EQUIVARIANT PERTURBATION IN GOMORY AND JOHNSON'S INFINITE GROUP PROBLEM

III. FOUNDATIONS FOR THE k-DIMENSIONAL CASE WITH APPLICATIONS TO $k = 2$

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ABSTRACT. We develop foundational tools for classifying the extreme valid functions for the k-dimensional infinite group problem. In particular, (1) we present the general regular solution to Cauchy's additive functional equation on bounded convex domains. This provides a k-dimensional generalization of the so-called interval lemma, allowing us to deduce affine properties of the function from certain additivity relations. (2) We study the discrete geometry of additivity domains of piecewise linear functions, providing a framework for finite tests of minimality and extremality. (3) We give a theory of non-extremality certificates in the form of perturbation functions.

We apply these tools in the context of minimal valid functions for the two-dimensional infinite group problem that are piecewise linear on a standard triangulation of the plane, under the assumption of a regularity condition called diagonal constrainedness. We show that the extremality of a minimal valid function is equivalent to the extremality of its restriction to a certain finite two-dimensional group problem. This gives an algorithm for testing the extremality of a given minimal valid function.

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Contents

1. Introduction

In 2013 and 2014 we celebrate the 42-year jubilee of Gomory and Johnson's papers [15, 16] introducing the *infinite group problem*, an elegant infinite-dimensional relaxation of integer linear optimization problems.[1] The motivation for studying it is the hope to find effective multi-row cutting plane procedures with better performance characteristics compared to the single-row cutting plane procedures in use today.

1.1. The group problem.

Gomory's *group problem* [14] is a central object in the study of strong cutting planes for integer linear optimization problems. One considers an abelian group G, written additively, and studies the set of functions $s\colon G \to \mathbb{R}$ satisfying the following constraints:

$$\begin{aligned} &\sum_{\mathbf{r}\in G} \mathbf{r}\, s(\mathbf{r}) \in \mathbf{f} + S \\ &s(\mathbf{r}) \in \mathbb{Z}_+ \quad \text{for all } \mathbf{r} \in G \\ &s \text{ has finite support,} \end{aligned} \tag{1}$$

where S is a subgroup of G and $\mathbf{f}$ is a given element in $G \setminus S$; so $\mathbf{f} + S$ is the coset containing the element $\mathbf{f}$. We will be concerned with the so-called *infinite group problem* [15, 16], where $G = \mathbb{R}^k$ is taken to be the group of real k-vectors under addition, and $S = \mathbb{Z}^k$ is the subgroup of the integer vectors. We are interested in studying the convex hull $R_{\mathbf{f}}(G, S)$ of all functions satisfying the constraints in (1). Observe that $R_{\mathbf{f}}(G, S)$ is a convex subset of the infinite-dimensional vector space $\mathcal{V}$ of functions $s\colon G \to \mathbb{R}$ with finite support.

A main focus of the research in this area is to give a description of $R_{\mathbf{f}}(\mathbb{R}, \mathbb{Z})$ as the intersection of halfspaces of $\mathcal{V}$. This makes a very useful connection between $R_{\mathbf{f}}(\mathbb{R}, \mathbb{Z})$ and traditional integer programming, both from a theoretical, as well as, practical point of view. This arises from the fact that important classes of cutting planes for general integer programs can be viewed as finite-dimensional restrictions of the linear inequalities used to describe $R_{\mathbf{f}}(\mathbb{R}, \mathbb{Z})$.

1.2. Valid inequalities and valid functions.

Any linear inequality in $\mathcal{V}$ is given by $\sum_{\mathbf{r}\in G} \pi(\mathbf{r})s(\mathbf{r}) \geq \alpha$ where π is a function $\pi\colon G \to \mathbb{R}$ and $\alpha \in \mathbb{R}$. The left-hand side of the inequality is a finite sum because s has finite support. Such an inequality is called a *valid inequality* for $R_{\mathbf{f}}(G, S)$ if $\sum_{\mathbf{r}\in G} \pi(\mathbf{r})s(\mathbf{r}) \geq \alpha$ for all $s \in R_{\mathbf{f}}(G, S)$. It is customary to concentrate on valid inequalities with $\pi \geq 0$; then we can choose, after a scaling, $\alpha = 1$. Thus, we only focus on valid inequalities of the form $\sum_{\mathbf{r}\in G} \pi(\mathbf{r})s(\mathbf{r}) \geq 1$ with $\pi \geq 0$. Such functions π will be termed *valid functions* for $R_{\mathbf{f}}(G, S)$.

As pointed out in [7], the nonnegativity assumption in the definition of a valid function might seem artificial at first. Although there exist valid inequalities $\sum_{r\in\mathbb{R}} \pi(r)s(r) \geq \alpha$ for $R_{\mathbf{f}}(\mathbb{R}, \mathbb{Z})$ such that $\pi(r) < 0$ for some $r \in \mathbb{R}$, it can be shown that π must be nonnegative over all *rational* $r \in \mathbb{Q}$. Since data in integer programs are usually rational, it is natural to focus on nonnegative valid functions.

1.3. Minimal functions.

Gomory and Johnson [15, 16] defined a hierarchy on the set of valid functions, capturing the strength of the corresponding valid inequalities, which we summarize now.

A valid function π for $R_{\mathbf{f}}(G, S)$ is said to be *minimal* for $R_{\mathbf{f}}(G, S)$ if there is no valid function $\pi' \neq \pi$ such that $\pi'(\mathbf{r}) \leq \pi(\mathbf{r})$ for all $\mathbf{r} \in G$. For every valid function π for $R_{\mathbf{f}}(G, S)$, there exists a minimal valid function π' such that $\pi' \leq \pi$ (cf. [6]), and thus non-minimal valid functions are redundant in the description of $R_{\mathbf{f}}(G, S)$. Minimal functions for $R_{\mathbf{f}}(G, S)$ were characterized by Gomory for the case where S has finite index in G in [14], and later for $R_{\mathbf{f}}(\mathbb{R}, \mathbb{Z})$ by Gomory and Johnson [15]. We state these results in a unified notation in the following theorem.

A function $\pi\colon G \to \mathbb{R}$ is *subadditive* if $\pi(\mathbf{x}+\mathbf{y}) \leq \pi(\mathbf{x})+\pi(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in G$. We say that π is *symmetric* if $\pi(\mathbf{x}) + \pi(\mathbf{f} - \mathbf{x}) = 1$ for all $\mathbf{x} \in G$.

Theorem 1.1 (Gomory and Johnson [15])**.** *Let $\pi\colon G \to \mathbb{R}$ be a nonnegative function. Then π is a minimal valid function for $R_{\mathbf{f}}(G, S)$ if and only if $\pi(\mathbf{z}) = 0$ for all $\mathbf{z} \in S$, π is subadditive, and π satisfies the symmetry condition. (The first two conditions imply that π is periodic modulo S, that is, $\pi(\mathbf{x}) = \pi(\mathbf{x}+\mathbf{z})$ for all $\mathbf{z} \in S$.)*

[1][15] was received by Mathematical Programming on April 6, 1971, and published 1972.

Remark 1.2. Note that this implies that one can view a minimal valid function π as a function from G/S to $\mathbb{R}$, and thus studying $R_{\mathbf{f}}(G, S)$ is the same as studying $R_{\mathbf{f}}(G/S, \mathbf{0})$. However, we avoid this viewpoint in this paper.

1.4. **Extreme functions and their classification.** In polyhedral combinatorics, one is interested in classifying the facet-defining inequalities of a polytope, which are the strongest inequalities and provide a finite minimal description. In the infinite group problem, the analogous notion is that of an *extreme function*.

A valid function π is *extreme* for $R_{\mathbf{f}}(G, S)$ if it cannot be written as a convex combination of two other valid functions for $R_{\mathbf{f}}(G, S)$, i.e., $\pi = \frac{1}{2}(\pi^1 + \pi^2)$ implies $\pi = \pi^1 = \pi^2$. Extreme functions are minimal.

Various sufficient conditions for extremality have been proved in the previous literature [6, 8, 10–12, 17, 20, 21]. In part I [4] of the present series of papers, the authors initiated the study of perturbation functions that are equivariant with respect to certain finitely generated reflection groups. This addressed an inherent previously unknown *arithmetic* (number-theoretic) aspect of the problem and allowed the authors to complete the classification of extreme functions that are piecewise linear functions with rational breakpoints.

Theorem 1.3 (Theorems 1.3 and 1.5 in [4])**.** *Consider the following problem.*

> *Given a minimal valid function π for $R_f(\mathbb{R}, \mathbb{Z})$ that is piecewise linear with a set of rational breakpoints with the least common denominator q, decide if π is extreme or not.*

(i) *There exists an algorithm for this problem that takes a number of elementary operations over the reals that is bounded by a polynomial in q.*
(ii) *If the function π is continuous, then π is extreme for $R_f(\mathbb{R}, \mathbb{Z})$ if and only if the restriction $\pi\big|_{\frac{1}{4q}\mathbb{Z}}$ is extreme for the finite group problem $R_f(\frac{1}{4q}\mathbb{Z}, \mathbb{Z})$.*

1.5. **Contributions and techniques of this paper.** In the present paper, we continue the program of [4]. We prove several general results that hold for arbitrary dimension k and then apply them to give a characterization of a large class of extreme functions for the case $k = 2$.

The main technique used to show a function π is extreme is to assume that $\pi = \frac{1}{2}(\pi^1 + \pi^2)$ where π^1, π^2 are valid functions, and then show that $\pi = \pi^1 = \pi^2$. We will use three important properties of π^1, π^2 in our proofs, which are summarized in the following lemma. These facts for the one-dimensional case can be found, for instance, in [4], and are easily extended to the general k-dimensional case.

Lemma 1.4. *Let π be minimal, $\pi = \frac{1}{2}(\pi^1 + \pi^2)$, and π^1, π^2 valid functions. Then the following hold:*

(i) *π^1, π^2 are minimal.*
(ii) *All subadditivity relations $\pi(\mathbf{x} + \mathbf{y}) \leq \pi(\mathbf{x}) + \pi(\mathbf{y})$ that are tight for π are also tight for π^1, π^2. That is, defining the* additivity domain *of π as*

$$E(\pi) := \{\, (\mathbf{x}, \mathbf{y}) \mid \Delta\pi(\mathbf{x}, \mathbf{y}) := \pi(\mathbf{x}) + \pi(\mathbf{y}) - \pi(\mathbf{x} + \mathbf{y}) = 0 \,\}, \tag{2}$$

we have $E(\pi) \subseteq E(\pi^1), E(\pi^2)$.
(iii) *If π is continuous and piecewise linear, then π, π^1, π^2 are all Lipschitz continuous.*

1.5.1. *Regular solutions to functional equations.* Utilizing the set $E(\pi)$ is fundamental in the literature to classifying extreme functions. In particular, much of the literature relies on a bounded version of a regularity result for the classical (additive) Cauchy functional equation

$$\theta(u) + \theta(v) = \theta(u + v), \tag{3}$$

where $u, v \in \mathbb{R}$ (see, e.g., [9, 19]). This result is known as the *interval lemma* in the integer programming community [17].

Lemma 1.5 (Interval lemma [3, 17])**.** *Let $\theta\colon \mathbb{R} \to \mathbb{R}$ be a function bounded on every bounded interval. Given real numbers $u_1 < u_2$ and $v_1 < v_2$, let $U = [u_1, u_2]$, $V = [v_1, v_2]$, and $U + V = [u_1 + v_1, u_2 + v_2]$. If $\theta(u) + \theta(v) = \theta(u + v)$ for every $(u, v) \in U \times V$, then θ is affine with the slope $c \in \mathbb{R}$ in each of the intervals U, V, and $U + V$.*

The interval lemma gives a powerful dimension reduction mechanism: where it applies, the infinite-dimensional space of functions on an interval is replaced by a finite-dimensional space. If this applies to all subintervals of a piecewise linear function, testing if this function is extreme can be reduced to finite-dimensional linear algebra.

For the k-dimensional case, various authors have attempted to give suitable generalizations of this lemma. References in the integer programming community include [6, 8, 10]; on the functional equations side, e.g., [19, section 13.5].

A main contribution in the present paper is a very general solution to the Cauchy functional equation (5) on convex domains, which we expect to be of independent interest. We first prove Theorem 2.2, which is a simple generalization for the one-dimensional case that allows for three functions instead of one. We then prove the following k-dimensional generalization, which pertains to convex sets U and V and their Minkowski sum $W = U + V$.

Theorem 1.6 (Higher-dimensional interval lemma, full-dimensional version)**.** *Let $f, g, h\colon \mathbb{R}^k \to \mathbb{R}$ be bounded functions. Let U and V be convex subsets of $\mathbb{R}^k$ such that $f(\mathbf{u}) + g(\mathbf{v}) = h(\mathbf{u} + \mathbf{v})$ for all $(\mathbf{u}, \mathbf{v}) \in U \times V$. Assume that $\operatorname{aff}(U) = \operatorname{aff}(V) = \mathbb{R}^k$. Then there exists a vector $\mathbf{c} \in \mathbb{R}^k$ such that f, g and h are affine over U, V and $W = U + V$, respectively, with the same gradient $\mathbf{c}$.*

The key generalization is to consider an additivity domain specified by a general convex set $F \subseteq \mathbb{R}^k \times \mathbb{R}^k$ instead the more restrictive setting of $F = U \times V$. We then prove a very general version of a "patching" principle previously used in the literature (Lemma 2.7), which allows us to prove the following general theorem.

Define the projections $p_1, p_2, p_3 \colon \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^k$ as

$$p_1(\mathbf{x}, \mathbf{y}) = \mathbf{x}, \quad p_2(\mathbf{x}, \mathbf{y}) = \mathbf{y}, \quad p_3(\mathbf{x}, \mathbf{y}) = \mathbf{x} + \mathbf{y}. \tag{4}$$

Theorem 1.7 (Convex additivity domain lemma, full-dimensional version)**.** *Let $f, g, h\colon \mathbb{R}^k \to \mathbb{R}$ be bounded functions. Let $F \subseteq \mathbb{R}^k \times \mathbb{R}^k$ be a full-dimensional convex set such that $f(\mathbf{u}) + g(\mathbf{v}) = h(\mathbf{u} + \mathbf{v})$ for all $(\mathbf{u}, \mathbf{v}) \in F$. Then there exists a vector $\mathbf{c} \in \mathbb{R}^k$ such that f, g and h are affine with the same gradient $\mathbf{c}$ over $\operatorname{int}(p_1(F))$, $\operatorname{int}(p_2(F))$ and $\operatorname{int}(p_3(F))$, respectively.*

It is notable that we can only deduce affine linearity over the *interiors* of the projections. This is best possible, as we illustrate by an example (Remark 2.12). Actually we prove both theorems in a significantly more general setting, which takes care of situations in which the set F is not full-dimensional (Theorems 2.5 and 2.11). We then deduce affine properties with respect to certain subspaces, which is important for the following results.

1.5.2. *The discrete geometry of additivity domains.* Piecewise linear functions form an important class of minimal valid functions. In fact, all classes of extreme functions described in the literature are piecewise linear, with the exception of a family of measurable functions constructed in [3].

In the one-dimensional case ($k = 1$), a continuous piecewise linear function π periodic modulo $\mathbb{Z}$ is given by a list of breakpoints in $[0, 1]$ and affine functions on the subintervals delimited by these breakpoints. In the higher-dimensional case ($k > 1$), it is not enough to give a list of breakpoints. We describe piecewise linear functions $\pi\colon \mathbb{R}^k \to \mathbb{R}$ by specifying a polyhedral complex $\mathcal{P}$ that covers all of $\mathbb{R}^k$ and affine functions on the cells of this complex. For example, we could specify a triangulation of $[0, 1]^k$ and then extend this complex periodically to cover $\mathbb{R}^k$.

Our second main contribution in the present paper is a detailed study of the discrete geometry of the additivity domain $E(\pi)$, as defined in (2), of a function π that is continuous piecewise linear on $\mathcal{P}$. This is missing from the previous literature on $R_{\mathbf{f}}(\mathbb{R}^k, \mathbb{Z}^k)$ for $k \geq 2$ and extends the discussion in the one-dimensional case in [4]. In section 3.2, we define a polyhedral complex $\Delta\mathcal{P}$ in $\mathbb{R}^k \times \mathbb{R}^k$, which arises from the polyhedral complex $\mathcal{P}$ of $\mathbb{R}^k$. The vertices of $\Delta\mathcal{P}$ hold information for necessary and sufficient conditions for minimality, as shown in Theorem 3.10, and for a necessary condition for extremality, as shown in Theorem 3.13.

Further, in section 3.4, we show that the additivity domain $E(\pi)$ can be combinatorialized by writing it as the union of the maximal faces F of $\Delta\mathcal{P}$ such that $F \subseteq E(\pi)$ (Lemma 3.12). We refer to these building blocks as *maximal additive faces.* Minimal functions can then be classified according to the types of maximal additive faces $F \in \Delta\mathcal{P}$ that appear. The generic case is that in which all maximal additive faces, with the possible exception of those corresponding to the symmetry condition, are full-dimensional in $\mathbb{R}^k \times \mathbb{R}^k$. In this case, the interval lemma (for $k = 1$) or the full-dimensional version of the Higher-dimensional interval lemma (for $k \geq 2$) are sufficient for proving extremality. All sufficient conditions for extremality studied in the previous literature fall into this class. Degenerate cases, in which some maximal additive faces are allowed to be lower-dimensional, require more machinery. For the case of functions for $k = 1$ with rational breakpoints, in [4] the authors interpret lower-dimensional maximal additive faces as translation and reflection operations on the real line. Using the structure of these operations, a special class of "perturbation" functions is invented in [4], which are used as certificates for the non-extremality of a given minimal function. Understanding the nature of these lower-dimensional maximal additive faces and their interaction with these perturbation functions was the key to breaking beyond the existing arguments from the literature which dealt with only full-dimensional maximal additive faces for the $k = 1$ case. For higher dimensions, degenerations of various types are possible and define a hierarchy of functions. In this paper, we initiate this higher-dimensional theory by studying the $k = 2$ case, for piecewise linear functions over a special triangulation of $\mathbb{R}^2$. We introduce this next and describe the hierarchy of minimal valid functions that arise (Figure 1).

1.5.3. *Characterization of extreme piecewise linear functions on a standard triangulation of the plane.* We consider the following well-known triangulation of $\mathbb{R}^2$. Let q be a positive integer. Consider the triangles

$$\mathbf{0}◺ = \frac{1}{q}\operatorname{conv}(\{\binom{0}{0}, \binom{1}{0}, \binom{0}{1}\}) \qquad \text{and} \qquad \mathbf{0}◹ = \frac{1}{q}\operatorname{conv}(\{\binom{1}{0}, \binom{0}{1}, \binom{1}{1}\}).$$

Translating these triangles by elements of the lattice $\frac{1}{q}\mathbb{Z}^2$ gives a triangulation of $\mathbb{R}^2$, which we denote by $\mathcal{P}_q$. More precise definitions appear in section 4. This triangulation is known as the K1 triangulation in the context of homotopy methods [13]. The triangulation $\mathcal{P}_q$ has convenient properties. In particular, Lemma 4.4 shows that the projections $p_i(F)$ of any face $F \in \Delta\mathcal{P}_q$ are faces of $\mathcal{P}_q$.

