On the Relation of Flow Cuts and Irreducible Infeasible Subsystems

Imke Joormann and Marc E. Pfetsch

Research Group Optimization, Dept. of Mathematics, TU Darmstadt, Germany,
{joormann,pfetsch}@opt.tu-darmstadt.de

March 2014

Abstract

Infeasible network flow problems with supplies and demands can be characterized via violated cut-inequalities of the classical Gale-Hoffman theorem. Written as a linear program, irreducible infeasible subsystems (IISs) provide a different means of infeasibility characterization. In this article, we answer a question left open in the literature, by showing a one-to-one correspondence between IISs and Gale-Hoffman-inequalities in which one side of the cut has to be weakly connected. We also give a polynomial-time algorithm that computes some IIS using a single max-flow computation and show strong NP-hardness of finding an IIS of minimal cardinality in this special case.

1 Introduction

Sometimes a linear program (LP) turns out to be infeasible, e.g., because of modeling errors or structural reasons. In this case, one would like to find the cause for its infeasibility. One way is to study irreducible infeasible subsystems (IISs), i.e., infeasible subsystems such that each proper subsystem is feasible. IISs might help to identify the reason of infeasibility and are basic structures in infeasibility analysis.

In this article, we study the special case of a flow system for a simple, directed graph $G = (V, E)$ with upper flow bounds $u \in \mathbb{R}^E$, lower flow bounds $\ell \in \mathbb{R}^E$, and a supply vector $b \in \mathbb{R}^V$. Thus, we consider the system

$$
\begin{align*}
x(\delta^+(v)) - x(\delta^-(v)) &= b(v) \quad \forall v \in V, \\
\ell \leq x \leq u,
\end{align*}
$$

where, for $S \subseteq V$ and $\bar{S} := V \setminus S$, we use the following standard notation: $\delta^+(S) := \{(v, w) \in E \mid v \in S, w \in \bar{S}\}$, $\delta^-(S) := \{(v, w) \in E \mid v \in \bar{S}, w \in S\}$, and $\delta(S) := \delta^+(S) \cup \delta^-(S)$; we also abbreviate $\delta^+(v) := \delta^+(\{v\})$ and similarly for $\delta^-(v)$ and $\delta(v)$. Moreover, for some vector $v \in \mathbb{R}^I$ with finite index sets $I$ and $I' \subseteq I$, we use $v(I') := \sum_{i \in I'} v_i$ and often write $v(i) := v(\{i\}) = v_i$. We assume throughout the article that $0 \leq \ell \leq u$ in order to avoid trivial infeasibilities.

A characterization of feasibility for this flow system is then well known:

**Theorem 1** (Gale and Hoffman [12, 20]). The network flow system $(F_\ell)$ is infeasible if and only if there exists $S \subseteq V$ such that

$$
b(S) > u(\delta^+(S)) - \ell(\delta^-(S)).
$$

1
A natural question is how IISs in the flow case and the *Gale-Hoffman-inequalities* (1) are related. In this article, we show that the IISs of $(F =)$ correspond to exactly those violated inequalities (1) for which the induced subgraph $G[S]$ is weakly connected. This implies, for instance, that there can be exponentially many IISs, see Remark 20.

The above question has been touched upon in the literature before: Greenberg [15] discusses the analysis of infeasible flow systems (see also [14]). He presents several heuristics to “localize” the cause of infeasibility, i.e., he tries to isolate small sets $S$ with violated Gale-Hoffman-inequalities. In [16], Greenberg further gives an example of a violated Gale-Hoffman-inequality that does not lead to an IIS and states that “there is presently no theory to construct an IIS from a violating cut, other than general methods […].” The missing link is connectivity of one of the sides of the cut corresponding to a Gale-Hoffman-inequality, see Section 2.

The problem of finding small sets $S$ with violated Gale-Hoffman-inequalities was investigated by Aggarwal et. al. [1]. They call $S$ a *witness* of infeasibility and show that the problem of finding a minimum witness (i.e., one of smallest cardinality) is strongly $NP$-hard. They further design an efficient algorithm to find a minimal witness (w.r.t. inclusion) based on preflow-push algorithms.

IISs and witnesses in flow networks can both be used to reveal a smaller portion of the network “witnessing” the infeasibility. For witnesses, the number of nodes is relevant, while for IISs the number of constraints corresponding to both nodes and arcs count. In Section 4, we further discuss the relation of IISs and witnesses. With respect to computational complexity, IISs have similar properties as witnesses: We show, in Section 5, that the minimum IIS problem, i.e., to find an IIS of smallest cardinality, is strongly $NP$-hard, extending the result that this problem is strongly $NP$-hard for linear inequality systems [3]. Some IIS, however, can be computed in polynomial time using one maximum flow computation, see Section 3.

For the analysis of general infeasible inequality systems, many approaches have been developed – we refer to the book of Chinneck [10] for an overview and only mention selected references here. The term IIS was coined by van Loon [28] as a means to analyze infeasibilities in linear programs. Gleeson and Ryan [13] gave a characterization of IISs related to an alternative polyhedron, see Theorem 2 below. The analysis of infeasible linear programs was further discussed by Greenberg and Murphy [17], including the case of flow networks. Dravnieks and Chinneck [11] and Chinneck [8] developed heuristics to isolate IISs (see also [6] for an application to flow systems). An overview of these approaches appeared in Chinneck [7]. Finally, Ryan [26] investigated combinatorial properties of IISs.

Related topics are methods to find maximum feasible subsystems, which are complementary to IIS-covers. Chinneck [5, 9] develops heuristics for this problem. An exact algorithm appears in [23], based on [3]. Moreover, McCormick [22] studies the related problem of finding least infeasible flows.

The remainder of this paper is structured as follows: In Section 2, we show the mentioned correspondence between IISs and Gale-Hoffman-inequalities. Section 3 shows that some IIS can be computed in polynomial time, using a single max-flow computation. In Section 4, we discuss the relation of minimum/minimal witnesses and IISs. Section 5 shows that it is $NP$-hard to compute an IIS of minimum cardinality. We close with an outlook for future research in Section 6.
2 IISs in Flow Networks

Sometimes flow systems are defined with inequalities instead of equalities:
\[ x(\delta^+(v)) - x(\delta^-(v)) \geq b(v) \quad \forall v \in V, \quad (F_{\geq}) \]
\[ \ell \leq x \leq u. \]

Note the reverse direction can be considered as well, see Section 2.3 below.

In fact, if \( b(V) = 0 \), the inequalities in \((F_{\geq})\) are always satisfied with equality by any feasible flow (see [16]). However, this statement does not help in our context since we are dealing with infeasibility, on the one hand, and subsystems, on the other hand. Nevertheless, we will show the claimed equivalence for these three cases, starting with \((F_{\geq})\) in the next two subsections.

We will use the following fundamental theorem characterizing IISs:

**Theorem 2** (Gleeson and Ryan [13]). The index sets of IISs of an infeasible inequality system \( Ax \leq b \), with \( A \in \mathbb{R}^{n \times m} \) and \( b \in \mathbb{R}^n \), are exactly the supports of vertices of the associated alternative polyhedron
\[
\{ y \in \mathbb{R}^n_+ \mid A^\top y = 0, \; b^\top y = -1 \}. 
\]

Note that the alternative polyhedron lies in the nonnegative orthant.

