ON THE IRREDUCIBILITY, LYAPUNOV RANK, AND AUTOMORPHISMS OF SPECIAL BISHOP-PHELPS CONES

M. SEETHARAMA GOWDA† AND D. TROTT‡

Abstract. Motivated by optimization considerations, we consider cones in $\mathbb{R}^n$ – to be called special Bishop-Phelps cones – of the form $\{(t, x): t \geq \|x\|\}$, where $\|\cdot\|$ is a norm on $\mathbb{R}^{n-1}$. We show that when $n \geq 3$, such cones are always irreducible. Defining the Lyapunov rank of a proper cone $K$ as the dimension of the Lie algebra of the automorphism group of $K$, we show that the Lyapunov rank of any special Bishop-Phelps polyhedral cone is one. Extending an earlier known result for the $l_1$-cone (which is a special Bishop-Phelps cone with 1-norm), we show that any $l_p$-cone, for $1 \leq p \leq \infty$, $p \neq 2$, has Lyapunov rank one. We also study automorphisms of special Bishop-Phelps cones, in particular giving a complete description of the automorphisms of the $l_1$-cone.

Key words. Complementarity set, Lyapunov rank, Bishop-Phelps cone, Irreducible cone

1. Introduction. For a proper cone $K$ in $\mathbb{R}^n$ with dual $K^*$, the complementarity set of $K$ is

$$C(K) := \{(x, s): x \in K, s \in K^*, \langle x, s \rangle = 0\}.$$ (1.1)

Such a set appears, for example, in complementarity problems [3], [13] and in primal and dual linear programming problems over a cone [12]. In various strategies for solving such problems, one tries to rewrite the complementarity/optimality conditions by replacing the complementarity constraints $x \in K, s \in K^*, \langle x, s \rangle = 0$ by linearly independent 'bilinear' relations. To elaborate, consider a complementarity problem corresponding to $K$ and a function $f: \mathbb{R}^n \to \mathbb{R}^n$, which is to find $x \in \mathbb{R}^n$ such that

$$x \in K, s = f(x) \in K^* \quad \text{and} \quad \langle x, s \rangle = 0.$$ 

Here, for the $2n$ variables $x \in K$ and $s \in K^*$, there are $n + 1$ equality relations, namely, $s = f(x)$ and $\langle x, s \rangle = 0$. So, to make this a square system, it is desirable to replace the single bilinear relation $\langle x, s \rangle = 0$ by an equivalent system of $n$ independent bilinear relations. This is clearly the case when $K = \mathbb{R}^n_+$ (the non-negative orthant in $\mathbb{R}^n$); here, the complementarity constraints are equivalently expressed as $x \geq 0, s \geq 0, x_i s_i = 0$ for $i = 1, 2, \ldots, n$. Motivated by this, to measure the number

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†Department of Mathematics and Statistics, University of Maryland Baltimore County, Baltimore, Maryland 21250, USA; E-mail: gowda@math.umbc.edu, URL: http://www.math.umbc.edu/~gowda
‡Department of Mathematics and Statistics, University of Maryland Baltimore County, Baltimore, Maryland 21250, USA; dtrott1@umbc.edu
of independent bilinear relations, Rudolf et al. [16], introduced the following: For a proper cone $K$ in $\mathbb{R}^n$, an $n \times n$ real matrix $Q$ is a bilinear complementarity relation if

$$(x, s) \in C(K) \Rightarrow \langle Q^T x, s \rangle = 0$$

and the bilinearity rank of $K$ is

$$\beta(K) := \dim Q(K),$$

where $Q(K)$ is the vector space of all bilinear complementarity relations on $K$. While cones with $\beta(K) = n$ lead to square systems and are desirable, for cones with $\beta(K) > n$, one gets an overdetermined system of bilinear relations. In many of these overdetermined systems, such as symmetric cones [4], one can still get a square system of bilinear relations [10]. In cones with $\beta(K) < n$, the complementarity system can never be written as a square system by means of bilinear complementarity relations alone and this may indicate or cause difficulty in reformulation and solvability of the problem. In [16], Rudolf et al., initiate the study of bilinearity rank and show that isomorphic cones have the same bilinearity rank, a proper cone and its dual have the same rank, and that the rank is additive on a Cartesian product. They also compute the bilinearity rank of certain cones.

A Lyapunov-like matrix/transformation on a proper cone $K$ satisfies the condition

$$(x, s) \in C(K) \Rightarrow \langle Qx, s \rangle = 0$$

and is thus the transpose of a bilinear complementarity relation. Lyapunov-like transformations were introduced in [8] as a generalization of the Lyapunov transformation $X \mapsto AX + XA^T$ that appears in linear dynamical systems theory. These are related to $Z$-matrices and have been the subject matter of several recent studies, see [8], [9], and [11]. As a consequence of a result in [17],

$A$ is Lyapunov-like on $K$ if and only if $e^{tA} \in Aut(K)$ for all $t \in \mathbb{R},$

where $Aut(K)$ denotes the automorphism group of $K$. Hence, Lyapunov-like transformations on $K$ are nothing but the elements of $\text{Lie}(Aut(K))$, the Lie algebra of the automorphism group of the cone $K$ [1]; thus, one may redefine the bilinearity rank of $K$ as

$$\beta(K) = \dim \text{Lie}(Aut(K)),$$

and (henceforth) call $\beta(K)$, the Lyapunov rank of $K$.