Figure 1 (on the right) shows a hierarchy of minimal valid functions π depending on the type of maximal additive faces of π. More precisely, the hierarchy is defined in terms of the dimensions of the possible projections $p_i(F)$ for maximal additive faces F of π (excluding the symmetry relations). The labeling of the class is meant to be self-explanatory in Figure 1; for example, in the lowest class "full-dimensionally constrained", all projections are 2-dimensional (triangles), in the class "full-dimensionally and point constrained", the projections are either 2-dimensional (triangles) or 0-dimensional (points), in the class "full-dimensionally, horizontally and point constrained" means the projections are either triangles, or horizontal edges, or points.

In this paper, we study the family of minimal valid functions that allows for two types of degenerations of the maximal additive faces, and characterize (in the sense of Theorems 1.8 and 1.9) the extreme functions within this family. Specifically, we assume that the maximal additive faces $F \in \Delta\mathcal{P}_q$ are so that its projections $p_i(F)$ are either full-dimensional (triangles ◺, ◹), points (·), or diagonal edges (⟍), but not horizontal or vertical edges. These full-dimensionally, diagonally, and point constrained minimal valid functions (Figure 1) will be called *diagonally constrained* minimal valid functions for brevity. The precise definition appears in section 4. In a similar way, *horizontally constrained* or *vertically constrained* minimal functions can be defined, and the proof can be easily adapted to these cases. The general case in which the restriction to diagonally constrained functions is removed and thus all degenerations of maximal additive faces are allowed requires the solutions of more general functional equations and leads to the construction of more complicated perturbation functions. We defer this discussion to the forthcoming paper [5].

Theorem 1.8. *Consider the following problem.*

> *Given a minimal valid function π for $R_{\mathbf{f}}(\mathbb{R}^2, \mathbb{Z}^2)$ that is piecewise linear continuous on $\mathcal{P}_q$ and diagonally constrained with $\mathbf{f} \in \operatorname{vert}(\mathcal{P}_q)$, decide if π is extreme.*

There exists an algorithm for this problem that takes a number of elementary operations over the reals that is bounded by a polynomial in q.

As a direct corollary of the proof of Theorem 1.8, we obtain the following result relating the finite and infinite group problems.

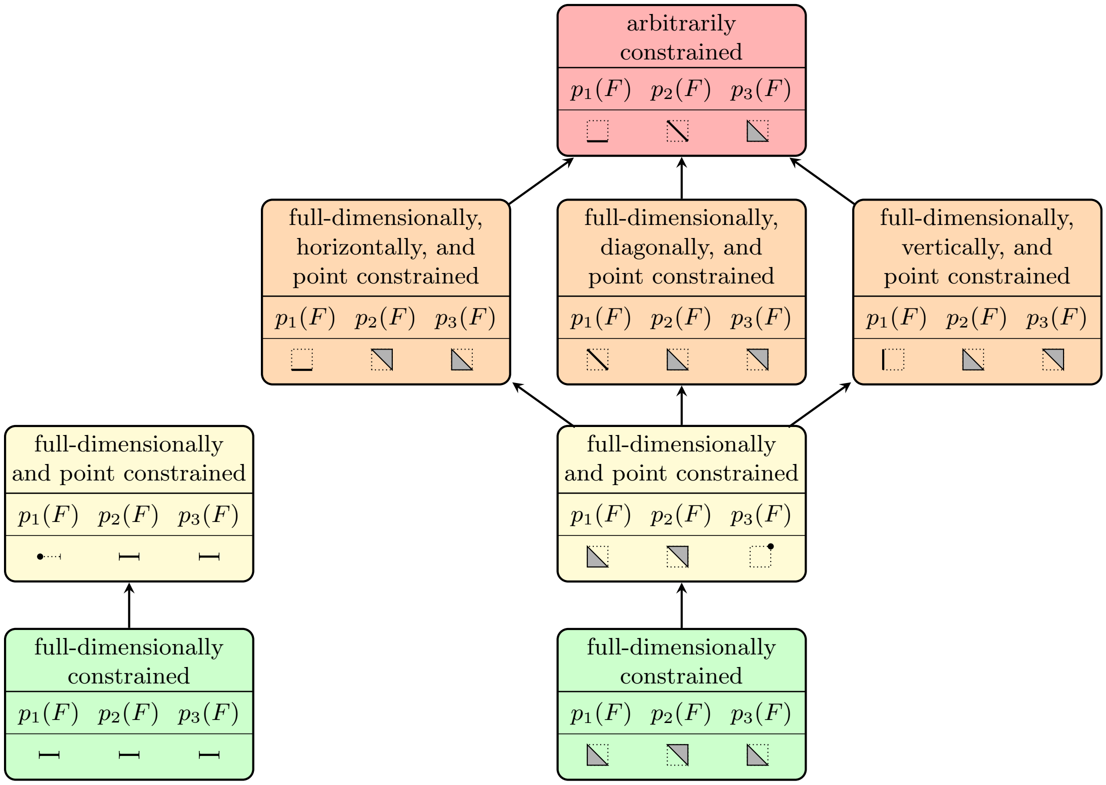


FIGURE 1. A hierarchy of minimal valid functions. At the top is the most general, at the bottom the least general class of functions. Each class is illustrated with the projections $p_1(F)$, $p_2(F)$, $p_3(F)$ of a maximal face $F \in \Delta\mathcal{P}$ with $F \subseteq E(\pi)$ that is allowed in this class, but not in the classes below. *Left*, case $k = 1$. *Right*, case $k = 2$ for the standard triangulation $\mathcal{P}_q$.

Theorem 1.9. *Let π be a minimal continuous piecewise linear function over $\mathcal{P}_q$ that is diagonally constrained and $\mathbf{f} \in \operatorname{vert}(\mathcal{P}_q)$. Fix $m \in \mathbb{Z}_{\geq 3}$. Then π is extreme for $R_{\mathbf{f}}(\mathbb{R}^2, \mathbb{Z}^2)$ if and only if the restriction $\pi\big|_{\frac{1}{mq}\mathbb{Z}^2}$ is extreme for $R_{\mathbf{f}}(\frac{1}{mq}\mathbb{Z}^2, \mathbb{Z}^2)$.*

We require the hypothesis of minimality in the two theorems above. In section 3.3, we show that it is a straightforward matter to check the minimality of a periodic piecewise linear function in light of Theorem 1.1. We also require that $\mathbf{f} \in \operatorname{vert}(\mathcal{P}_q)$. This turns out to be a natural assumption because for minimal functions that cannot be viewed as a lower-dimensional function, we must always have $\mathbf{f} \in \operatorname{vert}(\mathcal{P})$, Theorem B.11. Such functions are called *genuinely k-dimensional* and were studied in [6, 8]. We detail properties of these functions in Appendix B. In particular, we show that the study of continuous piecewise linear extreme functions can, under some mild assumptions, be reduced to the study of genuinely k-dimensional functions that are continuous and piecewise linear. We view the study of genuinely k-dimensional functions in Appendix B as path-clearing work in the study of extreme functions for Gomory and Johnson's infinite group problem.

2. Regular solutions to Cauchy's functional equation on bounded domains of $\mathbb{R}^k$

2.1. **Cauchy's functional equation.** As mentioned in the introduction, the standard technique for showing extremality of a minimal valid function $\pi\colon \mathbb{R}^k \to \mathbb{R}$ is as follows. Suppose that $\pi = \frac{1}{2}(\pi^1 + \pi^2)$, where π^1, π^2 are other (minimal) valid functions. One then studies the *additivity domain* $E(\pi)$. By Lemma 1.4,

$E(\pi) \subseteq E(\pi^1), E(\pi^2)$. One then considers π, π^1, π^2 as solutions to the *functional equation*

$$\theta(\mathbf{u}) + \theta(\mathbf{v}) = \theta(\mathbf{u} + \mathbf{v}), \quad (\mathbf{u}, \mathbf{v}) \in E, \tag{5}$$

where $E = E(\pi)$.

This equation is known as the *(additive) Cauchy functional equation.* Classically (see, e.g., [9, 19]), it is studied for functions $\theta\colon \mathbb{R} \to \mathbb{R}$, when the additivity domain E is the entire space $\mathbb{R} \times \mathbb{R}$:

$$\theta(u) + \theta(v) = \theta(u + v), \quad (u, v) \in E = \mathbb{R} \times \mathbb{R}. \tag{6}$$

In addition to the *regular solutions* to (6), which are the (homogeneous) linear functions $\theta(x) = cx$, there exist certain pathological solutions, which are highly discontinuous. In order to rule out these solutions, one imposes a regularity hypothesis. Various such regularity hypotheses have been proposed in the literature; for example, it is sufficient to assume that the function θ is bounded on bounded intervals [19].

2.2. **Convex additivity domains.** The additivity domain $E(\pi)$ of a subadditive function $\pi\colon \mathbb{R}^k \to \mathbb{R}$ can be a complicated set. It is convenient to break it into convex sets, which we then study independently. We discuss such a decomposition for piecewise linear functions π in Lemma 3.12. Let $F \subseteq \mathbb{R}^k \times \mathbb{R}^k$ be a convex set. Then we consider the solutions $\theta\colon \mathbb{R}^k \to \mathbb{R}$ to the functional equation

$$\theta(\mathbf{u}) + \theta(\mathbf{v}) = \theta(\mathbf{u} + \mathbf{v}), \quad (\mathbf{u}, \mathbf{v}) \in F. \tag{7}$$

In equation (7), the function θ is evaluated on the convex sets $p_i(F)$ for $i = 1, 2, 3$. In this section, we will show how to deduce affine properties of θ on these projections $p_i(F)$.

2.3. **Interval lemma in $\mathbf{R}^1$.** The so-called Interval Lemma was introduced by Gomory and Johnson in [17]. This result concerns the Cauchy functional equation (5) on a bounded domain, i.e., the arguments u, v, and $u + v$ come from bounded intervals U, V, and their sum $U + V$, rather than the entire real line. Additivity is on the set $F = U \times V$ and $p_1(F) = U$, $p_2(F) = V$, $p_3(F) = U + V$. In this case, we find that regular solutions are affine on these intervals; we lose homogeneity of the solutions. We first recall the notion of affine functions over a domain.

Definition 2.1. Let $U \subseteq \mathbb{R}^k$. We say $\pi\colon U \to \mathbb{R}$ is *affine* over U if there exists $\mathbf{c} \in \mathbb{R}^k$ such that for any $\mathbf{u}_1, \mathbf{u}_2 \in U$ we have

$$\pi(\mathbf{u}_2) - \pi(\mathbf{u}_1) = \mathbf{c} \cdot (\mathbf{u}_2 - \mathbf{u}_1).$$

We may also state specifically that π is affine over U with gradient $\mathbf{c}$.

We note that instead of equation (5), one can consider the more general equation $f(u) + g(v) = h(u + v)$, with three functions f, g, and h instead of one function θ.

Lemma 2.2 (Interval lemma)**.** *Given real numbers $u_1 < u_2$ and $v_1 < v_2$, let $U = [u_1, u_2]$, $V = [v_1, v_2]$, and $U + V = [u_1 + v_1, u_2 + v_2]$. Let $f\colon U \to \mathbb{R}$, $g\colon V \to \mathbb{R}$, $h\colon U + V \to \mathbb{R}$ be bounded functions.*
If $f(u) + g(v) = h(u + v)$ for every $(u, v) \in U \times V$, then there exists $c \in \mathbb{R}$ such that $f(u) = f(u_1) + c(u - u_1)$ for every $u \in U$, $g(v) = g(v_1) + c(v - v_1)$ for every $v \in V$, $h(w) = h(u_1 + v_1) + c(w - u_1 - v_1)$ for every $w \in U + V$. In other words, f, g and h are affine with gradient c over U, V, and $U + V$ respectively.

We include a proof of the result, in a more general setting with three functions, for the convenience of the reader. It is a modification of a proof from [3].

Proof. We first show the following.

Claim 1. Let $u \in U$, and let $\epsilon > 0$ such that $v_1 + \epsilon \in V$. For every nonnegative integer p such that $u + p\epsilon \in U$, we have $f(u + p\epsilon) - f(u) = p(g(v_1 + \epsilon) - g(v_1))$.

For $h = 1, \dots, p$, by hypothesis $f(u + h\epsilon) + g(v_1) = h(u + h\epsilon + v_1) = f(u + (h - 1)\epsilon) + g(v_1 + \epsilon)$. Thus $f(u + h\epsilon) - f(u + (h - 1)\epsilon) = g(v_1 + \epsilon) - g(v_1)$, for $h = 1, \dots, p$. By summing the above p equations, we obtain $f(u + p\epsilon) - f(u) = p(g(v_1 + \epsilon) - g(v_1))$. This concludes the proof of Claim 1.

Let $\bar{u}, \bar{u}' \in U$ such that $\bar{u} - \bar{u}' \in \mathbb{Q}$ and $\bar{u} > \bar{u}'$. Define $c := \frac{f(\bar{u}) - f(\bar{u}')}{\bar{u} - \bar{u}'}$.

Claim 2. For every $u, u' \in U$ such that $u - u' \in \mathbb{Q}$, we have $f(u) - f(u') = c(u - u')$.

We only need to show that, given $u, u' \in U$ such that $u - u' \in \mathbb{Q}$, we have $f(u) - f(u') = c(u - u')$. We may assume $u > u'$. Choose a positive rational ϵ such that $\bar{u} - \bar{u}' = \bar{p}\epsilon$ for some integer $\bar{p}$, $u - u' = p\epsilon$ for some integer p, and $v_1 + \epsilon \in V$. By Claim 1,

$$f(\bar{u}) - f(\bar{u}') = \bar{p}(g(v_1 + \epsilon) - g(v_1)) \quad \text{and} \quad f(u) - f(u') = p(g(v_1 + \epsilon) - g(v_1)).$$

Dividing the last equality by $u - u'$ and the second to last by $\bar{u} - \bar{u}'$, we get

$$\frac{g(v_1 + \epsilon) - g(v_1)}{\epsilon} = \frac{f(\bar{u}) - f(\bar{u}')}{\bar{u} - \bar{u}'} = \frac{f(u) - f(u')}{u - u'} = c.$$

Thus $f(u) - f(u') = c(u - u')$. This concludes the proof of Claim 2.

Claim 3. For every $u \in U$, $f(u) = f(u_1) + c(u - u_1)$.

Let $\delta(x) = f(x) - cx$ for all $x \in U$. We show that $\delta(u) = \delta(u_1)$ for all $u \in U$ and this proves the claim. Since f is bounded on U, δ is bounded over U. Let M be a number such that $|\delta(x)| \leq M$ for all $x \in U$.

Suppose by contradiction that, for some $u^* \in U$, $\delta(u^*) \neq \delta(u_1)$. Let N be a positive integer such that $|N(\delta(u^*) - \delta(u_1))| > 2M$.
By Claim 2, $\delta(u^*) = \delta(u)$ for every $u \in U$ such that $u^* - u$ is rational. Thus there exists $\bar{u}$ such that $\delta(\bar{u}) = \delta(u^*)$, $u_1 + N(\bar{u} - u_1) \in U$ and $v_1 + \bar{u} - u_1 \in V$. Let $\bar{u} - u_1 = \epsilon$. By Claim 1,

$$\begin{aligned}
\delta(u_1 + N\epsilon) - \delta(u_1) &= (f(u_1 + N\epsilon) - c(u_1 + N\epsilon)) - (f(u_1) - cu_1) \\
&= N(g(v_1 + \epsilon) - g(v_1)) - c(N\epsilon) \\
&= N(f(u_1 + \epsilon) - f(u_1)) - c(N\epsilon) \\
&= N(f(u_1 + \epsilon) - f(u_1) - c\epsilon) \\
&= N(\delta(u_1 + \epsilon) - \delta(u_1)) \\
&= N(\delta(\bar{u}) - \delta(u_1)).
\end{aligned}$$

Thus $|\delta(u_1+N\epsilon)-\delta(u_1)| = |N(\delta(\bar{u})-\delta(u_1))| = |N(\delta(u^*)-\delta(u_1))| > 2M$, which implies $|\delta(u_1+N\epsilon)|+|\delta(u_1)| > 2M$, a contradiction. This concludes the proof of Claim 3.

By symmetry between U and V, Claim 3 implies that there exists some constant c' such that, for every $v \in V$, $g(v) = g(v_1) + c'(v - v_1)$. We show $c' = c$. Indeed, given $\epsilon > 0$ such that $u_1 + \epsilon \in U$ and $v_1 + \epsilon \in V$, $c\epsilon = f(u_1 + \epsilon) - f(u_1) = g(v_1 + \epsilon) - g(v_1) = c'\epsilon$, where the second equality follows from Claim 1.
Therefore, for every $v \in V$, $g(v) = g(v_1) + cg(v - v_1)$. Finally, since $f(u) + g(v) = h(u + v)$ for every $u \in U$ and $v \in V$, we have that for every $w \in U + V$, $h(w) = h(u_1 + v_1) + c(w - u_1 - v_1)$. ☐

2.4. **Higher-dimensional interval lemma.** We now generalize the Interval Lemma (Lemma 2.2) presented in the previous section to the k-dimensional setting. The only known generalizations of Lemma 2.2 in the literature appear in [8, 10] for the case of $k = 2$ and in [6] for general k. The results in [6, 8] are special cases of our generalization that require one of the sets to intersect the origin. The result in [10] applies in $k = 2$ and allows for so-called *star-shaped* sets that also contain the origin; a similar proof to our generalization also yields a result on star-shaped sets, but we avoid this direction because we do not need this type of result.

We prove the result in a significantly more general setting, in which the additivity domain is $U \times V$ for convex sets $U \subseteq \mathbb{R}^k$ and $V \subseteq \mathbb{R}^k$, which are not necessarily of the same dimension. In this general setting we cannot expect to deduce that the solutions are affine over U, V, and $U + V$.

Remark 2.3. Indeed, if $U + V$ is a direct sum, i.e., for every $\mathbf{w} \in U + V$ there is a unique pair $\mathbf{u} \in U$, $\mathbf{v} \in V$ with $\mathbf{w} = \mathbf{u} + \mathbf{v}$, then $f(\mathbf{u}) + g(\mathbf{v}) = h(\mathbf{u} + \mathbf{v})$ merely expresses a form of separability of h with respect to certain subspaces, and f and g can be arbitrary functions; see Figure 2 (c).

