \bar{M}$, and $\bar{V} \in R^3$ be the projections of P, Q, R, K, L, M, and V on x-y plane respectively (Figure 3). Since T* is convex, the invisible region on KLM is the union of the projections of triangles on T. This is an important observation since if T* were not convex then we would also have to consider the projection of the earth (the object defined by $P\bar{P}Q\bar{Q}R\bar{R}$) below the BTCs.

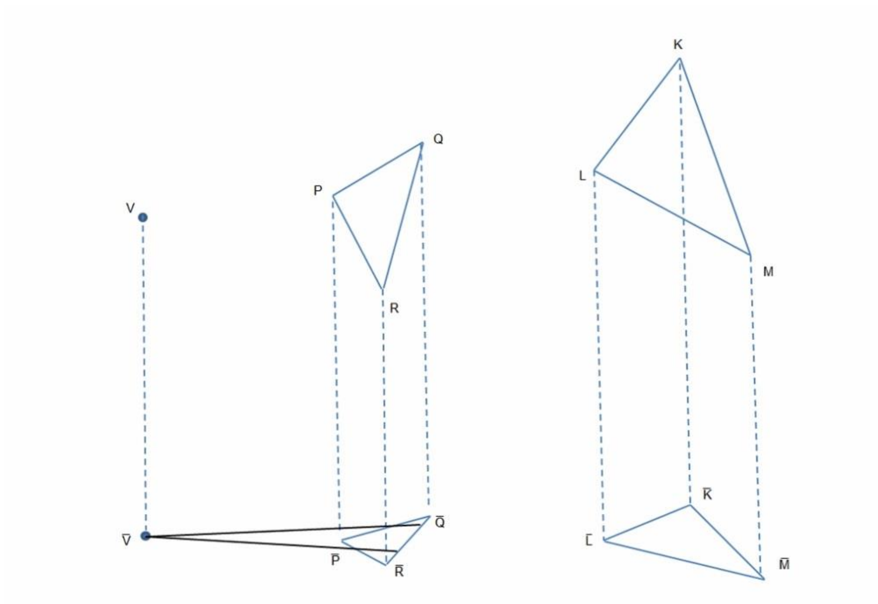


Figure 3. PQR may cast a shadow on KLM.

For any TT, BTCs can be found using T* as follows (Figure 4); First, we select two vertices of the $\bar{K}\bar{L}\bar{M}$ at a time and form a triangle with $\bar{V}$. Obviously, there can be at most three such triangles, $\bar{V}\bar{L}\bar{M}$, $\bar{V}\bar{K}\bar{M}$, and $\bar{V}\bar{L}\bar{K}$. We choose the triangle in which the angle associated with vertex $\bar{V}$ is largest. In Figure 4, this triangle is $\bar{V}\bar{K}\bar{M}$. A point $\bar{C}$ within the triangle $\bar{V}\bar{K}\bar{M}$ is the projection of a vertex C of a triangle on T and the triangle in which C is a vertex (ABC) is a BTC. Note that a point $\bar{C}$ is within $\bar{V}\bar{K}\bar{M}$ if it is a convex combination of the vertices $\bar{V}$, $\bar{K}$ and $\bar{M}$.

Let '**d**' be a vector normal to KLM. Since both $\vec{\mathbf{d}}$ and $-\vec{\mathbf{d}}$ are normal to KLM, we require $\vec{\mathbf{d}}$ to be directed toward the sky from the surface of KLM, i.e. $\vec{\mathbf{d}}$ must have an acute angle with the vector $\boldsymbol{e}_3=(0\ 0\ 1)^T$. Let N be a point on KLM. We note that when angle θ between $\overrightarrow{\mathbf{V}\text{-}\mathbf{N}}$ and $\vec{\mathbf{d}}$ is less than or equal to 90 degrees, V is able to see the upper side of KLM (if no blocking object would exist) and thus a projection analysis on KLM becomes meaningful (Figure 5). Note also that any point on KLM can be chosen as N. We formalize this discussion with the following lemma.

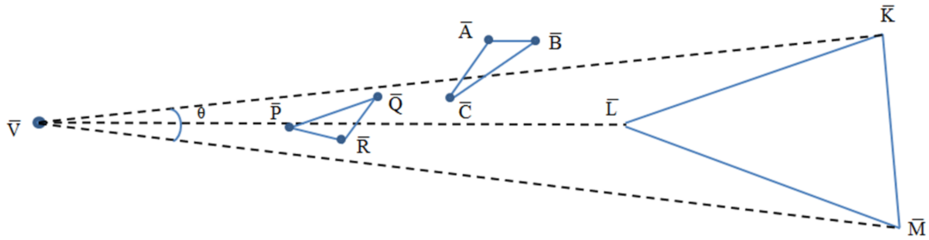


Figure 4. PQR and ABC are BTCs.

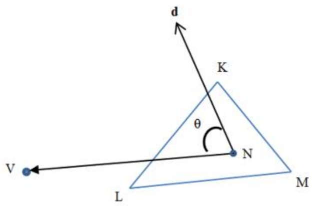


Figure 5. V is able to see point 'N' since angle θ is less than or equal to 90 degrees.