Gowda and Tao [10], following the work of [16], established several new results on the Lyapunov rank, and in particular, described the Lyapunov rank of an arbitrary symmetric cone. It was observed in [16] (see also [10], Example (1)), that the
Lyapunov rank of the $l_1$-cone in $\mathbb{R}^n$ is one, where the $l_1$-cone is defined by

$$l^n_{1,+} := \{(t, x) : t \geq ||x||_1\},$$

with $||x||_1$ denoting the 1-norm of the vector $x$ in $\mathbb{R}^{n-1}$. Since the Lyapunov rank is additive on a Cartesian product/sum, it follows that the $l_1$-cone is irreducible; see [7], Corollary 4.2.5 for an alternate proof. If the 1-norm is replaced by the 2-norm, the resulting $l_2$-cone

$$l^n_{2,+} = \{(t, x) : t \geq ||x||_2\}$$

is the so-called second-order cone (or the Lorentz cone or the ice-cream cone) in $\mathbb{R}^n$. This cone is irreducible and its Lyapunov rank is $n^2 - n + 2$, see [10], Section 5.

Motivated by the above results, we consider cones in $\mathbb{R}^n$ of the form

$$K = \{(t, x) : t \geq ||x||\},$$

where $|| \cdot ||$ is a norm on $\mathbb{R}^{n-1}$, $n > 1$. We will call these special Bishop-Phelps cones (abbreviated as special BP cones) as they are particular instances of the so-called Bishop-Phelps cones [5] given by

$$\{z \in \mathbb{R}^n : ||z|| \leq \phi(z)\},$$

where $|| \cdot ||$ is a norm on $\mathbb{R}^n$ and $\phi$ is a continuous linear functional on $\mathbb{R}^n$.

The above results on $l_1$ and $l_2$ cones motivate a number of interesting questions:

- Is every special BP cone irreducible?
- What is the Lyapunov rank of such a cone? What if this cone is polyhedral? What if the norm is the $p$-norm?
- Can one describe the automorphism group of such a cone?

Answering these, in this paper, we prove the following results for $n \geq 3$:

(i) Every special BP cone is irreducible.
(ii) Every polyhedral special BP cone has Lyapunov rank one.
(iii) The Lyapunov rank of the $l_p$-cone, for $1 \leq p \leq \infty$, $p \neq 2$, is one.
(iv) Every automorphism of the $l_1$-cone on $\mathbb{R}^n$ is of the form

$$\theta \begin{bmatrix} 1 & 0 \\ 0 & D \end{bmatrix},$$

where $\theta > 0$ and $D$ is a generalized permutation matrix (that is, it is a product of a permutation matrix and a diagonal matrix with diagonal entries $\pm 1$).
We remark that the above results (i) and (ii) do not extend to arbitrary Bishop-Phelps cones as every closed and pointed cone in $\mathbb{R}^n$ (in particular, $\mathbb{R}_+^n$) is a Bishop-Phelps cone and conversely [14]. However, Bishop-Phelps cones with strictly convex norm and $\|\phi\| > 1$ are irreducible, see [7], Example 4.1. We also note that the above results fail for special BP cones when $n = 2$.

The organization of the paper is as follows. In Section 2, we cover some basic material. Section 3 deals with the irreducibility issue. In Section 4, we consider the Lyapunov ranks of polyhedral special BP cones and $l_p$-cones. Our final section deals with automorphisms of special BP cones.

2. Preliminaries. Throughout this paper, $\mathbb{R}^n$ denotes the Euclidean $n$-space where the vectors are written as row vectors or column vectors depending on the context. The usual inner product is written as $\langle x, y \rangle$ or as $x^T y$. The standard unit vectors in $\mathbb{R}^n$ are denoted by $e_1, e_2, \ldots, e_n$; thus, $e_i$ has one in the $i$th slot and zeros elsewhere.

For a set $K$ in $\mathbb{R}^n$, $\text{int}(K)$ and $\overline{K}$ denote, respectively, the interior and closure of $K$. The subspace generated by $K$ is denoted by $\text{span}(K)$. We let

\[
\text{cone}(K) = \{ \lambda x : \lambda \geq 0, x \in K \}.
\]

The dual of $K$ is given by

\[
K^* := \{ y \in \mathbb{R}^n : \langle y, x \rangle \geq 0 \ \forall x \in K \}.
\]

A nonempty set $K$ is a cone if $K = \text{cone}(K)$. A closed convex cone $K$ in $\mathbb{R}^n$ is said to be, see [2],

(a) pointed if $K \cap -K = \{0\}$;

(b) proper if $K$ is pointed and has nonempty interior.

For a closed convex set $S$, a vector $x$ in $S$ is an extreme vector if $x = ty + (1-t)z$ with $0 < t < 1, y, z \in S$ holds only when $y = z = x$; we denote the set of all extreme vectors of $S$ by $\text{ext}(S)$. Note that when $S$ is also compact, by the (finite dimensional) Krein-Milman theorem, see Theorems 3.21 and 3.25 in [15], $S$ is the convex hull of $\text{ext}(S)$:

\[
S = \text{conv} (\text{ext}(S)).
\]

For a convex cone $K$, we say that a nonzero vector $x$ in $K$ is an extreme direction if the equality $x = y + z$ with $y, z \in K$ holds only when $y$ and $z$ are nonnegative multiples of $x$.

Given any norm $\| \cdot \|$ on $\mathbb{R}^{n-1}, n > 1$, consider the cone in (1.2). That this is a special case of a Bishop-Phelps cone (1.3) is seen by defining, on $\mathbb{R}^n$, the norm
\[(t, x) \mapsto |t| + |x|\] and the continuous linear functional \( \phi : (t, x) \mapsto 2t \). Bishop-Phelps cones are always closed and pointed, and proper when \( ||\phi|| > 1 \) (see Proposition 2.2 and Theorem 2.5 in [5]). Thus, \textit{any cone of the form (1.2) is proper}. If \( S \) denotes the closed unit ball in \( \mathbb{R}^{n-1} \) with respect to a norm \( || \cdot || \), we see that the cone \( K \) in (1.2) is also given by

\[ K = \text{cone} \left( \{1\} \times S \right) \]

and, as a consequence, every extreme direction of \( K \) is a positive multiple of \((1, x)\) for some \( x \in \text{ext}(S) \). In this setting, given \( x \in \text{ext}(S) \), we note that \(-x \in \text{ext}(S)\); We say that \((1, -x)\) is the \textit{conjugate} of \((1, x)\) and say that \((1, x)\) and \((1, -x)\) form a \textit{conjugate pair}. Corresponding to a norm \( || \cdot || \) on \( \mathbb{R}^{n-1} \), we define the \textit{dual norm} \( || \cdot ||_D \) on \( \mathbb{R}^{n-1} \) by

\[ ||x||_D = \max \{ (x, u) : ||u|| = 1 \}. \]

