A Robust Additive Multiattribute Preference Model using a Nonparametric Shape-Preserving Perturbation

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Abstract

This paper develops a multiattribute preference ranking rule in the context of utility robustness. A nonparametric perturbation of a given additive reference utility function is specified to solve the problem of ambiguity and inconsistency in utility assessments, while preserving the additive structure and the decision maker's risk preference under each criterion. A concept of robust preference value is defined using the worst expected utility of an alternative incurred by the perturbation, and we rank alternatives by comparing their robust preference values. An approximation approach is developed using Bernstein polynomials to solve the robust preference value. The constructed approximation problem is reformulated as a quadratic constrained linear program (QCP), and the bound of the approximation error is analyzed. An integrated energy distribution system planning problem is used to illustrate the usefulness of the robust ranking rule and the instability of the decision based on the expected utility theory.

Key Words: Multicriteria decision analysis, Multiattribute utility theory, Additive utility function, Nonparametric perturbation, Utility robustness

1 Introduction

An additive representation of the decision maker's multiattribute preference is the most common approach in the literature of utilitarianism (Dyer (2005)). Conventionally, the term value function refers to the decision maker's preference for certainty, and the term utility function is defined under risk (Keeney and Raiffa (1976)). Debreu (1960), Luce and Tukey (1964), and Gorman (1968) described the preference independence, with which a multiattribute value function has an additive form. Under risk, the existence of an additive utility function is ensured by the additive independence (Fishburn (1965)). Our study in this paper allows a value function to be treated as a special case of utility function. Hence, we only mention utility function for conciseness.

The additive structure greatly facilitates assessing a multiattribute utility function. We are allowed to separately elicit a conditional utility function for each attribute, and next assess the scaling constants (Keeney and Raiffa (1976)). The standard and paired gamble methods, such as preference comparison, probability equivalence, value equivalence, and certainty equivalence, are classical nonparametric single-attribute utility assessments (see e.g. Farquhar (1984), Wakker and Deneffe (1996) and references therein), while in parametric approach constant absolute risk aversion (CARA), relative risk aversion (CRRA), and hyperbolic absolute risk aversion (HARA) representations are often used to underlie the mathematical forms of utility functions. A shared concern in these methods is ambiguity and inconsistency in utility representation, which arise from cognitive difficulty and incomplete information (Karmarkar (1978); Weber (1987); Hu (2013)). Moreover, similar ambiguity and inconsistency in eliciting scaling constants with methods such as simple multiattribute rating technique and point allocation (Edwards (1977); von Winterfeldt and Edwards (1986)), the swing weighting (von Winterfeldt and Edwards (1986)), the trade-off weighting (Keeney and Raiffa (1976)), and the analytic hierarchy process Saaty (1980), give an additional difficulty in accurately assessing an additive utility function. The UTA method proposed by Jacquet-Lagrèze and Siskos (1982) considers a disaggregation paradigm, in which the decision maker is first asked to rank a small reference subset of alternatives and an ordinal regression approach is next used to address the additive utility function most qualified for representing the given ranking. However, only using the partial information, the UTA method may suggest many utility functions and leads to the ambiguity which a global decision making needs to face.

In the paper, we propose a multiattribute preference ranking rule based on a nonparametric perturbation of a given additive reference utility function. This perturbation is used to solve the problem of the ambiguity and inconsistency in the elicitation of both the condition utility functions and the scaling constants, while preserving the additive structure of the reference utility function and the decision maker's risk preference (risk-aversion, risk-neutralism, risk-loving, or S-shapedness) under every attribute. Alternatives are ranked in the context of robustness by comparing their worst expected utilities incurred by the perturbation. Our study in multicriteria decision analysis generalizes the research on a single-attribute case given in Hu (2013). To solve the ranking model, we develop an approximation approach using Bernstein polynomials. Compared to the Monte Carlo sampling method given in Hu (2013), the Bernstein polynomial based approach is more flexible and allow us to study alternatives with continuously distributed random attribute values. Analogous robust and regression decision models using a set of single-attribute utility functions were also studied by Boutilier et al. (2006) and Hu and Mehrotra (2012a,b), which specify boundary conditions on utility and marginal utility functions.

This paper is organized as follows. In Section 2 we specify a perturbation region of a given additive reference utility function, and based on the perturbation, develop the multiattribute preference ranking model. To solve this model, Section 3 develops an approximation approach using Bernstein polynomials. We reformulate the approximation problem by a quadratic constrained linear program (QCP), and give the bound of the approximation error with respect to the degrees of Bernstein polynomials. In Section 4 we study an integrated energy distribution system planning problem considering the future energy supply infrastructure for a newly develop suburb in Norway. A numerical test shows the instability of the decision based on the conventional expected utility theory and verifies the usefulness of the robust ranking rule. Section 5 gives the proofs of propositions and theorems in the paper, and Section 6 concludes.

2 Robust Preference Model and Perturbation Region

In this section, we first describe a reference utility function, and next specify a perturbation region of the given reference utility function. Based on this perturbation region, we finally develop a robust multiattribute preference ranking rule.

2.1 Reference Utility Function

Consider a multicriteria decision problem and let $\mathfrak{M} := \{1, \ldots, m\}$ be the index set of attributes. Let $u^r \in \mathfrak{U}$ be a given additive reference utility function, written as

$$u^r(t) := \sum_{i \in \mathfrak{M}} u_i^r(t_i), \tag{1}$$

where $u_i^r \in \mathfrak{U}_i$ is a single-attribute conditional utility function under the ith attribute. Let $\underline{\theta}_i$ and $\overline{\theta}_i$ represent the worst and best level of the ith attribute. Denote by $\Theta_i := [\underline{\theta}_i, \overline{\theta}_i]$ the evaluation region of the ith attribute, and by $\Theta := \prod_{i=1}^m \Theta_i$ a Cartesian product space of evaluation. $u_i^r(t)$ is increasing in Θ_i, and at points $\underline{\theta} := (\underline{\theta}_1, \ldots, \underline{\theta}_m)$ and $\overline{\theta} := (\overline{\theta}_1, \ldots, \overline{\theta}_m)$,

$$u^r(\underline{\theta}) = 0, \text{ and } u^r(\overline{\theta}) = 1. \tag{2}$$

Also in the study, u_i^r is assumed to be Lipschitz continuous in Θ_i with the Lipschitz modulus L_i^r, i.e.,

$$|u_i^r(t_i^1) - u_i^r(t_i^2)| \leq L_i^r |t_i^1 - t_i^2|, \quad t_i^1, t_i^2 \in \Theta_i. \tag{3}$$

The conventional additive form of $u^r(t)$ should be formulated as $u^r(t) := \sum_{i \in \mathfrak{M}} \lambda_i \bar{u}_i(t_i)$. Here, λ_i are the positive scaling constants satisfying $\sum_{i \in \mathfrak{M}} \lambda_i = 1$, and $\bar{u}_i(\cdot)$ are normalized conditional utility functions, i.e., $\bar{u}_i(\underline{\theta}_i) = 0$ and $\bar{u}_i(\overline{\theta}_i) = 1$. We can obtain the form (1) by defining $u_i^r = \lambda_i \bar{u}_i$. On the other hand, under the boundary condition (2), we can transform (1) to the conventional additive representation, using the normalization as $\lambda_i = u_i^r(\overline{\theta}_i)$ and $\bar{u}_i = u_i^r / \lambda_i$. Compared to the conventional form, form (1) facilitates the later study of a perturbation on not only the conditional utility functions $\bar{u}_i$ but also scaling constants λ_i.

Single-attribute utility functions are often categorized to be risk-averse, risk-neutral, and risk-loving, for characterizing the decision maker's different risk attitudes. Also, the Prospect theory given by Kahneman and Tversky (1979) conducts a S-shaped utility function which shows loss-aversion (risk-loving) for a negative outcome and the risk-aversion for a positive outcome. We let $\mathfrak{M}_a$, $\mathfrak{M}_n$ $\mathfrak{M}_l$, $\mathfrak{M}_s \subseteq \mathfrak{M}$ indicate index subsets of attributes under which the condition reference utility functions should be risk-averse, risk-neutral, risk-loving, and S-shaped, respectively. Implicitly, the subsets $\mathfrak{M}_a$, $\mathfrak{M}_n$, $\mathfrak{M}_l$, and $\mathfrak{M}_s$ are disjointed and their union covers $\mathfrak{M}$. For $i \in \mathfrak{M}_s$, denote by $\widehat{t}_i$ the inflection point of the S-shaped conditional utility function u_i^r.