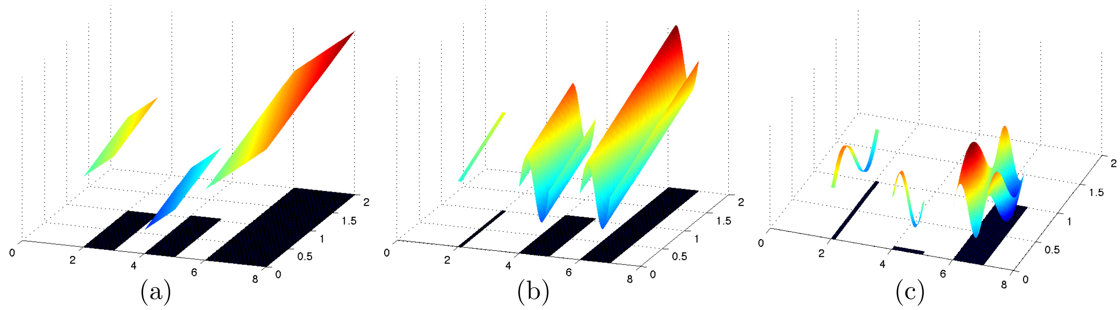


FIGURE 2. Cauchy's functional equation on bounded additivity domains $F = U \times V$. Each diagram shows $p_1(F) = U$ (*left black shadow*), $p_2(F) = V$ (*middle black shadow*), and $p_3(F) = U + V$ (*right black shadow*), and the graph (*colored by function values*) of an example function that is additive with respect to this domain. (a) Full-dimensional situation. (b) Sum of a one-dimensional and a two-dimensional set; not a direct sum. (c) Direct sum of (non-parallel) one-dimensional sets.

Definition 2.4. Let $U \subseteq \mathbb{R}^k$. Given a linear subspace $L \subseteq \mathbb{R}^k$, we say $\pi\colon U \to \mathbb{R}$ is *affine with respect to L over U* if there exists $\mathbf{c} \in \mathbb{R}^k$ such that $\pi(\mathbf{u}_2) - \pi(\mathbf{u}_1) = \mathbf{c} \cdot (\mathbf{u}_2 - \mathbf{u}_1)$ for any $\mathbf{u}^1, \mathbf{u}^2 \in U$ such that $\mathbf{u}^2 - \mathbf{u}^1 \in L$.

Theorem 2.5 (Higher-dimensional interval lemma)**.** *Let $f, g, h\colon \mathbb{R}^k \to \mathbb{R}$ be bounded functions. Let U and V be convex subsets of $\mathbb{R}^k$ such that $f(\mathbf{u}) + g(\mathbf{v}) = h(\mathbf{u} + \mathbf{v})$ for all $(\mathbf{u}, \mathbf{v}) \in F = U \times V$. Let L be a linear subspace of $\mathbb{R}^k$ such that $(L + U) \times (L + V) = (L \times L) + F \subseteq \operatorname{aff}(F) = \operatorname{aff}(U) \times \operatorname{aff}(V)$. Then there exists a vector $\mathbf{c} \in \mathbb{R}^k$ such that f, g and h are affine with respect to L over $p_1(F) = U$, $p_2(F) = V$ and $p_3(F) = U + V$ respectively, with gradient $\mathbf{c}$.*

For the proof, we will need the following notation and basic result. For any element $\mathbf{x} \in \mathbb{R}^k$, $k \geq 1$, $|\mathbf{x}|_p$ will denote the standard ℓ^p norm. We use $B^\infty(\mathbf{u}, r)$ to denote the open ℓ^∞ ball around $\mathbf{u} \in \mathbb{R}^k$ with radius $r \in \mathbb{R}_+$, i.e., $B^\infty(\mathbf{u}, r) = \{\, \mathbf{x} \in \mathbb{R}^k \mid |\mathbf{u} - \mathbf{x}|_\infty < r \,\}$.

Lemma 2.6. *Let $U \subseteq \mathbb{R}^k$ be a convex set and let L be a linear space such that $L + U \subseteq \operatorname{aff}(U)$. Then, for any $\mathbf{u} \in \operatorname{rel\,int}(U)$, there exists $r > 0$ such that $B^\infty(\mathbf{u}, r) \cap (\mathbf{u} + L) \subseteq U$.*

Proof. It suffices to show that for any $\mathbf{p} \in L$ there exists $\epsilon > 0$ such that $\mathbf{u} + \epsilon\mathbf{p} \in U$. One then can use a basis of L to find the desired $r > 0$.

Since $L + U \subseteq \operatorname{aff}(U)$, L is a subspace of $\operatorname{aff}(U) - \mathbf{u}$. Thus, $\mathbf{p} \in \operatorname{aff}(U) - \mathbf{u}$ and therefore, $\mathbf{u} + \mathbf{p} \in \operatorname{aff}(U)$. Since U is convex and $\mathbf{u} \in \operatorname{rel\,int}(U)$, there exists $\epsilon > 0$ such that $\mathbf{u} + \epsilon\mathbf{p} \in U$. ☐

Proof of Theorem 2.5. If $m := \dim(L) = 0$, there is nothing to prove. So we assume $m \geq 1$ and let $\mathbf{p}^1, \dots, \mathbf{p}^m$ be a basis for L (we obviously have $m \leq k$). Since U is convex and $L + U \subseteq \operatorname{aff}(U)$, by Lemma 2.6 for any vector $\mathbf{u}^0 \in \operatorname{rel\,int}(U)$, there exist real numbers $u_1^i < 0 < u_2^i$ such that the set $U_0 := \{\mathbf{u}^0 + \sum_{i=1}^m \lambda_i \mathbf{p}^i \mid u_1^i \leq \lambda_i \leq u_2^i \;\forall i = 1, \dots, m\} \subseteq U$. Similarly, for any vector $\mathbf{v}^0 \in \operatorname{rel\,int}(V)$, there exist real numbers $v_1^i < 0 < v_2^i$ such that the set $V_0 := \{\mathbf{v}^0 + \sum_{i=1}^m \mu_i \mathbf{p}^i \mid v_1^i \leq \mu_i \leq v_2^i \;\forall i = 1, \dots, m\} \subseteq V$.

Fix some $\mathbf{u}^0 \in \operatorname{rel\,int}(U)$, $\mathbf{v}^0 \in \operatorname{rel\,int}(V)$ and $i \in \{1, \dots, m\}$. Let $u_1^i \leq \bar\lambda_j \leq u_2^i$ and $v_1^i \leq \bar\mu_j \leq v_2^i$, for $j \neq i$, be real numbers. We consider the two line segments

$$\{\, \mathbf{u}^0 + \textstyle\sum_{j\neq i}^m \bar\lambda_j \mathbf{p}^j + \lambda_i \mathbf{p}^i \mid u_1^i \leq \lambda_i \leq u_2^i \,\} \subseteq U_0,$$
$$\{\, \mathbf{v}^0 + \textstyle\sum_{j\neq i}^m \bar\mu_j \mathbf{p}^j + \mu_i \mathbf{p}^i \mid v_1^i \leq \mu_i \leq v_2^i \,\} \subseteq V_0.$$

Let $f^i\colon [u_1^i, u_2^i] \to \mathbb{R}$ be defined by $f^i(\lambda) = f(\mathbf{u}^0 + \sum_{j\neq i}^m \bar\lambda_j \mathbf{p}^j + \lambda \mathbf{p}^i)$, $g^i\colon [v_1^i, v_2^i] \to \mathbb{R}$ be defined by $g^i(\lambda) = g(\mathbf{v}^0 + \sum_{j\neq i}^m \bar\mu_j \mathbf{p}^j + \lambda \mathbf{p}^i)$ and $h^i\colon [u_1^i + v_1^i, u_2^i + v_2^i] \to \mathbb{R}$ be defined by $h^i(\lambda) = h(\mathbf{u}^0 + \mathbf{v}^0 + \sum_{j\neq i}^m (\bar\lambda_j + \bar\mu_j)\mathbf{p}^j + \lambda \mathbf{p}^i)$.

Applying Lemma 2.2, there exists a constant $\hat{c}_i \in \mathbb{R}$ such that

$$\begin{aligned} f\big(\mathbf{u}^0 + \textstyle\sum_{j\neq i}^m \bar{\lambda}_j \mathbf{p}^j + \lambda \mathbf{p}^i\big) &= f\big(\mathbf{u}^0 + \textstyle\sum_{j\neq i}^m \bar{\lambda}_j \mathbf{p}^j\big) + \hat{c}_i \cdot \lambda \quad \text{for all } \lambda \in [u_1^i, u_2^i], \\ g\big(\mathbf{v}^0 + \textstyle\sum_{j\neq i}^m \bar{\mu}_j \mathbf{p}^j + \lambda \mathbf{p}^i\big) &= g\big(\mathbf{v}^0 + \textstyle\sum_{j\neq i}^m \bar{\mu}_j \mathbf{p}^j\big) + \hat{c}_i \cdot \lambda \quad \text{for all } \lambda \in [v_1^i, v_2^i]. \end{aligned} \tag{8}$$

Notice that this argument could be made with any other values of $\bar{\lambda}_j$, $j \neq i$ while using the same $\bar{\mu}_j$, $j \neq i$. Thus, $\hat{c}_i$ is independent of the values of $\bar{\lambda}_j$, $j \neq i$. Thus, we have m real numbers $\hat{c}_i$, $i = 1, \dots, m$, that only depend on f, g, h, L and the two points $\mathbf{u}^0 \in \operatorname{rel\,int}(U)$ and $\mathbf{v}^0 \in \operatorname{rel\,int}(V)$, and (8) holds for any values of $u_1^j \leq \bar{\lambda}_j \leq u_2^j$, $j \neq i$.

We choose $\mathbf{c} \in \mathbb{R}^k$ satisfying $\mathbf{c} \cdot \mathbf{p}^i = \hat{c}_i$ for all $i = 1, \dots, m$ (this can be done since $\mathbf{p}^1, \dots, \mathbf{p}^m$ are linearly independent). Now for any $\mathbf{p} \in L$ such that $\mathbf{u}^0 + \mathbf{p} \in U_0$, we can represent $\mathbf{p} = \sum_{i=1}^m \lambda_i \mathbf{p}^i$ for some $u_1^i \leq \lambda_i \leq u_2^i$, $i = 1, \dots, m$. Thus, $f(\mathbf{u}^0 + \mathbf{p}) = f(\mathbf{u}^0 + \sum_{i=1}^m \lambda_i \mathbf{p}^i)$.

Now using (8) with $i = m$ we have

$$\begin{aligned} f(\mathbf{u}^0 + \textstyle\sum_{i=1}^m \lambda_i \mathbf{p}^i) &= f(\mathbf{u}^0 + \textstyle\sum_{i=1}^{m-1} \lambda_i \mathbf{p}^i + \lambda_m \mathbf{p}^m) \\ &= f(\mathbf{u}^0 + \textstyle\sum_{i=1}^{m-1} \lambda_i \mathbf{p}^i) + \hat{c}_m \cdot \lambda_m, \end{aligned}$$

which follows because the $\hat{c}_i$'s do not depend on the particular values λ_i, $i \neq m$. By applying this argument iteratively, we find that

$$\begin{aligned} f(\mathbf{u}^0 + \mathbf{p}) &= f(\mathbf{u}^0 + \textstyle\sum_{i=1}^m \lambda_i \mathbf{p}^i) \\ &= f(\mathbf{u}^0) + \textstyle\sum_{i=1}^m \hat{c}_i \cdot \lambda_i \\ &= f(\mathbf{u}^0) + \textstyle\sum_{i=1}^m \lambda_i \mathbf{c} \cdot \mathbf{p}^i \\ &= f(\mathbf{u}^0) + \mathbf{c} \cdot \textstyle\sum_{i=1}^m \lambda_i \mathbf{p}^i \\ &= f(\mathbf{u}^0) + \mathbf{c} \cdot \mathbf{p}. \end{aligned}$$

Thus, $f(\mathbf{u}^0 + \mathbf{p}) = f(\mathbf{u}^0) + \mathbf{c} \cdot \mathbf{p}$ for all $\mathbf{p}$ such that $\mathbf{u}^0 + \mathbf{p} \in U_0$, i.e., f is affine with respect to L over U_0 with gradient $\mathbf{c}$. This argument can also be used to show that g is affine with respect to L over V_0 with the same gradient $\mathbf{c}$ (the relations in (8) will now be used on g, keeping $\bar{\lambda}_j, j \neq i$ fixed and allowing $\bar{\mu}_j, j \neq i$ to vary).

Finally, we do one more step to show that f is affine with respect to L over all of U with gradient $\mathbf{c}$. Let $\mathbf{u}^1, \mathbf{u}^2 \in U$ such that $\mathbf{u}^2 - \mathbf{u}^1 = \mathbf{p}' \in L$. Let $v_1^0 < v_2^0 \in \mathbb{R}$, $i = 1, \dots, m$ be such that $\{\, \mathbf{v}^0 + \lambda \mathbf{p}' \mid v_1^0 \leq \lambda \leq v_2^0 \,\} \subseteq V_0$.

Let $f^0 \colon [0,1] \to \mathbb{R}$ be defined by $f^0(\lambda) = f(\mathbf{u}^1 + \lambda \mathbf{p}')$, $g^0 \colon [v_1^0, v_2^0] \to \mathbb{R}$ be defined by $g^0(\lambda) = g(\mathbf{v}^0 + \lambda \mathbf{p}')$ and $h^0 \colon [0 + v_1^0, 1 + v_2^0] \to \mathbb{R}$ be defined by $h^0(\lambda) = h(\mathbf{u}^1 + \mathbf{v}^0 + \lambda \mathbf{p}')$. Applying Lemma 2.2 to f^0, g^0 and h^0, there exists a constant $\hat{c}_0 \in \mathbb{R}$ such that

$$f(\mathbf{u}^1 + \lambda \mathbf{p}') = f(\mathbf{u}^1) + \hat{c}_0 \cdot \lambda \qquad \text{for all } \lambda \in [0,1], \tag{9a}$$

$$g(\mathbf{v}^0 + \lambda \mathbf{p}') = g(\mathbf{v}^0) + \hat{c}_0 \cdot \lambda \qquad \text{for all } \lambda \in [v_1^0, v_2^0]. \tag{9b}$$

Since g is affine over V_0 with gradient $\mathbf{c}$, $g(\mathbf{v}^0 + \lambda \mathbf{p}') = g(\mathbf{v}^0) + \lambda(\mathbf{c} \cdot \mathbf{p}')$ for all $\lambda \in [v_1^0, v_2^0]$. Thus, $\hat{c}_0 = \mathbf{c} \cdot \mathbf{p}'$. Using (9a), we get $f(\mathbf{u}^1 + \mathbf{p}') = f(\mathbf{u}^1) + \hat{c}_0 = f(\mathbf{u}^1) + \mathbf{c} \cdot \mathbf{p}'$. Therefore, $f(\mathbf{u}^2) - f(\mathbf{u}^1) = \mathbf{c} \cdot \mathbf{p}'$ as required. The same argument applies for proving g is affine with respect to L over V with gradient $\mathbf{c}$. Finally, since $h(\mathbf{x} + \mathbf{y}) = f(\mathbf{x}) + g(\mathbf{y})$ for all $\mathbf{x} \in U$, $\mathbf{y} \in V$, it follows that h is affine with respect to L over $U + V$ with gradient $\mathbf{c}$. □

2.5. **Cauchy functional equation on convex additivity domains in $\mathbf{R}^k$.** We now prove a technical lemma which can be used to transfer affine properties using small "patches" within a larger domain. This will allow us to connect local applications of the Higher-dimensional Interval Lemma (Theorem 2.5) within convex sets. This lemma generalizes techniques explicitly used in [4, 6, 8] and implicitly in many other references in the literature, for example in [10, 12, 15–17, 20].

Lemma 2.7 (Patching lemma)**.** *Let $U \subseteq \mathbb{R}^k$ be a convex subset. Let $\pi \colon U \to \mathbb{R}$ be any function. Suppose $r \colon U \to \mathbb{R}$ is a function such that for every $\mathbf{u} \in U$,*

(i) $r(\mathbf{u}) > 0$, *and*
(ii) π *is affine on* $B^\infty(\mathbf{u}, r(\mathbf{u})) \cap U$.

Then π *is affine on all of* U.

Proof. If U is empty there is nothing to show. Fix any $\mathbf{u}^0 \in U$. Since π is affine on $B^\infty(\mathbf{u}^0, r(\mathbf{u}^0)) \cap U$, there exists $\mathbf{c} \in \mathbb{R}^k$ such that $\pi(\mathbf{u}) - \pi(\mathbf{u}^0) = \mathbf{c} \cdot (\mathbf{u} - \mathbf{u}^0)$ for every $\mathbf{u} \in B^\infty(\mathbf{u}^0, r(\mathbf{u}^0)) \cap U$. We claim that $\pi(\mathbf{u}) - \pi(\mathbf{u}^0) = \mathbf{c} \cdot (\mathbf{u} - \mathbf{u}^0)$ for every $\mathbf{u} \in U$. This will establish the lemma. Indeed, consider $\mathbf{u}^1, \mathbf{u}^2 \in U$. $\pi(\mathbf{u}^2) - \pi(\mathbf{u}^1) = (\pi(\mathbf{u}^2) - \pi(\mathbf{u}^0)) + (\pi(\mathbf{u}^0) - \pi(\mathbf{u}^1)) = \mathbf{c} \cdot (\mathbf{u}^2 - \mathbf{u}^0) - \mathbf{c} \cdot (\mathbf{u}^1 - \mathbf{u}^0) = \mathbf{c} \cdot (\mathbf{u}^2 - \mathbf{u}^1)$.

Consider any arbitrary $\mathbf{u} \in U$ and the line segment $[\mathbf{u}, \mathbf{u}^0] \subseteq U$. For every $\mathbf{x} \in [\mathbf{u}, \mathbf{u}^0]$, consider $B^\infty(\mathbf{x}, r(\mathbf{x}))$. Since $r(\mathbf{x}) > 0$ for all $\mathbf{x} \in U$, $\bigcup_{\mathbf{x} \in [\mathbf{u}, \mathbf{u}^0]} B^\infty(\mathbf{x}, r(\mathbf{x}))$ is an open cover of $[\mathbf{u}, \mathbf{u}^0]$. Thus, there exists a finite subcover from this open cover. In particular, there exist points $\mathbf{x}^0, \mathbf{x}^1, \dots, \mathbf{x}^n \in [\mathbf{u}, \mathbf{u}^0]$ such that the following hold:

(i) $\mathbf{u}^0 \in B^\infty(\mathbf{x}^0, r(\mathbf{x}^0)) \cap U$,
(ii) $\mathbf{u} \in B^\infty(\mathbf{x}^n, r(\mathbf{x}^n)) \cap U$, and
(iii) $(B^\infty(\mathbf{x}^{i-1}, r(\mathbf{x}^{i-1})) \cap U) \cap (B^\infty(\mathbf{x}^i, r(\mathbf{x}^i)) \cap U) \neq \emptyset$ for every $i = 1, \dots, n$.