The alternative polyhedron associated to \((F_{\geq})\) is
\[
P_{\geq} := \left\{ \begin{pmatrix} y \\ p \\ q \end{pmatrix} \in \mathbb{R}^{V \times E \times E} \left| \begin{bmatrix} -A^\top & I & -I \\ -b^\top & u^\top & -\ell^\top \end{bmatrix} \begin{pmatrix} y \\ p \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right. \right\} \quad (2)
\]
where \( A = (A_{ve})_{v \in V, e \in E} \in \{0, \pm 1\}^{V \times E} \) is the node-arc-incidence matrix of \( G \), and \( I \) denotes the identity matrix of appropriate size. We will write
\[ A := \begin{bmatrix} -A^\top \\ -b^\top \end{bmatrix}, \; \mathcal{U} := \begin{bmatrix} I \\ u^\top \end{bmatrix}, \; \text{and} \; \mathcal{L} := \begin{bmatrix} -I \\ -\ell^\top \end{bmatrix}. \]

Moreover, we use \( n := |V|, \; m := |E| \), and denote the all-ones vector of suitable dimension by 1. We write \( [k] := \{1, \ldots, k\} \) for \( k \in \mathbb{Z}_{>0} \).

**Definition 3** (GH-cuts). For \( S \subseteq V \), a cut \( \delta(S) \) is called \( GH \)-cut if \( b(S) > u(\delta^+(S)) - \ell(\delta^-(S)) \). A \( GH \)-cut for which \( G[S] := (S, \mathcal{E}[S]), \; \mathcal{E}[S] := \{(u, v) \in E \mid u, v \in S\} \), is weakly connected is called connected \( GH \)-cut. For a \( GH \)-cut \( \delta(S) \), we call the system
\[
\begin{align*}
x(\delta^+(v)) - x(\delta^-(v)) & \geq b(v) \quad \forall v \in S, \\
x_e & \geq \ell_e \quad \forall e \in \delta^-(S), \\
x_e & \leq u_e \quad \forall e \in \delta^+(S),
\end{align*}
\]
\( GH \)-subsystem and \( S \) the associated \( GH \)-set.

In the remainder of this section, we will show that connected \( GH \)-cut systems and IISs are equivalent (see Theorem 14 below). In order to simplify notation, we will identify the constraints of a \( GH \)-subsystem or IIS with the corresponding columns of the matrix \([A, \mathcal{U}, \mathcal{L}]\).

For a node-arc-incidence matrix \( A \) of a connected, directed graph, it is well-known that \( \text{rank}(A) = n - 1 \) (see, e.g., [2]). We will need the following variant of the statement:
Lemma 4. Let $A$ be a node-arc-incidence matrix of a connected directed graph. Then
\[ \ker(A^\top) := \{ y \mid A^\top y = 0 \} = \{ cI \mid c \in \mathbb{R} \}. \]

Proof. Using the rank-nullity theorem, $n - 1 = \text{rank}(A) = \text{rank}(A^\top) = n - \text{dim}(\ker(A^\top))$. Hence, $\text{dim}(\ker(A^\top)) = 1$. Due to the structure of incidence matrices, obviously $A^\top I = 0$, which shows the statement. \qed

2.1 Connected GH-Cuts Give Rise to IISs

We will use the following notation: For $S \subseteq V$, we denote by $A_S$ the matrix with rows indexed by $S$. We also identify a row $A_i$ of the matrix with the constraint $A_i x \geq b_i$ it describes. However, when discussing the alternative polyhedron, e.g., if we are dealing with $[A, U, \mathcal{L}]$, an indexed matrix refers to the respective column subset.

Proposition 5. Every GH-subsystem $J$ of a connected GH-cut $\delta(S)$ for $(F_\geq)$ is an IIS.

Proof. We will show that the index set of $J$ is the support of a vertex of the alternative polyhedron $P_\geq$ given in (2), by extending $J$ to a feasible basis in three steps. We start by determining the rank of the system.

Observe that the first $m$ rows of $[A, U, \mathcal{L}]$ are linearly independent since they contain an identity matrix; thus, $\text{rank}([A, U, \mathcal{L}]) \geq m$. Assume now that $\text{rank}([A, U, \mathcal{L}]) = m$. Then the last row must be representable by the first $m$ rows, i.e., there exists a nonzero solution $h \in \mathbb{R}^m$ of
\[ h^\top [-A^\top, I, -I] = [-b^\top, u^\top, -\ell^\top]. \]

Regarding only the submatrix $U$ yields $h^\top I = u^\top$, so $h = u$. Then, for the submatrix $\mathcal{L}$, we obtain $-h^\top I = -\ell^\top$, so $u \equiv \ell$. Consequently, for the submatrix $A$, $-u^\top A^\top = -b^\top$, or equivalently, $Au = b$. But then $J$ has a feasible solution, namely $x = u = \ell$ (and $\delta(S)$ is not a GH-cut). Therefore, $\text{rank}([A, U, \mathcal{L}]) = m + 1$.

We proceed by finding $m + 1$ linearly independent columns to obtain a basis (see Figure 1 for an example). Throughout the proof, we will show the independency of the respective columns by demonstrating that the nullspace of the corresponding submatrix is trivial.

Step 1. We start with the columns corresponding to $J$. These columns without the last row, $[-A^\top, I, -I]_J$, form a transposed incidence matrix: Every row contains exactly one 1 and one -1 since for each edge $e \in \delta(S)$, the missing head or tail node entry is supplied by the column $U_e$ or $\mathcal{L}_e$, respectively. The corresponding graph is connected by assumption. Therefore, by Lemma 4, the only solution of $[-A^\top, I, -I]_J d = 0$ is $d = cI$ for $c \in \mathbb{R}$. This implies $c(-b(S) + u(\delta^+(S)) - \ell(\delta^-(S))) = 0$ for the last row. By the GH-cut property $b(S) > u(\delta^+(S)) - \ell(\delta^-(S))$. Thus, $c = 0$, and the columns are linearly independent.

Step 2. Let $(S, T)$ be a spanning tree in $G[S]$. We extend the partial basis by the columns $U_e$ for every $e \in E[S] \setminus T$. The resulting columns are linearly independent: Consider
\[ [-A_S^\top, I_{\delta^+(S) \cup E[S] \setminus T}, -I_{\delta^-(S)}] d = 0. \]

Regarding only the rows corresponding to $T$, the resulting matrix is again a transposed incidence matrix of a connected graph (with additional all-zero columns), since only columns $U_e$ for $e \notin T$ were added. Thus, we know that $d_S = cI$ (for some $c \in \mathbb{R}$) by Lemma 4. In particular, we have determined the multiplier $d_v$ for every node $v \in S$ due to the spanning feature of the tree.
Considering the rows corresponding to \( e \in \delta(S) \), we see that they still contain only two nonzero entries, since w.r.t. the new columns \( U_e \), \( E[S] \cap \delta(S) = \emptyset \). With the same argumentation as in Step 1, we also have \( d_{\delta^-(S)} = d_{\delta^+(S)} = c1 \).

The rows corresponding to \( E[S] \setminus T \) have exactly three nonzero entries: The constraint for \( e = (v,w) \in E[S] \setminus T \) is \(-d_v + d_w + d_e = 0\). Since by the above, \( d_v = d_w = c \) for all \( v,w \in S \), it follows that \( d_e = 0 \) for all \( e \in E[S] \setminus T \). Combining all these multipliers yields for the last row \( c ( -b(S) + u(\delta^+(S)) - \ell(\delta^-(S)) ) + 0 \cdot u(E[S] \setminus T) = 0 \). Thus, as above, \( c = 0 \), and the columns are linearly independent.