2.2 Perturbation Region

We now specify a perturbation region of the reference utility function u^r. For each $i \in \mathfrak{M}$, we denote by $\mathfrak{U}_i$ the set of increasing single-attribute utility functions defined on Θ_i, which are concave if $i \in \mathfrak{M}_a$, linear if $i \in \mathfrak{M}_n$, convex if $i \in \mathfrak{M}_l$, and otherwise, S-shaped. Since a perturbation incurring an abrupt but huge change on the reference utility function may be unreasonable in a real decision case, we assume that all $u_i \in \mathfrak{U}_i$ are Lipschitz continuous with the Lipschitz modulus L_i, i.e.,

$$|u_i(t_i^1) - u_i(t_i^2)| \leq L_i |t_i^1 - t_i^2|, \quad t_i^1,\ t_i^2 \in \Theta_i,\ u \in \mathfrak{U}_i,\ i \in \mathfrak{M}. \tag{4}$$

For a given constant $\kappa \geq 1$, we may choose $L_i := \kappa L_i^r$. In addition, when the decision maker delimits the loss and gain, the inflection point of the S-shaped utility function associated with her preference can be precisely prescribed. Hence, we assume that, for $i \in \mathfrak{M}_s$, $\widehat{t}_i$ is the common inflection point of all $u_i \in \mathfrak{U}_i$.

Let $(\Theta, \mathfrak{G}, \mu)$ be a complete measure space where $\mathfrak{G}$ is the smallest σ-algebra of Θ and μ is a positive signed measure on Θ with $\mu(\Theta) = 1$. Using the measure μ, we denote by $\mathfrak{U}(\epsilon)$ ($\epsilon = (\epsilon_1, \epsilon_2) \in [0,1]^2$ is a given vector describing the levels of perturbation tolerance and asymmetry) the set of multiattribute utility functions u satisfying the following conditions:

- Additivity: there are some $u_i \in \mathfrak{U}_i$ such that

$$u(t) = \sum_{i \in \mathfrak{M}} u_i(t_i), \quad t = (t_1, \ldots, t_m) \in \Theta. \tag{5}$$

- Boundedness:

$$u(\underline{\theta}) = 0, \quad u(\overline{\theta}) = 1. \tag{6}$$

- Perturbation tolerance:

$$\psi_1(u) := \int_{\Theta} (u(t) - u^r(t))^2 \mu(dt) \leq \epsilon_1^2, \tag{7}$$

- Perturbation symmetry:

$$\psi_2(u) := \left| \int_{\Theta} u(t) - u^r(t)\ \mu(dt) \right| \leq \epsilon_2. \tag{8}$$

The additivity condition (5) ensures that all utility functions in $\mathfrak{U}(\epsilon)$ have the additive form. The boundary condition (6) is a commonly used order-preserving normalization, since the additive utility function is unique up to a positive linear transformation. The tolerance condition (7) specifies a perturbation neighborhood of u^r using $\mathcal{L}_2$ norm metric. The symmetry condition (8) requires that a perturbation should be symmetrically around u^r at both up and down sides. In conditions (7) and (8) μ is used to measure the relative importance of different subregions of Θ. A particular case is that μ is a discrete measure. Actually, nonparametric utility assessments can only generate finitely many points of utility value corresponding to the decision maker's answers to a questionnaire, and use a piecewise linear curve to link the all points. In this case, we may define a perturbation region based on these discrete points rather than the entire piecewise linear utility function. Let $(t^j, u^r(t^j)) \in \Theta \times [0,1]$, $j = 1, \ldots, J$, be the points of utility value. The measure μ should be assigned on t^j, i.e.,

$$\mu(t^j) > 0,\ j = 1, \ldots, J, \text{ and } \mu(\Theta \setminus \{t^1, \ldots, t^J\}) = 0.$$

Conditions (7) and (8) are represented as

$$\sum_{j=1}^{J} \mu(t^j)(u(t^j) - u^r(t^j))^2 \leq \epsilon_1^2,$$

and

$$\left| \sum_{j=1}^{J} \mu(t^j)(u(t^j) - u^r(t^j)) \right| \leq \epsilon_2.$$

Conditions (7) and (8) are first introduced by Hu (2013) to specify a perturbation of a single-attribute utility function. In this paper we consider an extension in multicriteria decision analysis. The following proposition shows that $\mathfrak{U}(\epsilon)$ is a convex set and the proof is given in Section 5.1.

Proposition 2.1 $\mathfrak{U}(\epsilon)$ *is a nonempty convex set.*

2.3 Robust Preference with Respect to the Reference Utility Function

Let $\mathfrak{K}$ be the index set of available alternatives. $X^k := (X_1^k, \ldots, X_m^k)$, for $k \in \mathfrak{K}$, represents a random attribute-value vector of alternative k, where X_i^k is the random attribute value of alternative k under attribute $i \in \mathfrak{M}$. Assume that the support of X^k for every $k \in \mathfrak{K}$ is contained in Θ. Based on the perturbation $\mathfrak{U}(\epsilon)$ of the reference utility function u^r, we define the robust preference value of alternative k as

$$\pi(k) := \min_{u \in \mathfrak{U}(\epsilon)} \mathbb{E}[u(X^k)]. \tag{9}$$

The robust preference value $\pi(k)$ is the worst expected utility of alternative k under the perturbation. Ranking alternatives by comparing the preference values exhibits a preference relationship in the context of robustness. That is, alternative k is said to be robust preferred to alternative ℓ with respect to the reference utility function u^r if and only if $\pi(k) \geq \pi(\ell)$.

3 Approximation Using Bernstein Polynomials

The robust preference value π is defined as the optimal value of the optimization model (9) built on a functional decision space. In this section we use Bernstein polynomials to address an approximation approach to solve π.

3.1 Bernstein Polynomial Based Approximation Function

Utility functions exhibit different geometrical properties in regard to risk attitudes. We consider Bernstein polynomial based approximations for risk aversion, risk loving, risk neutralism, and S-shapedness, respectively. For the different cases, we first discuss the specific constructions of approximation base functions. Thus, the approximation of a utility function can be represented as a linear combination of these bases.

A standard Bernstein polynomial base is defined as

$$b(y, j, n) := \binom{n}{j} y^j (1-y)^{n-j},$$

for some $j \in \{1, \dots, n\}$. For $i \in \mathfrak{M}_a \cup \mathfrak{M}_\ell \cup \mathfrak{M}_n$, we denote Bernstein polynomial base functions with degree n_i in approximation as

$$\varphi_i(t_i, j, n_i) := b\left(\frac{t_i - \underline{\theta}_i}{\overline{\theta}_i - \underline{\theta}_i}, j, n_i\right), \quad t_i \in \Theta_i,\ j = 0, \dots, n_i.$$

It is worth mentioning that, for the case of risk neutralism, the perturbation described in the specification of the set $\mathfrak{U}(\epsilon)$ changes the scaling constant but keeps the linearity of utility function such that, for $i \in \mathfrak{M}_n$, we can fix $n_i \equiv 1$. The S-shapedness is an exception. Since the position of the inflection point should be fixed in the perturbation, to approximate a S-shaped utility function, we need to give different notions of base functions on left and right sides of the inflection point. For $i \in \mathfrak{M}_s$, let

$$\widehat{n}_i := \left\lfloor \frac{n_i(\widehat{t}_i - \underline{\theta}_i)}{\overline{\theta}_i - \underline{\theta}_i} \right\rfloor, \tag{10}$$

where $\lfloor y \rfloor$ is the floor function which returns the largest integer not greater than y. Let

$$\varphi_i(t_i, j, n_i) := \begin{cases} b\left(\frac{t_i - \underline{\theta}_i}{\widehat{t}_i - \underline{\theta}_i}, j, \widehat{n}_i\right) \mathbf{1}\left\{t_i \le \widehat{t}_i\right\}, & j = 0, \dots, \widehat{n}_i - 1, \\ \left(\frac{t_i - \underline{\theta}_i}{\widehat{t}_i - \underline{\theta}_i}\right)^{\widehat{n}_i} \mathbf{1}\left\{t_i \le \widehat{t}_i\right\} + \left(1 - \frac{t_i - \widehat{t}_i}{\overline{\theta}_i - \widehat{t}_i}\right)^{n_i - \widehat{n}_i} \mathbf{1}\left\{t_i > \widehat{t}_i\right\}, & j = \widehat{n}_i \\ b\left(\frac{t_i - \widehat{t}_i}{\overline{\theta}_i - \widehat{t}_i}, j - \widehat{n}_i, n_i - \widehat{n}_i\right) \mathbf{1}\left\{t_i > \widehat{t}_i\right\}, & j = \widehat{n}_i + 1, \dots, n_i, \end{cases}$$

where $\mathbf{1}\{\cdot\}$ is an indicate function.