Lemma 4.1. Let V be a viewpoint and KLM be a triangle on T. Suppose, without loss of generality, that KL is the first edge (possibly one of at most two edges) that intersects a ray shot from V toward KLM. If $\frac{(\boldsymbol{V}-\boldsymbol{K})^T\boldsymbol{d}}{\|\mathbf{V}-\mathbf{K}\|\|\mathbf{d}\|} < 0$, then all points on KLM except (possibly) those on KL are invisible to V.

4.2. Calculation of the Invisible Region on a TT caused by a BTC

Algorithm by De Floriani and Magillo (1997), which is also discussed in De Floriani et al. (1994), projects the vertices of the horizon onto the supporting plane of the target triangle and then connect the projections with an edge to obtain the visible region (see Figure 6). Black region in figure 6 is assumed to be invisible and the region above the projected horizon is visible. However as shown in figure 7, the ray passing through V and vertex 'a' hits the supporting plane of the triangle at 'b', which is not between V and the target triangle. Therefore, the visibility information obtained using such an approach is wrong. Below, we give an alternative correct model in which we project a BTC directly onto the target triangle itself.

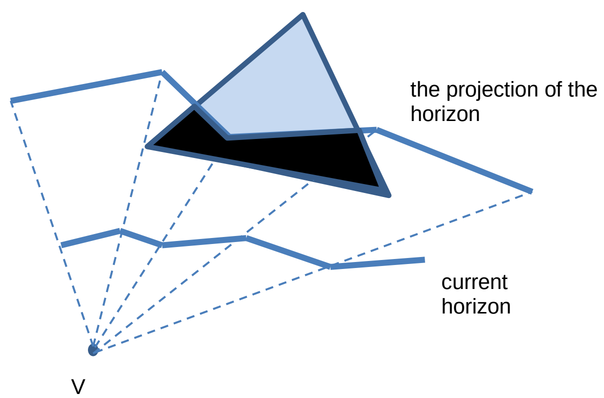


Figure 6. Projection of the vertices of the horizon, which is not correct for all cases.

Let PQR be a BTC and KLM be the TT. We assume that $\frac{(\boldsymbol{V}-\boldsymbol{K})^{\mathbf{T}}\boldsymbol{d}}{\|\mathbf{V}-\mathbf{K}\|\|\mathbf{d}\|} \geq 0$, where $\boldsymbol{d}$ is as defined in section 4.1. It is true that if PQR blocks the visibility of KLM from V then PQR has a projection (casts a shadow) on KLM. PQR has a projection on KLM if and only if there exist points on PQR that satisfy the set of equations given by (1) and (2) (figure 8). We note that the invisible region on KLM is given by $\alpha_1\mathbf{K}+\alpha_2\mathbf{L}+\alpha_3\mathbf{M}$ with $\sum_{i=1}^{3}\alpha_i=1$, and $\alpha_i\geq 0$, i=1,2,3.

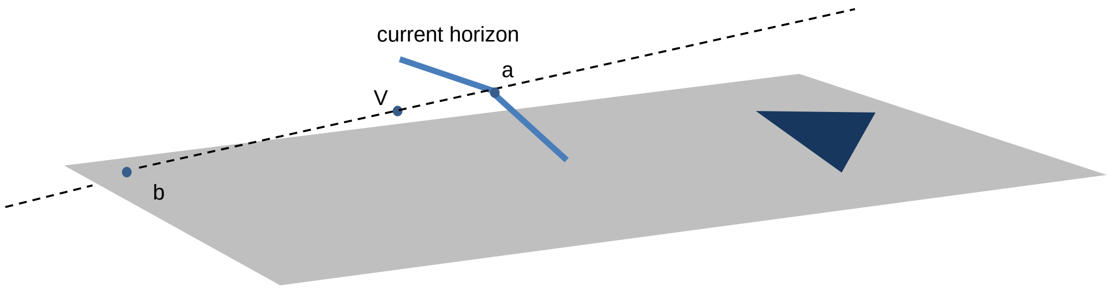


Figure 7. Projection of a point may lie behind V.

$$\text{For } \mu\geq 1,\ \mathbf{V}+\mu(\lambda_1\mathbf{P}+\lambda_2\mathbf{Q}+\lambda_3\mathbf{R}-\mathbf{V})=(\alpha_1\mathbf{K}+\alpha_2\mathbf{L}+\alpha_3\mathbf{M}) \quad (1)$$

$$\sum_{i=1}^{3}\lambda_i=1,\ \ \sum_{i=1}^{3}\alpha_i=1,\quad \lambda_i, \alpha_i\geq 0,\ i=1,2,3. \quad (2)$$