**Step 3.** We append columns \( L_e \) for each \( e \in E[S] \). The column for any \( e \in E[S] \) has exactly one \(-1\) in the row corresponding to \( e \), \(-\ell_e\) in the last row, and zeros everywhere else. Since \( e \notin E[S] \cup \delta(S) \), row \( e \) has zeros in all columns chosen in Step 1 and 2, respectively. It follows that by adding columns \( L_e \), we maintain a linearly independent matrix.

We have now chosen \((|S| + |\delta^+(S)| + |\delta^-(S)|) + (|E[S]| - (|S| - 1)) + |E[S]| = m + 1\) columns and have indeed constructed a basis. It remains to show that there exists a feasible basic solution with the claimed support \( J \) for the alternative polyhedron. To this end, let \( c = -1/(-b(S) + u(\delta^+(S)) - \ell(\delta^-(S))) \), and

\[
y_e = \begin{cases} 
c, & \text{for } v \in S, \\
0, & \text{otherwise,}
\end{cases}
\]

\[
p_e = \begin{cases} 
c, & \text{for } e \in \delta^+(S), \\
0, & \text{otherwise,}
\end{cases}
\]

\[
q_e = \begin{cases} 
c, & \text{for } e \in \delta^-(S), \\
0, & \text{otherwise.}
\end{cases}
\]

Note that \( c > 0 \), because \( S \) defines a GH-cut. Consequently, \( y, p, q \) are feasible: The supported columns and the first \( m \) rows belong to an incidence matrix as discussed in Step 1 and hence, Lemma 4 is applicable. Moreover, the last row yields \(-1\) by construction. Together with the constructed basis from above, this yields a feasible basic solution. Hence, \( J \) forms the support of a vertex of the alternative polyhedron. We can now use Theorem 2.

### 2.2 IISs Give Rise to Connected GH-Cuts

We will proceed to show the reverse direction, i.e., that every IIS must have the specific form of a connected GH-system by a series of claims. This direction seems to require
more work – one reason might be degeneracy, see Remark 15.

In the following, we need some more notation: Let $\sigma_v$ refer to the flow conservation constraint for a node $v \in V$, $\mu_e$ to the constraint $x_e \leq u_e$, and $\lambda_e$ to the constraint $x_e \geq \ell_e$ for $e \in E$. For an IIS $\mathcal{I}$, let $S_\mathcal{I} := \{v \in V \mid \sigma_v \in \mathcal{I}\}$.

**Lemma 6.** If $\mathcal{I}$ is an IIS, then $G' = (S_\mathcal{I}, E[S_\mathcal{I}])$ is connected.

**Proof.** Let $\mathcal{I}$ be an IIS, and suppose $G'$ is disconnected. Then there exists a nontrivial partition $S_1 \cup S_2 = S_\mathcal{I}$ and $E[S_1] \cup E[S_2] = E[S_\mathcal{I}]$. Letting $E := E \setminus (E[S_1] \cup E[S_2] \cup \delta(S_1) \cup \delta(S_2))$, we observe that after possible reordering $A_{S_\mathcal{I}}$ has the following structure:

$$
\begin{bmatrix}
E[S_1] & E[S_2] & \delta(S_1) & \delta(S_2) & \bar{E} \\
S_1 & * & 0 & * & 0 \\
S_2 & 0 & * & 0 & *
\end{bmatrix}
$$

Furthermore, consider the systems $\mathcal{I}_1 := \{\sigma_v \mid v \in S_1\} \cup \{\lambda_e \in \mathcal{I}\} \cup \{\mu_e \in \mathcal{I}\}$ and $\mathcal{I}_2 := \{\sigma_v \mid v \in S_2\} \cup \{\lambda_e \in \mathcal{I}\} \cup \{\mu_e \in \mathcal{I}\}$ with $S_{\mathcal{I}_1} = S_1$ and $S_{\mathcal{I}_2} = S_2$. Since $\mathcal{I}_1$ and $\mathcal{I}_2$ are proper subsets of $\mathcal{I}$, they must be feasible. Consequently

$$
\exists x^1 \in \mathbb{R}^E : A_{S_1} x^1 \geq b_{S_1}, \quad \ell_e \leq x^1_e \forall \lambda_e \in \mathcal{I}, \quad x^1_e \leq u_e \forall \mu_e \in \mathcal{I},
$$

$$
\exists x^2 \in \mathbb{R}^E : A_{S_2} x^2 \geq b_{S_2}, \quad \ell_e \leq x^2_e \forall \lambda_e \in \mathcal{I}, \quad x^2_e \leq u_e \forall \mu_e \in \mathcal{I}.
$$

Let $x^* \in \mathbb{R}^E$ be defined by $x^*_e := x^1_e$ for $e \in E[S_1] \cup \delta(S_1)$, $x^*_e := x^2_e$ for $e \in E[S_2] \cup \delta(S_2)$, $x^*_e$ arbitrary for $e \in \bar{E}$. By the structure of $A_{S_\mathcal{I}}$ displayed above, we have $A_{S_1} x^* \geq b_{S_1}$ and $A_{S_2} x^* \geq b_{S_2}$. Moreover, $x^*$ obeys all bounds in $\mathcal{I}$ by construction. Thus, we found a feasible solution for the system $\mathcal{I}$, which is a contradiction.

We note that if $\mathcal{I}$ is infeasible, $S_\mathcal{I} \neq \emptyset$, because otherwise only flow bounds would remain, yielding a feasible system by the assumption $\ell \leq u$.

**Lemma 7.** If $\mathcal{I}$ is an IIS, then $\mu_e, \lambda_e \notin \mathcal{I}$ for all $e \in E[S_\mathcal{I}]$.

**Proof.** Suppose there is an $\hat{e} \in E[S_\mathcal{I}]$ with $\mu_{\hat{e}} \in \mathcal{I}$. With $\mathcal{I}_1 := \mathcal{I} \setminus \{\mu_{\hat{e}}\}$, $\mathcal{I}_2 := \{\mu_{\hat{e}}\}$, we can proceed as in the proof of Lemma 6: There exists a solution $x^1$ to $\mathcal{I}_1$ and we define $x^*$ by $x^*_e := x^1_e$ for $e \neq \hat{e}$ and $x^*_e := u_{\hat{e}}$. Then $x^*$ is a solution to $\mathcal{I}$, which contradicts infeasibility.

The proof for $\lambda_e \in \mathcal{I}$ follows analogously.

We will need the following general lemma, which was mentioned by Chinneck [8] without a proof.

**Lemma 8.** Let $Ax \leq b$ be a general system with $A \in \mathbb{R}^{s \times r}$, $b \in \mathbb{R}^s$, and let $\ell_i \leq x_i$ for some variable index $i \in [r]$. Then there cannot exist an IIS $\{Ax \leq b, \ x_i \geq \ell_i, \ x_i \leq u_i\}$.

**Proof.** Suppose there exists an IIS $\mathcal{I} := \{Ax \leq b, \ x_i \geq \ell_i, \ x_i \leq u_i\}$. Then there are solutions $x^1$ and $x^2$ for $Ax \leq b$ with $x^1_i < \ell_i$ and $x^2_i > u_i$. But every convex combination of $x^1$ and $x^2$ is also a solution, so $\mathcal{I}$ is feasible.

In fact, this can easily be generalized to the following lemma, which we state for completeness.