It is easily seen that the dual cone of \( K = \{(t, x) : t \geq ||x||\} \) is

\[ K^* = \{(t, x) : t \geq ||x||_D \}. \]

For \( 1 \leq p \leq \infty \) and \( x \in \mathbb{R}^{n-1} \), the \( p \)-norm is \( ||x||_p := \left( \sum_{i=1}^{n-1} |x_i|^p \right)^{1/p} \) when \( p < \infty \) and \( ||x||_\infty = \max |x_i| \). The dual norm of \( || \cdot ||_p \) is \( || \cdot ||_q \) where \( \frac{1}{p} + \frac{1}{q} = 1 \). We define the \( l^p \)-cone as

\[ l^p = \{(t, x) : t \geq ||x||_p \}. \]

\textbf{3. Irreducibility.} Given a closed convex cone \( K \) in \( \mathbb{R}^n \), we say that it is \textit{reducible} if there exist nonempty sets \( K_1 \neq \{0\} \) and \( K_2 \neq \{0\} \) such that

\[ K = K_1 + K_2, \quad \text{span}(K_1) \cap \text{span}(K_2) = \{0\}. \]

(As in [7], it can be shown that \( K_1 \) and \( K_2 \) are then closed convex cones in \( \mathbb{R}^n \).) In this case, we say that \( K \) is a \textit{direct sum} of \( K_1 \) and \( K_2 \). A closed convex cone that is not reducible is said to be \textit{irreducible}.

\textbf{Theorem 3.1.} In \( \mathbb{R}^n \), for \( n \geq 3 \), every special BP cone is proper and irreducible.

\textbf{Proof.} The properness of \( K \) has already been noted. Let \( S \) denote the closed unit ball in \( (\mathbb{R}^{n-1}, || \cdot ||) \) so that \( K = \text{cone} \left( \{1\} \times S \right) \). As all norms are equivalent on \( \mathbb{R}^{n-1} \), we see that the compact convex set \( S \) has nonempty interior. Since \( \text{conv} (\text{ext}(S)) = S \), \( \text{ext}(S) \) must contain \( n-1 \) linearly independent vectors, say, \( z_1, z_2, \ldots, z_{n-1} \). Now let \( T : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1} \) be a matrix/linear transformation with \( T(z_i) = e_i \) for all \( i = 1, 2, \ldots, n-1 \), where (we recall that) \( e_1, e_2, \ldots, e_{n-1} \) are the standard unit vectors in \( \mathbb{R}^{n-1} \). Clearly, \( T \) is invertible. Define a new norm \( || \cdot ||_* \) on \( \mathbb{R}^{n-1} \) by

\[ ||x||_* = ||T^{-1}x|| \quad (x \in \mathbb{R}^{n-1}). \]
Then the closed unit ball corresponding to $|| \cdot ||_*$ is $S_* = T(S)$ and the corresponding norm induced cone is

$$\text{cone} \left( \{1\} \times S_* \right) = \begin{bmatrix} 1 & 0 \\ 0 & T \end{bmatrix} \left( \text{cone} \left( \{1\} \times S \right) \right).$$

Note that the cones induced by $|| \cdot ||$ and $|| \cdot ||_*$ are isomorphic and irreducibility of one implies that of the other. So, we may assume without loss of generality that the closed unit ball $S$ of the given norm $|| \cdot ||$ on $\mathbb{R}^{n-1}$ contains $e_1, e_2, \ldots, e_{n-1}$ as extreme vectors and, as $x \in \text{ext}(S) \Rightarrow -x \in \text{ext}(S)$, write

$$(3.1) \quad E := \{\pm e_1, \pm e_2, \ldots, \pm e_{n-1}\} \subseteq \text{ext}(S).$$

Now suppose, if possible, that $K$ is reducible: $K = K_1 + K_2$, where $K_1$ and $K_2$ are closed convex cones with $K_1 \neq \{0\}, K_2 \neq \{0\}, \text{span}(K_1) \cap \text{span}(K_2) = \{0\}$. Define, for $i = 1, 2$,

$$S_i = \{x \in S : (1, x) \in K_i\}.$$

Clearly, these sets are compact, convex, disjoint, and $S_1 \cup S_2 \subseteq S$. We claim that

$$(3.2) \quad \text{ext}(S) \subseteq \text{ext}(S_1) \cup \text{ext}(S_2).$$

To see this, let $x \in \text{ext}(S)$ so that $||x|| = 1$. Then $(1, x) \in K_1 + K_2$ and we may write

$$(1, x) = (\lambda_1, x_1) + (\lambda_2, x_2),$$

where $(\lambda_i, x_i) \in K_i$ for $i = 1, 2$. Then $\lambda_i \geq ||x_i||$ for $i = 1, 2, 1 = \lambda_1 + \lambda_2$, and $x = x_1 + x_2$. Now,

$$1 = ||x|| \leq ||x_1|| + ||x_2|| \leq \lambda_1 + \lambda_2 = 1$$

implies that $||x_i|| = \lambda_i$ for $i = 1, 2$. If one $\lambda_i$ is zero, say $\lambda_1 = 0$, then $x_1 = 0$ and so $(1, x) = (1, x_2) \in K_2, x \in S_2$. As $x \in \text{ext}(S)$ and $S_1 \cup S_2 \subseteq S$, we must have $x \in \text{ext}(S_2)$. If both $\lambda_1$ and $\lambda_2$ are nonzero (that is, positive), then the equality

$$x = (\frac{x_1}{\lambda_1})\lambda_1 + (\frac{x_2}{\lambda_2})\lambda_2$$

says that $x$ is a convex combination of two unit vectors. Since $x \in \text{ext}(S)$, we must have $x = \frac{x_1}{\lambda_1} = \frac{x_2}{\lambda_2}$ which further implies that for $i = 1, 2$,

$$(1, x) = (1, \frac{x_i}{\lambda_i}) = \frac{1}{\lambda_i} (\lambda_i, x_i) \in K_i.$$