First, because of (i) and the facts that π is affine on $B^\infty(\mathbf{x}^0, r(\mathbf{x}^0)) \cap U$ and π is affine on $B^\infty(\mathbf{u}^0, r(\mathbf{u}^0)) \cap U$ with gradient $\mathbf{c}$, we conclude that π is affine with gradient $\mathbf{c}$ on $B^\infty(\mathbf{x}^0, r(\mathbf{x}^0)) \cap U$. From (iii), we know that $(B^\infty(\mathbf{x}^{i-1}, r(\mathbf{x}^{i-1})) \cap U) \cap (B^\infty(\mathbf{x}^i, r(\mathbf{x}^i)) \cap U) \neq \emptyset$. Since π is affine on $B^\infty(\mathbf{x}^0, r(\mathbf{x}^0)) \cap U$ with gradient $\mathbf{c}$ and π is affine over $B^\infty(\mathbf{x}^1, r(\mathbf{x}^1)) \cap U$, we conclude π is affine over $B^\infty(\mathbf{x}^1, r(\mathbf{x}^1)) \cap U$ with gradient $\mathbf{c}$. Applying this argument repeatedly, we have that π is affine on each $B^\infty(\mathbf{x}^i, r(\mathbf{x}^i)) \cap U$ with the same gradient $\mathbf{c}$. Choose $\mathbf{y}^i$, $i = 1, \dots, n$ as points in $(B^\infty(\mathbf{x}^{i-1}, r(\mathbf{x}^{i-1})) \cap U) \cap (B^\infty(\mathbf{x}^i, r(\mathbf{x}^i)) \cap U)$. Therefore, since $\mathbf{y}^{i+1}, \mathbf{y}^i \in B^\infty(\mathbf{x}^i, r(\mathbf{x}^i)) \cap U$ for every $i = 1, \dots, n-1$, we have

$$\pi(\mathbf{y}^{i+1}) - \pi(\mathbf{y}^i) = \mathbf{c} \cdot (\mathbf{y}^{i+1} - \mathbf{y}^i).$$

Also, from (i) and (ii), we have

$$\pi(\mathbf{y}^1) - \pi(\mathbf{u}^0) = \mathbf{c} \cdot (\mathbf{y}^1 - \mathbf{u}^0), \qquad \pi(\mathbf{u}) - \pi(\mathbf{y}^n) = \mathbf{c} \cdot (\mathbf{u} - \mathbf{y}^n).$$

Adding these equalities, together, we obtain $\pi(\mathbf{u}) - \pi(\mathbf{u}^0) = \mathbf{c} \cdot (\mathbf{u} - \mathbf{u}^0)$. ☐

The Higher-dimensional Interval Lemma will be used to deduce affine properties from more complicated convex sets. Since we do not always have additivity on all of $U \times V$, we prove affine properties on smaller cross products and then patch them together.

We will need the following basic lemma from convex analysis.

Lemma 2.8 (Theorem 6.6 in [22]). *Let* C *be a convex set in* $\mathbb{R}^n$ *and let* A *be a linear transformation from* $\mathbb{R}^n$ *to* $\mathbb{R}^m$. *Then*

$$A \operatorname{rel\,int}(C) = \operatorname{rel\,int}(AC).$$

Lemma 2.9 (Relative interior lemma). *Let* $F \subseteq \mathbb{R}^k \times \mathbb{R}^k$ *be a convex set. For any* $\mathbf{x} \in \operatorname{rel\,int}(p_1(F))$, *there exist* $\mathbf{y} \in \operatorname{rel\,int}(p_2(F))$ *such that* $(\mathbf{x}, \mathbf{y}) \in \operatorname{rel\,int}(F)$ *and* $p_3(\mathbf{x}, \mathbf{y}) = \mathbf{x} + \mathbf{y} \in \operatorname{rel\,int}(p_3(F))$. *Similarly, for any* $\mathbf{y} \in \operatorname{rel\,int}(p_2(F))$, *there exist* $\mathbf{x} \in \operatorname{rel\,int}(p_1(F))$ *such that* $(\mathbf{x}, \mathbf{y}) \in \operatorname{rel\,int}(F)$ *and* $p_3(\mathbf{x}, \mathbf{y}) = \mathbf{x} + \mathbf{y} \in \operatorname{rel\,int}(p_3(F))$.

Proof. Since $p_i \colon \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}^k$ are linear transformations for $i = 1, 2, 3$, by Lemma 2.8, we have $p_i(\operatorname{rel\,int}(F)) = \operatorname{rel\,int}(p_i(F))$. Therefore, $p_i \colon \operatorname{rel\,int}(F) \to \operatorname{rel\,int}(p_i(F))$ is a well defined surjective map.

We only prove the first claim as the second has a similar proof. Let $\mathbf{x} \in \operatorname{rel\,int}(p_1(F)) = p_1(\operatorname{rel\,int}(F))$. Hence, there exists a point $\mathbf{y} \in \mathbb{R}^k$ such that $(\mathbf{x}, \mathbf{y}) \in \operatorname{rel\,int}(F)$. Then, for $i = 2, 3$, $p_i(\mathbf{x}, \mathbf{y}) \in p_i(\operatorname{rel\,int}(F)) = \operatorname{rel\,int}(p_i(F))$, that is, $\mathbf{y} \in \operatorname{rel\,int}(p_2(F))$ and $\mathbf{x} + \mathbf{y} \in \operatorname{rel\,int}(p_3(F))$. ☐

Definition 2.10. For a linear space $L \subseteq \mathbb{R}^k$ and a set $U \subseteq \mathbb{R}^k$ such that for some $\mathbf{u} \in \mathbb{R}^k$ we have $\operatorname{aff}(U) \subseteq L + \mathbf{u}$, we will denote by $\operatorname{int}_L(U)$ the interior of U in the relative topology of $L + \mathbf{u}$.

Note that $\operatorname{int}_L(U)$ is well defined because either $\operatorname{aff}(U) = L + \mathbf{u}$, or $\operatorname{int}_L(U) = \emptyset$. We now prove our most general theorem relating to equation 5 on a convex domain.

Theorem 2.11 (Convex additivity domain lemma)**.** *Let $f, g, h\colon \mathbb{R}^k \to \mathbb{R}$ be bounded functions. Let $F \subseteq \mathbb{R}^k \times \mathbb{R}^k$ be a convex set such that $f(\mathbf{u}) + g(\mathbf{v}) = h(\mathbf{u}+\mathbf{v})$ for all $(\mathbf{u}, \mathbf{v}) \in F$. Let L be a linear subspace of $\mathbb{R}^k$ such that $L \times L + F \subseteq \operatorname{aff}(F)$. Let $(\mathbf{u}^0, \mathbf{v}^0) \in \operatorname{rel\,int}(F)$. Then there exists a vector $\mathbf{c} \in \mathbb{R}^k$ such that f, g and h are affine with gradient $\mathbf{c}$ over $\operatorname{int}_L((\mathbf{u}^0 + L) \cap p_1(F))$, $\operatorname{int}_L((\mathbf{v}^0 + L) \cap p_2(F))$ and $\operatorname{int}_L((\mathbf{u}^0 + \mathbf{v}^0 + L) \cap p_3(F))$, respectively.*

Proof. If $\dim(L) = 0$, there is nothing to prove. So we assume $\dim(L) \geq 1$. Let $I = p_1(F)$, $J = p_2(F)$, $K = p_3(F)$.

For $\mathbf{u} \in \operatorname{rel\,int}(I)$, define

$$r(\mathbf{u}) = \sup\left\{ \frac{r}{2} \in \mathbb{R} \;\middle|\; \exists \mathbf{v} \in \mathbb{R}^k \text{ such that } B^\infty((\mathbf{u}, \mathbf{v}), r) \cap \big((\mathbf{u}, \mathbf{v}) + L \times L\big) \subseteq F \right\}.$$

By Lemma 2.9, for any $\mathbf{u} \in \operatorname{rel\,int}(I)$, there exists $\mathbf{v} \in \operatorname{rel\,int}(J)$ such that $(\mathbf{u}, \mathbf{v}) \in \operatorname{rel\,int}(F)$. Since $\dim(L) \geq 1$, Lemma 2.6 implies that $r(\mathbf{u}) > 0$ for every $\mathbf{u} \in \operatorname{rel\,int}(I)$. Let $\mathbf{v} \in F$ such that $B^\infty((\mathbf{u}, \mathbf{v}), r(\mathbf{u})) \cap \big((\mathbf{u}, \mathbf{v}) + L \times L\big) \subseteq F$ and let

$$\begin{aligned} U &= p_1\left(B^\infty((\mathbf{u}, \mathbf{v}), r(\mathbf{u})) \cap \big((\mathbf{u}, \mathbf{v}) + L \times L\big)\right) = B^\infty(\mathbf{u}, r(\mathbf{u})) \cap (\mathbf{u} + L) \text{ and} \\ V &= p_2\left(B^\infty((\mathbf{u}, \mathbf{v}), r(\mathbf{u})) \cap \big((\mathbf{u}, \mathbf{v}) + L \times L\big)\right) = B^\infty(\mathbf{v}, r(\mathbf{u})) \cap (\mathbf{v} + L). \end{aligned}$$

Notice that

$$U \times V = B^\infty((\mathbf{u}, \mathbf{v}), r(\mathbf{u})) \cap \big((\mathbf{u}, \mathbf{v}) + L \times L\big) \subseteq F.$$

Hence, applying Theorem 2.5 with U and V, we obtain that f is affine over U. Thus, we satisfy the hypotheses of Lemma 2.7 and f is affine over $\operatorname{int}_L((\mathbf{u} + L) \cap I)$ for every $\mathbf{u} \in \operatorname{rel\,int}(I)$. This argument can be repeated to show that g is affine over $\operatorname{int}_L((\mathbf{v} + L) \cap J)$ for every $\mathbf{v} \in \operatorname{rel\,int}(J)$.

For the pair $(\mathbf{u}^0, \mathbf{v}^0) \in \operatorname{rel\,int}(F)$, by Lemma 2.6, there exists $r > 0$ such that $B^\infty((\mathbf{u}^0, \mathbf{v}^0), r) \cap \big((\mathbf{u}^0, \mathbf{v}^0) + L \times L\big) \subseteq F$. Then for $U_0 = B^\infty(\mathbf{u}^0, r(\mathbf{u})) \cap (\mathbf{u}^0 + L)$ and $V_0 = B^\infty(\mathbf{v}^0, r(\mathbf{u}^0)) \cap (\mathbf{v}^0 + L)$, we have $U_0 \times V_0 \subseteq F$ and Theorem 2.5 also tells us that f and g have the same gradient $\mathbf{c}$ in U_0 and V_0, respectively. Since f and g are affine in $\operatorname{int}_L((\mathbf{u}^0 + L) \cap I)$ and $\operatorname{int}_L((\mathbf{v}^0 + L) \cap J)$, respectively, we have that f and g are affine with the same gradient $\mathbf{c}$ over all $\operatorname{int}_L((\mathbf{u}^0 + L) \cap I)$ and $\operatorname{int}_L((\mathbf{v}^0 + L) \cap J)$, respectively. Finally, since $f(\mathbf{u}) + g(\mathbf{v}) = h(\mathbf{u} + \mathbf{v})$ for all $(\mathbf{u}, \mathbf{v}) \in F$, it follows that h is affine over $\operatorname{int}_L((\mathbf{u}^0 + \mathbf{v}^0 + L) \cap K)$. This finishes the proof. ☐

Remark 2.12 (Comparing Theorem 2.5 and Theorem 2.11)**.** The reader might think that the Higher-dimensional Interval Lemma (Theorem 2.5) could be obtained as a corollary of Convex Additivity Domain Lemma (Theorem 2.11), by setting $F = U \times V$. However, the Higher-dimensional Interval Lemma shows that under the appropriate additivity conditions over U and V, we can obtain affine properties over *all* of U and V (with respect to L); whereas, the Convex Additivity Domain Lemma derives affine properties only over the interiors with respect to L. We now show that this cannot be avoided. In particular, we give an example satisfying the hypotheses of Convex Additivity Domain Lemma where the functions are affine over the interiors, but not on the boundaries. Let

$$F = \operatorname{conv}\left(\begin{pmatrix}\mathbf{u}\\ \mathbf{v}\end{pmatrix} = \begin{pmatrix}3\\4\\5\\0\end{pmatrix}, \begin{pmatrix}1\\4\\4\\1\end{pmatrix}, \begin{pmatrix}0\\4\\5\\2\end{pmatrix}, \begin{pmatrix}0\\3\\5\\2\end{pmatrix}\right) \in \mathbb{R}^2 \times \mathbb{R}^2.$$

This is a full-dimensional simplex of $\mathbb{R}^2 \times \mathbb{R}^2$, which has the projections

$$\begin{aligned} U &= p_1(F) = \operatorname{conv}\big(\mathbf{u} = \tbinom{3}{4}, \tbinom{0}{4}, \tbinom{0}{3}\big), \\ V &= p_2(F) = \operatorname{conv}\big(\mathbf{v} = \tbinom{5}{0}, \tbinom{5}{2}, \tbinom{4}{1}\big), \\ W &= p_3(F) = \operatorname{conv}\big(\mathbf{w} = \tbinom{8}{4}, \tbinom{5}{6}, \tbinom{5}{5}\big). \end{aligned}$$

See Figure 3. Now define $f, g, h\colon \mathbb{R}^2 \to \mathbb{R}$ in the following way:

$$f(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} = \mathbf{u}, \\ 0 & \text{otherwise,} \end{cases} \qquad g(\mathbf{x}) = \begin{cases} 2 & \text{if } \mathbf{x} = \mathbf{v}, \\ 0 & \text{otherwise,} \end{cases} \qquad h(\mathbf{x}) = \begin{cases} 3 & \text{if } \mathbf{x} = \mathbf{w}, \\ 0 & \text{otherwise.} \end{cases}$$

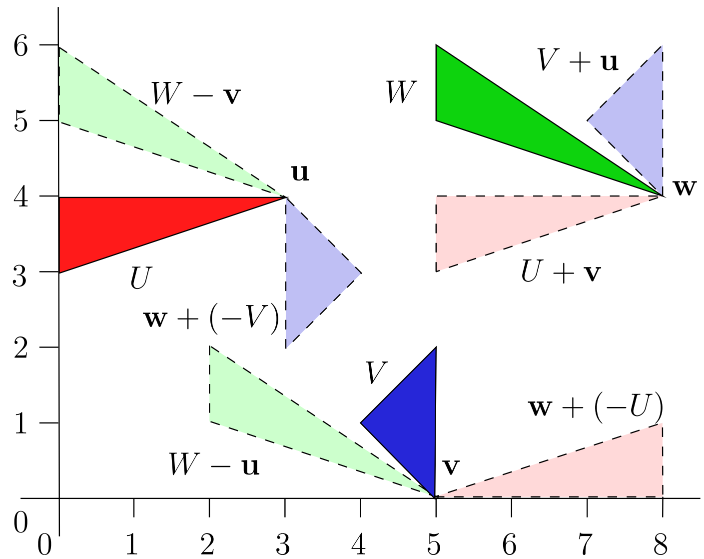


FIGURE 3. An illustration of the counterexample of Remark 2.12. The 4-dimensional simplex F projects to the three closed triangles $U = p_1(F)$, $V = p_2(F)$, $W = p_3(F)$. The points $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are additive, i.e., $\mathbf{u} + \mathbf{v} = \mathbf{w}$, but none of them is additive with any other points. To see this, we plot the sums $U + \mathbf{v}$, $V + \mathbf{u}$, $\mathbf{w} + (-U)$, $W - \mathbf{u}$, $\mathbf{v} + (-V)$, and $W - \mathbf{v}$ and show that these sets intersect U, V, and W only at the points $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$.

Claim 1. $f(\mathbf{x}) + g(\mathbf{y}) = h(\mathbf{x} + \mathbf{y})$ *for all* $(\mathbf{x}, \mathbf{y}) \in F$.

Clearly this equation holds whenever $\mathbf{x} \neq \mathbf{u}$, $\mathbf{y} \neq \mathbf{v}$, $\mathbf{x}+\mathbf{y} \neq \mathbf{w}$. So suppose $\mathbf{x} = \mathbf{u}$. Since $(\mathbf{u}+V) \cap W = \{\mathbf{w}\}$ and $(W - \mathbf{u}) \cap V = \{\mathbf{v}\}$, the only choice for $\mathbf{y}$ is $\mathbf{v}$. Similarly, if we choose $\mathbf{y} = \mathbf{v}$, the only choice for $\mathbf{x}$ is $\mathbf{u}$ or if we choose $\mathbf{x} + \mathbf{y} = \mathbf{w}$, the only choices for $\mathbf{x}$ and $\mathbf{y}$ are $\mathbf{u}$ and $\mathbf{v}$. See Figure 3 to see these arguments illustrated. Therefore, the claim holds if and only if

$$f(\mathbf{x}) + g(\mathbf{y}) = h(\mathbf{x} + \mathbf{y}) \text{ for all } \mathbf{x} \in U \setminus \{\mathbf{u}\},\ \mathbf{y} \in V \setminus \{\mathbf{v}\},\ \mathbf{x} + \mathbf{y} \in W \setminus \{\mathbf{w}\},$$

and

$$f(\mathbf{u}) + g(\mathbf{v}) = h(\mathbf{w}).$$

Since all these equations hold, the claim is proved.

Observe that, since F is full-dimensional, Theorem 2.11 applies with $L = \mathbb{R}^2$. We are ensured affine properties over the interiors of $p_1(F) = U$, $p_2(F) = V$ and $p_3(F) = W$. This shows that Theorem 2.11 cannot be extended to deduce affine properties on all of U, V, W, unless we require further restrictions on the types of convex sets F that we consider.

Remark 2.13 (Extension to boundary)**.** The one-dimensional interval lemma, Lemma 2.2, includes affine properties on the boundaries. Using this, it is easy to prove that a similar interval lemma holds on all non-degenerate intervals $U, V \subseteq \mathbb{R}$ that are any of open, half-open, or closed. Only in special cases in higher dimensions is it possible to extend affine properties in Theorem 2.11 to the boundary. For example, affine properties are extended to any point $(\mathbf{x}, \mathbf{y}) \in F \subseteq \mathbb{R}^k \times \mathbb{R}^k$ such that there exists a vector $\mathbf{p} \in \mathbb{R}^k \setminus \{\mathbf{0}\}$ such that $(\mathbf{x}+\mathbf{p}, \mathbf{y}+\mathbf{p}) \in \operatorname{rel\,int}(F)$. This can be shown by first applying applying Theorem 2.11 in k dimensions to deduce affine properties on the relative interiors of the projections of F. Next, we consider a parametrized one-dimensional space in the direction of $\mathbf{p}$ and use the one-dimensional interval lemma, Lemma 2.2, to extend these affine properties to the point $(\mathbf{x}, \mathbf{y})$.

Of course, if we use the stronger regularity assumption that f, g, and h are continuous functions (rather than merely bounded functions), then the affine properties extend to the boundary as well.

Corollary 2.14 (Convex additivity domain lemma for continuous functions)**.** *Let $f, g, h\colon \mathbb{R}^k \to \mathbb{R}$ be continuous functions. Let $F \subseteq \mathbb{R}^k \times \mathbb{R}^k$ be a convex set such that $f(\mathbf{u}) + g(\mathbf{v}) = h(\mathbf{u} + \mathbf{v})$ for all $(\mathbf{u}, \mathbf{v}) \in F$. Let L be a linear subspace of $\mathbb{R}^k$ such that $L \times L + F \subseteq \operatorname{aff}(F)$. Let $(\mathbf{u}^0, \mathbf{v}^0) \in \operatorname{rel\,int}(F)$. Then there exists a vector $\mathbf{c} \in \mathbb{R}^k$ such that f, g and h are affine with gradient $\mathbf{c}$ over $(\mathbf{u}^0 + L) \cap p_1(F)$, $(\mathbf{v}^0 + L) \cap p_2(F)$ and $(\mathbf{u}^0 + \mathbf{v}^0 + L) \cap p_3(F)$, respectively.*

3. Discrete geometry of piecewise linear minimal valid functions and their additivity domains

3.1. **Polyhedral complexes and piecewise linear functions.** We introduce the notion of polyhedral complexes, which serves two purposes in our paper. First, it provides a framework to define piecewise linear functions. Second, it is a tool for studying subadditivity and additivity relations of these functions.