**Lemma 9.** Let $A \in \mathbb{R}^{s \times r}$, $b \in \mathbb{R}^s$, $\alpha \in \mathbb{R}^r$ and $\beta' \leq \beta \in \mathbb{R}$. Then there cannot exist an IIS $\{Ax \leq b, \ \alpha^\top x \geq \beta', \ \alpha^\top x \leq \beta\}$. 


Lemma 10. If $\mathcal{I}$ is an IIS, then $\mu_e, \lambda_e \notin \mathcal{I}$ for all $e \in E[S_T]$.

Proof. Let $\mathcal{I}$ be an IIS and suppose $\mu_e \in \mathcal{I}$ for $\hat{e} \in E[S_T]$. Then there must be a feasible basic solution for the alternative polyhedron given in (2) with $p_{\hat{e}} \neq 0$, according to Theorem 2. Furthermore, $-A_{S_T}^T$ is part of the basis.

By Lemma 6, $G[S_T]$ is connected. Now assume that for some spanning tree $T$ in $G[S_T]$, no column $\mathcal{U}_e$ for $e \in T$ is included in the basis. Therefore, only columns corresponding to $e \notin T$ can be in the basis. We have seen in Step 2 of the proof of Proposition 5 that in this case all components of a possible basic solution corresponding to $e \notin T$ are 0. Thus, if $\hat{e} \in T$, it cannot be in the basis, and if $\hat{e} \notin T$, it cannot be in the support. Therefore, no such spanning tree exists.

Hence, the basis contains arc bounds $\mathcal{U}_e$ for at least one arc per spanning tree, i.e., it contains bounds $\mathcal{U}_e$ corresponding to a spanning tree cover $C \subseteq E[S_T]$. But removing the arcs of a spanning tree cover splits $G[S_T]$ into (at least) two disjoint components. Regarding the system

$\begin{bmatrix}
-A_{S_T}^T & I_C \\
-b^T_{S_T} & u^T_C
\end{bmatrix} \begin{bmatrix} d \end{bmatrix} = 0,
$}

we see that the first $m$ rows consist of two different kinds of constraints:

\begin{align*}
-d_v + d_w + d_e &= 0 & \forall e = (v, w) \in C, \\
-d_v + d_w &= 0 & \forall e = (v, w) \in E[S_T] \setminus C.
\end{align*}

Since the graph $(S_T, E[S_T] \setminus C)$ is disconnected by construction, let $S_T = S_1 \cup \cdots \cup S_k$ denote the node sets of the connected components. We need the notation

$[U : W] := \{e = (u, w) \in E \mid u \in U, w \in W\}$

for $U, W \subseteq V$.

Note that $e = (v, w) \in E[S_T] \setminus C$ iff $v$ and $w$ are in the same component $S_i, i \in [k]$. Thus, by Lemma 4,

$\begin{align*}
d_v &= c_i, & \forall v \in S_i, \\
d_e &= c_i - c_j, & \forall e \in E[S_i : S_j],
\end{align*}$

yields a feasible solution for (5), where $c_i \in \mathbb{R}$ is arbitrary for each $i \in [k]$. In fact, we can assume that $c_i := \alpha$ for $i \in [k - 1]$ and $c_k := \beta$. Then, with $\bar{S} := S_1 \cup \cdots \cup S_{k-1}$, calculating the last row of (4) yields:

$-\alpha b(\bar{S}) - \beta b(S_k) + \alpha u(C \setminus ([\bar{S} : S_k] \cup [S_k : \bar{S}])) + (\alpha - \beta) u([\bar{S} : S_k]) + (\beta - \alpha) u([S_k : \bar{S}]) = 0.$

A nonzero solution is given by:

$\beta = \alpha \frac{-b(\bar{S}) + u(C \setminus ([\bar{S} : S_k] \cup [S_k : \bar{S}])) + u([\bar{S} : S_k]) - u([S_k : \bar{S}])}{b(S_k) + u([S_k : \bar{S}]) - u([S_k : \bar{S}])},$

except if the denominator is 0, in which case we set $\alpha = 0, \beta \neq 0$ arbitrarily. In total, we have found a nonzero solution for (4), and the corresponding columns are thus linearly dependent. Hence, the feasible basis corresponding to $\mathcal{I}$ cannot contain a spanning tree cover, and we have shown: $\mu_e \notin \mathcal{I}$ for all $e \in E[S_T]$.

For $\lambda_e \in \mathcal{I}$, the proof runs analogously. Moreover, the case $e \in E[S_T]$ with $\mu_e \in \mathcal{I}$ and $\lambda_e \in \mathcal{I}$ is impossible by Lemma 8. \qed
Lemma 11. Let \( \mathcal{I} \) be an IIS, then \( \mu_e \in \mathcal{I} \) for all \( e \in \delta^+(S_I) \) and \( \lambda_e \in \mathcal{I} \) for all \( e \in \delta^-(S_I) \).

Proof. Let \( \mathcal{I} \) be an IIS. Therefore, according to Theorem 2, there must be a solution of \( P_\geq \). For the row corresponding to an arc \( e = (v, w) \in \delta(S_I) \), we have

\[
-y_v + y_w + p_e - q_e = 0 \\
y_v, y_w, p_e, q_e \geq 0.
\]

For \( e \in \delta^+(S_I) \), we have \( v \in S_I, w \notin S_I \). Thus, \( y_v > 0 \) and \( y_w = 0 \). But then \( p_e > 0 \), and therefore \( \mu_e \in \mathcal{I} \). Similarly, for \( (v, w) = e \in \delta^-(S_I) \), we have \( y_w = 0 \) and \( y_v > 0 \). Thus, \( q_e > 0 \) and \( \lambda_e \in \mathcal{I} \).

Lemma 12. Let \( \mathcal{I} \) be an IIS, then \( \mu_e \notin \mathcal{I} \) for all \( e \in \delta^-(S_I) \) and \( \lambda_e \notin \mathcal{I} \) for all \( e \in \delta^+(S_I) \).

Proof. The result follows immediately from Lemma 8 and Lemma 11.

Proposition 13. If \( \mathcal{I} \) is an IIS, then \( \mathcal{I} \) is a GH-subsystem of a connected GH-cut.

Proof. An IIS \( \mathcal{I} \) contains flow conservation constraints for a connected subset by Lemma 6. Furthermore, \( \mathcal{I} \) contains the upper bounds of outgoing and the lower bounds of ingoing arcs (Lemma 11). Moreover, \( \mathcal{I} \) contains no bounds beyond that (see Lemma 7 for \( e \in E[S_I] \), Lemma 10 for \( e \in E[S_I] \), and Lemma 12 for \( e \in \delta(S_I) \)). Hereafter, the claim follows.

Thus, Proposition 5 and Proposition 13 show the following theorem:

Theorem 14. A subsystem \( \mathcal{I} \) of the network flow problem \( (F_\geq) \) is an IIS if and only if \( \mathcal{I} \) is the GH-subsystem of a connected GH-cut.

Remark 15. Theorem 14 shows that the alternative polyhedron \( P_\geq \) can be highly degenerate: Consider a node \( v \) with supply \( b(v) = 2 \), which has a single incident arc \( e = (v, w) \) with \( \ell_e = 0, u_e = 1 \). Then \( \delta^+(v) \) is a GH-cut, so the corresponding vertex of the alternative polyhedron has a support of size two. The alternative system has \( m + 1 \) rows. Thus, for large \( m \), the alternative polyhedron can be “arbitrarily degenerate”.