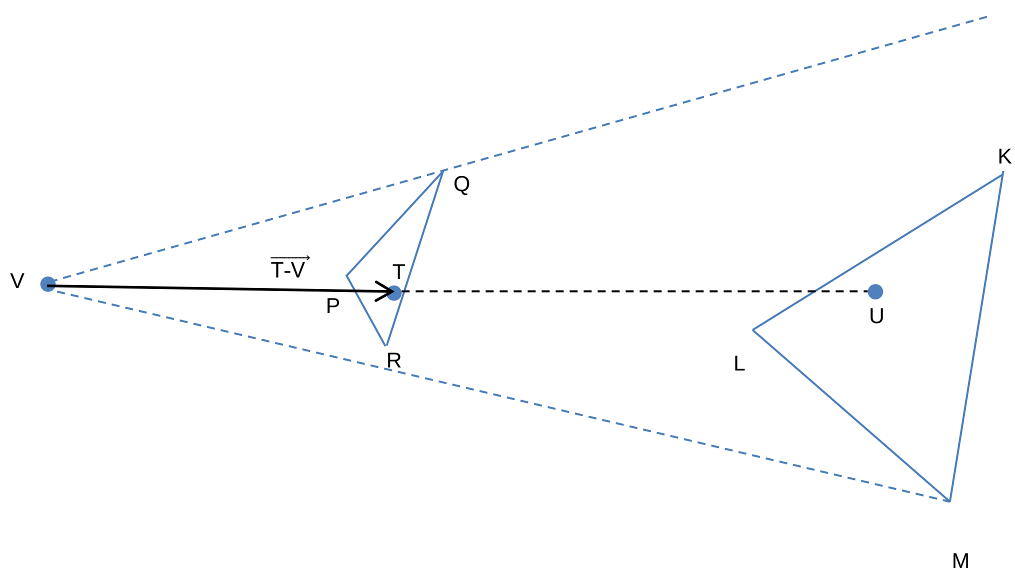


Figure 8. U is the projection of T on KLM. Note that T and U can be written as convex combinations of the vertices of PQR and KLM respectively. Also $\mathbf{U}=\mathbf{V}+\mu(\mathbf{T}-\mathbf{V})$, for some $\mu\geq 1$.

Or,

$$\mu\lambda_1\mathbf{P}+\mu\lambda_2\mathbf{Q}+\mu\lambda_3\mathbf{R}-\alpha_1\mathbf{K}-\alpha_2\mathbf{L}-\alpha_3\mathbf{M} = \mu\mathbf{V}-\mathbf{V} \qquad (3)$$

$$\lambda_1 + \lambda_2 + \lambda_3 = 1 \qquad (4)$$

$$\alpha_1 + \alpha_2 + \alpha_3 = 1 \qquad (5)$$

$$\lambda_1,\ \lambda_2,\ \lambda_3,\ \alpha_1,\ \alpha_2,\ \alpha_3 \geq 0,\quad \mu\geq 1 \qquad (6)$$

Note that equation (3) is nonlinear and we use the following change of variable to linearize the equation;

We define $\lambda_i' = \mu\lambda_i$, i=1,2,3. Then $\lambda_1+\lambda_2+\lambda_3= 1$ gives $1/\mu\ (\lambda_1'+\lambda_2'+\lambda_3')=1$ or $\lambda_1'+\lambda_2'+\lambda_3'= \mu$.

We now have;

$$\lambda_1'\mathbf{P}+\lambda_2'\mathbf{Q}+\lambda_3'\mathbf{R}-\alpha_1\mathbf{K}-\alpha_2\mathbf{L}-\alpha_3\mathbf{M} = \mu\mathbf{V}-\mathbf{V} \qquad (7)$$

$$\lambda_1' + \lambda_2' + \lambda_3' = \mu \qquad (8)$$

$$\alpha_1 + \alpha_2 + \alpha_3 = 1 \qquad (9)$$

$$\lambda_1',\ \lambda_2',\ \lambda_3',\ \alpha_1,\ \alpha_2,\ \alpha_3 \geq 0,\quad \mu\geq 1 \qquad (10)$$

Let $\mathbf{A_1}$ be the matrix [**P-V**, **Q-V**, **R-V**] and $\mathbf{A_2}$ be the matrix [**K**, **L**, **M**]. Then we obtain the following equivalent system;

$$\mathbf{A_1}\boldsymbol{\lambda}' - \mathbf{A_2}\boldsymbol{\alpha} = -\mathbf{V} \qquad (11)$$

$$\boldsymbol{e}^{\boldsymbol{T}}\boldsymbol{\lambda}' \geq 1 \qquad (12)$$

$$\boldsymbol{e}^{\mathbf{T}}\boldsymbol{\alpha} = 1 \qquad (13)$$

$$\boldsymbol{\lambda}', \boldsymbol{\alpha} \geq \boldsymbol{0} \qquad (14)$$

where $\boldsymbol{\lambda}' = \begin{pmatrix} \lambda_1' \\ \lambda_2' \\ \lambda_3' \end{pmatrix}$, $\boldsymbol{\alpha} = \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix}$, and $\boldsymbol{e}^{\mathbf{T}} = (1\ 1\ 1)$.

Suppose that $\mathbf{A_1}$ is invertible. $\mathbf{A_1}$ is not invertible when V, P, Q, and R lie on the same plane and this implies that PQR does not block the visibility from V due to theorem 2.1. Premultiplying (11) by $\mathbf{A_1^{-1}}$ on both sides we obtain; $\boldsymbol{\lambda}' = \left(\mathbf{A_1^{-1}A_2}\right)\boldsymbol{\alpha} - \mathbf{A_1^{-1}V}$, and since we require $\boldsymbol{\lambda}' \geq \boldsymbol{0}$, we obtain the following equivalent system;

$$\left(\mathbf{A_1^{-1}A_2}\right)\ \boldsymbol{\alpha} \geq \mathbf{A_1^{-1}V} \qquad (15)$$