Clearly this cannot happen as $\text{span}(K_1) \cap \text{span}(K_2) = \{0\}$. We thus have our claim. Recalling the definition of $E$ from (3.1) let, for $i = 1, 2$, $E_i := E \cap S_i$. We claim that
$E_1$ and $E_2$ are nonempty. To see this, assume the contrary and suppose (without loss of generality) $E_2 = \emptyset$ so that, by (3.2), $E \subseteq S_1$. Then, $\{(1, \pm e_i) : i = 1, 2, \ldots, n-1\} \subseteq K_1$. As the set $\{(1, e_1), (1, e_2), \ldots, (1, e_{n-1}), (1, -e_1)\}$ forms a basis of $\mathbb{R}^n$, we see that

$$\mathbb{R}^n = \text{span}(\{(1, \pm e_i) : i = 1, 2, \ldots, n-1\}) \subseteq \text{span}(K_1).$$

This means that $\text{span}(K_2) = \{0\}$, leading to a contradiction.

Thus, $E_1$ and $E_2$ are nonempty and $E = E_1 \cup E_2$. Let

$$E_1 = \{u_1, u_2, \ldots, u_k\} \quad \text{and} \quad E_2 = \{v_1, v_2, \ldots, v_l\}$$

so that $k + l = 2(n - 1)$.

Let $C_1 := \{(1, u_1), (1, u_2), \ldots, (1, u_k)\}$ and $C_2 = \{(1, v_1), (1, v_2), \ldots, (1, v_l)\};$ we note that $C_i \subset K_i$ so that

$$\text{span}(C_1) \cap \text{span}(C_2) = \{0\}.$$

Now for any given element $(1, x)$ in $\{1\} \times E$, we recall that $(1, -x)$ is the conjugate of $(1, x)$ and $(1, x)$ and $(1, -x)$ form a conjugate pair. As every element of $E$ is of the form $\pm e_i$ for some $i$, the conjugate of any element in $C_1$ (likewise $C_2$) is either in $C_1$ or in $C_2$. We now consider the following cases:

1. Both $C_1$ and $C_2$ contain some conjugate pairs.
2. Both $C_1$ and $C_2$ are without conjugate pairs.
3. Only $C_1$ (say) contains conjugate pairs.

We show that each case leads to a contradiction.

**Case 1:** Suppose that $(1, e_i), (1, -e_i) \in C_1$ and $(1, e_j), (1, -e_j) \in C_2$ for some $i \neq j$. In this case, $(1, e_i) + (1, -e_i) = (2, 0) = (1, e_j) + (1, -e_j) \in \text{span}(C_1) \cap \text{span}(C_2) = \{0\}$ which is not possible.

**Case 2:** In this case, the conjugate of any element of $C_1$ (of $C_2$) is found in $C_2$ (respectively, in $C_1$). This sets up a one-to-one correspondence between elements of $C_1$ and $C_2$ showing that the cardinalities of $C_1$ and $C_2$ are equal, that is, $k = l$. Since these cardinalities add up to $2(n - 1)$, we must have $k = l = n - 1$. As there are no conjugate pairs in $C_1$ and in $C_2$, both $C_1$ and $C_2$ are linearly independent sets in $\mathbb{R}^n$.

Thus, $\dim(\text{span}(C_i)) = n - 1$ for $i = 1, 2$. Since $\text{span}(C_1) \cap \text{span}(C_2) = \{0\}$, we must have $n \geq (n - 1) + (n - 1)$, that is, $n \leq 2$. This cannot happen, as we have assumed that $n \geq 3$.

**Case 3:** In this case, we write $C_1$ and $C_2$ in terms of distinct elements:

$C_1 = \{(1, w_1), \ldots, (1, w_m), (1, -w_1), \ldots, (1, -w_m), (1, z_1), \ldots, (1, z_r)\}$ and $C_2 = \{(1, -z_1), \ldots, (1, -z_r)\}$. (Note that $(1, z_1), \ldots, (1, z_r)$ are elements in $C_1$ whose conjugates are not in $C_1$ but in $C_2$.) It follows that $r = l$ and $k = 2m + r = 2m + l$. Since $k + l = 2(n - 1)$, we must have $m + l = n - 1$ or $m + l + 1 = n$. Since the subset
\{(1, w_1), \ldots, (1, w_m), (1, -w_1), (1, z_1), \ldots, (1, z_r)\} \text{ of } C_1 \text{ is linearly independent and its cardinality is } n, \ \text{span}(C_1) = \mathbb{R}^n. \text{ This leads to } K_2 = \{0\} \text{ and to a contradiction.}

We have thus proved that the reducibility of } K \text{ leads to a contradiction. Hence the theorem.}

\textbf{Remark (1).} The following examples show that for general BP cones or for special BP cones with } n = 2, \text{ the above theorem may not hold.}

For } n \geq 2, \text{ consider the BP cone}

\{x \in \mathbb{R}^n : ||x||_1 \leq \phi(x)\},

where ||x||_1 \text{ is the 1-norm of } x \text{ and } \phi(x) = \langle x, e \rangle, \text{ with } e \text{ denoting the vector of ones.}

This cone, being } \mathbb{R}_+^n, \text{ is reducible.