This fact has consequences on the enumeration of IISs: Note that enumerating the vertices of an (unbounded) polyhedron is hard, see Khachiyan et al. [21]. However, if the polyhedron is non-degenerate (simple), the vertices can be enumerated in output-polynomial time, i.e., in the size of the input and output, adapting the reverse search method by Avis and Fukuda [4]; see [19] for a discussion.

Remark 16. Moreover, note that the results in this paper, and Theorem 14 in particular, remain valid if we require the flow to be integral and the lower and upper bounds are integral because of the total unimodularity of the system.

2.3 Demand and Equality Forms

A flow problem can also be formulated in \( \leq \)-form. In this case, the GH-inequalities have the complementary form:

\[
-b(S) \geq u(\delta^-(S)) - \ell(\delta^+(S)).
\]
We call this \textit{demand form}, since it belongs to the subset with an unfulfilled demand, while the form stated in Theorem 1 describes the subset with an unmet supply. The \textit{GH-demand-subsystem} is defined analogously to Definition 3:

\[
x(\delta^+(v)) - x(\delta^-(v)) \leq b(v) \quad \forall v \in S,
\]
\[
x_e \geq \ell_e \quad \forall e \in \delta^+(S),
\]
\[
x_e \leq u_e \quad \forall e \in \delta^-(S).
\]

The above proofs can be easily adapted by exchanging $A$ with $-A$ and adapting the respective chosen columns, proving the following counterpart of Theorem 14:

\textbf{Corollary 17.} A subsystem $\mathcal{I}$ of the network flow problem in $\leq$-form is an IIS if and only if $\mathcal{I}$ is the GH-demand-subsystem of a connected GH-demand-cut.

We will now return to the equality-form ($F_\pm$). In this case, both supply- and demand-form of the GH-inequalities indicate infeasibility of the system. In fact, if for a subset $S \subseteq V$ the supply form is violated, $\bar{S}$ violates the demand form, and vice versa. However, $S$ might be connected, while $\bar{S}$ is not, and conversely. This means that we have to take both of them into account when determining IISs.

We will proceed to show that every IIS w.r.t. $Ax = b$, where $A$ is a node-arc-incidence matrix of a directed graph, corresponds to an IIS w.r.t. either $Ax \leq b$ or $Ax \geq b$.

\textbf{Lemma 18.} Each IIS $\mathcal{I}$ of the network system

\[
Ax \leq b, \quad Ax \geq b, \quad \ell \leq x, \quad x \leq u.
\]

can either contain constraints exclusively from $Ax \geq b$ or exclusively from $Ax \leq b$.

\textbf{Proof.} Let $\mathcal{I}$ be an IIS. The alternative polyhedron for the given system is

\[
P_\pm := \left\{ \begin{pmatrix} z \\ y \\ p \\ q \end{pmatrix} \in \mathbb{R}_+^{V \times V \times E} \ : \ \begin{bmatrix} A^T & -A^T \\ b & -b^T \\ I & -I \end{bmatrix} \begin{pmatrix} z \\ y \\ p \\ q \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right\}. \tag{6}
\]

One of the columns $[-A_v, -b_v]^T$ and $[A_v, b_v]^T$ must be part of the associated basis for each $v \in S_\mathcal{I}$, otherwise the system cannot be infeasible since we assume the bounds to be consistent. Additionally, these columns are linearly dependent, and cannot both be present simultaneously.

Suppose first that for some $\hat{v} \in S_\mathcal{I}$, the columns in the basis are $\hat{A} = [-A_{S_\mathcal{I}\setminus\{\hat{v}\}}, A_{\hat{v}}^T]$. Since $\hat{A}$ is a transposed incidence matrix with one column multiplied with $-1$, by Lemma 4 any solution for $\hat{A}d = 0$ is of the form $d_{S_\mathcal{I}\setminus\{\hat{v}\}} = c\mathbf{1}$, $d_{\hat{v}} = -c$, for some $c \in \mathbb{R} \setminus \{0\}$. But this is not feasible for (6) because $d \not\geq 0$.

Thus, in order to get a nonnegative solution of (6) as guaranteed by Theorem 2, the corresponding basis needs to include at least one additional column with a nonzero entry for every nonzero entry in $A_{\hat{v}}^T$. The only possible columns with this property are $U_e$ (or $L_e$) for $e \in \delta(\hat{v})$ since no further node constraint may have a nonzero multiplier. The corresponding constraints for all $e \in \delta(\hat{v})$, $w \in e$, have the form $\pm d_w \pm d_{\hat{v}} + d_e = 0$; the first two signs depend on the direction of $e$ and the last on whether $U_e$ or $L_e$ has been used. Fixing all $d_w = c$, yields a solution for all constraints for $e \in E[S \setminus \{\hat{v}\}]$, and we see that $d_e = \pm c \pm d_{\hat{v}}$ for all $e \in \delta(\hat{v})$; so all $d_e$ are equal and depend uniquely on $d_{\hat{v}}$. Thus, plugging all these expressions into the last row, we obtain an affine expression depending
on $d_i$. If the coefficient for $d_i$ is 0, the columns will be linearly dependent since summing
up the columns $[A_i, b_i] \top$ and $U_e$ or $L_e$, respectively, for $e \in \delta(\hat{v})$, provides 0. Otherwise,
we can determine $d_i$ yielding a zero value for the last row, which again shows linear dependency.

The proof can be easily extended to the case of an arbitrary proper mixture of positive
and negative columns.

Herewith we have obtained the equality-form of Theorem 14:

**Theorem 19.** A subsystem $I$ of the network flow problem $(F_m)$ is an IIS if and only
if $I$ is the GH-subsystem of a connected GH-cut (in either demand or supply form).

**Proof.** Every GH-cut in supply form is an IIS in $\geq$-form by Theorem 14, every GH-
cut in demand-form is an IIS in $\leq$-form by Corollary 17, and therefore an IIS of $(F_m)$.
With Lemma 18, every IIS $I$ is an IIS in either $\leq$- or $\geq$-form and therefore a connected
GH-cut. □

**Remark 20.** Theorem 19 has the following consequences: Wallace and Wets [29] proved
that the GH-inequalities are nonredundant if and only if both sides of the cut are weakly
connected. Moreover, they showed that there can be exponentially (in the number of
nodes) many nonredundant GH-cuts. Hence, there can be exponentially many IISs
of $(F_m)$.

The technique of splitting equations into two inequalities also works for general linear
systems, as we will illustrate now. We need the following claim.

**Lemma 21.** Let $Ax \leq b$ be an IIS for some $A \in \mathbb{R}^{s \times r}$, $b \in \mathbb{R}^s$. Then any proper
reorientation of the inequalities yields a feasible system.

**Proof.** Suppose that we reorient (at least) the $i$-th inequality, $i \in [s]$. In [3] it is proven
that $A'x = b'$ has a solution $x^*$, where $A' \in \mathbb{R}^{(s-1) \times r}$ and $b' \in \mathbb{R}^{s-1}$ are obtained by
removing row $i$ of $A$ and component $i$ of $b$, respectively. Thus, $x^*$ satisfies the reversed
$i$-th inequality (otherwise $Ax \leq b$ would be feasible), and any reorientation of $A'x \leq b'$
can be used. □

Consider $J = \{Ax \leq b, \, Dx = d\}$ with $A \in \mathbb{R}^{s \times r}$, $b \in \mathbb{R}^s$, $D \in \mathbb{R}^{t \times r}$, $d \in \mathbb{R}^t$. For $S \subseteq [s]$ and $T \subseteq [t]$, define the subsystem $J(S,T) := \{Ax \leq b, \, Dx = d_T\}$. Moreover,
for $T^1$, $T^2 \subseteq [t]$, $J \subseteq (S,T^1,T^2) := \{Ax \leq b, \, D_{T^1} x \leq d_{T^1}, \, D_{T^2} x \geq d_{T^2}\}$.