We now use the linear combination of base functions φ_i for $i \in \mathfrak{M}$ to approximate a utility function in $\mathfrak{U}_i$. Denote a vector of base function as $\phi^i_{n_i}(t_i) := (\varphi_i(t_i, 0, n_i),\ \dots,\ \varphi_i(t_i, n_i, n_i))^T$, and write the linear combination of base functions as

$$f^i_{n_i}(t_i; c^i) := {c^i}^T \phi^i_{n_i}(t_i), \quad t_i \in \Theta_i, \tag{11}$$

where $c^i := (c^i_0, \cdots, c^i_{n_i})^T$ represents a vector of coefficient. For example, to approximate a S-shaped utility function u_i, let $c^i_j = u_i\left(\underline{\theta}_i + \frac{j(\widehat{t}_i - \underline{\theta}_i)}{\widehat{n}_i}\right)$ for $j = 0, \dots, \widehat{n}_i$, and $c^i_j = u_i\left(\widehat{t}_i + \frac{(j - \widehat{n}_i)(\overline{\theta}_i - \widehat{t}_i)}{n_i - \widehat{n}_i}\right)$ for $j = \widehat{n}_i + 1, \cdots, n_i$. Function $f^i_{n_i}(t_i; c^i)$ is two pieces of Bernstein polynomials which are used to approximate $u_i(t_i)$ on $[\underline{\theta}_i, \widehat{t}_i]$ and $[\widehat{t}_i, \overline{\theta}_i]$, respectively. Let $n := (n_1, \dots, n_m)$, $c := ({c^1}^T, \dots, {c^m}^T)^T$, and $\phi_n(t) := ((\phi^1_{n_1}(t_1))^T, \cdots, (\phi^m_{n_m}(t_m))^T)^T$. We further denote the sum of functions $f^i_{n_i}$ as

$$S_n(t; c) := \sum_{i \in \mathfrak{M}} f^i_{n_i}(t_i; c^i) = c^T \phi_n(t), \quad t = (t_1, \dots, t_m) \in \Theta. \tag{12}$$

By the above discussion, the sum function S_n can be used to approximate an additive multiattribute utility function.

3.2 Perturbation Region Associated with the Approximation Functions

The following proposition describes the boundedness, increasing, concavity (convexity, S-shapedness), and Lipschitz continuity of function $f^i_{n_i}$. The proof is given in Section 5.2.

Proposition 3.1

(i). For $i \in \mathfrak{M}$, if

$$c^i_{j+1} - c^i_j \geq 0 \quad j = 0, \ldots, n_i - 1, \; i \in \mathfrak{M}, \tag{13}$$

then $f^i_{n_i}(t_i; c^i)$ is increasing in Θ_i.

(ii). For $i \in \mathfrak{M}_a$, if

$$c^i_j + c^i_{j+2} \leq 2c^i_{j+1}, \quad j = 0, \ldots, n_i - 2, \tag{14}$$

then $f^i_{n_i}(t_i; c^i)$ is concave in Θ_i.

(iii). For $i \in \mathfrak{M}_l$, if

$$c^i_j + c^i_{j+2} \geq 2c^i_{j+1}, \quad j = 0, \ldots, n_i - 2, \tag{15}$$

then $f^i_{n_i}(t_i; c^i)$ is convex in Θ_i.

(iv). For $i \in \mathfrak{M}_s$, if

$$c^i_j + c^i_{j+2} \geq 2c^i_{j+1}, \quad j = 0, \ldots, \widehat{n}_i - 2, \tag{16}$$

$$c^i_j + c^i_{j+2} \leq 2c^i_{j+1}, \quad j = \widehat{n}_i, \ldots, n_i - 2, \tag{17}$$

then $f^i_{n_i}(t_i; c^i)$ is a S-shaped function with the inflection point $\widehat{t}_i$.

(v). Suppose that, for $i \in \mathfrak{M}_a \cup \mathfrak{M}_\ell \cup \mathfrak{M}_n$,

$$c^i_{j+1} - c^i_j \leq \frac{L_i(\overline{\theta}_i - \underline{\theta}_i)}{n_i}, \quad j = 0, \ldots, n_i - 1, \tag{18}$$

and for $i \in \mathfrak{M}_s$,

$$c^i_{j+1} - c^i_j \leq \frac{L_i(\widehat{t}_i - \underline{\theta}_i)}{\widehat{n}_i}, \quad j = 0, \ldots, \widehat{n}_i - 1, \tag{19}$$

$$c^i_{j+1} - c^i_j \leq \frac{L_i(\overline{\theta}_i - \widehat{t}_i)}{n_i - \widehat{n}_i}, \quad j = \widehat{n}_i, \ldots, n_i - 1. \tag{20}$$

Then, $f^i_{n_i}(t_i; c^i)$ is Lipschitz continuous in Θ_i with the Lipschitz modulus L_i.

(vi). If

$$\sum_{i \in \mathfrak{M}} c^i_0 = 0, \quad \sum_{i \in \mathfrak{M}} c^i_{n_i} = 1 \tag{21}$$

then $S_n(\underline{\theta}; c) = 0$ and $S_n(\overline{\theta}; c) = 1$.

Proposition 3.1 gives sufficient conditions for $S_n(\cdot; c)$ satisfying the additive condition (5) and the boundary condition (6). We now consider the perturbation tolerance and symmetry conditions (7) and (8) associated with the Bernstein polynomial bases. Let μ_i be the marginal measure for the ith attribute as

$$\mu_i((y_1, y_2]) = \mu\left((y_1, y_2] \times \prod_{k \in \mathfrak{M},\; k \neq i} \Theta_k\right),$$

and μ_{ij} be the marginal measure for the ith and jth attributes as

$$\mu_{ij}((y_1, y_2],\; (z_1, z_2]) = \mu\left((y_1, y_2] \times (z_1, z_2] \times \prod_{k \in \mathfrak{M},\; k \neq i,j} \Theta_k\right),$$

for any measurable $(y_1, y_2] \subseteq \Theta_i$ and $(z_1, z_2] \subseteq \Theta_j$. For all $i, j \in \mathfrak{M}$, we denote matrixes

$$A^{ij} := \begin{cases} \int_{\Theta_i} \phi^i_{n_i}(t_i) \left(\phi^i_{n_i}(t_i)\right)^T \mu_i(dt_i), & i = j, \\ \int_{\Theta_i} \int_{\Theta_j} \phi^i_{n_i}(t_i) \left(\phi^j_{n_j}(t_j)\right)^T \mu_{ij}(dt_i, dt_j), & i \neq j, \end{cases}$$

vectors

$$g^i := -2 \int_{\Theta_i} u^r_i(t_i)\phi^i_{n_i}(t_i)\mu_i(dt_i) - 2 \sum_{\substack{w \in \mathfrak{M} \\ w \neq i}} \int_{\Theta_i} \int_{\Theta_w} u^r_w(t_w)\phi^i_{n_i}(t_i)\mu_{iw}(dt_i, dt_w),$$