For } n = 2, \text{ consider the special BP cone}

K = \{(t, x) : t \geq |x|\}.

This is isomorphic to the nonnegative orthant in } \mathbb{R}^2 \text{ and hence reducible.

4. The Lyapunov rank. Recall that given a proper cone } K \text{ in } \mathbb{R}^n, \text{ the Lyapunov rank of } K \text{ is the dimension of the space of all Lyapunov-like matrices on } K.

It has been shown in [10], Theorem 3, that the Lyapunov rank of a polyhedral cone in } \mathbb{R}^n \text{ can be any natural number } m \text{ with } 1 \leq m \leq n, m \neq n - 1. \text{ In particular, the Lyapunov rank of the nonnegative orthant in } \mathbb{R}^n \text{ is } n. \text{ In this section, we consider cones of the form (1.2).

**Theorem 4.1.** In } \mathbb{R}^n, \text{ for } n \geq 3, \text{ every polyhedral special BP cone has Lyapunov rank one.

The result follows immediately from Theorem 3.1 (of the previous section) and Corollary 5 of [10] that says that for any polyhedral proper cone, the Lyapunov rank is one if and only if it is irreducible. Below, we offer a direct and elementary proof.

**Proof.** Let } n \geq 3 \text{ and } K \text{ given by (1.2) be polyhedral. We show that every Lyapunov-like matrix on } K \text{ is a multiple of the identity matrix, thus proving the result. As done in the proof of Theorem 3.1, we may assume that } \pm e_i, \ i = 1, 2, \ldots, n - 1 \text{ are extreme vectors of the closed unit ball of } \mathbb{R}^{n-1} \text{ under the given norm. Then } (1, \pm e_i), \ i = 1, 2, \ldots, n - 1, \text{ are extreme directions of } K. \text{ Assuming that vectors in } \mathbb{R}^n \text{ are now written as column vectors, consider a Lyapunov-like matrix given by}

\[ A = \begin{bmatrix} a & b^T \\ c & D \end{bmatrix}, \]

where } a \in \mathbb{R}, \ b, c \in \mathbb{R}^{n-1}, \text{ and } D \text{ is an } (n-1) \times (n-1) \text{ matrix. As } K \text{ is a polyhedral cone, by Theorem 2 in [10], every (column) vector } [1 \ e_i]^T, \ i = 1, 2, \ldots, n - 1, \text{ is an}
eigenvector of $A$. Thus, there exist real numbers $\lambda_i$ and $\mu_i$, $i = 1, 2, \ldots, n - 1$, such that
\[
A \begin{bmatrix} 1 \\ e_i \end{bmatrix} = \lambda_i \begin{bmatrix} 1 \\ e_i \end{bmatrix} \quad \text{and} \quad A \begin{bmatrix} 1 \\ -e_i \end{bmatrix} = \mu_i \begin{bmatrix} 1 \\ -e_i \end{bmatrix},
\]
for all $i = 1, 2, \ldots, n - 1$. From these, we get
\[
a + \langle b, e_i \rangle = \lambda_i, \ a - \langle b, e_i \rangle = \mu_i, \ c + De_i = \lambda_i e_i, \ c - De_i = -\mu_i e_i
\]
for all $i = 1, 2, \ldots, n - 1$. These lead to $a = \frac{\lambda_i + \mu_i}{2}, \ De_i = \frac{\lambda_i + \mu_i}{2} e_i = a e_i$, $2c = (\lambda_i - \mu_i)e_i$, and $2\langle b, e_i \rangle = \lambda_i - \mu_i$ for all $i = 1, 2, \ldots, n - 1$. As $n \geq 3$, the conditions $2c = (\lambda_i - \mu_i)e_i$ for all $i = 1, 2, \ldots, n - 1$ imply that $c = 0$ and $\lambda_i = \mu_i$ for all $i$. We see that $D = a I_{n-1}$, where $I_{n-1}$ is the identity matrix of size $n - 1$ and $b = 0$. From these we see that $A = a I_n$. Thus, multiples of identity are the only Lyapunov-like matrices on $K$. Hence the Lyapunov rank of $K$ is one.

**Proof.** The Minkowski functional of $S$ is a norm whose closed unit ball is $S$ [15]. The corresponding cone induced by this norm is $K$. Thus, $K$ is a polyhedral special BP cone and the result follows from the above theorem.

**Theorem 4.3.** Let $n \geq 3$. For any $p$ with $1 \leq p \leq \infty$, $p \neq 2$, the Lyapunov rank of $l^p_{p,+}$ is one.

**Proof.** For $p = 1, \infty$, the cone $l^p_{p,+}$ is polyhedral; hence the result follows from the previous theorem. We assume $1 < p < \infty$, $p \neq 2$, and define $q$ by $\frac{1}{p} + \frac{1}{q} = 1$. Consider a matrix
\[
A = \begin{bmatrix} a & b^T \\ c & D \end{bmatrix},
\]
which is Lyapunov-like on $l^p_{p,+}$, where $a \in \mathbb{R}$, $D$ is an $(n - 1) \times (n - 1)$ matrix, etc. Our goal is to show that $A = a I$. Now, for each $x \in \mathbb{R}^{n-1}$ with $||x||_p = 1$, define $s \in \mathbb{R}^{n-1}$ by
\[
s = \text{sgn}(x) \ast |x|^{\frac{p}{q}},
\]
whose $i$th component is $s_i = \text{sgn}(x_i) |x_i|^{\frac{p}{q}}$, where $\text{sgn}(\alpha)$ is $1, 0, -1$ according as whether the number $\alpha$ is positive, zero, or negative. Then, $||s||_q = 1$ and $\langle x, s \rangle = 1$. Now viewing vectors in $\mathbb{R}^n$ as column vectors, we see that $u = [1 x]^T \in l^p_{p,+}$, $v =
\[ [1 - s]^T \in l_{q,+}^n, \text{ and } \langle u, v \rangle = 0. \text{ Since } A \text{ is Lyapunov-like, we have } \langle Au, v \rangle = 0. \text{ This leads to } \]

\[ a + \langle b, x \rangle - \langle c, s \rangle - \langle Dx, s \rangle = 0. \]