**Lemma 22.** For each IIS $J(S,T)$, where $S \subseteq [s]$ and $T \subseteq [t]$, there exists a partition
$T^1 \cup T^2 = T$ such that $J \subseteq (S,T^1,T^2)$ is an IIS. Conversely, each IIS $J \subseteq (S,T^1,T^2)$
yields an IIS $J(S,T^1 \cup T^2)$.

**Proof.** Let $J(S,T)$ be an IIS. Clearly, $J \subseteq (S,T^1,T^2)$ with $T^1$, $T^2 \subseteq T$ exists. Then $T^1 \cap T^2 = \emptyset$ must hold, otherwise the corresponding columns in a basis for the alternative system would be linearly dependent.

Now suppose $T^1 \cup T^2 \subseteq T$. Since $J(S,T^1 \cup T^2)$ is infeasible, this would contradict the fact that $J(S,T)$ is an IIS. Hence, $T^1 \cup T^2 = T$.

Conversely, let $J \subseteq (S,T^1,T^2)$ be an IIS. Again, $J(S,T^1 \cup T^2)$ is infeasible. Suppose
there exists an IIS $J(S,T)$ with $T \subseteq T^1 \cup T^2$. By the forward direction, there exists
an IIS $J \subseteq (S,T^1,T^2)$ with $T = T^1 \cup T^2$. Consequently, a reorientation of $J \subseteq (S,T^1,T^2)$
contains an infeasible subsystem which contradicts Lemma 21. □

**Remark 23.** The message of Lemma 18 is that in the case of flow systems, no reorien-
tation is needed, i.e., one can always choose $T^1 = \emptyset$ or $T^2 = \emptyset$ in Lemma 22. It seems
that Lemma 22 has not been stated before.
3 Computing IISs in Flow Networks

An IIS of a linear program can be computed in polynomial time by finding a vertex of the alternative polyhedron. For network flow problems, this can be done even in strongly polynomial time, see Tardos [27]. In the following, we present a polynomial time combinatorial algorithm for this task, which has to perform a single max-flow computation.

First suppose that a GH-cut $\delta(S)$ has been found. If $S$ is weakly connected, we have found an IIS by Theorem 14; otherwise, we can extract an IIS in roughly the time needed to compute the connected components:

**Proposition 24.** Let $\mathcal{I}(S)$ be a GH-subsystem, where $S$ is disconnected. Then there exists a connected component $(S_1, E[S_1])$ with $S_1 \subset S$ such that $\mathcal{I}(S_1)$ is an IIS.

**Proof.** Let $\mathcal{I}(S)$ be a GH-subsystem, and suppose there are $k \geq 2$ connected components $(S_1, E[S_1]), \ldots, (S_k, E[S_k])$ with $S = S_1 \cup \cdots \cup S_k$. W.l.o.g., we assume

$$b(S_1) - u(\delta^+(S_1)) + \ell(\delta^-(S_1)) \geq b(S_i) - u(\delta^+(S_i)) + \ell(\delta^-(S_i))$$

for all $i = 2, \ldots, k$. Since by assumption $\delta(S_i) \cap \delta(S_j) = \emptyset$ for all $i \neq j$, we have

$$0 < b(S) - u(\delta^+(S)) + \ell(\delta^-(S))$$

$$= \sum_{i=1}^{k} (b(S_i) - u(\delta^+(S_i)) + \ell(\delta^-(S_i)))$$

$$\leq k(b(S_1) - u(\delta^+(S_1)) + \ell(\delta^-(S_1))).$$

Thus, we have $b(S_1) > u(\delta^+(S_1)) - \ell(\delta^-(S_1))$ and $\mathcal{I}(S_1)$ is an IIS by Theorem 19. For $\mathcal{I}(S)$ in demand form, the proof follows analogously.

Therefore, to obtain an IIS, we essentially just need a violated GH-inequality. It is well known that this can be done by computing a maximum $(s-t)$-flow for the following extended network $G' = (V', E')$ with source $s$, sink $t$, and capacities $u'$ (see, e.g., [2] or any other textbook on flow networks): The nodes are $V' := V \cup \{s, t\}$. For each $v \in V$, let

$$d_v := b_v - \sum_{e \in \delta^+(v)} \ell_e + \sum_{e \in \delta^-(v)} \ell_e.$$

The arcs are

$$E' := E \cup \{(s, v) \mid d_v > 0\} \cup \{(v, t) \mid d_v < 0\}.$$

For each arc of the form $e = (s, v)$ define $u'_e := d_v > 0$. For each arc $e = (v, t)$ define $u'_e := -d_v > 0$. For every arc $e \in E$, set $u'_e := u_e - \ell_e$. All lower bounds in $G'$ are 0.

Let $D := \sum_{v \in V : d_v \geq 0} d_v$. If the maximum $(s-t)$-flow in $G'$ has value $< D$, the original instance is infeasible. In fact, we show that every GH-set in $G$ corresponds to a cut in $G'$ with capacity $< D$. Aggarwal et al. proved a similar result for the witness problem, see [1, Lemma 1]. Our result is an extension for lower bounds and holds w.r.t. IISs.

**Lemma 25.** There is a one-to-one correspondence between GH-sets $S \subseteq V$ in $G$ and sets $S' = S \cup \{s\} \subseteq V'$ with a cut value $u'(\delta^+(S')) < D$ in $G'$.
Proof. In the following, we will use the notation $S^+ := \{v \in S \mid d_v > 0\}$ and $\bar{S}^+ := \{v \in V \setminus S \mid d_v > 0\}$, and analogously $S^-$, $\bar{S}^-$ for the demand nodes.

Let $S \subseteq V$ and $S' := S \cup \{s\}$. Then

$$u'(\delta^+(S')) = u'(\delta^+(S)) + u'|\{s : \bar{S}^+\}$$

$$= u(\delta^+(S)) - \ell(\delta^+(S)) - d(S^-) + d(S^+).$$

In the last line, all values are with respect to the graph $G$.

Note that for every $U \subseteq V$, we have $d(U) = b(U) - \ell(\delta^+(U)) + \ell(\delta^-(U))$. Moreover, $D = d(S^+) + d(S^-)$, i.e., $d(S^+) = D - d(S^-) = D - b(S^+) + \ell(\delta^+(S^+)) - \ell(\delta^-(S^+))$. Thus,

$$u'(\delta^+(S')) = u(\delta^+(S)) - \ell(\delta^+(S)) - b(S^-) + \ell(\delta^-(S^-)) - \ell(\delta^-(S^-)) + D - b(S^+) + \ell(\delta^+(S^+)) - \ell(\delta^-(S^+)).$$

We observe that

$$\ell(\delta^+(S^-)) - \ell(\delta^-(S^-)) + \ell(\delta^+(S^+)) - \ell(\delta^-(S^+)) = \ell(\delta^+(S)) - \ell(\delta^-(S)).$$

Since $b(S^+) + b(S^-) = b(S)$, this leads to

$$u'(\delta^+(S')) = u(\delta^+(S)) - \ell(\delta^-(S)) + D - b(S).$$

Thus, $u'(\delta^+(S')) < D$ if and only if $u(\delta^+(S)) - \ell(\delta^-(S)) < b(S)$, which concludes the proof. \qed

We have thus shown that we can find GH-cuts with a max-flow algorithm:

**Corollary 26.** An IIS for a network flow problem can be computed in the time needed to compute a maximum flow.

In Proposition 24 we have seen that every (disconnected) GH-cut yields an IIS for at least one connected component. It turns out that every connected component yields an IIS if the GH-cut is computed using a max-flow algorithm.