Definition 3.1. A (locally finite) *polyhedral complex* is a collection $\mathcal{P}$ of polyhedra in $\mathbb{R}^k$ such that:

(i) $\emptyset \in \mathcal{P}$,
(ii) if $I \in \mathcal{P}$, then all faces of I are in $\mathcal{P}$,
(iii) the intersection $I \cap J$ of two polyhedra $I, J \in \mathcal{P}$ is a face of both I and J,
(iv) Any compact subset of $\mathbb{R}^k$ intersects only finitely many faces in $\mathcal{P}$.

A polyhedron I from $\mathcal{P}$ is called a *face* of the complex. A polyhedral complex $\mathcal{P}$ is called *pure* if all its maximal faces (with respect to set inclusion) have the same dimension. In this case, we call the maximal faces of $\mathcal{P}$ the *cells* of $\mathcal{P}$. A polyhedral complex $\mathcal{P}$ is *complete* if the union of all faces of the complex is $\mathbb{R}^k$. The reader can find examples illustrating this and the following definitions in section 4.

Given a pure and complete polyhedral complex $\mathcal{P}$, we call a function $\pi\colon \mathbb{R}^k \to \mathbb{R}$ *continuous piecewise linear over* $\mathcal{P}$ if it is affine over each of the cells of $\mathcal{P}$. We introduce the following notation for a continuous piecewise linear function π over $\mathcal{P}$.

Motivated by Gomory–Johnson's characterization of minimal valid functions (Theorem 1.1), we are interested in functions $\pi\colon \mathbb{R}^k \to \mathbb{R}$ that are periodic modulo $\mathbb{Z}^k$, i.e., for all $\mathbf{x} \in \mathbb{R}^k$ and all vectors $\mathbf{t} \in \mathbb{Z}^k$, we have $\pi(\mathbf{x} + \mathbf{t}) = \pi(\mathbf{x})$. If π is periodic modulo $\mathbb{Z}^k$ and continuous piecewise linear over a pure and complete complex $\mathcal{P}$, then we will usually assume that $\mathcal{P}$ is also *periodic modulo* $\mathbb{Z}^k$, i.e., for all $I \in \mathcal{P}$ and all vectors $\mathbf{t} \in \mathbb{Z}^k$, the translated polyhedron $I + \mathbf{t}$ also is a face of $\mathcal{P}$.

Remark 3.2. Under these assumptions it is clear that there are various ways to make the description of π finite. For example, $\tilde{D} := [0, 1)^k$ is a fundamental domain (system of unique representatives) of $\mathbb{R}^k$ with respect to the natural action of $\mathbb{Z}^k$, and so it suffices to know the values of π on $\tilde{D}$. However, it is inconvenient that $\tilde{D}$ is not closed. On the other hand, if we use instead its closure, $D := [0, 1]^k$, we lose uniqueness. Another viewpoint, considering polyhedral complexes of the torus $\mathbb{R}^k/\mathbb{Z}^k$, would require more complicated definitions. Thus, in most of this paper, we find it most convenient and natural to work with periodic functions and infinite periodic complexes.

3.2. **The extended complex $\boldsymbol{\Delta\mathcal{P}}$.** Let $\mathcal{P}$ be a pure, complete polyhedral complex of $\mathbb{R}^k$ and let π be a continuous piecewise linear function over $\mathcal{P}$. For any $I, J, K \subseteq \mathbb{R}^k$, we define the set

$$F(I, J, K) = \{\, (\mathbf{x}, \mathbf{y}) \in \mathbb{R}^k \times \mathbb{R}^k \mid \mathbf{x} \in I,\ \mathbf{y} \in J,\ \mathbf{x} + \mathbf{y} \in K \,\}.$$

In the specific case where I, J, K are polyhedra, $F(I, J, K)$ is also a polyhedron. In order to study the additivity domain $E(\pi)$, we define the following family of polyhedra in $\mathbb{R}^k \times \mathbb{R}^k$,

$$\Delta\mathcal{P} = \{\, F(I, J, K) \mid I, J, K \in \mathcal{P} \,\}.$$

First, we present formulas for the projections p_1, p_2, p_3 of $F(I, J, K)$, as defined in (4), in terms of I, J and K.

Proposition 3.3. *Let $I, J, K \subseteq \mathbb{R}^k$. Then*

$$\begin{aligned} p_1(F(I,J,K)) &= (K + (-J)) \cap I, \\ p_2(F(I,J,K)) &= (K + (-I)) \cap J, \\ p_3(F(I,J,K)) &= (I + J) \cap K. \end{aligned}$$

Proof. First of all, we have

$$\begin{aligned} p_1(F(I,J,K)) &= \{\, \mathbf{x} \in I \mid \exists \mathbf{y} \in J, \mathbf{z} \in K \text{ such that } \mathbf{x} + \mathbf{y} = \mathbf{z} \,\} \\ &= \{\, \mathbf{x} \in \mathbb{R}^k \mid \exists \mathbf{y} \in J, \mathbf{z} \in K \text{ such that } \mathbf{x} + \mathbf{y} = \mathbf{z} \,\} \cap I \\ &= \{\, \mathbf{z} - \mathbf{y} \mid \mathbf{y} \in J, \mathbf{z} \in K \,\} \cap I \\ &= (K + (-J)) \cap I. \end{aligned}$$

A similar calculation shows $p_2(F(I,J,K)) = (K + (-I)) \cap J$. Finally,

$$\begin{aligned} p_3(F(I,J,K)) &= \{\, \mathbf{z} \in K \mid \exists \mathbf{x} \in I, \mathbf{y} \in J \text{ such that } \mathbf{x} + \mathbf{y} = \mathbf{z} \,\} \\ &= \{\, \mathbf{z} \in \mathbb{R}^k \mid \exists \mathbf{x} \in I, \mathbf{y} \in J \text{ such that } \mathbf{x} + \mathbf{y} = \mathbf{z} \,\} \cap K \\ &= \{\, \mathbf{x} + \mathbf{y} \mid \mathbf{x} \in I, \mathbf{y} \in J \,\} \cap K \\ &= (I + J) \cap K. \end{aligned}$$

□

Remark 3.4. Note that in general, $p_1(F(I,J,K)) \subsetneq I$, $p_2(F(I,J,K)) \subsetneq J$, $p_3(F(I,J,K)) \subsetneq K$. Consider $I = [0,1], J = [0,1], K = [1.5, 2.5]$. Then $F(I,J,K)$ is the triangle $\operatorname{conv}\{(1,0.5),(1,1),(0.5,1)\}$, so $p_1(F(I,J,K)) = [0.5,1]$, $p_2(F(I,J,K)) = [0.5,1]$ and $p_3(F(I,J,K)) = [1.5,2]$.

The next lemma explains the tight relation between F and its projections $p_1(F), p_2(F)$ and $p_3(F)$.

Lemma 3.5. *Let $I, J, K \subseteq \mathbb{R}^k$ and let $F = F(I,J,K)$. Let $I' = p_1(F)$, $J' = p_2(F)$, and $K' = p_3(F)$. Then $F = F(I', J', K')$.*

Proof. By definition of I', J', K' it follows that $I' \subseteq I$, $J' \subseteq J$, $K' \subseteq K$. Therefore $F(I',J',K') \subseteq F(I,J,K)$.

Observe that for every $F \subseteq \mathbb{R}^k \times \mathbb{R}^k$, $F \subseteq F(p_1(F), p_2(F), p_3(F))$. Thus,

$$F(I,J,K) \subseteq F\big(p_1\big(F(I,J,K)\big), p_2\big(F(I,J,K)\big), p_3\big(F(I,J,K)\big)\big) = F(I',J',K').$$

Therefore, $F(I,J,K) = F(I',J',K')$. □

The next lemma shows that $\Delta\mathcal{P}$ is a polyhedral complex, which follows from the fact that $\mathcal{P}$ is a polyhedral complex. We include a detailed proof of the following lemma for the convenience of the reader.

Lemma 3.6. *If $\mathcal{P}$ is a pure, complete polyhedral complex in $\mathbb{R}^k$, then $\Delta\mathcal{P}$ is a pure, complete polyhedral complex in $\mathbb{R}^k \times \mathbb{R}^k$.*

Proof. We show the 4 conditions of Definition 3.1.

(i) Since $\emptyset \in \mathcal{P}$, we have $F(\emptyset,\emptyset,\emptyset) = \emptyset \in \Delta\mathcal{P}$.

(ii) Let $I, J, K \in \mathcal{P}$. Let $\hat{F}$ be a face of $F(I,J,K)$. Write I, J, K as inequality systems as $A_I\mathbf{x} \leq \mathbf{b}_I$, $A_J\mathbf{x} \leq \mathbf{b}_J$, $A_K\mathbf{x} \leq \mathbf{b}_K$. Then

$$F(I,J,K) = \{\, (\mathbf{x},\mathbf{y}) \mid A_I\mathbf{x} \leq \mathbf{b}_I,\ A_J\mathbf{y} \leq \mathbf{b}_J,\ A_K(\mathbf{x}+\mathbf{y}) \leq \mathbf{b}_K \,\}.$$

The face $\hat{F}$ is obtained by setting certain inequalities to equalities. This corresponds to restricting to faces of I, J, K. Therefore, there exist $I', J', K' \in \mathcal{P}$ such that $F(I',J',K') = \hat{F}$. Therefore $\hat{F} \in \Delta\mathcal{P}$.

(iii) Let $I, J, K, I', J', K' \in \mathcal{P}$. Then $F(I,J,K) \cap F(I',J',K') = F(I \cap I', J \cap J', K \cap K')$. Since $\mathcal{P}$ is closed under intersection, $I \cap I', J \cap J', K \cap K' \in \mathcal{P}$. Therefore $F(I \cap I', J \cap J', K \cap K') \in \Delta\mathcal{P}$.

(iv) Since $\mathcal{P}$ is locally finite, it follows that $\Delta\mathcal{P}$ is locally finite.

Hence, $\Delta\mathcal{P}$ is a polyhedral complex. Finally, consider any $(\mathbf{x},\mathbf{y}) \in \mathbb{R}^k \times \mathbb{R}^k$. Let $I, J, K \in \mathcal{P}$ such that $\mathbf{x} \in I$, $\mathbf{y} \in J$, $\mathbf{x}+\mathbf{y} \in K$. These faces I, J, K exist since $\mathcal{P}$ is complete in $\mathbb{R}^k$. Therefore, $(\mathbf{x},\mathbf{y}) \in F(I,J,K) \in \Delta\mathcal{P}$. Thus, $\Delta\mathcal{P}$ is complete. Since it is a locally finite complete polyhedral complex, it is also pure. □

We will study the function $\Delta\pi\colon \mathbb{R}^k \times \mathbb{R}^k \to \mathbb{R}$, as defined in Lemma 1.4, which measures the slack in the subadditivity constraints.

Lemma 3.7. *$\Delta\pi$ is continuous piecewise linear over $\Delta\mathcal{P}$.*

Proof. First, $\Delta\pi$ is continuous since it is the sum of continuous functions.

For any $F(I, J, K) \in \Delta\mathcal{P}$, $\Delta\pi|_{F(I,J,K)}(\mathbf{x}, \mathbf{y}) = \pi|_I(\mathbf{x}) + \pi|_J(\mathbf{y}) - \pi|_K(\mathbf{x} + \mathbf{y})$. Since $\pi|_I$, $\pi|_J$, $\pi|_K$ are all affine, it follows that $\Delta\pi|_{F(I,J,K)}$ is affine. Therefore $\Delta\pi$ is affine over every face in $\Delta\mathcal{P}$, i.e., $\Delta\pi$ continuous piecewise linear over $\Delta\mathcal{P}$. □

Remark 3.8. If π and $\mathcal{P}$ are periodic modulo $\mathbb{Z}^k$, then $\Delta\pi$ and $\Delta\mathcal{P}$ are periodic modulo $\mathbb{Z}^k \times \mathbb{Z}^k$. Indeed, let $F \in \Delta\mathcal{P}$, so $F = F(I, J, K)$ for some $I, J, K \in \mathcal{P}$. Then for $(\mathbf{u}, \mathbf{v}) \in \mathbb{Z}^k \times \mathbb{Z}^k$ we have $F + (\mathbf{u}, \mathbf{v}) = F(I+\mathbf{u}, J+\mathbf{v}, K+\mathbf{u}+\mathbf{v}) \in \Delta\mathcal{P}$. In order to make the description of $\Delta\pi$ finite, we can choose a fundamental domain (system of unique representatives) of $\mathbb{R}^k \times \mathbb{R}^k$ with respect to the action of $\mathbb{Z}^k \times \mathbb{Z}^k$, for example $\Delta\tilde{D} := [0, 1)^k \times [0, 1)^k$.

Remark 3.9. We remark that $\Delta\pi(\mathbf{x}, \mathbf{y})$ is also invariant under exchanging $\mathbf{x}$ and $\mathbf{y}$. This can be expressed as an action of the symmetric group S_2. Together we obtain the action of the group $\mathbb{Z}^k \wr S_2$, a wreath product, and so we would be able to choose a smaller fundamental domain, corresponding to the action of this group.

3.3. **Finite test for minimality of piecewise linear functions.** By Theorem 1.1, we can test whether a function is minimal by testing subadditivity and the symmetry condition. These properties are easy to test when the function is continuous piecewise linear. The first of such tests came from Gomory and Johnson [17, Theorem 7] for the case $k = 1$.[2] Richard, Li, and Miller [21, Theorem 22] gave a similar superadditivity test for discontinuous piecewise linear functions. In [4], the authors gave a minimality test for discontinuous piecewise linear functions for the $k = 1$ case. In the present paper, we give a similar test for continuous piecewise linear functions for general k. As in [4], we do not claim novelty for these ideas. Since our focus of this paper is classifying extreme functions and our theorems only consider minimal functions, we present these minimality tests to give a complete picture.

We assume that the function given to us is periodic and is described by a pure and complete polyhedral complex $\mathcal{P}$ where every cell in $\mathcal{P}$ is bounded and therefore each cell is the convex hull of its vertices. We explain this assumption in section B.4 of the Appendix. In particular, we show that every continuous minimal piecewise linear function $\pi\colon \mathbb{R}^k \to \mathbb{R}$ that satisfies a certain regularity condition called *genuinely k-dimensional* has the property that if $\mathcal{P}$ is periodic modulo $\mathbb{Z}^k$, then every cell of $\mathcal{P}$ is bounded (Lemma B.15). The precise definition of *genuinely k-dimensional* functions is given in Definition B.1. Furthermore, if the function π is not genuinely k-dimensional, then we can project it into a lower dimension and study it there (Proposition B.9, Remark B.10), and thus, this assumption is not very restrictive.

We use $\operatorname{vert}(\cdot)$ to denote the set of vertices of a polyhedron or polyhedral complex. For a polyhedral complex $\mathcal{P}$ in $\mathbb{R}^k$ and a set $S \subseteq \mathbb{R}^k$, we define $S \cap \mathcal{P} := \{\, S \cap F \mid F \in \mathcal{P} \,\}$. When S is a polyhedron in $\mathbb{R}^k$, the collection $S \cap \mathcal{P}$ is again a polyhedral complex. We write $\mathbf{1}$ to denote the all one's vector, and $\pmod{\mathbf{1}}$ to denote componentwise equivalence modulo 1.

Theorem 3.10 (Minimality test)**.** *Let $\mathcal{P}$ be a pure, complete, polyhedral complex in $\mathbb{R}^k$ that is periodic modulo $\mathbb{Z}^k$ and every cell of $\mathcal{P}$ is bounded. Let $\pi\colon \mathbb{R}^k \to \mathbb{R}$ be a continuous function that is periodic modulo $\mathbb{Z}^k$ and that is piecewise linear function over $\mathcal{P}$. Let $\Delta\tilde{D} = [0, 1)^k \times [0, 1)^k$ or another fundamental domain as described in Remarks 3.8 and 3.9. Then π is minimal for $R_{\mathbf{f}}(\mathbb{R}^k, \mathbb{Z}^k)$ if and only if the following conditions hold:*

(1) $\pi(\mathbf{0}) = 0$,
(2) Subadditivity test: $\Delta\pi(\mathbf{u}, \mathbf{v}) \geq 0$ *for all* $(\mathbf{u}, \mathbf{v}) \in \Delta\tilde{D} \cap \operatorname{vert}(\Delta\mathcal{P})$.
(3) Symmetry test: $\pi(\mathbf{f}) = 1$ *and*

$$\Delta\pi(\mathbf{u}, \mathbf{v}) = 0 \quad \textit{for all} \quad (\mathbf{u}, \mathbf{v}) \in \Delta\tilde{D} \cap \operatorname{vert}(\Delta\mathcal{P} \cap \{\, (\mathbf{u}, \mathbf{v}) \mid \mathbf{u} + \mathbf{v} \equiv \mathbf{f} \pmod{\mathbf{1}} \,\}).$$

[2]Note that in [17], the word "minimal" needs to be replaced by "satisfies the symmetry condition" throughout the statement of their theorem and its proof.

Proof. We use the characterization of minimal functions given by Theorem 1.1. Clearly these conditions are necessary. We will show that they are sufficient.

Since every cell of $\mathcal{P}$ is bounded, the cells of $\Delta\mathcal{P}$ are also bounded. By Lemma 3.7, $\Delta\pi$ is continuous piecewise linear over $\Delta\mathcal{P}$, and therefore the $\Delta\pi$ is completely determined by the values on $\operatorname{vert}(\Delta\mathcal{P})$.

Let $(\mathbf{x},\mathbf{y}) \in \mathbb{R}^k \times \mathbb{R}^k$. For subadditivity, we need to show that $\Delta\pi(\mathbf{x},\mathbf{y}) \geq 0$. Let $F \in \Delta\mathcal{P}$ such that $(\mathbf{x},\mathbf{y}) \in F$. Consider any vertex $(\mathbf{u},\mathbf{v}) \in \operatorname{vert}(F)$. Since $\Delta\tilde{D}$ is a fundamental domain for $\mathbb{Z}^k \times \mathbb{Z}^k$, and $\Delta\mathcal{P}$ is periodic modulo $\mathbb{Z}^k \times \mathbb{Z}^k$, there exists a point $(\mathbf{w},\mathbf{z}) \in \mathbb{Z}^k \times \mathbb{Z}^k$ such that $(\mathbf{u}+\mathbf{w},\mathbf{v}+\mathbf{z}) \in \Delta\tilde{D} \cap \operatorname{vert}(\Delta\mathcal{P})$. Since $\Delta\pi$ is periodic modulo $\mathbb{Z}^k \times \mathbb{Z}^k$ and is nonnegative on $(\mathbf{u}+\mathbf{w},\mathbf{v}+\mathbf{z})$, we have that $\Delta\pi$ is also nonnegative on $(\mathbf{u},\mathbf{v})$. Therefore $\Delta\pi$ is nonnegative on all of $\operatorname{vert}(F)$, and since $\Delta\pi|_F$ is affine, by convexity it follows that $\Delta\pi(\mathbf{x},\mathbf{y}) \geq 0$. Therefore π is subadditive.