$$\boldsymbol{e}^{\mathbf{T}}\left(\mathbf{A_1^{-1}A_2}\right)\ \boldsymbol{\alpha} \geq 1 + \boldsymbol{e}^{\mathbf{T}}\mathbf{A_1^{-1}V} \qquad (16)$$

$$\boldsymbol{e}^{\mathbf{T}}\boldsymbol{\alpha} = 1 \qquad (17)$$

$$\boldsymbol{\alpha} \geq \boldsymbol{0} \qquad (18)$$

For each $\boldsymbol{\alpha}$ that solves (15)-(18), we put $\boldsymbol{\lambda}' = \left(\mathbf{A_1^{-1}A_2}\right)\boldsymbol{\alpha} - \mathbf{A_1^{-1}V}$, then compute μ from $\mu = \lambda_1' + \lambda_2' + \lambda_3'$, and $\boldsymbol{\lambda}$ from $\boldsymbol{\lambda} = \frac{1}{\mu}\ \boldsymbol{\lambda}'$. The values we get for $\boldsymbol{\alpha}$**,** $\boldsymbol{\lambda}$ and μ satisfy all requirements in the initial system. Note that the set of solutions to the system of equations (15)-(18) is a bounded polyhedron (a polytope), which we name $\boldsymbol{\Omega}$. The preceding analysis establishes the following theorem.

Theorem 4.1. Suppose $\mathbf{A_1}$ is invertible. Then PQR has a projection (blocks the visibility of) on

KLM if and only if **Ω** is nonempty.

For each $\boldsymbol{\alpha}$ in **Ω** there is a corresponding point on KLM, given by $\mathbf{A_2}\boldsymbol{\alpha}$. Let $\boldsymbol{\Pi}=\{\boldsymbol{y}: \boldsymbol{y}=\mathbf{A_2}\boldsymbol{\alpha}, \boldsymbol{\alpha} \in \Omega\}$, i.e. **Π** is the invisible region on KLM caused by PQR and is also a bounded polyhedron.

Lemma 4.2. Let $\boldsymbol{\alpha^*}$ be a point in **Ω** and $\boldsymbol{y^*}=\mathbf{A_2}\boldsymbol{\alpha^*}$. Then $\boldsymbol{\alpha^*}$ is the unique solution to the system of equations given by $\boldsymbol{y^*}=\mathbf{A_2}\boldsymbol{x}$, $\sum_{i=1}^{3} x_i = 1$, $\boldsymbol{x} \geq \boldsymbol{0}$ and $\boldsymbol{x} \in \boldsymbol{\Omega}$.

Proof. Let $\mathbf{D}=\begin{pmatrix}\vec{K} & \vec{L} & \vec{M} \\ 1 & 1 & 1\end{pmatrix}$, $\mathbf{E} = \begin{pmatrix}y^* \\ 1\end{pmatrix}$. **D** is a 4 by 3 matrix whose coloumns are the vertices of KLM with 1's at the 4th row. Similarly **E** is a 4x1 matrix with components of $\boldsymbol{y^*}$ and a 1 at the last row. We note that $\boldsymbol{\alpha^*}$ is a solution to $\mathbf{D}\boldsymbol{x}=\mathbf{E}$, $\boldsymbol{x} \geq \boldsymbol{0}$, $\boldsymbol{x} \in \mathbf{R}^3$. We will show that the columns of **D** are linearly independent and this will establish the uniqueness of $\boldsymbol{\alpha^*}$ by the result given in Bertsimas and Tsitsiklis (1997, theorem 2.2). Suppose the coloumns are linearly dependent and suppose without loss of generality that $\begin{pmatrix}\vec{M} \\ 1\end{pmatrix}$ is a linear combination of $\begin{pmatrix}\vec{K} \\ 1\end{pmatrix}$ and $\begin{pmatrix}\vec{L} \\ 1\end{pmatrix}$, i.e.

$\exists\, \xi, \beta \in$ R such that $\xi\begin{pmatrix}\vec{K} \\ 1\end{pmatrix} + \beta\begin{pmatrix}\vec{L} \\ 1\end{pmatrix} = \begin{pmatrix}\vec{M} \\ 1\end{pmatrix}$. Last row implies $\xi + \beta = 1$, which implies $\xi\mathbf{K} + (1-\xi)\mathbf{L}=\mathbf{M}$. However, last equality implies that M is on the line passing through K and L, contradicting the fact that KLM is a triangle. Hence $\boldsymbol{\alpha^*}$ is unique ■

Let $\boldsymbol{\alpha^*}, \boldsymbol{\alpha^1}, \boldsymbol{\alpha^2} \in \Omega$ and $\boldsymbol{y^*}=\mathbf{A_2}\boldsymbol{\alpha}^*$, $\boldsymbol{y^1}=\mathbf{A_2}\boldsymbol{\alpha^1}$, $\boldsymbol{y^2}=\mathbf{A_2}\boldsymbol{\alpha^2}$. Lemma 4.2 implies that if $\boldsymbol{\alpha^*} \neq \boldsymbol{\alpha^1} \neq \boldsymbol{\alpha^2}$ then $\boldsymbol{y^*} \neq \boldsymbol{y^1} \neq \boldsymbol{y^2}$. Since a nonempty and bounded polyhedra have at least one extreme point (Bertsimas and Tsitsiklis 1997, corollary 2.2), we conclude that if **Ω** is nonempty then both **Ω** and **Π** have at least one extreme point.