$$h^i := \int_{\Theta_i} \phi^i_{n_i}(t_i)\mu_i(dt_i),$$

and scalars

$$r := \sum_{w\in\mathfrak{M}} \int_{\Theta_w} (u^r_w(t_w))^2 \mu_w(dt_w) + \sum_{\substack{w,\, v \in \mathfrak{M} \\ w \neq v}} \int_{\Theta_w} \int_{\Theta_v} u^r_w(t_w)u^r_v(t_v)\mu_{wv}(dt_w, dt_v),$$

$$s := \sum_{w\in\mathfrak{M}} \int_{\Theta_w} u^r_w(t_w)\mu_w(dt_w).$$

Using the above definitions, we further construct vectors $g := ({g^1}^T, \cdots, {g^m}^T)^T$ and $h := ({h^1}^T, \cdots, {h^m}^T)^T$, and matrix $A := (A^{ij})$. The perturbation tolerance condition can be specified in the standard form of quadratic condition as

$$\begin{aligned} \psi_1(S_n(\cdot; c)) &= \int_\Theta (S_n(t; c) - u^r(t))^2 \mu(dt) \\ &= \sum_{i,j\in\mathfrak{M}} {c^i}^T A^{ij} c^j + \sum_{i\in\mathfrak{M}} {g^i}^T c^i + r \\ &= c^T A c + g^T c + r \\ &\leq \epsilon_1^2. \end{aligned} \tag{22}$$

By definition A is a positive semidefinite symmetric matrix, and hence, condition (22) is convex. Similarly, we write the perturbation symmetry condition as

$$\psi_2(S_n(\cdot; c)) = \left| \int_\Theta S_n(t; c) - u^r(t)\mu(dt) \right| = \left| h^T c - s \right| \leq \epsilon_2, \tag{23}$$

which can be reformulated as two linear conditions $h^T c - s \geq -\epsilon_2$ and $h^T c - s \leq \epsilon_2$.

We now describe a set of coefficients as

$$\mathfrak{C}_n(\epsilon) := \{c \in \mathbb{R}^{\sum_{i\in\mathfrak{M}} n_i} : c \text{ satisfies conditions (13) - (21) and (22) - (23)}\}, \tag{24}$$

and correspondingly, the set of the linear combinations of all Bernstein polynomial base functions is

$$\mathfrak{S}_n(\epsilon) := \{S_n(\cdot; c) \ : \ c \in \mathfrak{C}_n(\epsilon)\}. \tag{25}$$

The above discussion implies the relationship of the sets $\mathfrak{S}_n(\epsilon)$ and $\mathfrak{U}(\epsilon)$ given in the following proposition.

Proposition 3.2 $\mathfrak{S}_n(\epsilon) \subseteq \mathfrak{U}(\epsilon)$.

3.3 Approximation of the Robust Preference Value

We now address an approximation approach to the robust preference value as

$$\pi_n(k) := \min_{s \in \mathfrak{S}_n(\epsilon)} \mathbb{E}[s(X^k)] = \min_{c \in \mathfrak{C}_n(\epsilon)} \mathbb{E}[S_n(X^k; c)] = \min_{c \in \mathfrak{C}_n(\epsilon)} c^T \mathbb{E}[\phi_n(X^k)]. \tag{26}$$

The optimization problem (26) is a quadratic constrained linear program (QCP), where the objective is a linear function of c, and the feasible set $\mathfrak{C}(\epsilon)$ consists of the quadratic constraint (22) and the linear constraints (13) - (21) and (23). We have mentioned that the matrix A in the quadratic constraint (22) is positive semidefinite. Hence, (26) is a convex programming problem. The QCP has been well studied in the literature (see e.g. van de Panne (1966); Martein and Schaible (1987); Boyd and Vandenberghe (2004), and references therein). We will use the CPLEX QCP solver for an integrated energy distribution system planning problem studied in Section 4. In what follows, we give the bound of the approximation error of π_n. The proof is given in Section 5.3.

Theorem 3.3 *For $n = (n_1, \ldots, n_m)$, denote*

$$\Delta(n) := \sum_{i \in \mathfrak{M}_a \cup \mathfrak{M}_\ell} \frac{L_i}{\sqrt{n_i}} \left(1 + \frac{\overline{\theta}_i - \underline{\theta}_i}{2}\right) + \sum_{i \in \mathfrak{M}_s} \max\left\{ \frac{L_i}{\sqrt{\widehat{n}_i}} \left(1 + \frac{\widehat{t}_i - \underline{\theta}_i}{2}\right), \ \frac{L_i}{\sqrt{n_i - \widehat{n}_i}} \left(1 + \frac{\overline{\theta}_i - \widehat{t}_i}{2}\right) \right\}.$$

For a given $\delta \in (0,1)$, we choose n such that $n_i = 1$ for $i \in \mathfrak{M}_n$ and

$$\Delta(n) \le \frac{\delta}{2} \min\{\epsilon_1, \epsilon_2\},$$

Then, for any alternative $k \in \mathfrak{K}$,

$$\pi(k) \le \pi_n(k) \le \pi(k) + \delta.$$

4 Case Study: Integrated Energy Distribution System Planning

In this section we address an integrated energy distribution system planning project, presented in Botterud et al. (2005) and Løken (2007), which decides the future energy supply infrastructure for a newly developed suburb in Norway with 2000 households and possible additional industrial demand. The robust preference is used as the ranking rule. The results show advantages of the robust preference model, compared with the classical expect utility approach.

4.1 Problem Description

Five experts with background in energy research and industry are invited to investigate four investment plans and evaluate their potential economical and environmental impacts. Table 1 given by Botterud et al. (2005) summarizes these four plans. The first plan P_1 reinforces the existing electrical grid with a new supply line to meet the entire local stationary energy demand in the area. The other three plans deliver electricity through the electrical grid but build a district heating network and a combined heat and power (CHP) plant to serve the heat demand. In addition, there is a need to build a gas boiler to meet the peak demand for district heating. The new heating network is designed in Plan P_2 to also supply heat to an industrial site outside the residential area. The CHP plant is set up there to replace a diesel boiler currently used to produce heat for the industrial area. Plans P_3 and P_4 place the CHP plant nearby the residential area. In comparison, Plan P_4 intends to build a larger CHP plant facilitating generation of more electricity, which is

Table 1: Alternative plans (Botterud et al. (2005))

Plans	New el line	DH network	CHP plant	Gas boiler
P_1	yes	no	no	no
P_2	no	large	3.6 MW	5.0 MW
P_3	no	small	3.6 MW	5.0 MW
P_4	no	small	5.0 MW	5.0 MW

Table 2: Potential performances of the investment plans under the attributes (Botterud et al. (2005) and Løken (2007))

Plan	Scen.	Prob.	Annual operating cost [-MNOK/year]	Annual inv. cost [-MNOK/year]	Annual CO_2 emissions [-tons/year]	Annual NO_x emissions [-tons/year]	Annual heat dump [-MWh/year]
P_1	Low	0.25	-14.9	-2.87	-41600	0	0
	Medium	0.5	-21.2	-2.87	-51325	0	0
	High	0.25	-27.6	-2.87	-61590	0	0
P_2	Low	0.25	-12.9	-6.85	-32902	-44.7	0
	Medium	0.5	-15.8	-6.85	-37440	-45.4	-377
	High	0.25	-18.6	-6.85	-41974	-45.5	-468
P_3	Low	0.25	-13.8	-5.46	-36188	-36.8	0
	Medium	0.5	-17.0	-5.46	-40170	-46.2	-4547
	High	0.25	-19.9	-5.46	-44665	-47.0	-5082
P_4	Low	0.25	-13.7	-6.31	-35662	-42.6	-821
	Medium	0.5	-16.5	-6.31	-38701	-60.8	-11319
	High	0.25	-18.6	-6.31	-41917	-62.7	-12604

Table 3: Coefficients of absolute risk aversion (Botterud et al. (2005))

	A	B	C	D	E
γ_1	1.12	0.7	0.99	0.99	0.7
γ_2	1.65	0.7	2.24	1.65	0.7
γ_3	0.79	-1.26	-1.61	N/A	0
γ_4	-4.24	-1.95	2.02	-1.59	0
γ_5	0.45	N/A	2.48	N/A	N/A

N/A means that the experts consider the objective irrelevant.