Similarly, to show symmetry, we need to show that $\Delta\pi(\mathbf{x},\mathbf{y}) = 0$ for all $\mathbf{x},\mathbf{y} \in \mathbb{R}^k$ such that $\mathbf{x}+\mathbf{y} \equiv \mathbf{f} \pmod{\mathbf{1}}$. Observe that $\Delta\mathcal{P} \cap \{\, (\mathbf{u},\mathbf{v}) \mid \mathbf{u}+\mathbf{v} \equiv \mathbf{f} \pmod{\mathbf{1}} \,\}$ is a polyhedral complex. Let $(\mathbf{x},\mathbf{y}) \in \mathbb{R}^k$ such that $\mathbf{x}+\mathbf{y} \equiv \mathbf{f} \pmod{\mathbf{1}}$. By letting $F \in \Delta\mathcal{P} \cap \{\, (\mathbf{u},\mathbf{v}) \mid \mathbf{u}+\mathbf{v} \equiv \mathbf{f} \pmod{\mathbf{1}} \,\}$ such that $(\mathbf{x},\mathbf{y}) \in F$, the same argument as above shows that $\Delta\pi = 0$ for all vertices of F, and by convexity, $\Delta\pi|_F = 0$. Therefore $\Delta\pi(\mathbf{x},\mathbf{y}) = 0$ and we conclude that π is symmetric. □

Remark 3.11 (Symmetry test simplified)**.** Suppose $\mathcal{P}$ is pure, complete polyhedral complex that is periodic modulo $\mathbb{Z}^k$ and contains $\{\mathbf{f}\} \in \mathcal{P}$. Then $\Delta\tilde{D} \cap \operatorname{vert}(\Delta\mathcal{P} \cap \{\, (\mathbf{u},\mathbf{v}) \mid \mathbf{u}+\mathbf{v} \equiv \mathbf{f} \pmod{\mathbf{1}} \,\}) \subseteq \Delta\tilde{D} \cap \operatorname{vert}(\Delta\mathcal{P})$. In particular, the symmetry test (3 in Theorem 3.10) then reduces to checking on vertices $(\mathbf{u},\mathbf{v}) \in \Delta\tilde{D}$ of $\Delta\mathcal{P}$ such that $\mathbf{u}+\mathbf{v} \equiv \mathbf{f} \pmod{\mathbf{1}}$.

To see this, consider any face $F(I,J,K) \in \Delta\mathcal{P}$, where $I,J,K \in \mathcal{P}$, and any $\mathbf{z} \in \mathbb{Z}^k$. Then $F(I,J,K) \cap \{(\mathbf{x},\mathbf{y}) \in \mathbb{R}^k \times \mathbb{R}^k \mid \mathbf{x}+\mathbf{y} = \mathbf{f}+\mathbf{z}\} = F(I,J,\{\mathbf{f}+\mathbf{z}\} \cap K)$. Since $\mathcal{P}$ is periodic modulo $\mathbb{Z}^k$ and $\{\mathbf{f}\} \in \mathcal{P}$, we have $\{\mathbf{f}+\mathbf{z}\} \in \mathcal{P}$. Since $\mathcal{P}$ is a polyhedral complex, $\{\mathbf{f}+\mathbf{z}\} \cap K \in \mathcal{P}$. Therefore, $F(I,J,\{\mathbf{f}+\mathbf{z}\} \cap K) \in \Delta\mathcal{P}$. Therefore, $\operatorname{vert}(\Delta\mathcal{P} \cap \{\, (\mathbf{u},\mathbf{v}) \mid \mathbf{u}+\mathbf{v} \equiv \mathbf{f} \pmod{\mathbf{1}} \,\}) \subseteq \operatorname{vert}(\Delta\mathcal{P})$. Intersecting both sides with $\Delta\tilde{D}$ maintains the containment relationship.

3.4. **Combinatorializing the additivity domain.** Let $\pi\colon \mathbb{R}^k \to \mathbb{R}$ be a continuous piecewise linear function over a pure, complete polyhedral complex $\mathcal{P}$. Recall the definition of the *additivity domain* of π,

$$E(\pi) = \bigl\{\, (\mathbf{x},\mathbf{y}) \bigm| \Delta\pi(\mathbf{x},\mathbf{y}) = 0 \,\bigr\}.$$

We now give a combinatorial representation of this set using the faces of $\mathcal{P}$; this extends a technique in [4]. Let

$$E(\pi,\mathcal{P}) = \bigl\{\, F \in \Delta\mathcal{P} \bigm| \Delta\pi|_F = 0 \,\bigr\}.$$

We consider $E(\pi,\mathcal{P})$ to include $F = \emptyset$, on which $\Delta\pi|_F = 0$ holds trivially. Then $E(\pi,\mathcal{P})$ is another polyhedral complex, a subcomplex of $\Delta\mathcal{P}$. As mentioned, if π is continuous, then $\Delta\pi$ is continuous. Under this continuity assumption, we can consider only the set of maximal faces in $E(\pi,\mathcal{P})$. We define

$$E_{\max}(\pi,\mathcal{P}) = \bigl\{\, F \in E(\pi,\mathcal{P}) \bigm| F \text{ is a maximal face by set inclusion in } E(\pi,\mathcal{P}) \,\bigr\}.$$

Lemma 3.12.

$$E(\pi) = \bigcup\{F \in E(\pi,\mathcal{P})\} = \bigcup\{F \in E_{\max}(\pi,\mathcal{P})\}.$$

Proof. Clearly $E(\pi) \supseteq \bigcup\{F \in E(\pi,\mathcal{P})\} \supseteq \bigcup\{F \in E_{\max}(\pi,\mathcal{P})\}$. We show the reverse inclusions. Suppose $(\mathbf{x},\mathbf{y}) \in E(\pi)$. Since $\Delta\mathcal{P}$ is a polyhedral complex that covers all of $\mathbb{R}^k \times \mathbb{R}^k$, there exists a face $F \in \Delta\mathcal{P}$ such that $(\mathbf{x},\mathbf{y}) \in \operatorname{rel\,int}(F)$. Since $\Delta\pi$ is affine in F, $\Delta\pi \geq 0$, $(\mathbf{x},\mathbf{y}) \in \operatorname{rel\,int}(F)$, $\Delta\pi(\mathbf{x},\mathbf{y}) = 0$ and therefore $\Delta\pi|_F = 0$. Therefore, $F \in E(\pi,\mathcal{P})$ and $(\mathbf{x},\mathbf{y}) \in F$ is contained in the first right hand side. Clearly, if F is not maximal in $E(\pi,\mathcal{P})$, then it is contained in a maximal face $F' \in E_{\max}(\pi,\mathcal{P})$, and hence the reverse inclusions also hold. □

This combinatorial representation can then be made finite by choosing representatives under the action of $\mathbb{Z}^k \times \mathbb{Z}^k$, which leaves $E(\pi)$ and thus $E(\pi,\mathcal{P})$ and $E_{\max}(\pi,\mathcal{P})$ invariant, as in Remark 3.8.

3.5. **Non-extremality via perturbation functions.** We now give a method of showing π is not extreme when we are given a certain piecewise linear perturbation function $\bar\pi$.

Theorem 3.13 (Perturbation)**.** *Let $\mathcal{P}$ be a pure, complete, polyhedral complex in $\mathbb{R}^k$ that is periodic modulo $\mathbb{Z}^k$ and every cell of $\mathcal{P}$ is bounded. Suppose π is minimal and continuous piecewise linear over $\mathcal{P}$. Suppose $\bar\pi \not\equiv 0$ is continuous piecewise linear over $\mathcal{P}$, is periodic modulo $\mathbb{Z}^k$ and satisfies $E(\pi) \subseteq E(\bar\pi)$ and $\bar\pi(\mathbf{f}) = 0$. Then π is not extreme. Furthermore, given $\bar\pi$, there exists an $\epsilon > 0$ such that $\pi^1 = \pi + \epsilon\bar\pi$ and $\pi^2 = \pi - \epsilon\bar\pi$ are distinct minimal functions that are continuous piecewise linear over $\mathcal{P}$ such that $\pi = \frac{1}{2}(\pi^1 + \pi^2)$.*

Proof. Let $\Delta\tilde{D} = [0,1)^k \times [0,1)^k$ (or any other fundamental domain as in Remarks 3.8 and 3.9). Let

$$\epsilon = \frac{1}{2}\frac{\min(\Delta\pi(\mathbf{x},\mathbf{y}) \mid (\mathbf{x},\mathbf{y}) \in \Delta\tilde{D} \cap \operatorname{vert}(\Delta\mathcal{P}), \Delta\pi(\mathbf{x},\mathbf{y}) \neq 0)}{\max(|\Delta\bar\pi(\mathbf{u},\mathbf{v})| \mid (\mathbf{u},\mathbf{v}) \in \Delta\tilde{D} \cap \operatorname{vert}(\Delta\mathcal{P}), \Delta\bar\pi(\mathbf{u},\mathbf{v}) \neq 0)}.$$

Note that ϵ exists and $\epsilon > 0$ since $\Delta\pi$ and $\Delta\bar\pi$ are non-zero somewhere, $\Delta\pi$ is a nonnegative function because π is minimal, and $\operatorname{vert}(\Delta\mathcal{P}) \neq \emptyset$ since $\Delta\mathcal{P}$ is a collection of bounded polyhedra.

Setting $\pi^1 = \pi + \epsilon\bar\pi, \pi^2 = \pi - \epsilon\bar\pi$, we show that π^1, π^2 satisfy conditions (1), (2), and (3) of Theorem 3.10 to show that π^1, π^2 are minimal functions. Since $\epsilon > 0$ and $\bar\pi \not\equiv 0$, π^1, π^2 are then distinct minimal functions that show that π is not extreme.

We use the assumption that $E(\pi) \subseteq E(\bar\pi)$, which implies that $\Delta\bar\pi(\mathbf{x},\mathbf{y}) = 0$ whenever $\Delta\pi(\mathbf{x},\mathbf{y}) = 0$.

First, $\Delta\pi(\mathbf{0},\mathbf{0}) = \pi(\mathbf{0}) + \pi(\mathbf{0}) - \pi(\mathbf{0}) = \pi(\mathbf{0}) = 0$, therefore $0 = \Delta\bar\pi(\mathbf{0},\mathbf{0}) = \bar\pi(\mathbf{0})$. Therefore $\pi^1(\mathbf{0}) = \pi^2(\mathbf{0}) = 0$. Since $\bar\pi(\mathbf{f}) = 0$ and $\pi(\mathbf{f}) = 1$, it follows that $\pi^1(\mathbf{f}) = \pi^2(\mathbf{f}) = 1$. These results along with $E(\pi) \subseteq E(\bar\pi)$ satisfy conditions (1) and (3) and Theorem 3.10.

Next, for any $(\mathbf{x},\mathbf{y}) \in \Delta\tilde{D} \cap \operatorname{vert}(\Delta\mathcal{P})$, from the definition of ϵ and the fact that $E(\pi) \subseteq E(\bar\pi)$, which implies that $\Delta\bar\pi(\mathbf{x},\mathbf{y}) = 0$ whenever $\Delta\pi(\mathbf{x},\mathbf{y}) = 0$, we have

$$\Delta\pi(\mathbf{x},\mathbf{y}) \pm \epsilon\Delta\bar\pi(\mathbf{x},\mathbf{y}) \geq \Delta\pi(\mathbf{x},\mathbf{y}) - \epsilon|\Delta\bar\pi(\mathbf{x},\mathbf{y})| \geq \tfrac{1}{2}\Delta\pi(\mathbf{x},\mathbf{y}) \geq 0.$$

Therefore π^1, π^2 satisfy also condition (2) of Theorem 3.10, and we are done. ☐

4. A class of minimal valid functions defined over $\mathbb{R}^2$

We now define the class of *diagonally constrained* functions $\pi\colon \mathbb{R}^2 \to \mathbb{R}$. We first introduce a special two-dimensional polyhedral complex. The functions will be continuous piecewise linear over this complex.

4.1. **The standard triangulations $\mathcal{P}_q$ of $\mathbf{R}^2$ and their geometry.** Let q be a positive integer. Consider the arrangement $\mathcal{H}_q$ of all hyperplanes (lines) of $\mathbb{R}^2$ of the form $\binom{0}{1} \cdot \mathbf{x} = b$, $\binom{1}{0} \cdot \mathbf{x} = b$, and $\binom{1}{1} \cdot \mathbf{x} = b$, where $b \in \frac{1}{q}\mathbb{Z}$. The complement of the arrangement $\mathcal{H}_q$ consists of two-dimensional cells, whose closures are the triangles

$$_{\mathbf{0}}◺ = \tfrac{1}{q}\operatorname{conv}(\{\tbinom{0}{0}, \tbinom{1}{0}, \tbinom{0}{1}\}) \qquad \text{and} \qquad _{\mathbf{0}}◹ = \tfrac{1}{q}\operatorname{conv}(\{\tbinom{1}{0}, \tbinom{0}{1}, \tbinom{1}{1}\})$$

and their translates by elements of the lattice $\frac{1}{q}\mathbb{Z}^2$.

We denote by $\mathcal{P}_q$ the collection of these triangles and the vertices and edges that arise as intersections of the triangles, and the empty set. Thus $\mathcal{P}_q$ is a locally finite polyhedral complex that is periodic modulo $\mathbb{Z}^2$. Since all nonempty faces of $\mathcal{P}_q$ are simplices, it is a triangulation of the space $\mathbb{R}^2$.

Example 4.1. Figure 4 shows the complex $\mathcal{P}_5$ with an example of a minimal valid continuous piecewise linear function on $\mathcal{P}_5$ with $\mathbf{f} = \binom{2/5}{2/5}$ that is periodic modulo $\mathbb{Z}^2$. The function is uniquely determined by its values on the vertices of $\mathcal{P}_5$ that lie within the fundamental domain $\tilde{D} = [0,1)^2$. Note that, due the periodicity of the function modulo $\mathbb{Z}^2$, the values of the function on the left and the right edge (and likewise on the bottom and the top edge) of $D = [0,1]^2$ match.

Within the polyhedral complex $\mathcal{P}_q$, let $\mathcal{P}_{q,\cdot}$ be the set of 0-faces (vertices), $\mathcal{P}_{q,|\,\_\,\diagdown}$ be the set of 1-faces (edges), and $\mathcal{P}_{q,◺◹}$ be the set of 2-faces (triangles). The sets of diagonal, vertical, and horizontal edges will be denoted by $\mathcal{P}_{q,\diagdown}$, $\mathcal{P}_{q,|}$, and $\mathcal{P}_{q,\_}$, respectively. We also use abbreviations such as $\mathcal{P}_{q,\cdot\diagdown} = \mathcal{P}_{q,\cdot} \cup \mathcal{P}_{q,\diagdown}$, $\mathcal{P}_{q,\cdot\diagdown◺◹} = \mathcal{P}_{q,\cdot} \cup \mathcal{P}_{q,\diagdown} \cup \mathcal{P}_{q,◺◹}$, etc.

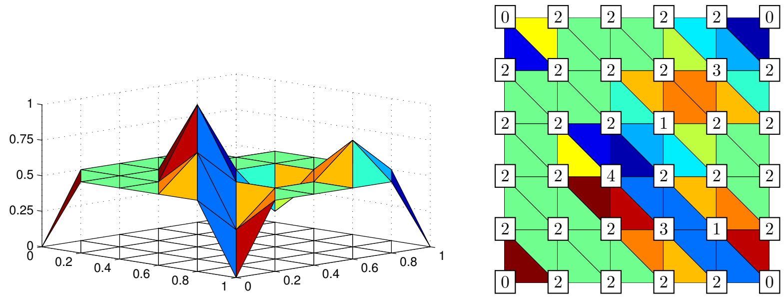


FIGURE 4. A minimal valid, continuous, piecewise linear function over the polyhedral complex $\mathcal{P}_5$, which is diagonally constrained (subsection 4.2). *Left*, the three-dimensional plot of the function on the closure $D = [0,1]^2$ of the fundamental domain $\tilde{D} = [0,1)^2$. *Right*, the complex $\mathcal{P}_5$, restricted to D and colored according to slopes to match the 3-dimensional plot, and decorated with values v at each vertex of $\mathcal{P}_5$ where the function takes value $\frac{v}{4}$.

Remark 4.2. Let

$$A = \begin{bmatrix} 1 & -1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & -1 & 1 & -1 \end{bmatrix}^T.$$

Then for every face $I \in \mathcal{P}_q$, there exists a vector $\mathbf{b} \in \frac{1}{q}\mathbb{Z}^6$ such that $I = \{\, \mathbf{x} \mid A\mathbf{x} \leq \mathbf{b} \,\}$. Furthermore, for every vector $\mathbf{b} \in \frac{1}{q}\mathbb{Z}^6$, the set $\{\, \mathbf{x} \mid A\mathbf{x} \leq \mathbf{b} \,\}$ is a union of faces of $\mathcal{P}_q$ (possibly empty), since each inequality corresponds to a hyperplane in the arrangement $\mathcal{H}_q$.

The matrix A is totally unimodular. Thus the specific choice of the triangulation $\mathcal{P}_q$ lends itself to strong unimodularity properties that reveal structure in the complex. These properties provide an easy method to compute $E(\pi, \mathcal{P}_q)$ and test if a function is diagonally constrained by using simple arithmetic and set membership operations on vertices of $\mathcal{P}_q$; see the thesis [18] for details. More importantly, they allow us to develop a simple theory of extremality, in which all relevant properties of the function can be expressed using the faces of the original complex $\mathcal{P}_q$.

Lemma 4.3. *Let $I, J \in \mathcal{P}_q$. Then $-I$ and $I + J$ are unions of faces in $\mathcal{P}_q$.*

Proof. If $I = \{\, \mathbf{x} \in \mathbb{R}^2 \mid A\mathbf{x} \leq \mathbf{b} \,\}$ for some $\mathbf{b} \in \frac{1}{q}\mathbb{Z}^6$, then $-I = \{\, \mathbf{x} \in \mathbb{R}^2 \mid -A\mathbf{x} \leq \mathbf{b} \,\}$. Since $-A$ has the same rows as A (with a permutation), by Remark 4.2, $-I$ is a union of faces of $\mathcal{P}_q$.

We now show that the Minkowski sum $I + J$ is a union of faces in $\mathcal{P}_q$.