Theorem 4.2. Suppose $\mathbf{A_1}$ is invertible and **Ω** is nonempty. Let $\boldsymbol{\alpha}^*$ be a point in **Ω** and $\boldsymbol{y^*}= \mathbf{A_2}\boldsymbol{\alpha}^*$. Then $\boldsymbol{y^*}$ is an extreme point of **Π** if and only if $\boldsymbol{\alpha}^*$ is an extreme point of **Ω**.

Proof (by contradiction). Suppose $\boldsymbol{\alpha}^*$ is an extreme point of $\mathbf{\Omega}$ but $\boldsymbol{y}^*$ is not an extreme point of $\mathbf{\Pi}$. Then there exists a $\lambda' \in (0,1)$, $\boldsymbol{y}^1$ and $\boldsymbol{y}^2 \in \mathbf{\Pi}$ such that $\boldsymbol{y}^1 \neq \boldsymbol{y}^2 \neq \boldsymbol{y}^*$ and $\boldsymbol{y}^* = \lambda'\boldsymbol{y}^1 + (1-\lambda')\boldsymbol{y}^2$. Since $\boldsymbol{y}^1$ and $\boldsymbol{y}^2 \in \mathbf{\Pi}$, there exist $\boldsymbol{\alpha}^1$ and $\boldsymbol{\alpha}^2 \in \mathbf{\Omega}$ such that $\boldsymbol{y}^1 = \mathbf{A}_2\boldsymbol{\alpha}^1$ and $\boldsymbol{y}^2 = \mathbf{A}_2\boldsymbol{\alpha}^2$. Note that $\boldsymbol{y}^1 \neq \boldsymbol{y}^2 \neq \boldsymbol{y}$ implies $\boldsymbol{\alpha}^1 \neq \boldsymbol{\alpha}^2 \neq \boldsymbol{\alpha}^*$. $\boldsymbol{y}^* = \lambda'\boldsymbol{y}^1 + (1-\lambda')\boldsymbol{y}^2$ implies $\mathbf{A}_2\boldsymbol{\alpha}^* = \lambda'\mathbf{A}_2\boldsymbol{\alpha}^1 + (1-\lambda')\mathbf{A}_2\boldsymbol{\alpha}^2 = \mathbf{A}_2(\lambda'\boldsymbol{\alpha}^1 + (1-\lambda')\boldsymbol{\alpha}^2)$, which implies, by Lemma 4.2., $\boldsymbol{\alpha}^* = \lambda'\boldsymbol{\alpha}^1 + (1-\lambda')\boldsymbol{\alpha}^2$. But this implies $\boldsymbol{\alpha}^*$ is not an extreme point of $\mathbf{\Omega}$, a contradiction. The other direction follows similarly ■

Theorem 4.2. shows that there is one-to-one correspondence between the extreme points of $\mathbf{\Omega}$ and $\mathbf{\Pi}$. A point in a polyhedron is an extreme point if and only if it is a basic feasible solution and basic feasible solutions can be found using the constraints that define the polyhedron (Bertsimas and Tsitsiklis 1997, Definition 2.9 and Theorem 2.3). The boundary of $\mathbf{\Pi}$ is given by edges connecting the neighboring basic feasible solutions since a bounded polyhedron is the convex hull of its extreme points (Bertsimas and Tsitsiklis 1997, Theorem 2.9). To summarize; to find the region on KLM blocked by PQR, we first find the extreme points of $\mathbf{\Omega}$, and then, using these points, find the extreme points (basic feasible solutions) of $\mathbf{\Pi}$, and finally connect the neighboring extreme points of $\mathbf{\Pi}$.

4.3. Finding the boundary of the invisible region on KLM

For a point V on T, let us denote the projection of (invisible region caused by) triangle ‘i’ on TT ‘j’ by $\Pi_i(j)$. Then the invisible region on j, IR(j), is the union of $\Pi_i(j)$ over all i, i.e. $IR(j)=\cup_i \Pi_i(j)$. Our goal is to find IR(KLM), and we proceed as follows;

First we form two sets: First set, denoted by TR, is the the set of projections on KLM of all triangles associated with blocking triangles, i.e., $TR=\{\Pi_1,.., \Pi_r\}$. Due to the analysis in section 4.4 we have the exact coordinates of the extreme points (ext.pts.) of Π_i, i=1,...,r. The elements of TR will not change during the execution of the algorithm. Second set, denoted by BP (set of boundary points), is composed of points that mark the boundary of current invisible region with edges existing between each consecutive point in BP. Initially, only the ext. pts. of the first triangle is in BP and the boundary of the initial invisible region (the projection of the triangle) is formed by the edges that connect the ext.pts. of the projection of the first triangle.