Table 4: Scaling constants of the additive utility functions (Botterud et al. (2005))

	A	B	C	D	E
λ_1	0.6	0.71	0.46	0.73	0.66
λ_2	0.1	0.14	0.14	0.13	0.13
λ_3	0.14	0.09	0.04	0	0.07
λ_4	0.14	0.05	0.23	0.14	0.14
λ_5	0.03	0	0.14	0	0

profitable when sold to the electricity market. However, the greater electricity generation results in dumping excessive heat to local surroundings.

Botterud et al. (2005) and Løken (2007) both use the multiattribute utility theory to decide the preference ordering of the investment plans. Annual operating cost, investment cost, CO_2 emissions, NO_x emissions, and heat dump are suggested to be critical attributes, and the total utility function follows the additive structure. The potential performances of the investment plans under these attributes given in Table 2 are considered as random quantities embodying uncertainty arising from the fluctuating market price of electricity and possible technical development. Note that, in Table 2, we change the sign of the positive performance values given in Botterud et al. (2005) and Løken (2007). The original project actually uses disutility functions requiring that a better plan should have a smaller loss under these attributes. To simplify the discussion on risk loving (loss aversion) and risk aversion (loss loving), we adjust the data for uniformly using utility functions. The experts independently answer two types of questionnaires in order to elicit their individual single-attribute utility functions and scaling constants. The single-attribute utility functions are assumed to be CARA functions most commonly used in industry. For expert $\ell \in \{A, B, C, D, E\}$, Botterud et al. (2005) and Løken (2007) formulate the elicited conditional utility functions as

$$u_i^\ell(t_i) := \frac{\lambda_i^\ell(1 - e^{-\gamma_i^\ell(t_i - \underline{\theta}_i)/(\overline{\theta}_i - \underline{\theta}_i)})}{1 - e^{-\gamma_i^\ell}}. \tag{27}$$

where $\underline{\theta}_i$ and $\overline{\theta}_i$ are the worst and best performances given in Table 2, γ_i^ℓ is the coefficient of absolute risk aversion given in Table 3, λ_i^ℓ is the scaling constant given in Table 4. A positive γ_i^ℓ implies that expert ℓ is risk averse for attribute i, whereas a negative γ_i^ℓ shows a risk loving attitude. In particular, choosing $\gamma_3^E = \gamma_4^E = 0$, expert E thinks that CO_2 and NO_x emissions are risk neutral.

4.2 Result Analysis

We now discuss the use the robust preference as the ranking rule in this project. The reference utility function of expert ℓ is given as

$$u^\ell(t) = \sum_{i=1}^{5} u_i^\ell(t_i). \tag{28}$$

We fix $\epsilon_2 = 10^{-10}$, and choose the measure μ to be the uniform distribution on Θ. Then the marginal measure μ_i is the uniform distribution on Θ_i for each attribute $i \in \mathfrak{M} = \{1, \ldots, 5\}$. In computation, we use the approximation approach given in (26) and choose the degree of Bernstein polynomial base functions to be $n_i = 200$ for all $i \in \mathfrak{M} \setminus \mathfrak{M}_n$ and $n_i = 1$ for $i \in \mathfrak{M}_n$ in the approximation.

Table 5: Robust preference values and rankings of the investment plans for the five experts

(a) Expert A

Plan	ϵ_1 = 0	0.01	0.05	0.1
P_1	0.631 (4)	0.583 (4)	0.474 (4)	0.427 (2)
P_2	0.675 (2)	0.620 (3)	0.504 (3)	0.391 (4)
P_3	0.679 (1)	0.654 (1)	0.583 (1)	0.505 (1)
P_4	0.660 (3)	0.629 (2)	0.527 (2)	0.409 (3)

(b) Expert B

Plan	ϵ_1 = 0	0.01	0.05	0.1
P_1	0.565 (4)	0.518 (4)	0.431 (4)	0.420 (4)
P_2	0.682 (2)	0.633 (3)	0.525 (3)	0.424 (3)
P_3	0.685 (1)	0.657 (1)	0.591 (1)	0.529 (1)
P_4	0.676 (3)	0.642 (2)	0.544 (2)	0.442 (2)

(c) Expert C

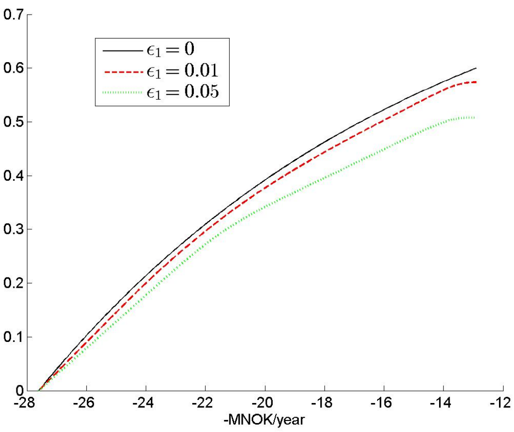
Plan	ϵ_1 = 0	0.01	0.05	0.1
P_1	0.743 (1)	0.703 (1)	0.611 (2)	0.544 (2)
P_2	0.676 (3)	0.619 (3)	0.488 (3)	0.373 (3)
P_3	0.716 (2)	0.693 (2)	0.624 (1)	0.555 (1)
P_4	0.541 (4)	0.506 (4)	0.408 (4)	0.308 (4)

(d) Expert D

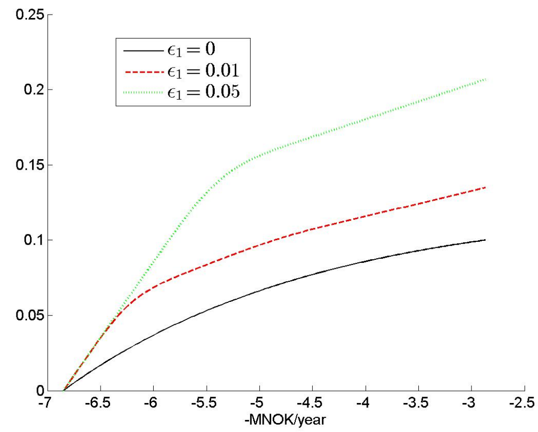
Plan	ϵ_1 = 0	0.01	0.05	0.1
P_1	0.639 (4)	0.605 (3)	0.521 (3)	0.475 (2)
P_2	0.655 (2)	0.603 (4)	0.504 (4)	0.403 (4)
P_3	0.683 (1)	0.658 (1)	0.601 (1)	0.536 (1)
P_4	0.654 (3)	0.627 (2)	0.544 (2)	0.448 (3)

(e) Expert E

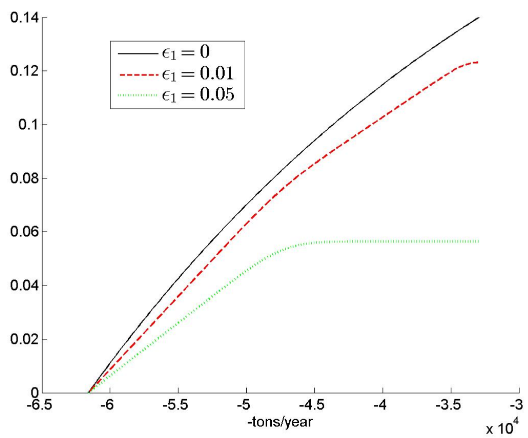
Plan	ϵ_1 = 0	0.01	0.05	0.1
P_1	0.617 (4)	0.569 (4)	0.483 (4)	0.452 (2)
P_2	0.657 (2)	0.605 (2)	0.499 (3)	0.412 (4)
P_3	0.666 (1)	0.640 (1)	0.576 (1)	0.521 (1)
P_4	0.632 (3)	0.600 (3)	0.507 (2)	0.417 (3)