Since this equation is valid if we replace \( x \) by \(-x\) and \( s \) by \(-s\), we must have \( \langle b, x \rangle - \langle c, s \rangle = 0 \) and \( \langle (D - aI)x, s \rangle = 0 \). We specialize \( x \) and \( s \) to show that \( b = c = 0 \) and \( D = aI \).

(i) By taking \( x = s = e_i, i = 1, 2, \ldots, n - 1 \), we see that \( b = c \) and that any diagonal element of \( D - aI \) is zero.

(ii) Recalling that \( n \geq 3 \), for any \( \varepsilon_i = \pm 1 \), we let \( x = (\frac{1}{n-1})^\frac{1}{n} \sum_{i=1}^{n-1} \varepsilon_i e_i \) and \( s = (\frac{1}{n-1})^\frac{1}{n} \sum_{i=1}^{n-1} \varepsilon_i e_i \). Then with \( b = c \) and \( p \neq q \), \( \langle b, x \rangle - \langle c, s \rangle = 0 \) leads to \( \sum_{i=1}^{n-1} b_i \varepsilon_i = 0 \). Since \( \varepsilon_i = \pm 1 \) are arbitrary, we deduce that \( b = 0 \).

(iii) For any \( t, 0 < t < 1 \), we let \( x_1 = t^{\frac{1}{n}} \), \( x_2 = (1-t)^{\frac{1}{n}} \), \( x_3 = x_4 = \cdots = x_{n-1} = 0 \), and \( s_1 = t^{\frac{1}{n}} \), \( s_2 = (1-t)^{\frac{1}{n}} \), \( s_3 = s_4 = \cdots = s_{n-1} = 0 \). Putting these in \( \langle (D - aI)x, s \rangle = 0 \) and simplifying, we deduce that the leading \( 2 \times 2 \) principal submatrix of \( D - aI \) is zero. By a similar argument, we show that any \( 2 \times 2 \) principal submatrix of \( D - aI \) is also zero. We conclude that \( D - aI = 0 \).

Thus we have proved that \( A = aI \). Hence, the Lyapunov rank of \( l_{p,+}^n \) is one.

**Remark (2).** For \( n = 2 \), consider the special BP cone \( K = \{(t, x) : t \geq |x|\} \). This, being isomorphic to the nonnegative orthant in \( \mathbb{R}^2 \), has Lyapunov rank \( 2 \).

5. **Automorphisms.** Given a proper cone \( K \) in \( \mathbb{R}^n \), we say that an \( n \times n \) matrix \( A \) is an automorphism of \( K \) and write \( A \in \text{Aut}(K) \) if \( A \) is nonsingular and \( A(K) = K \).

As noted in the Introduction, if \( A \) is Lyapunov-like on \( K \), then \( e^{tA} \in \text{Aut}(K) \) for all \( t \in \mathbb{R} \). When \( \beta(K) = 1 \), multiples of the identity matrix are the only Lyapunov-like matrices. Motivated by these, we raise the question of describing \( \text{Aut}(K) \), when \( K \) is a special BP cone. While this remains an open problem, we describe some special automorphisms that are induced by isometries of the given norm on \( \mathbb{R}^{n-1} \). As a special case, we completely describe the automorphisms of the \( l_1 \)-cone.

Given a norm on \( \mathbb{R}^{n-1} \), \( n > 1 \), with the corresponding closed unit ball \( S \), we consider the special BP cone \( K \) defined by (1.2). Relative to this \( K \), we say that an \( n \times n \) real matrix \( A \) is conjugate-pair-preserving if for any \( x \in \text{ext}(S) \) and \( \lambda > 0 \)

\[ A \begin{bmatrix} 1 \\ x \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ y \end{bmatrix} \Rightarrow A \begin{bmatrix} 1 \\ -x \end{bmatrix} = \mu \begin{bmatrix} 1 \\ -y \end{bmatrix}. \]

Recall that a matrix \( D \) on \( \mathbb{R}^{n-1} \) is an isometry of \( ||·|| \) if \( ||Dx|| = ||x|| \) for all \( x \in \mathbb{R}^{n-1} \).

**Theorem 5.1.** For \( n \geq 3 \), consider a special BP cone given by (1.2). Then for
any \( \theta > 0 \) and an isometry \( D \) of \( \| \cdot \| \), the matrix

\[
\theta \begin{bmatrix} 1 & 0 \\ 0 & D \end{bmatrix},
\]

is a conjugate-pair-preserving automorphism of \( K \). Conversely, every conjugate-pair-preserving automorphism of \( K \) arises this way.

**Proof.** The first part of the theorem is easily verified. For the second part, we take a conjugate-pair-preserving automorphism \( A \) of \( K \) and show that it is of the specified form. We write \( A \) in the form

\[
A = \begin{bmatrix} a & b^T \\ c & D \end{bmatrix},
\]

where \( a \in \mathbb{R}, D \) is an \((n-1) \times (n-1)\)-matrix, etc. Since the vector \( u = [1 \ 0]^T \) in \( \mathbb{R}^n \) is in the interior of \( K \), the first column of \( A \), namely \( Au \), is also in the interior of \( K \). This means that \( a > ||c|| \). Thus, by scaling \( A \) if necessary (which results in \( \theta = a \)), we may assume that

\[
A = \begin{bmatrix} 1 & b^T \\ c & D \end{bmatrix}.
\]

Our immediate goal is to show that \( c = 0 = b \).