We find the union of the current invisible region and the projection of the next triangle in TR, and then update BP. Once all elements in TR is considered for analysis, we will have obtained IR(KLM) and its boundary. Let IR_k denote the invisible region obtained after projecting the k^{th} triangle on KLM. Then, $IR_k = IR_{(k-1)} \cup \Pi_k$. Note that $IR(KLM)=IR_r$. Let BP_k denote similarly the boundary points obtained after inserting the k^{th} projection. We note that the number of ext.pts. of a projection of a triangle on KLM can be at most 6. Suppose the k^{th} element (Π_k) in TR is considered for analysis. We study two mutually exclusive cases;

a. No edge of Π_k intersects, at its interior point, the edges of $IR_{(k-1)}$

Then it must be true that one of the following three cases exists; (1) $IR_{(k-1)} \subseteq \Pi_k$ (2) $\Pi_k \subseteq IR_{(k-1)}$ or (3) $IR_{(k-1)} \cap int(\Pi_k) = \emptyset$, where $int(\Pi_k)$ denotes the interior of Π_k (Figure 9). For each preceding case it is true that (1) $IR_k = \Pi_k$ (2) $IR_k = IR_{(k-1)}$ and (3) $IR_k = IR_{(k-1)} \cup \Pi_k$. Also, BP_k is composed of the ext.pts. of Π_k for case (1), $BP_k = BP_{(k-1)}$ for case (2) and BP_k is $BP_{(k-1)}$ and the ext.pts. of Π_k for case (3). We can determine which case is obtained after an insertion of Π_k as follows; if all boundary points of $IR_{(k-1)}$ lie within Π_k we are faced with case (1), if the ext.pts. of Π_k lie within $IR_{(k-1)}$ we obtain case (2) and if neither is true we obtain case (3). Note that this operation is done after ensuring that no edge intersection (at an interior point) exists.

b. *At least one edge of Π_k intersects at least one edge of $IR_{(k-1)}$ at its interior point.*

Suppose that an edge (c,d) of Π_k intersects an edge (e,f) of $IR_{(k-1)}$ at p and suppose, wlog, that p is an interior point of (c,d) (p ≠ c and p ≠ d), f is in Π_k, p ≠ f, and d is in $IR_{(k-1)}$ (Figure 10 (a)). We note that c and d are ext.pts. of Π_k. Then we remove f from BP_k, add c and p to BP_k. The boundary of IR_k becomes (e,p,c) for that part. In general, to find IR_k and its boundary, we remove all boundary points of $IR_{(k-1)}$ which remain in Π_k from BP_k, include all intersection points in BP_k, and remove from BP_k all ext.pts. of Π_k which remain in $IR_{(k-1)}$. Then, connecting the points in BP_k we obtain the boundary of IR_k (Figure 12 (b)). We observe that those points of IR(KLM) (the dashed gray boundary of IR(KLM) in Figure 12(b)) which belong to interior of KLM are visible due to Theorem 2.1.

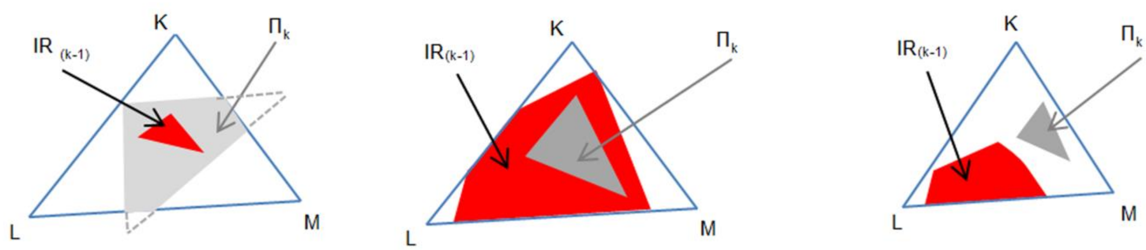


Figure 9. Three cases that may result when the edges a projection of a triangle do not intersect with edges of the current invisible region.

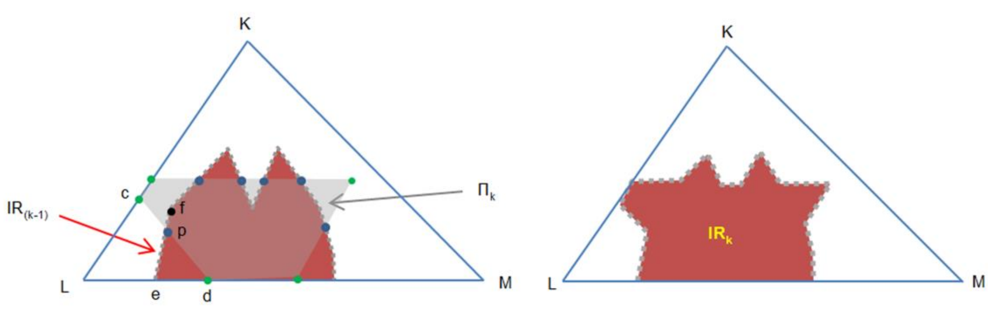


Figure 10. Finding the union of the current invisible region and a projection of a triangle.

4.4. Pseudocode of the algorithm viewshed and an application

1. Initialize: T (Triangulated Terrain), V (viewpoint), N (# of triangles), InvisibleRegion(V)=∅, *VS*(V) (The viewshed of V)

2. for j=1 to N do;

   calculate the vector '***d***' normal to triangle 'j' which has an acute angle with $\boldsymbol{e}^{\boldsymbol{T}}$=(0 0 1),

   check if $\vec{\boldsymbol{d}}$ normal to triangle 'j' (KLM) has an acute angle with $\overrightarrow{\boldsymbol{V}\text{-}\boldsymbol{K}}$.