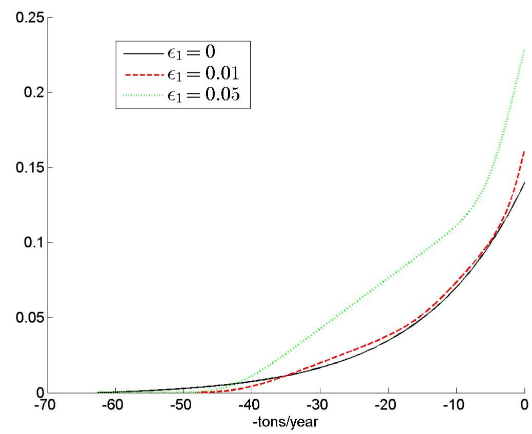
(a) Annual operating cost



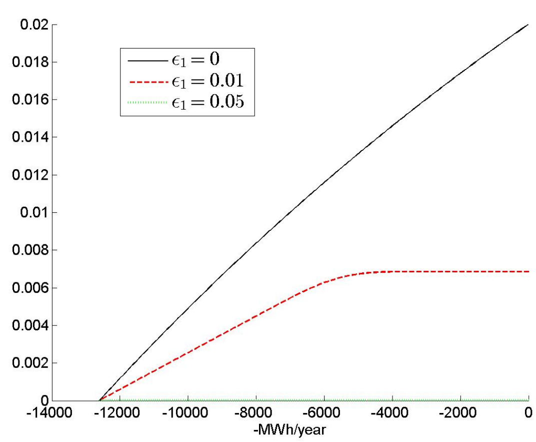
(b) Annual investment cost



(c) Annual CO_2 emissions



(d) Annual NO_x emissions



(e) Annual heat dump

Figure 1: The worst conditional utility function of the robust preference model for Plan P_2

We test the impact of different tolerance levels of the perturbation by adjusting ϵ_1. Table 5 gives the robust preference values and rankings of the investment plans for every expert. The experts' original preferences are given for $\epsilon_1 = 0$. This case indicates the expected utilities of the plans without perturbation, while the case of the biggest perturbation given at $\epsilon_1 = 0.1$ reports the view of robustness. It is not surprising that the robust preference values decrease in ϵ_1. The ranking orders of the plans recommended by Expert A are $P_3 - P_2 - P_4 - P_1$ without considering perturbation. A very small perturbation for $\epsilon_1 = 0.01$ switches the orders of P_2 and P_4, and under the biggest perturbation, P_1 is substituted for P_4 in the second position. Similar changes can also be observed for the ranking orders given by Experts B, C, D, and E. P_3 is the best plan in all cases except that Expert C prefers P_1 to P_3 when $\epsilon_1 = 0$ and 0.01. Under the perturbation, P_3 keeps a very good and stable performance. All experts except Expert C recommend P_2 as the second best plan. However, this plan is the worst in the view of robustness. In comparison, although P_1 is acknowledged to be the worst by almost all experts, this plan becomes more appreciated at the growth of the tolerance level of the perturbation.

Running the robust preference model (9) for Expert A to test Plan P_3, we obtain the worst conditional utility functions (optimal solutions of model (9)) with respect to $\epsilon_1 = 0$, 0.01, and 0.05 shown in Figure 1. The perturbation enhances the importance of the annual investment cost and NO_x emissions, for which the conditional utility functions are increased. On the other hand, the roles of the annual operating cost, CO_2 emissions, and heat dump are weakened. Considering the relative changes, the conditional utility function for the annual operating cost is the most stable in the perturbation test. For the annual CO_2 and NO_x emissions, the relative changes are slight for the small perturbation but obvious for the big perturbation. In comparison, for the annual improvement cost and heat dump, the condition utility functions are the least stable. In particular, when $\epsilon_1 = 0.05$, the annual heat dump becomes irrelevant in the evaluation of Plan P_3.

Table 6: Robust preference values with respect to degree of Bernstein polynomial base functions

Plan	n_i $(i = 1, \ldots, 5)$			
	10	50	100	200
P_1	0.478	0.475	0.474	0.474
P_2	0.519	0.507	0.505	0.504
P_3	0.587	0.584	0.584	0.583
P_4	0.531	0.527	0.527	0.527

We also test the accuracy of the approximation. Using Expert A as an example and choosing $\epsilon_1 = 0.05$, we solve the approximation model (26) for different degrees of the Bernstein polynomial base functions. In the test, we give all n_i, for $i = 1, \ldots, 5$, the same values, which are from 10 to 200. Table 6 lists the robust preference values of the plans with respect to the degree of the Bernstein polynomial base functions. The largest relative difference, which is (0.519 - 0.507) / 0.519 = 2.3%, results from increasing the degree from 10 to 50. The approximation approach shows a very good accuracy even when the degree of the Bernstein polynomial base functions is very small.

5 Proofs of Theorems and Propositions

5.1 Proof of Proposition 2.1.

Since $u^r \in \mathfrak{U}(\epsilon)$, $\mathfrak{U}(\epsilon)$ is nonempty. We now check its convexity. For $u^1,\ u^2 \in \mathfrak{U}(\epsilon)$, let $u := \lambda u^1 + (1-\lambda)u^2$ for $\lambda \in [0,1]$. Obviously, u satisfies the boundary condition (6). By definition, there are some $u_i^j \in \mathfrak{U}_i$ ($j = 1$ or 2) such that we can write $u^j(t) = \sum_{i\in\mathfrak{M}} u_i^j(t_i)$ for all $t = (t_1,\ldots,t_m) \in \Theta$. Since a positive linear combination preserves the all properties of the single-attribute utility functions in $\mathfrak{U}_i$, $u_i := \lambda u_i^1 + (1-\lambda)u_i^2$ belongs to $\mathfrak{U}_i$. It implies that u satisfies the additivity condition (5) since $u(t) = \sum_{i\in\mathfrak{M}} u_i(t_i)$.

We next check the perturbation tolerance and symmetry conditions (7) and (8). Since $\psi_1(u)$ is a convex functional of u, we have

$$\psi_1(u) \le \lambda\psi_1(u^1) + (1-\lambda)\psi_1(u^2) = \epsilon_1^2,$$

and similarly, $\psi_2(u) \le \epsilon_2$. Thus, $u \in \mathfrak{U}(\epsilon)$, and hence $\mathfrak{U}(\epsilon)$ is convex. □

5.2 Proof of Proposition 3.1.

(i). Theorem 7.1.3 in Philips (2003) shows that, for $i \in \mathfrak{M}_a \cup \mathfrak{M}_\ell \cup \mathfrak{M}_n$,

$$\frac{df_{n_i}^i(t_i;c^i)}{dt_i} = \frac{n_i}{\overline{\theta}_i - \underline{\theta}_i}\sum_{j=0}^{n_i-1}(c_{j+1}^i - c_j^i)\varphi^i(t_i,j,n_i-1), \tag{29}$$

and for $i \in \mathfrak{M}_s$,

$$\begin{aligned}\frac{df_{n_i}^i(t_i;c^i)}{dt_i} &= \frac{\widehat{n}_i}{\widehat{t}_i - \underline{\theta}_i}\sum_{j=0}^{\widehat{n}_i-1}(c_{j+1}^i - c_j^i)b\left(\frac{t_i-\underline{\theta}_i}{\widehat{t}_i-\underline{\theta}_i}, j, \widehat{n}_i - 1\right)\mathbf{1}\left\{t_i \le \widehat{t}_i\right\}\\ &\quad + \frac{n_i-\widehat{n}_i}{\overline{\theta}_i - \widehat{t}_i}\sum_{j=\widehat{n}_i}^{n_i-1}(c_{j+1}^i - c_j^i)b\left(\frac{t_i-\widehat{t}_i}{\overline{\theta}_i-\widehat{t}_i}, j-\widehat{n}_i, n_i-\widehat{n}_i - 1\right)\mathbf{1}\left\{t_i > \widehat{t}_i\right\}.\end{aligned} \tag{30}$$

Condition (13) and equation (29) implies (i) for the case of $i \in \mathfrak{M}_a \cup \mathfrak{M}_\ell \cup \mathfrak{M}_n$. Condition (13) and equation (30) implies (i) for the case of $i \in \mathfrak{M}_s$.