**Theorem 27.** Let $x$ be a maximum $(s-t)$-flow in $G'$ with value $< D$, and let $G'_2$ be its residual graph. Define $S := \{v \in V \mid v$ reachable from $s$ in $G'_2\}$. Suppose that $S$ has $k$ weakly connected components $S_1, \ldots, S_k$ with respect to $G[S]$. Then every $I(S_1), \ldots, I(S_k)$ is an IIS.

Proof. By Proposition 24, we can assume that $I(S_1)$ is an IIS and $k \geq 2$. We first observe that the demand form of the GH-inequalities is irrelevant in this setting: By construction of $G'$ and $S_i$, we have $d(S_i) > 0$. Thus, for every $i \in [k]$

$$0 < d(S_i) = b(S_i) - \ell(\delta^+(S_i)) + \ell(\delta^-(S_i)) \leq b(S_i) - \ell(\delta^+(S_i)) + u(\delta^-(S_i)),$$

i.e., the demand form cannot be violated (see Section 2.3).

Hence, $b(S_1) > u(\delta^+(S_1)) - \ell(\delta^-(S_1))$. Now assume that $I(S_i)$, for some $i \in \{2, \ldots, k\}$, is not an IIS. Since $S_i$ is connected, $I(S_i)$ must be feasible, i.e., $b(S_i) \leq u(\delta^+(S_i)) - \ell(\delta^-(S_i))$.

Consider the arcs $e = (v, w) \in \delta^+(S_i)$ in $G$. Since $S_i$ is a connected component, $w \notin S$. Thus, $x_e = u_e$, i.e., $x(\delta^+(S_i)) = u(\delta^+(S_i))$. Similarly, for arcs $e = (v, w) \in \delta^-(S_i), v \notin S$.\]
If \( e \in E[V \setminus S : S_i] \), we have \( x_e = 0 \). Moreover, since \( S_i \) is reachable from \( s \) in \( G'_x \), not all arcs \((s, w)\) for \( w \in S_i \) are saturated, i.e., \( x(\delta^- (S_i)) < d(S_i) \). We obtain

\[
0 = x(\delta^-(S_i)) - x(\delta^+(S_i)) = x(\delta^-(S_i)) - u(\delta^+(S_i)) < d(S_i) - u(\delta^+(S_i)).
\]

This yields:

\[
0 < b(S_i) - \ell(\delta^+(S_i)) + \ell(\delta^-(S_i)) - u(\delta^+(S_i)) \implies u(\delta^+(S_i)) - \ell(\delta^- (S_i)) < b(S_i) - \ell(\delta^+(S_i)) \leq b(S_i),
\]

which is a contradiction.

### 4 A Comparison of IISs and Witnesses

IISs in flow networks and the witness concept share certain similarities, especially since they are both intended to highlight a smaller portion of the network “witnessing” the infeasibility, and rely on the GH-inequalities. The difference is that in the witness problem, one minimizes the number of nodes (in minimum or minimal meaning), while the IIS minimizes both nodes and arcs. In the following proposition, we will further explore their connection.

**Proposition 28.**

1. \( I(W) \) is infeasible for every witness \( W \), but \( S_I \) is not necessarily a witness for every infeasible \( I \).
2. \( I(W) \) is an IIS for every minimal witness \( W \), but \( S_I \) is not necessarily a minimal witness for an IIS \( I \).
3. For a minimum witness \( W \), \( I(W) \) is not necessarily a minimum IIS and for a minimum IIS \( I \), \( S_I \) is not necessarily a minimum witness.

**Proof.**

1. Note that the definition of witness \( W \) asks for a violated GH-inequality, so \( I(W) \) is a GH-subsystem per definition and therefore infeasible. Conversely, an infeasible \( I \) can contain arbitrarily more node constraints, whence \( b(S_I) \leq u(\delta^+(S_I)) - \ell(\delta^-(S_I)) \) might hold.
2. Let \( W \) be a minimal witness; then \( b(W) > u(\delta^+(W)) - \ell(\delta^-(W)) \) holds. Suppose \( I(W) \) is not an IIS. Thus, \( (W, E[W]) \) must be disconnected. By Proposition 24, there exists \( U \subset W \) with \( b(U) > u(\delta^+(U)) - \ell(\delta^-(U)) \), which contradicts the minimality of \( W \). An example for the second statement is given in Figure 2a.
3. An example in which the minimum IIS and the minimum witness are different is given in Figure 2b.

### 5 MinIISs in Flow Networks

Greenberg [15] pointed out that, in a practical application, the chance to understand the cause for the infeasibility is increased if an IIS is small. In our setting, it is thus interesting to ask for a minimum cardinality IIS (minIIS). Such minimum IISs can be characterized as follows:
On the Relation of Flow Cuts and IIIs

Corollary 29. A minimum IIS of the network flow problem ($F_\leq$) is given by the GH-subsystem $\mathcal{I}(S)$, where $S$ is a solution of

\[
\min_{S \subseteq V} |S| + |\delta(S)| \tag{8}
\]

s.t. $b(S) > u(\delta^+(S)) - \ell(\delta^-(S)) \lor -b(S) > u(\delta^-(S)) - \ell(\delta^+(S))$.

Proof. By Theorem 19, a minimum IIS is the GH-subsystem of a connected GH-cut with a minimum number of nodes plus arcs. Furthermore, any $S$ optimal for (8) is necessarily connected: Suppose there exists a proper connected component $T \subset S$ with $\delta(T) \cap \delta(S \setminus T) = \emptyset$, which, by Proposition 24, induces an IIS $\mathcal{I}(T)$. Then $|\mathcal{I}(T)| = |\delta(T)| + |T| < |\delta(S)| + |S| = |\mathcal{I}(S)|$, contradicting the optimality of $S$.

We can use this characterization for a reduction from the maximum clique problem on regular graphs (more precisely, the respective decision problem). Aggarwal et al. [1] showed that the clique problem remains strongly $\mathcal{NP}$-hard when restricted to regular graphs, by the observation that it is equivalent to the independent set problem on the complement graph, and independent set is $\mathcal{NP}$-hard even on planar cubic graphs.

Theorem 30. Given a network flow problem and a positive integer $\tilde{k}$, it is $\mathcal{NP}$-complete in the strong sense to decide whether an IIS of size at most $\tilde{k}$ exists.

Proof. Note first that the problem is in $\mathcal{NP}$ by Theorem 19: we can check in polynomial time whether a given subsystem $\mathcal{I}$ has size at most $\tilde{k}$, whether $\mathcal{I}$ has the form of a GH-subsystem, and whether the induced graph of $S_{\mathcal{I}}$ is connected.