        If no then do

            InvisibleRegion(V) = InvisibleRegion(V) ∪ j

        Else

            For i = 1 to N

                if i is a BTC

                    find the projection (from V) $\Pi_i(j)$ of 'i' on 'j'

                    Invisible region on 'j', IR(j) = $\bigcup_{i=1}^{N} \Pi_i(j)$

            End

        InvisibleRegion(V) = InvisibleRegion(V) ∪ IR(j)

   End

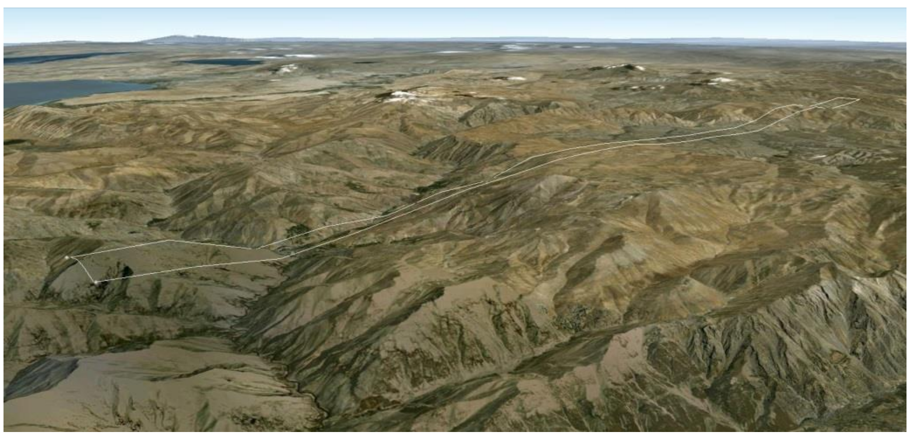
3. Output: *VS*(V) = T \ InvisibleRegion (V)

The outer for loop considers one triangle at a time and the inner for loop calculates the projections of other triangles on the triangle. Since there are O(n) projections for a single triangle the running time of the algorithm is $O(n^2)$.

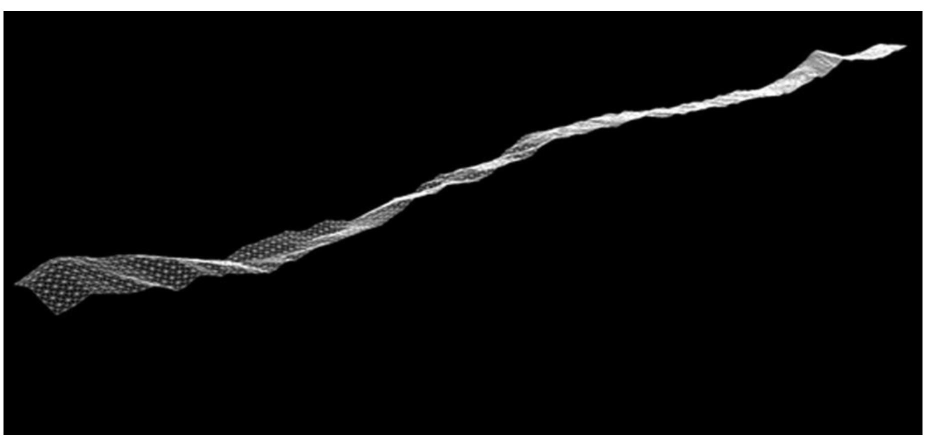
We apply our algorithm to a real terrain (Figure 13 (a)), which is 50 km in length and 1.6 km in width and is triangulated to obtain 6972 triangles (Figure 13 (b)). The algorithm was coded in JAVA and run on an Intel Core i7-2630QM CPU 2.00 GHz 8.00 GB RAM PC. Those portions of the terrain seen by V are painted green and the invisible regions are painted red (Figure 13 (c)).

5. Conclusions

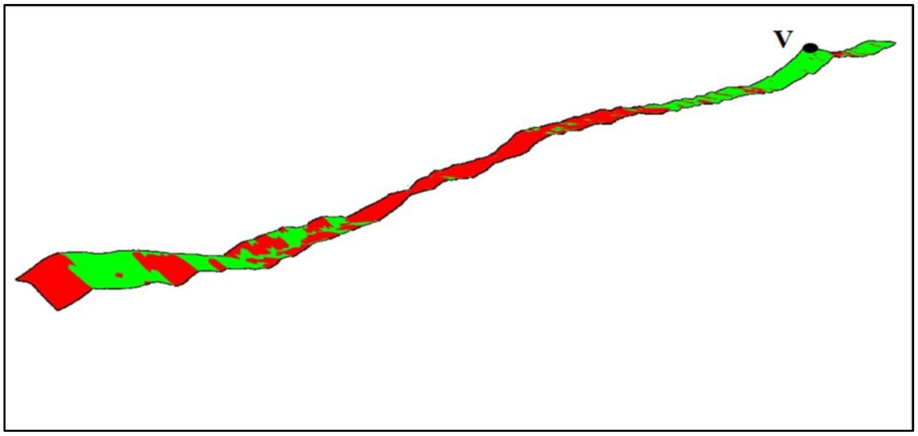
We have given a comprehensive theoretical analysis of the viewshed of a given point and proposed a polynomial-time algorithm, which is easy to implement and has an optimal worst-case complexity of $O(n^2)$, for finding the viewshed of the point. We conclude that, with the algorithm presented in this paper, optimization problems on terrains, such as those given in Lee (1991) and Kaučič and Žalık (2004), can be modelled exactly.