Let \( u_i, \) \( i = 1, 2, \ldots, n - 1 \), be linearly independent vectors in \( \text{ext}(S) \), where \( S \) is the closed unit ball in \( \mathbb{R}^{n-1} \). As \( n \geq 3 \), we have at least two (different) vectors \( u_1 \) and \( u_2 \). Now, \( A \) is nonsingular and maps extreme directions of \( K \) to extreme directions of \( K \); so, we have

\[
(5.2) \quad A \begin{bmatrix} 1 \\ u_1 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ x \end{bmatrix}, \quad A \begin{bmatrix} 1 \\ u_2 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ w \end{bmatrix},
\]

where \( \lambda, \alpha > 0 \) and \( ||x|| = 1 = ||w|| \). Since \( A \) is conjugate-pair-preserving, we must have

\[
(5.3) \quad A \begin{bmatrix} 1 \\ -u_1 \end{bmatrix} = \mu \begin{bmatrix} 1 \\ -x \end{bmatrix}, \quad A \begin{bmatrix} 1 \\ -u_2 \end{bmatrix} = \beta \begin{bmatrix} 1 \\ -w \end{bmatrix},
\]

where \( \mu, \beta > 0 \). Expanding these we get

\[1 + \langle b, u_1 \rangle = \lambda, \quad c + Du_1 = \lambda x, \quad 1 - \langle b, u_1 \rangle = \mu, \quad c - Du_1 = -\mu x\]

with similar statements for \( u_2 \) in place of \( u_1 \). These yield

\[\lambda + \mu = 2, \quad 2c = (\lambda - \mu)x, \quad \alpha + \beta = 2, \quad 2c = (\alpha - \beta)w.\]
Now suppose, to get a contradiction, that \( c \neq 0 \). As \( ||x|| = 1 = ||w|| \), the equality \( (\lambda - \mu)x = (\alpha - \beta)w \) implies that \( |\lambda - \mu| = |\alpha - \beta| \) and \( x = \pm w \). From these and the equality \( \lambda + \mu = 2 = \alpha + \beta \), we get the following two cases:

(i) \( \lambda = \alpha, \mu = \beta, x = w \).

(ii) \( \lambda = \beta, \mu = \alpha, x = -w \).

From (5.2) and (5.3), along with the invertibility of \( A \), the first case leads to 
\[
\begin{bmatrix}
1 & u_1^T \\
0 & D
\end{bmatrix}
\] 
and the second case leads to 
\[
\begin{bmatrix}
1 & -u_2^T \\
0 & D
\end{bmatrix}
\] 
Clearly, these cannot happen. Hence \( c = 0 \). From \( 2c = (\lambda - \mu)x \), we get \( \lambda = \mu \) or \( x = 0 \). Now, \( x \neq 0 \) as the vector \([1 \ u_1]^T\), which is on the boundary of \( K \), cannot map to \( \lambda[1, 0]^T \), which is in the interior of \( K \). Thus, we must have \( \lambda = \mu \). But then,
\[
1 + \langle b, u_1 \rangle = \lambda, \quad 1 - \langle b, u_1 \rangle = \mu \Rightarrow \langle b, u_1 \rangle = 0.
\]
Likewise, \( \langle b, u_2 \rangle = 0 \). By similar considerations, we arrive at \( \langle b, u_i \rangle = 0 \) for all \( i = 1, 2, \ldots, n - 1 \), yielding \( b = 0 \). Thus,
\[
A = \begin{bmatrix}
1 & 0 \\
0 & D
\end{bmatrix}.
\]

We now claim that \( D \) is an isometry. Let \( u \) be any unit vector in \( \mathbb{R}^{n-1} \). Then, the vector \([1 \ u]^T\) is on the boundary of \( K \). Hence \( A[1 \ u]^T \) is a positive multiple of a vector of the form \([1 \ v]^T\), where \( ||v|| = 1 \). This leads to \( Du = v \) and to \( ||Du|| = ||v|| = 1 \). Thus, \( D \) is an isometry. This completes the proof.

In the result below, we say that a square matrix is a *generalized permutation matrix* if it is the product of a permutation matrix and a diagonal matrix with diagonal entries \( \pm 1 \).

**Theorem 5.2.** For \( n \geq 3 \), every matrix in \( Aut(l_{1,+}^n) \) is of the form
\[
\theta \begin{bmatrix}
1 & 0 \\
0 & D
\end{bmatrix},
\]
where \( \theta > 0 \) and \( D \) is a generalized permutation matrix.

**Proof.** It is clear that every matrix of the form (5.4) is an automorphism of the \( l_1 \)-cone. Now we prove the converse. Let \( A \in Aut(l_{1,+}^n) \). We first claim that \( A \) is conjugate-pair-preserving. If \( S \) denotes the closed unit ball of \( l_1 \)-norm on \( \mathbb{R}^{n-1} \), then \( \text{ext}(S) = \{ \pm e_i : i = 1, 2, \ldots, n - 1 \} \). As the \( l_1 \)-cone is cone \((\{1\} \times S)\), we note that the extreme directions of the \( l_1 \)-cone are given by
\[
\left\{ \begin{bmatrix}
1 \\
\pm e_i
\end{bmatrix} : i = 1, \ldots, n - 1 \right\}.
\]
Now, let $A \in \text{Aut}(l^n_{1,+})$. As in the proof of the previous theorem, we see that the $(1, 1)$ entry of $A$ is positive; thus, we can scale $A$ and assume without loss of generality that $A$ is in the form

$$A = \begin{bmatrix} 1 & b^T \\ c & D \end{bmatrix}.$$ 

Now, $A$ is nonsingular and maps extreme directions to extreme directions; so, we have