Let $\mathbf{a}^i$ be the i^{th} row vector of A. Then there exist vectors $\mathbf{b}^1, \mathbf{b}^2$ such that $I = \{\, \mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}^1 \,\}$, $J = \{\, \mathbf{y} \mid A\mathbf{y} \leq \mathbf{b}^2 \,\}$. Moreover, due to the total unimodularity of the matrix A, the right-hand side vectors $\mathbf{b}^1, \mathbf{b}^2$ can be chosen so that $\mathbf{b}^1, \mathbf{b}^2$ are tight, i.e.,

$$\max_{\mathbf{x} \in I} \mathbf{a}^i \cdot \mathbf{x} = \mathbf{b}^1_i, \quad \max_{\mathbf{y} \in J} \mathbf{a}^i \cdot \mathbf{y} = \mathbf{b}^2_i, \tag{10}$$

and $\mathbf{b}^1, \mathbf{b}^2 \in \frac{1}{q}\mathbb{Z}^6$.

We claim that $I + J = \{\, \mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}^1 + \mathbf{b}^2 \,\}$. Clearly $I + J \subseteq \{\, \mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}^1 + \mathbf{b}^2 \,\}$. We show the reverse direction. Let K' be a facet (edge) of $I + J$. Then $K' = I' + J'$, where I' is a face of I and J' is a face of J. Without loss of generality, assume that I' is an edge; then J' is either a vertex or an edge. By well-known properties of Minkowski sums, the normal cone of K' is the intersection of the normal cones of I' in I and J' in J. Thus K' has the same normal direction as the facet (edge) I'. (This argument relied on the fact that we are in dimension two.) This proves that $I + J = \{\, \mathbf{x} \mid A\mathbf{x} \leq \mathbf{b} \,\}$ for some vector $\mathbf{b}$.

Let $\mathbf{x}^*$, $\mathbf{y}^*$ be maximizers in (10). Then $\mathbf{x}^* + \mathbf{y}^* \in I + J$, and thus

$$\mathbf{b}^1_i + \mathbf{b}^2_i = \mathbf{a}^i \cdot \mathbf{x}^* + \mathbf{a}^i \cdot \mathbf{y}^* \leq \max_{\mathbf{z} \in I+J} \mathbf{a}^i \cdot \mathbf{z} \leq \max_{\mathbf{x} \in I} \mathbf{a}^i \cdot \mathbf{x} + \max_{\mathbf{y} \in J} \mathbf{a}^i \cdot \mathbf{y} = \mathbf{b}^1_i + \mathbf{b}^2_i.$$

Therefore, $\max_{\mathbf{z}\in I+J} \mathbf{a}^i \cdot \mathbf{z} = \mathbf{b}^1_i + \mathbf{b}^2_i$, which shows that every constraint $\mathbf{a}^i \cdot \mathbf{z} \le \mathbf{b}^1_i$ is met at equality, and therefore $I + J = \{\, \mathbf{x} \mid A\mathbf{x} \le \mathbf{b}^1 + \mathbf{b}^2 \,\}$ and we conclude that $I + J$ is a union of subsets in $\mathcal{P}_q$. □

This result has an important consequence for the complex $\Delta\mathcal{P}_q$, allowing component projections of the faces of $\Delta\mathcal{P}_q$ to be faces of $\mathcal{P}_q$.

Lemma 4.4. (i) *Let $F \in \Delta\mathcal{P}_q$. Then the projections $p_1(F)$, $p_2(F)$, and $p_3(F)$ are faces in the complex $\mathcal{P}_q$.*
(ii) *In particular, let $(\mathbf{x}, \mathbf{y})$ be a vertex of $\Delta\mathcal{P}_q$. Then $\mathbf{x}, \mathbf{y}$ are vertices of the complex $\mathcal{P}_q$, i.e., $\mathbf{x}, \mathbf{y} \in \frac{1}{q}\mathbb{Z}^2$.*

Proof. By definition of $\Delta\mathcal{P}_q$, there exist $I, J, K \in \mathcal{P}_q$ such that $F = F(I, J, K)$. Let $I' = p_1(F)$, $J' = p_2(F)$, and $K' = p_3(F)$. By Proposition 3.3,

$$\begin{aligned} I' &= p_1(F) = (K + (-J)) \cap I, \\ J' &= p_2(F) = (K + (-I)) \cap J, \\ K' &= p_3(F) = (I + J) \cap K, \end{aligned}$$

and thus, by Lemma 4.3, I', J', and K' are faces of $\mathcal{P}_q$. □

Theorem 4.5 (Simplified minimality test)**.** *Let $\pi\colon \mathbb{R}^2 \to \mathbb{R}$ be a continuous piecewise linear function over $\mathcal{P}_q$ that is periodic modulo $\mathbb{Z}^2$. Suppose $\mathbf{f} \in \operatorname{vert}(\mathcal{P}_q)$. Then π is minimal for $R_{\mathbf{f}}(\mathbb{R}^2, \mathbb{Z}^2)$ if and only if the following conditions hold.*

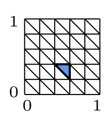
(1) $\pi(\mathbf{0}) = 0$.
(2) Subadditivity test: $\pi(\mathbf{x}) + \pi(\mathbf{y}) \ge \pi(\mathbf{x} + \mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \frac{1}{q}\mathbb{Z}^2 \cap [0, 1)^2$.
(3) Symmetry test: $\pi(\mathbf{x}) + \pi(\mathbf{f} - \mathbf{x}) = 1$ for all $\mathbf{x} \in \frac{1}{q}\mathbb{Z}^2 \cap [0, 1)^2$.

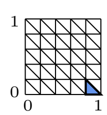
Proof. Since $\{\mathbf{f}\} \in \mathcal{P}_q$, the result follows by applying Theorem 3.10 with $\Delta\tilde{D} = [0, 1)^2 \times [0, 1)^2$ and using Remark 3.11 and Lemma 4.4 to show that the vertices that need to be considered are vertices $(\mathbf{x}, \mathbf{y}) \in \frac{1}{q}\mathbb{Z}^2 \times \frac{1}{q}\mathbb{Z}^2 \cap \Delta\tilde{D}$. □

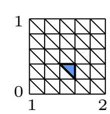
Example 4.6 (Example 4.1, continued)**.** We now visualize the additive faces $F \in E(\pi, \mathcal{P}_q)$ of Example 4.1 (Figure 4); following Lemma 3.12, we are particularly interested in the maximal additive faces $\bar{F} \in E_{\max}(\pi, \mathcal{P}_q)$. Following Remark 3.8, $E(\pi, \mathcal{P}_q)$ is invariant under the action of $\mathbb{Z}^k \times \mathbb{Z}^k$. By the construction of $\mathcal{P}_q$, we can always choose a representative $\bar{F} \in E_{\max}(\pi, \mathcal{P}_q)$ that is a subset of the closure $\Delta D = [0, 1]^2 \times [0, 1]^2$ of the fundamental domain. Then all faces $F \in E(\pi, \mathcal{P}_q)$ with $F \subseteq \bar{F}$ also are subsets of ΔD.

By Lemma 3.5, each $F \in E(\pi, \mathcal{P}_q)$ is determined by its projections $I = p_1(F)$, $J = p_2(F)$, $K = p_3(F)$ as $F = F(I, J, K)$. Due to the choice of triangulation $\mathcal{P}_q$, by Lemma 4.4, I, J, and K are faces of $\mathcal{P}_q$. When $F \subseteq \Delta D = [0, 1]^2 \times [0, 1]^2$, we have $I, J \subseteq D = [0, 1]^2$ and $K \subseteq 2D = [0, 2]^2$.

Thus we can visualize faces $F \subseteq \Delta D$ by showing three diagrams, corresponding to its projections $p_i(F) \in \mathcal{P}_q$, where $p_1(F), p_2(F) \subseteq D$ and $p_3(F) \subseteq 2D$ as follows. For example, consider the face $\bar{F}$ with

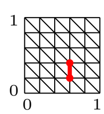
$p_1(\bar{F}) =$ 

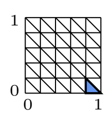
, $p_2(\bar{F}) =$ 

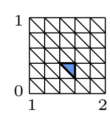
, $p_3(\bar{F}) =$ 

.

It is a maximal additive face. It has of course many smaller included faces, for example F given by

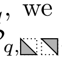
$p_1(F) =$ 

, $p_2(F) =$ 

, $p_3(F) =$ 

.

Here, $p_1(F) \in \mathcal{P}_{q,}$, $p_2(F) \in \mathcal{P}_{q,}$, and $p_3(F) \in \mathcal{P}_{q,}$.

Since π is a minimal valid function, the symmetry condition implies that for any face $I \in \mathcal{P}_q$, we have $F(I, \mathbf{f} - I, \{\mathbf{f}\}) \in E(\pi, \mathcal{P}_q)$; but these are not necessarily maximal additive faces, even when $I \in \mathcal{P}_{q,}$. We illustrate this in Figure 5, which shows a face $F = F(I, \mathbf{f} - I, \{\mathbf{f}\})$ with $I = p_1(F) \in \mathcal{P}_{q,}$ with a containing maximal additive face $\bar{F}$ and the poset of the faces of F.

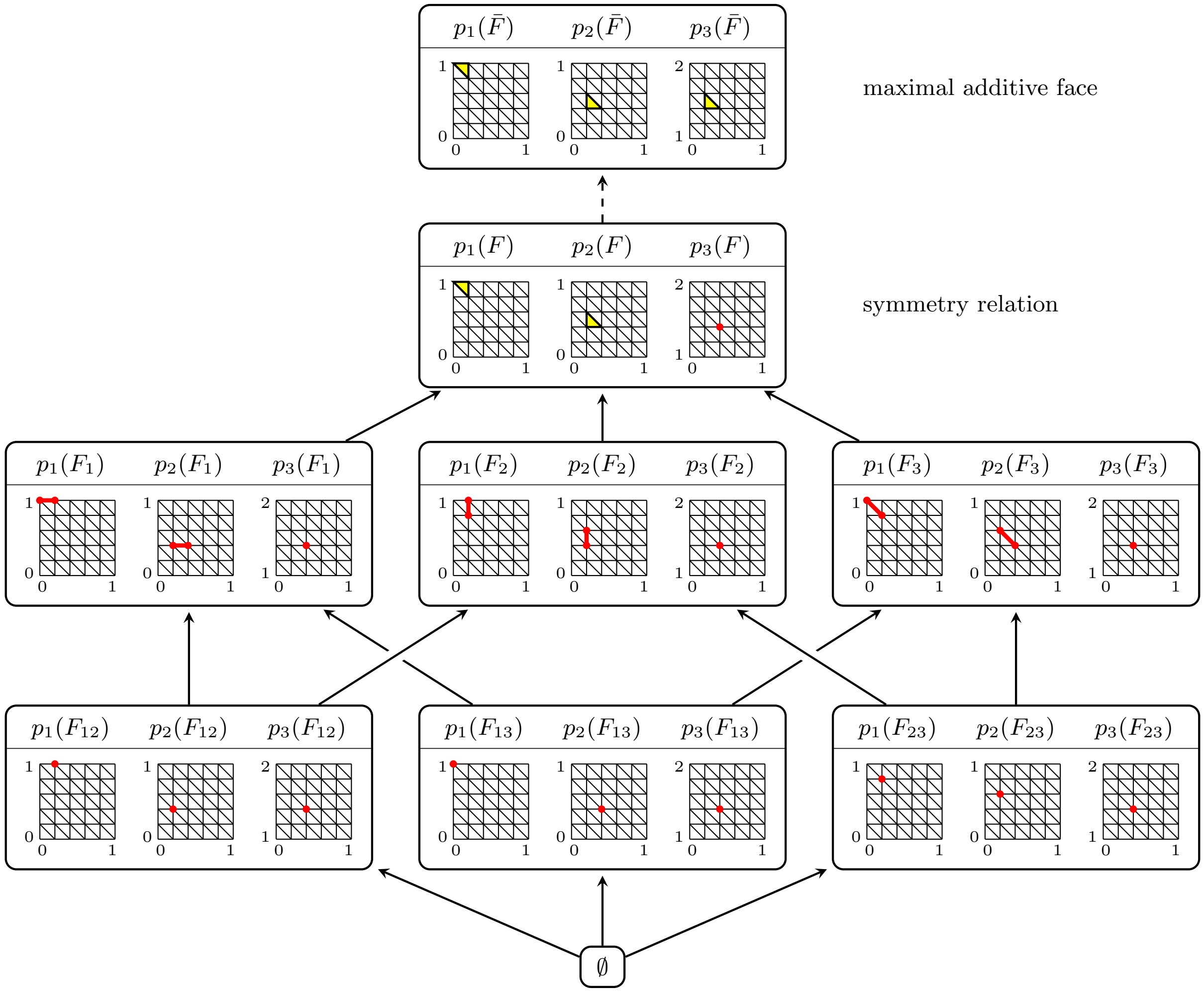


FIGURE 5. A non-maximal additive face $F \in E(\pi, \mathcal{P}_q)$ corresponding to a symmetry relation, the poset of its faces, and a maximal additive face $\bar{F} \in E_{\max}(\pi, \mathcal{P}_q)$ with $F \subset \bar{F}$. The triangles in these diagrams are colored yellow, matching Figure 4, while points and edges are colored red.

Table 1 shows all maximal additive faces $F \in E_{\max}(\pi, \mathcal{P}_q)$ after all the faces arising from the symmetry condition have been removed. Following Remark 3.9, $F(I, J, K) \in E_{\max}(\pi, \mathcal{P}_q)$ if and only if $F(J, I, K) \in E_{\max}(\pi, \mathcal{P}_q)$, so we have also removed the redundancy of swapping I and J by choosing either one of the two representatives.

4.2. **Diagonally constrained functions on $\mathcal{P}_q$.** In the present paper, we will restrict ourselves to a setting determined by the types of faces $F \in E_{\max}(\pi, \mathcal{P}_q)$.

Definition 4.7. A continuous piecewise linear function π on $\mathcal{P}_q$ is called *diagonally constrained* if whenever $F \in E_{\max}(\pi, \mathcal{P}_q)$, then $p_i(F) \in \mathcal{P}_{q,\cdot\diagdown\triangle\triangledown}$ for $i = 1, 2, 3$.

There are many examples of diagonally constrained functions.

Example 4.8 (Example 4.1, continued)**.** Since no relations appearing in the list of all maximal additive faces (Table 1) involve a vertical or horizontal edge, the function is diagonally constrained. Note that there are relations derived from two triangles and one diagonal edge. These relations create affine properties as

TABLE 1. All maximal faces $F \in E_{\max}(\pi, \mathcal{P}_q)$ of the function π from Example 4.1, except for the faces corresponding to the symmetry condition. The triangles in these diagrams are colored to match Figure 4, while points and edges are just colored red. Type numbers refer to Lemma 4.9.

described in Figure 2 (b), and demonstrate that this function is more complicated than full-dimensionally constrained functions, as discussed in section 1.

The following lemma characterizes the types of possible maximal additive faces that can exist for a valid function that is diagonally constrained.

Lemma 4.9. *Suppose π is continuous piecewise linear over $\mathcal{P}_q$ and is diagonally constrained. Suppose that $F \in E_{\max}(\pi, \mathcal{P}_q)$. Let $I = p_1(F), J = p_2(F), K = p_3(F)$. Then one of the following is true.*

(Type 1) $I, J, K \in \mathcal{P}_{q,\bullet\square\diagdown}$.
(Type 2) $I, J, K \in \mathcal{P}_{q,\blacktriangle\blacktriangledown}$.
(Type 3) One of I, J, K is in $\mathcal{P}_{q,\bullet\square}$, while the other two are in $\mathcal{P}_{q,\blacktriangle\blacktriangledown}$.
(Type 4) One of I, J, K is in $\mathcal{P}_{q,\diagdown}$, while the other two are in $\mathcal{P}_{q,\blacktriangle\blacktriangledown}$.

All of these types of maximal additive faces appear in the function from Example 4.8: Maximal faces corresponding to the symmetry condition are of Type 3, whereas Types 1, 2, and 4 appear in Table 1.

Proof. By definition of diagonally constrained functions, $I, J, K \in \mathcal{P}_{q,\bullet\square} \cup \mathcal{P}_{q,\diagdown} \cup \mathcal{P}_{q,\blacktriangle\blacktriangledown}$. Elementary counting reveals that there are 27 possible ways to put I, J, K into those three sets, whereas 15 possibilities are described above. We will show that the 12 remaining cases not listed above are not possible because I, J, K are projections of F.

(1) Suppose $I, J \in \mathcal{P}_{q,\bullet\square\diagdown}$, $K \in \mathcal{P}_{q,\blacktriangle\blacktriangledown}$. By Proposition 3.3, $K \subseteq I + J$. But this is not possible because $I + J$ is one-dimensional while K is two-dimensional.
(2) Suppose $I, K \in \mathcal{P}_{q,\bullet\square\diagdown}$, $J \in \mathcal{P}_{q,\blacktriangle\blacktriangledown}$. By Proposition 3.3, $J \subseteq K + (-I)$. But again, this is not possible because $K + (-I)$ is one-dimensional while J is two-dimensional.
(3) Suppose $J, K \in \mathcal{P}_{q,\bullet\square\diagdown}$, $I \in \mathcal{P}_{q,\blacktriangle\blacktriangledown}$. This is similar to the last case. ☐

4.3. **Affine properties of π^i on projections of faces in $E(\pi, \mathcal{P}_q)$.** Let π be a minimal valid function that is continuous piecewise linear over $\mathcal{P}_q$. The lemmas of this subsection will be used to deduce affine properties of valid functions π^1, π^2 when $\pi = \frac{1}{2}(\pi^1 + \pi^2)$ by using Lemma 1.4. Here we will apply Corollary 2.14 to conclude affine properties on faces of $\mathcal{P}_q$. By using Corollary 2.14, we are using the continuity of the function to extend affine properties to the boundaries of faces. We note that the condition in Remark 2.13 that allows Theorem 2.11 to extend to the boundary can be shown to hold for full-dimensional faces $F \in \Delta\mathcal{P}_q$. We do not utilize this fact in this paper.

Lemma 4.10. *Suppose $\theta\colon \mathbb{R}^2 \to \mathbb{R}$ is a continuous function and let $F \in E(\theta, \mathcal{P}_q)$ such that $p_i(F) \in \mathcal{P}_{q,\blacktriangle\blacktriangledown}$ for $i = 1, 2, 3$. Then θ is affine in $p_i(F)$ for $i = 1, 2, 3$ with the same gradient.*

Proof. We apply Corollary 2.14 to F with $f, g, h = \theta$ and $L = \mathbb{R}^2$. Since $p_1(F), p_2(F) \in \mathcal{P}_{q,\blacktriangle\blacktriangledown}$, and triangles are two-dimensional objects, we have $L \times L + F \subseteq \operatorname{aff}(F)$. The conclusion of the corollary then says that θ is affine over $p_i(F)$ for $i = 1, 2, 3$ with the same gradient. ☐

Lemma 4.11. *Let $\theta\colon \mathbb{R}^2 \to \mathbb{R}$ be a continuous function. Let $F \in E(\theta, \mathcal{P}_q)$ such that $p_1(F), p_2(F) \in \mathcal{P}_{q,\blacktriangle\blacktriangledown}$ and $p_3(F) \in \mathcal{P}_{q,|\square\_\diagdown}$ (resp., $p_1(F), p_3(F) \in \mathcal{P}_{q,\blacktriangle\blacktriangledown}$ and $p_2(F) \in \mathcal{P}_{q,|\square\_\diagdown}$). Let L be the linear space such that $\operatorname{aff}(p_3(F))$ (resp., $\operatorname{aff}(p_2(F))$) is a translate of L. Then there exists $\mathbf{c} \in \mathbb{R}^2$ such that θ is affine with respect to L over $p_1(F), p_2(F), p_3(F)$ with the gradient $\mathbf{c}$.*

Proof. We only give the proof for $p_1(F), p_2(F) \in \mathcal{P}_{q,\blacktriangle\blacktriangledown}$ and $p_3(F) \in \mathcal{P}_{q,|\square\_\diagdown}$. The other case is similar.