We reduce the strongly $\mathcal{NP}$-complete regular maximum clique problem: Given an $r$-regular, undirected graph $G' = (V', E')$, with $|V'| = n$, $|E'| = m$, and a positive integer $k$, does there exist a clique $C \subseteq V'$ (i.e., $G'[C]$ is a complete graph) such that $|C| \geq k$? For the reduction, we will construct a network flow problem instance $(V, E_0, E, b, u, \ell)$ that has an IIS $\mathcal{I}$ of size at most $\tilde{k} := k + 2k(r - k + 1) + kn^2$ if and only if $G'$ has a clique of size $k$. Note that $k \leq r + 1 \leq n$.

First, we take all nodes in $V'$ and assign them a supply of $r - k + 3$ each, and we replace every arc in $E'$ by a pair of oppositely directed arcs with an upper bound of 1 and a lower bound of 0. We need additional $n^4$ copies of $V'$ yielding a set of transition nodes $V_a$, each copy connected by forward arcs $E_3$. The first $n^2$ nodes in $V_a$ are connected to the corresponding node in $V'$ by arcs $E_2$. Moreover, we add a sink node $t$ connected to the last copy of $V'$; see Figure 3 for an illustration of the construction.
Then, since \( |I| \leq k \), we again necessarily have \( u(\delta^+(S)) \geq n(r - k + 3) \geq b(S) \). It follows that \( S \subseteq V' \).

Figure 3: Sketch of the construction for the reduction; \( \ell_e = 0, u_e = n(r - k + 3) \) for solid, 1 for dotted, and \( 1/n^2 \) for dashed arcs \( e \).

The resulting directed graph \( G = (V,E) \) is formally defined by its arc set \( E := E_1 \cup E_2 \cup E_3 \) and node set \( V := V' \cup (V' \times [n^4]) \cup \{t\} \), where we write \( (v \times j) \) for the nodes in \( V' \times [n^4] \), with

\[
E_1 := \{(v,w) \mid (v,w) \in E' \} \cup \{(w,v) \mid (v,w) \in E' \},
E_2 := \{(v, (v \times j)) \mid v \in V', j \in [n^2]\},
E_3 := \{((v \times j), (v \times j + 1)) \mid v \in V', j \in [n^4 - 1]\} \cup \{((v \times n^4), t) \mid v \in V'\}.
\]

Furthermore, let \( \ell := 0 \) and

\[
b_v := \begin{cases} 
    r - k + 3, & v \in V', \\
    -n(r - k + 3), & v = t,
    0, & \text{otherwise},
\end{cases}
\]

\[
u_e := \begin{cases} 
    1, & e \in E_1, \\
\frac{1}{n^2}, & e \in E_2, \\
    n(r - k + 3), & e \in E_3.
\end{cases}
\]

A clique \( C \) with \( |C| = k \) in an \( r \)-regular graph has \( |\delta(C)| = k(r - k + 1) \) (every clique node has \( r \) incident edges, from which \( k - 1 \) are connected to nodes inside the clique). Herewith, the “if”-direction is easy: Consider a clique \( C \subseteq V' \) in the constructed graph \( G \).

Then

\[
b(C) = k(r - k + 3) > k(r - k + 1) + kn^2 \frac{1}{n^2} - 0 = u(\delta^+(C)) - \ell(\delta^-(C)).
\]

Thus, since \( G[C] \) is connected, \( \mathcal{I}(C) \) is an IIS by Theorem 14. It has size \( \tilde{k} = k + 2k(r - k + 1) + kn^2 \), since we have \( k \) nodes, \( k(r - k + 1) \) out- and ingoing arcs each, and for every node, there are \( n^2 \) arcs in \( E_2 \).

For the converse, suppose there exits an IIS \( \mathcal{I} \) with \( |\mathcal{I}| \leq \tilde{k} \). First, assume that \( t \in S := S_\mathcal{I} \), the only possibility for the demand case. Then \( V' \times \{j\} \subset S \) for all \( j \in [n^4] \), since otherwise, there would exist \( e \in \delta^-(S) \cap E_3 \) with \( u_e = n(r - k + 3) \geq -b_q \), rendering the subsystem feasible (by Theorem 19, an IIS must yield a violated GHi-inequality). Then \( |\mathcal{I}| > n^5 + 1 \geq (n + 1)^4/8 \geq \tilde{k} \), as can be easily verified; this is a contradiction. Consequently, \( t \notin S \), and only the supply form can occur.

Moreover, if there exists a node \( (v \times j) \in S \) for \( v \in V' \) and \( j \in [n^4] \), we again necessarily have \( u(\delta^+(S)) \geq n(r - k + 3) \geq b(S) \). It follows that \( S \subseteq V' \).
Let $k' := |S|$. If $k' > k$, then 
\[ |\mathcal{I}| > k' (n^2 + 1) = (k' - k)(n^2 + 1) + k(n^2 + 1) > n^2 + 1 + k(n^2 + 1) \]
\[ > 2k(r - k + 1) + k(n^2 + 1) = \tilde{k}, \]
since $2k(r - k + 1) \leq 2k(n - 1 - k + 1) \leq n^2$, which can be seen by some easy calculations. This is again a contradiction, and we conclude that $k' \leq k$. Suppose that $k' < k$. It then holds that 
\[ u(\delta^+(S)) \geq k' \frac{n^2}{n^2} + k'(r - k' + 1) = k'(r - k' + 2) \geq k'(r - k + 3) = b(S), \]
which contradicts the infeasibility of $\mathcal{I}$.

Eventually, consider $k' = k$, and suppose that no clique of size at least $k$ exists in $G'$. Hence, there must be at least one more pair of oppositely directed arcs in $\delta(S)$ than for a clique. Consequently, 
\[ |\mathcal{I}| = k + |\delta(S)| \geq k + 2k(r - k + 1) + 2 + kn^2 > \tilde{k}, \]
again a contradiction. In conclusion, a clique of size at least $k$ has to exist.

Finally note that the encoding length of the resulting instance is clearly polynomial in that of the graph $G'$. In fact, all numbers occurring in the instance are bounded by a polynomial in $n$, which proves strong $\mathcal{NP}$-completeness.

**Corollary 31.** Computing a min$IIS$ in flow networks is strongly $\mathcal{NP}$-hard.

**Remark 32.** Aggarwal et al. [1] used the regular clique problem to show $\mathcal{NP}$-hardness of the minimum witness problem; the above proof relies on their idea and uses that the number of arcs leaving a clique is constant in a regular graph. Our auxiliary graph, however, needs to be much bigger than theirs, since we cover the demand form, while the witness problem is conveniently defined only w.r.t. to the positive subset of a GH-cut.

### 6 Outlook

The results of this paper characterize IISs in terms of connected Gale-Hoffman-inequalities. This can possibly be used for general mathematical programs that contain flows as a substructure. In fact, one motivation for this article was the analysis of infeasible systems arising in stationary gas transportation, where the systems are nonlinear, nonconvex and can contain discrete variables, see, e.g., [24]. For instance, in such general systems on networks, the components corresponding to IISs are necessarily connected, see Lemma 6 for the corresponding result in the flow context. Other generalizations will be the topic of future research.

The infeasibility characterizations developed in this paper might also be relevant for network reliability (see, e.g., Provan [25]) and survivable network design (see, e.g., Grötschel et al. [18]).

Another open issue is to obtain inapproximability results for determining a minimum IIS. For the general case, strong inapproximability results exist, see [3]. Moreover, the related problem to compute an IIS cover of smallest cardinality is interesting as well.
References