(ii) - (iii). Theorem 7.1.3 in Philips (2003) shows that, for $i \in \mathfrak{M}_a \cup \mathfrak{M}_\ell$,

$$\frac{d^2 f_{n_i}^i(t_i;c^i)}{dt_i^2} = \frac{n_i(n_i-1)}{(\overline{\theta}_i - \underline{\theta}_i)^2}\sum_{j=0}^{n_i-2}(c_{j+2}^i + c_j^i - 2c_{j+1}^i)\varphi^i(t_i,j,n_i-2).$$

Condition (14) implies the concavity of function $f_{n_i}^i$ for $i \in \mathfrak{M}_a$, and condition (15) implies the convexity of function $f_{n_i}^i$ for $i \in \mathfrak{M}_\ell$.

(iv). Similarly as the proof of (ii) - (iii), for $i \in \mathfrak{M}_s$, conditions (16) and (17) imply that function $f_{n_i}^i$ is convex on $[\underline{\theta}_i, \widehat{t}_i]$ and concave on $[\widehat{t}_i, \overline{\theta}_i]$.

(v). For the case of $i \in \mathfrak{M}_a \cup \mathfrak{M}_\ell \cup \mathfrak{M}_n$, using condition (18) and equation (29) , we have that, for all $t_i \in \Theta_i$,

$$\frac{df_{n_i}^i(t_i;c^i)}{dt_i} \le L_i \sum_{j=0}^{n_i-1}\varphi^i(t_i,j,n_i-1) = L_i.$$

Similarly, conditions (19), (20), and equation (30) implies $\frac{df_{n_i}^i(t_i;c^i)}{dt_i} \le L_i$ for $t_i \in (\underline{\theta}_i, \widehat{t}_i) \cup (\widehat{t}_i, \overline{\theta}_i)$. Then, the Lipschitz continuity of $b_{n_i}^i$ follows.

(vi). It directly follows by the definition of function S_n.

□

5.3 Proof of Theorem 3.3.

5.3.1 Preparatory Material.

To prove theorem 3.3, we need two technical lemmas. By adjusting Theorem 2.1 in Rivlin (1891), we obtain Lemma 5.1 which gives a bound for the error of approximating a function, defined on a general bounded interval, by the corresponding Bernstein polynomial. The result straightforwardly follows the proof of Theorem 2.1 in Rivlin (1891), and hence we omit the proof. Lemma 5.2 gives some conditions with which the error of approximating an additive utility function by the corresponding sum function S_n is less than a given small number. For a utility function u satisfying $u = \sum_{i\in\mathfrak{M}} u_i$ for some $u_i \in \mathfrak{U}_i$, let

$$\rho_i(u_i) := \begin{cases} \left(u_i(\underline{\theta}_i), u_i\left(\underline{\theta}_i + \frac{\overline{\theta}_i-\underline{\theta}_i}{n_i}\right), \cdots, u_i(\overline{\theta}_i)\right)^T, & i \in \mathfrak{M}_a \cup \mathfrak{M}_\ell \cup \mathfrak{M}_n, \\ \left(u_i(\underline{\theta}_i), u_i\left(\underline{\theta}_i + \frac{\widehat{t}_i-\underline{\theta}_i}{\widehat{n}_i}\right), \cdots, u_i(\widehat{t}_i), u_i\left(\widehat{t}_i + \frac{\overline{\theta}_i-\widehat{t}_i}{n_i-\widehat{n}_i}\right), \cdots, u_i(\overline{\theta}_i)\right)^T, & i \in \mathfrak{M}_s, \end{cases}$$

and

$$\rho(u) := \left(\rho_1(u_1)^T, \ldots, \rho_m(u_m)^T\right)^T.$$

Lemma 5.1 (Theorem 1.2 in Rivlin (1891)) *Let g be a continuous function defined on $[\underline{\theta}, \overline{\theta}]$. For $\delta > 0$, the modulus of continuity of f is*

$$\omega(\delta) := \sup_{\substack{y_1,\, y_2\, \in\, [\underline{\theta}, \overline{\theta}] \\ |y_1 - y_2| \le \delta}} |g(y_1) - g(y_2)|.$$

Then,

$$\left\| g - \sum_{j=0}^{n} f\left(\underline{\theta} + \frac{j(\overline{\theta}-\underline{\theta})}{n}\right) b\left(\frac{y-\underline{\theta}}{\overline{\theta}-\underline{\theta}}, j, n\right) \right\|_\infty \le \omega\left(\frac{1}{\sqrt{n}}\right)\left(1 + \frac{\overline{\theta}-\underline{\theta}}{2}\right).$$

Lemma 5.2 *Let $u \in \mathfrak{U}$ be represented as $u = \sum_{i\in\mathfrak{M}} u_i$ where $u_i \in \mathfrak{U}_i$. For $\delta > 0$, if*

$$\Delta(n) \le \delta,$$

then $\|u - S_n(\cdot;\rho_n(u))\|_\infty \le \delta$. Moreover, if $u \in \mathfrak{U}(\epsilon)$, $\rho_n(u) \in \mathfrak{C}_n(\epsilon + \delta e)$ (e denotes the vector whose elements are all 1's).

Proof: By the Lipschitz continuity required by condition (4), it follows by Lemma 5.1 that, for $i \in \mathfrak{M}_a \cup \mathfrak{M}_\ell$,

$$\|u_i - f^i_{n_i}(\cdot;\rho_i(u_i))\|_\infty \le \frac{L_i}{\sqrt{n_i}}\left(1 + \frac{\overline{\theta}_i - \underline{\theta}_i}{2}\right).$$

Similarly, for $i \in \mathfrak{M}_s$, we have

$$\|u_i - f^i_{n_i}(\cdot;\rho_i(u_i))\|_\infty \le \max\left\{\frac{L_i}{\sqrt{\widehat{n}_i}}\left(1 + \frac{\widehat{t}_i - \underline{\theta}_i}{2}\right),\ \frac{L_i}{\sqrt{n_i - \widehat{n}_i}}\left(1 + \frac{\overline{\theta}_i - \widehat{t}_i}{2}\right)\right\}.$$

Also note that for $i \in \mathfrak{M}_n$, the risk neutral utility function u_i is equal to $f^i_{n_i}(\cdot;\rho_i(u_i))$ for any $n_i \geq 1$, i.e,

$$\|u_i - f^i_{n_i}(\cdot;\rho_i(u_i))\|_\infty = 0.$$

Hence,

$$\|u - S_n(\cdot;\rho(u))\|_\infty \leq \sum_{i\in\mathfrak{M}} \left\|u_i - f^i_{n_i}(\cdot;\rho_i(u_i))\right\|_\infty \leq \Delta(n) \leq \delta.$$

The definition of the $\mathfrak{U}(\epsilon)$ implies that $\rho_n(u)$ for $u \in \mathfrak{U}(\epsilon)$ satisfies conditions (13) - (21). Also, the perturbation conditions (22) and (23) can be verified as

$$\begin{aligned}(\psi_1(S_n(\cdot;\rho_n(u))))^{1/2} &\leq (\psi_1(u))^{1/2} + \left(\int_\Theta (S_n(t;\rho_n(u)) - u(t))^2\mu(dt)\right)^{1/2} \\ &\leq \epsilon_1 + \|u - S_n(\cdot;\rho(u))\|_\infty \\ &\leq \epsilon_1 + \delta,\end{aligned}$$

and

$$\begin{aligned}\psi_2(S_n(\cdot;\rho_n(u))) &\leq \psi_2(u) + \left|\int_\Theta S_n(t;\rho_n(u)) - u(t)\mu(dt)\right| \\ &\leq \epsilon_2 + \int_\Theta |S_n(t;\rho_n(u)) - u(t)|\mu(dt) \\ &\leq \epsilon_2 + \|u - S_n(\cdot;\rho(u))\|_\infty \\ &\leq \epsilon_2 + \delta.\end{aligned}$$