(a)



(b)



(c)

Figure 13. (a) The real terrain (50km x 1.6km) used for analysis. The picture is obtained from Google Earth® (b) TIN representation of the terrain (c) The viewshed of V, the regions painted green are visible and those painted red are invisible.

REFERENCES

Ben-Moshe B., Carmi P., and Katz, M.J., 2008. Approximating the visible region of a point on a terrain, *Geoinformatica*, 12, 21-36.

Bertsimas, D. and Tsitsiklis, J.N., 1997. *Introduction to linear optimization*. Belmont, MA: Athena Scientific.

Bittner, J. and Wonka P., 2003. *Visibility in computer graphics, Environment and Planning B :Planning and Design*, 30, 729-755.

Cole, R. and Sharir, M., 1989. Visibility problems for polyhedral terrains, *Journal of Symbolic Computation*, 7, 11-30.

De Berg, M. and Gray, C., 2010. Computing the visibility map of fat objects, *Computational Geometry: Theory and Applications*, 43, 410–418.

De Berg, M. *et al.*, 1997. *Computational geometry, algorithms and applications*. Berlin: Springer-Verlag.

De Floriani, L. and Magillo,P., 1994. Visibility algorithms on triangulated digital terrain models, *International Journal of Geographical Information Systems*, 8, 13-41.

De Floriani, L., Montani, C., and Scopigno, R., 1994. Parallelizing visibility computations on triangulated terrains, *International Journal of Geographical Information Systems*, 8, 515-532.

De Floriani, L. and Magillo,P., 1997. Visibility computations on hierarchial triangulated terrain models, *Geoinformatica*, 1, 219-250.

De Floriani, L. and Magillo,P., 2003. Algorithms for visibility computation on terrains: a survey, *Environment and Planning B: Planning and Design*, 30, 709-728.

Dercole, F., 2003. Remarks on the computation of the horizon of a digital terrain, *Applied Mathematics and Computation*, 146, 627-641.

Edelsbrunner, H., Guibas, L.J. and Sharir, M., 1989. The upper envelope of piecewise linear functions: algorithms and applications, Discrete and Computational Geometry, 4, 311-336.

Goodchild, M.F. and Lee, J., 1989. Coverage problems and visibility regions on topographic surfaces, *Annals of Operations Research*, 18, 175-186.

Katz, M.J., Overmars, M.H., and Sharir, M. 1992. Efficient hidden surface removal for objects with small union size, *Computational Geometry: Theory and Applications*, 2, 223-234

Kaučič, B. and Žalık, B., 2004. *K*-guarding of polyhedral terrain, *International Journal of Geographical Information Science*, 18 (7), 709–718.

Lake, M.W., Woodman, P.E., and Mithen, S.J., 1998. Tailoring GIS software for archaeological applications: an example concerning viewshed analysis, *Journal of Archaeological Science*, 25, 27–38.

Lee, J., 1991. Analyses of visibility sites on topographic surfaces, *International Journal of Geographical Information Systems*, 5, 413-429.

Li, Z., Zu, Q., and Gold, C, 2005. *Digital terrain modelling: principles and methodology*, Washington, D.C.: CRC press.

Magoč, T., Kassin, A., and Romero, R., 2010. A line of sight algorithm using fuzzy measures. In: *2010 Annual meeting of the North American fuzzy information processing society (NAFIPS)*, 12–14 July 2010, Toronto, Canada. Piscataway, NJ: The Institute of Electrical and Electronics Engineers (IEEE).

Marcos, L., 2007. Modeling visibility through vegetation, *International Journal of Geographical Information Science*, 21 (7), 799–810.

Mc Kenna, M., 1987. Worst-Case Optimal Hidden-Surface Removal. *ACM Transactions on Graphics*, 6 (1), 19-28.

Nagy, G., 1994. Terrain visibility, *Computers & Graphics*, 6, 763-773.

Pompa-García, M., Solís-Moreno, R., Rodríguez-Téllez, E., Pinedo-Álvarez, A., Avila-Flores, D., Hernández-Díaz, C., and Velasco-Bautista, E., 2010. Viewshed analysis for improving the effectiveness of watchtowers, in the North of Mexico, *The Open Forest Science Journal*, 3, 17-22.

Ray, C.K., Franklin, W.R., and Mehta, S., 1992. Geometric algorithms for siting of air defense missile batteries, Technical report 2756, Rensselaer Polytechnic Institute, USA.

Reif, J.H. and Sen, S., 1995. An Efficient output-sensitive hidden-surface removal algorithm for polyhedral terrains, *Mathematical and Computer Modelling*, 21 (5), 89-104.

Riggs, P.D. and Dean, D.J., 2007. An investigation into the causes of errors and inconsistencies in predicted viewsheds, *Transactions in GIS*, 11 (2), 175-196.

Stewart, J.A., 1998. Fast horizon computation at all points of a terrain with visibility and shading applications, *IEEE Transactions On Visualization and Computer Graphics*, 4(1), 82-93.