$$A \begin{bmatrix} 1 \\ e_1 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ x \end{bmatrix}, \quad A \begin{bmatrix} 1 \\ -e_1 \end{bmatrix} = \mu \begin{bmatrix} 1 \\ y \end{bmatrix},$$

where $\lambda, \mu > 0$ and $x, y \in \{\pm e_i : i = 1, 2, \ldots, n-1\}$. Since $2u = [1 \ e_1]^T + [1 - e_1]^T$ is in the interior of $l^n_{1,+}$, $A(2u)$ is in the interior of $l^n_{1,+}$. From the above relations, we see that $\lambda + \mu = |\lambda + \mu| > ||\lambda x + \mu y||_1$. Since $x, y \in \{\pm e_i : i = 1, 2, \ldots, n-1\}$, using the definition of $l_1$-norm, we see that $y = -x$. This proves that $A$ is conjugate-pair-preserving. By the previous result,

$$A = \begin{bmatrix} 1 & 0 \\ 0 & D \end{bmatrix},$$

where $D$ is an isometry of the $l_1$-norm. Since the isometries of the $l_1$-norm are generalized permutations, see [6], we have the stated result.

**Remark (3).** That $D$ is a generalized permutation in (5.7) can be shown in a different way (without using a result of [6]): Using (5.7) in (5.6), we get

$$1 = \lambda, \quad De_1 = \lambda x.$$ 

As $x \in \{\pm e_i : i = 1, 2, \ldots, n-1\}$, we see that $De_1 \in \{\pm e_i : i = 1, 2, \ldots, n-1\}$. More generally, $De_j \in \{\pm e_i : i = 1, 2, \ldots, n-1\}$ for any $j$. Note that such an inclusion is valid for $D^{-1}$ in place of $D$ as $A^{-1}$ is also an automorphism. Thus,

$$D(\{\pm e_i : i = 1, 2, \ldots, n-1\}) = \{\pm e_i : i = 1, 2, \ldots, n-1\}.$$ 

This shows that $D$ is a generalized permutation.

**Remark (4).** For $n = 2$, let $K = \{(t, x) : t \geq |x|\}$. Then the matrix

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

is an automorphism of $K$ which is clearly not of the form given in the above theorem.

**Remark (5).** For any proper cone $K$, $A \in \text{Aut}(K)$ if and only if $A^T \in \text{Aut}(K^*)$. Thus, knowing the automorphisms of the $l_1$-cone, one can describe the automorphisms of the $l_\infty$-cone.
6. **Concluding Remarks.** In this paper, we have studied the so-called special Bishop-Phelps cones and described some results pertaining to irreducibility, Lyapunov rank, and automorphisms. We end this paper by noting a characterization result on self-dual special Bishop-Phelps cones and raising a question on the homogeneity property. The following result provides a simple answer for the self-duality property (which is likely to be known).

**Theorem 6.1.** For \( n \geq 2 \), the special BP cone \( K \) in \( \mathbb{R}^n \) given by (1.2) is self-dual in \( \mathbb{R}^n \), that is, \( K = K^* \) if and only if the norm \( \| \cdot \| \) on \( \mathbb{R}^{n-1} \) is the 2-norm.

**Proof.** When the norm is the 2-norm, the corresponding special BP-cone is either the second-order cone \( l^n_{2,+} \) (see Section 1) or the cone \( K = \{(t,x) : t \geq |x|\} \) in \( \mathbb{R}^2 \). These cones are self-dual. Now suppose that \( K \) is self-dual so that \( K = K^* \). We recall that

\[
K^* = \{(s,y) : s \geq \|y\|_D\},
\]

where \( \|y\|_D \) denotes the dual norm of \( y \). Now for any \( x \in \mathbb{R}^{n-1} \),

\[
(||x||, x) \in K = K^*
\]

implies that \( ||x|| \geq ||x||_D \). Similarly, the inclusion \( (||x||_D, x) \in K^* = K \) implies that \( ||x||_D \geq ||x|| \). Hence, \( ||x|| = ||x||_D \) for all \( x \in \mathbb{R}^{n-1} \). Now,

\[
||x||^2 = (x,x) \leq ||x|| ||x||_D = ||x||^2.
\]

Thus, \( ||x|| \leq ||x|| \) for all \( x \in \mathbb{R}^{n-1} \). Finally, by definition of the dual norm, for any \( x \in \mathbb{R}^{n-1} \), there exists a vector \( u \) with \( ||u|| = 1 \) such that \( ||x||_D = ||(x,u)|| \). Thus,

\[
||x|| = ||x||_D = ||(x,u)|| \leq ||x||_2 ||u||_2 \leq ||x||_2 ||u|| \leq ||x||_2.
\]

We conclude that \( ||x|| = ||x||_2 \) for all \( x \in \mathbb{R}^{n-1} \). This completes the proof.

We say that a proper cone \( K \) is **homogeneous** [18] if for any two elements \( x, y \in \text{int} K \), there exists \( A \in \text{Aut}(K) \) such that \( A(x) = y \). A self-dual homogeneous cone is said to be a **symmetric cone** [4]. It is known that every symmetric cone is the cone of squares in some Euclidean Jordan algebra (and conversely). The second order cone \( l^n_{2,+} \) is a symmetric cone. It is easily seen, from Theorem 5.2, that the cone \( l^n_{1,+} \) \( (n \geq 3) \) is not homogeneous. (If not, any element of the open unit ball of \( (\mathbb{R}^{n-1}, ||\cdot||_1) \) can be mapped onto any another in the open unit ball by a generalized permutation.) These two examples motivate the following problem:

**Which special Bishop-Phelps cones are homogeneous? In particular, is \( l^n_{p,+} \) non-homogeneous for \( p \neq 2 \)?**
REFERENCES