Hence, it shows that $\rho_n(u) \in \mathfrak{S}(\epsilon + \delta)$. □

5.3.2 Proof of Theorem 3.3

Let u^* be the minimizer of the robust preference model (9). We define $u_\delta := \frac{\delta}{2}u^r + \left(1 - \frac{\delta}{2}\right)u^*$, and have

$$\|u_\delta - u^*\|_\infty = \frac{\delta}{2}\|u_\delta - u^*\|_\infty \leq \frac{\delta}{2}.$$

Using the fact $u^r \in \mathfrak{U}(\epsilon)$, the convexity of the set $\mathfrak{U}(\epsilon)$ shown in Proposition 2.1 indicates that $u_\delta \in \mathfrak{U}(\epsilon)$. Since

$$\psi_1(u_\delta) = \left(1 - \frac{\delta}{2}\right)^2 \int_\Theta (u^*(t) - u^r(t))^2\mu(dt) \leq \left(\left(1 - \frac{\delta}{2}\right)\epsilon_1\right)^2 \leq \left(\epsilon_1 - \frac{\delta}{2}\min\{\epsilon_1,\epsilon_2\}\right)^2,$$

and

$$\psi_2(u_\delta) = \left(1 - \frac{\delta}{2}\right)\left|\int_\Theta u^*(t) - u^r(t)\mu(dt)\right| \leq \left(1 - \frac{\delta}{2}\right)\epsilon_2 \leq \epsilon_2 - \frac{\delta}{2}\min\{\epsilon_1,\epsilon_2\},$$

we have $u_\delta \in \mathfrak{U}\left(\epsilon - \frac{\delta e}{2}\min\{\epsilon_1,\epsilon_2\}\right)$. By Lemma 5.2 and the choice of n in the assumption, we have $\rho_n(u_\delta) \in \mathfrak{C}(\epsilon)$ and

$$\|S_n(\cdot;\rho_n(u_\delta)) - u_\delta\|_\infty \leq \frac{\delta}{2}\min\{\epsilon_1,\epsilon_2\} \leq \frac{\delta}{2}.$$

It follows that

$$\|S_n(\cdot;\rho_n(u_\delta)) - u^*\|_\infty \leq \|S_n(\cdot;\rho_n(u_\delta)) - u_\delta\|_\infty + \|u_\delta - u^*\|_\infty \leq \delta.$$

Hence, for alternative $k \in \mathfrak{K}$,

$$\pi_n(k) \le \mathbb{E}[S_n(X^k;\rho_n(u_\delta))] \le \mathbb{E}[u^*(X^k)] + \delta = \pi(k) + \delta.$$

On the other hand, Proposition 3.2 implies that

$$\pi(x) \le \pi_n(x).$$

6 Conclusions

This paper develops a multiattribute preference ranking rule based on a nonparametric perturbation of a given additive reference utility function. The concept of robust preference value is defined using the worst expected utility of an alternative under the perturbation. Alternatives are ranked by comparing their robust preference values.

The perturbation is specified using tolerance and symmetric conditions. The tolerance condition describes a neighborhood of the reference utility on the functional $\mathcal{L}_2$ metric space, while the symmetric condition requires that the perturbation should be symmetrically around the reference utility at both up and down sides. We use the perturbation to solve the problem of ambiguity and inconsistency in utility assessments but preserve the additive structure of the reference utility and the decision maker's risk preference (risk-aversion, risk-neutralism, risk-loving, or S-shapedness) under every attribute. An approximation approach using Bernstein polynomials is developed to solve the robust preference model. We reformulate the constructed approximation problem as a QCP problem, and analyze the bound of the approximation error with respect to the degrees of Bernstein polynomials.

The robust ranking rule is applied in an integrated energy distribution system planning project. In this project, five experts evaluate four energy supply plans for a newly developed suburb in Norway, by considering potential economical and environmental impacts. We compare the robust ranking model with the decision based on the conventional expected utility theory. The results show that a small perturbation on the reference utility may result in a large changes on the ranking orders of the plans. We also analyze the worst-case utility function under the perturbation, and test the accuracy of the approximation approach when using different degrees of Bernstein polynomials.

References

A. Botterud, M. Catrinu, O. Wolfgang, and A. T. Holen. Integrated energy distribution system planning: a multi-criteria approach. In *Proceedings In 15th Power Systems Computation Conference*, Liege, Belgium, August 2005.

C. Boutilier, R. Patrascu, P. Poupart, and D. Schuurmans. Constraint-based optimization and utility elicitation using the minimax decision criterion. *Artificial Intelligence*, 170(8-9):686–713, 2006.

S. Boyd and L. Vandenberghe. *Convex Optimization.* Cambridge University Press, 2004.

G. Debreu. Topological methods in cardinal utility theory. In K. Arrow, S. Karlin, and P. Suppes, editors, *Mathematical Methods in the Social Sciences*, pages 16–26. Stanford University Press, 1960.

J. S. Dyer. Maut - multiattribute utility theory. In J. Figueira, S. Greco, and M. Ehrgott, editors, *Multiple Criteria Decision Analysis: State of the Art Surveys*, chapter 7, pages 265–295. Springer, 2005.

W. Edwards. How to use multiattribute utility measurement for social decision making. *IEEE Transactions on Systems Man and Cybernetics*, 7(5):326–340, 1977.

P. H. Farquhar. Utility assessment methods. *Management Science*, 30(11):1283–1300, 1984.

P. C. Fishburn. Independence in utility theory with whole product sets. *Operations Research*, 13 (1):28–45, 1965.

W. Gorman. The structure of utility functions. *Review of Economic Studies*, 35:367–390, 1968.

J. Hu. Decision making based on a nonparametric shape-preserving perturbation of a reference utility function. Depart. of IMSE, University of Michigan - Dearborn, `http://www.optimization-online.org/DB_HTML/2013/11/4135.html`, 2013.

J. Hu and S. Mehrotra. Robust decision making using a risk averse utility set with an application to portfolio optimization. Submitted to IIE Transaction `http://www.optimization-online.org/DB_FILE/2012/03/3412.pdf`, 2012a.

J. Hu and S. Mehrotra. Robust decision making using a general utility set. under second review for SIAM Journal on Optimization `http://www.optimization-online.org/DB_FILE/2012/07/3527.pdf`, 2012b.

E. Jacquet-Lagrèze and Y. Siskos. Assessing a set of additive utility functions for multicriteria decision making: The uta method. *European Journal of Operational Research*, 10(2):151–164, 1982.

D. Kahneman and A. Tversky. Prospect theory: An analysis of decision under risk. *Econometrica*, 47(2):263–291, 1979.

U. S. Karmarkar. Subjectively weighted utility: A descriptive extension of the expected utility model. *Organizational Behavior and Human Performance*, 21(1):61–72, 1978.

R. L. Keeney and H. Raiffa. *Decisions with multiple objectives: preferences and value tradeoffs*. John Wiley & Sons, New York, 1976.

E. Løken. *Multi-Criteria Planning of Local Energy Systems with Multiple Energy Carriers*. PhD thesis, Norwegian University of Science and Technology, Trondheim, Norway, April 2007.

R. Luce and J. Tukey. Simultaneous conjoint measurement: A new type of fundamental measurement. *Journal of Mathematical Psychology*, 1(1):1–27, 1964.

L. Martein and S. Schaible. On solving a linear program with one quadratic constraint. *Rivista di matematica per le scienze economiche e sociali*, 10(1-2):75–90, 1987.

G. M. Philips. *Interpolation and Approximation by Polynomials*. Springer-Verlag, New York, 2003.

T. J. Rivlin. *An Introduction to the Approximation of Functions*. Dover, New York, 1891.

T. L. Saaty. *The analytc hierarchy process*. McGraw-Hill, 1980.

C. van de Panne. Programming with a quadratic constraint. *Management Science*, 12(11):798–815, 1966.

D. von Winterfeldt and W. Edwards. *Decision Analysis and Behavioral Research*. Cambridge University Press, 1986.

P. Wakker and D. Deneffe. Eliciting von neumann-morgenstern utilities when probabilities are distorted or unknown. *Management Science*, 42(8):1131–1150, 1996.

M. Weber. Decision making with incomplete information. *European Journal of Operational Research*, 28(1):44–57, 1987.