Consider any $(\mathbf{u}^1, \mathbf{v}^1), (\mathbf{u}^2, \mathbf{v}^2) \in \operatorname{rel\,int}(F)$. By applying Corollary 2.14 with F and L we see that there exist vectors $\mathbf{c}^1, \mathbf{c}^2 \in \mathbb{R}^k$ such that θ is affine with gradient $\mathbf{c}^i$ over $(\mathbf{u}^i + L) \cap p_1(F)$, $(\mathbf{v}^i + L) \cap p_2(F)$ and $(\mathbf{u}^i + \mathbf{v}^i + L) \cap p_3(F)$ for $i = 1, 2$. Let $\bar{\mathbf{c}}^1, \bar{\mathbf{c}}^2$ be the orthogonal projections of $\mathbf{c}^1$ and $\mathbf{c}^2$, respectively, onto the linear space L. Therefore, θ is affine with gradient $\bar{\mathbf{c}}^i$ over $(\mathbf{u}^i + L) \cap p_1(F)$, $(\mathbf{v}^i + L) \cap p_2(F)$ and $(\mathbf{u}^i + \mathbf{v}^i + L) \cap p_3(F)$ for $i = 1, 2$. Then, since $(\mathbf{u}^1 + \mathbf{v}^1 + L) \cap p_3(F) = (\mathbf{u}^2 + \mathbf{v}^2 + L) \cap p_3(F) = p_3(F)$, we have $\bar{\mathbf{c}}^1 = \bar{\mathbf{c}}^2$. Therefore, we obtain that θ is affine with respect to L with gradient $\bar{\mathbf{c}} = \bar{\mathbf{c}}^1 = \bar{\mathbf{c}}^2$ over $p_i(F)$ for $i = 1, 2, 3$. ☐

Definition 4.12. Define

$$L_{\diagdown} = \{\, \mathbf{x} \in \mathbb{R}^2 \mid \mathbf{1} \cdot \mathbf{x} = 0 \,\} = \{\, \lambda \tbinom{-1}{1} \mid \lambda \in \mathbb{R} \,\}.$$

Lemma 4.13 (Geometric adjacent transference)**.** *Let $I, J \in \mathcal{P}_{q,\triangle\triangledown}$ be triangles such that $I \cap J \in \mathcal{P}_{q,|\,\_}$. Let π be a continuous function defined on $I \cup J$ satisfying the following properties:*

(i) *π is affine on I.*
(ii) *π is affine with respect to the linear space $L_{\diagdown}$ (the diagonal direction) on J.*

Then π is affine on J.

Proof. Let $e = I \cap J \in \mathcal{P}_{q,|\,\_}$ be the common edge of I and J. We assume that e is horizontal (the argument for vertical edges is exactly the same) and let $\mathbf{v}^0 \in \mathbb{R}^2$ be the vertex of e such that the other vertex is $\mathbf{v}^0 + \binom{0}{1/q}$. Since π is affine with respect to the linear space $L_{\diagdown}$ on J, there exists $c \in \mathbb{R}$ such that $\pi(\mathbf{x} + \lambda\binom{-1}{1}) = \pi(\mathbf{x}) + c \cdot \lambda$ for all $\mathbf{x} \in J$ and $\lambda \in \mathbb{R}$ such that $\mathbf{x} + \lambda\binom{-1}{1} \in J$. Since π is affine on I, there exists $c' \in \mathbb{R}$ such that $\pi(\mathbf{v}^0 + \lambda\binom{0}{1}) = \pi(\mathbf{v}^0) + c' \cdot \lambda$ for all $0 \le \lambda \le 1$.

Now observe that any point in J can be written as $\mathbf{v}^0 + \mu_1\binom{0}{1} + \mu_2\binom{-1}{1}$ with $0 \le \mu_1, \mu_2 \le 1$ and therefore, $\pi(\mathbf{v}^0 + \mu_1\binom{0}{1} + \mu_2\binom{-1}{1}) = \pi(\mathbf{v}^0 + \mu_1\binom{0}{1}) + c \cdot \mu_2$ (using (ii) in the hypothesis) and $\pi(\mathbf{v}^0 + \mu_1\binom{0}{1}) + c \cdot \mu_2 = \pi(\mathbf{v}^0) + c' \cdot \mu_1 + c \cdot \mu_2$. Thus, π is affine over J. □

5. Proof of the main results for the two-dimensional case

In this section we prove our main results for continuous piecewise linear functions over $\mathcal{P}_q$.

Assumption 5.1. For the remainder of the paper, we assume that π is a minimal valid function that is continuous piecewise linear over $\mathcal{P}_q$.

Definition 5.2. (a) For any $I \in \mathcal{P}_q$, if π is affine in I and if for all valid functions π^1, π^2 such that $\pi = \frac{1}{2}(\pi^1 + \pi^2)$ we have that π^1, π^2 are affine in I, then we say that π is *affine imposing in I.*

(b) For any $I \in \mathcal{P}_q$, if π is affine with respect to $L_{\diagdown}$ over I and if for all valid functions π^1, π^2 such that $\pi = \frac{1}{2}(\pi^1 + \pi^2)$ we have that π^1, π^2 are both affine with respect to $L_{\diagdown}$ over I, then we say that π is *diagonally affine imposing in I.*

(c) For a collection $\mathcal{P} \subseteq \mathcal{P}_q$, if for all $I \in \mathcal{P}$, π is affine imposing (or diagonally affine imposing) in I, then we say that π is *affine imposing (diagonally affine imposing) in $\mathcal{P}$.*

We either show that π is affine imposing in $\mathcal{P}_q$ (subsection 5.2) or construct a continuous piecewise linear perturbation (subsection 5.1) that proves π is not extreme (subsections 5.3 and 5.4). If π is affine imposing in $\mathcal{P}_q$, we set up a system of linear equations to decide if π is extreme or not (subsection 5.5). This implies Theorem 1.8 stated in the introduction.

5.1. Perturbation functions.

In this section we study functions $\psi\colon \mathbb{R}^2 \to \mathbb{R}$ that satisfy entire classes of additivity relations that appear in $E(\pi, \mathcal{P}_q)$. These will be used to construct perturbation functions $\bar\pi$ such that $E(\pi) \subseteq E(\bar\pi)$. We may then leverage Theorem 3.13 to show that π is not extreme.

For $m \ge 3$, we define $\psi^m_{q,\cdot}\colon \mathbb{R}^2 \to \mathbb{R}$ that is piecewise linear over $\mathcal{P}_{mq}$ as follows: at all vertices of $\mathcal{P}_{mq}$ that lie on the boundary of ${}_{\mathbf{0}}\triangle$, let $\psi^m_{q,\cdot}$ take the value 0, and at all vertices of $\mathcal{P}_{mq}$ that lie in the interior of ${}_{\mathbf{0}}\triangle$, we assign $\psi^m_{q,\cdot}$ to have the value 1. Interpolate these values linearly to define $\psi^m_{q,\cdot}$ on all of ${}_{\mathbf{0}}\triangle$. For every point $\mathbf{x}$ in ${}_{\mathbf{0}}\triangledown$ define $\psi^m_{q,\cdot}(\mathbf{x}) = -\psi^m_{q,\cdot}(\binom{1/q}{1/q} - \mathbf{x})$. Finally, for any $\mathbf{y} \in \mathbb{R}^2$, let $\mathbf{x} \in [0, \frac{1}{q}]^2$ and $\mathbf{t} \in \frac{1}{q}\mathbb{Z}^2$ be vectors such that $\mathbf{y} = \mathbf{x} + \mathbf{t}$; define $\psi^m_{q,\cdot}(\mathbf{y}) = \psi^m_{q,\cdot}(\mathbf{x})$. Since $\psi^m_{q,\cdot}$ vanishes on the boundary of $[0, \frac{1}{q}]^2$, this periodic extension is well-defined. The function for $m = 3$ is shown in Figure 6 (left).

The following result is quite easy to verify from the definition of $\psi^m_{q,\cdot}$; formally, we appeal to a result from Appendix A whose proof uses more general tools which, in our opinion, are of independent interest.

Lemma 5.3. *For every $m \ge 3$, the function $\psi^m_{q,\cdot}\colon \mathbb{R}^2 \to \mathbb{R}$ constructed above has the following properties:*

(i) $\psi^m_{q,\cdot}\big|_I = 0$ *on all edges and vertices $I \in \mathcal{P}_{q,\cdot\,|\,\_\,\diagdown}$.*

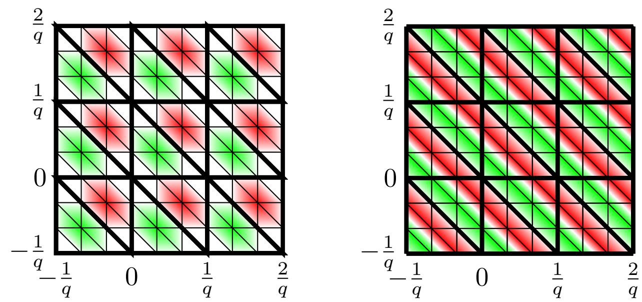


FIGURE 6. Perturbation functions $\psi^m_{q,\bullet\square}$ (*left*) and $\psi^m_{q,\diagdown}$ (*right*) for $m = 3$. Colors indicate whether the value of the function is negative (*red*), positive (*green*), zero (*white*). Two polyhedral complexes are drawn: $\mathcal{P}_q$ (*thick lines*) and its refinement $\mathcal{P}_{mq}$ (*thin lines*).

(ii) *Let $i = 1, 2$ or 3, and let $F \in \Delta\mathcal{P}_q$ be such that $p_i(F) \in \mathcal{P}_{q,\bullet\square}$. Then, $F \subseteq E(\psi^m_{q,\bullet\square})$.*
(iii) *$\psi^m_{q,\bullet\square}$ is continuous piecewise linear over $\mathcal{P}_{mq}$.*

Proof. The assertions follow from (i), (iv) and (v) in Lemma A.7 in Appendix A. □

We will also need another class of functions $\psi^m_{q,\diagdown}\colon \mathbb{R}^2 \to \mathbb{R}$ parametrized by $m \geq 3$. Define $\psi^m_{q,\diagdown}\colon \mathbb{R}^2 \to \mathbb{R}$ is a piecewise linear function over $\mathcal{P}_{mq}$ defined in the following way. The values on the vertices of $\mathcal{P}_{mq}$ are given as follows: for any $\mathbf{x} \in \operatorname{vert}(\mathcal{P}_{mq})$,

$$\psi^m_{q,\diagdown}(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{1}\cdot\mathbf{x} \equiv \frac{i}{mq} \pmod{\frac{1}{q}} \text{ for any } 1 \leq i < \frac{m}{2}, i \in \mathbb{Z}, \\ -1 & \text{if } \mathbf{1}\cdot\mathbf{x} \equiv \frac{j}{mq} \pmod{\frac{1}{q}} \text{ for any } \frac{m}{2} < i \leq m-1, i \in \mathbb{Z}, \\ 0 & \text{if } \mathbf{1}\cdot\mathbf{x} \equiv 0 \text{ or } \frac{1}{2q} \pmod{\frac{1}{q}}. \end{cases}$$

The function $\psi^m_{q,\diagdown}$ is then uniquely extended to $\mathbb{R}^2$ continuously by interpolation on the faces of $\mathcal{P}_{mq}$. The function is shown for $m = 3$ in Figure 6 (right).

The next result can again be easily verified from the definition of the function $\psi^m_{q,\diagdown}$; formally, we again appeal to results from Appendix A.

Lemma 5.4. *The function $\psi^m_{q,\diagdown}\colon \mathbb{R}^2 \to \mathbb{R}$ constructed above is well-defined and has the following properties:*

(i) *$\psi^m_{q,\diagdown}\big|_I = 0$ on all edges and vertices $I \in \mathcal{P}_{q,\bullet\square\diagdown}$.*
(ii) *Let $i = 1, 2,$ or 3 and let $F \in \Delta\mathcal{P}_q$ be such that $p_i(F) \in \mathcal{P}_{q,\bullet\square\diagdown}$. Then, $F \subseteq E(\psi^m_{q,\diagdown})$.*
(iii) *$\psi^m_{q,\diagdown}$ is continuous piecewise linear over $\mathcal{P}_{mq}$.*

Proof. The assertions follow from (i), (iv) and (v) in Lemma A.8 in Appendix A. □

5.2. **Imposing affine linearity on faces of $\mathcal{P}_q$.**

5.2.1. *Covered triangles.* We now consider faces of $\mathcal{P}_{q,\blacktriangle\blacktriangledown}$ on which we will can deduce affine properties.

$$\mathcal{P}^1_{q,\blacktriangle\blacktriangledown} = \{\, I, J \in \mathcal{P}_{q,\blacktriangle\blacktriangledown} \mid \exists K \in \mathcal{P}_{q,\diagdown}, F \in E(\pi, \mathcal{P}_q) \text{ with } (I, J, K) = (p_1(F), p_2(F), p_3(F)) \\ \text{or } (I, K, J) = (p_1(F), p_2(F), p_3(F)) \,\},$$

$$\mathcal{P}^2_{q,\blacktriangle\blacktriangledown} = \{\, I, J, K \in \mathcal{P}_{q,\blacktriangle\blacktriangledown} \mid \exists F \in E(\pi, \mathcal{P}_q) \text{ with } (I, J, K) = (p_1(F), p_2(F), p_3(F)) \,\}.$$

It follows from Lemma 4.10 that π is affine imposing in $\mathcal{P}^2_{q,\blacktriangle\blacktriangledown}$ and from Lemma 4.11 that π is diagonally affine imposing in $\mathcal{P}^1_{q,\blacktriangle\blacktriangledown}$. The superscripts here correspond to the dimension of the linear space on which π is affine imposing on a face.

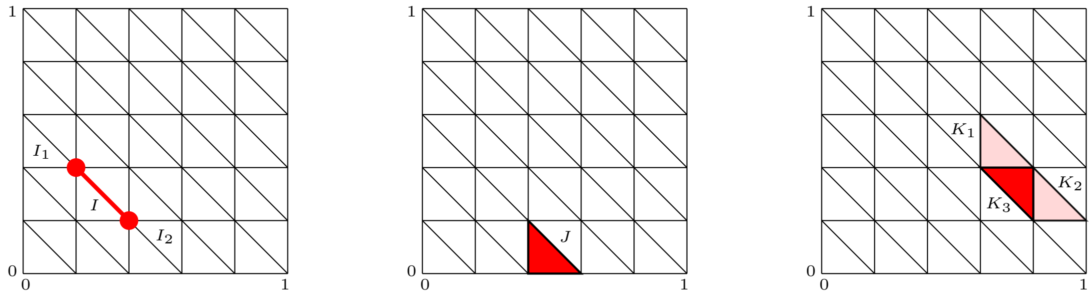


FIGURE 7. An example of an important edge connection in $\mathcal{E}_{\diagdown}$ that is not captured with $\mathcal{E}_{\boxdot}$. For a given minimal valid function π, we could have $F(I, J, K_1), F(I, J, K_2), F(I, J, K_3) \in E(\pi, \mathcal{P}_q)$. Therefore, $\{[J], [K_1]\}, \{[J], [K_2]\}, \{[J], [K_3]\} \in \mathcal{E}_{\diagdown}$. Thus, $[J], [K_1], [K_2], [K_3]$ are connected in the graph $\mathcal{G}(\mathcal{P}_{q,\triangle\triangledown}/\mathbb{Z}^2, \mathcal{E})$. Notice however that already the smaller faces $F(I_1, J, K_1) \subseteq F(I, J, K_1)$ and $F(I_2, J, K_2) \subseteq F(I, J, K_2)$ (corresponding to the vertices I_1 and I_2 of the one-dimensional face I) induce $F(I_1, J, K_1), F(I_2, J, K_2) \in E(\pi, \mathcal{P}_q)$. Thus $\{[J], [K_1]\}, \{[J], [K_2]\} \in \mathcal{E}_{\boxdot}$, and so we can realize these two edges in the graph without using $\mathcal{E}_{\diagdown}$. But the third edge, $\{[J], [K_3]\}$, is only realized in $\mathcal{E}_{\diagdown}$. Therefore, recording $\mathcal{E}_{\diagdown}$ is crucial to the construction of $\mathcal{G}(\mathcal{P}_{q,\triangle\triangledown}/\mathbb{Z}^2, \mathcal{E})$.

5.2.2. *Finite graph.* Next we will define a finite graph $\mathcal{G}$ whose nodes correspond to the two-dimensional faces (triangles) in $\mathcal{P}_{q,\triangle\triangledown}$. To make this graph finite, we will use the periodicity of the function π and of the complex $\mathcal{P}_q$ modulo $\mathbb{Z}^2$. By $\mathcal{P}_{q,\triangle\triangledown}/\mathbb{Z}^2$ we denote the set of equivalence classes

$$[I] = \{\, \tau_{\mathbf{s}}(I) = I + \mathbf{s} \mid \mathbf{s} \in \mathbb{Z}^2 \,\} \tag{11}$$

of two-dimensional faces (triangles) $I \in \mathcal{P}_{q,\triangle\triangledown}$ modulo translations by integer vectors $\mathbf{s} \in \mathbb{Z}^2$. We can identify an equivalence class with its unique representative that is a triangle contained in $[0, 1]^2$.

Definition 5.5. Let $\mathcal{G} = \mathcal{G}(\mathcal{P}_{q,\triangle\triangledown}/\mathbb{Z}^2, \mathcal{E})$ be the finite undirected graph with node set $\mathcal{P}_{q,\triangle\triangledown}$ and edge set $\mathcal{E} = \mathcal{E}_{\boxdot} \cup \mathcal{E}_{\diagdown}$ where $\{[I], [J]\} \in \mathcal{E}_{\boxdot}$ (resp., $\{[I], [J]\} \in \mathcal{E}_{\diagdown}$) if and only if $[I] \neq [J]$ and for some $K \in \mathcal{P}_{q,\boxdot}$ (resp., $K \in \mathcal{P}_{q,\diagdown}$) and $F \in E(\pi, \mathcal{P}_q)$, we have one of the following cases:

(Case a.) $(I, J, K) = (p_1(F), p_2(F), p_3(F))$, which implies $F' := F(J, I, K) \in E(\pi, \mathcal{P}_q)$ with $(J, I, K) = (p_1(F'), p_2(F'), p_3(F'))$, or
(Case b.) $(I, K, J) = (p_1(F), p_2(F), p_3(F))$, or
(Case c.) $(J, K, I) = (p_1(F), p_2(F), p_3(F))$.

Therefore we record an edge between two cells in $\mathcal{P}_{q,\triangle\triangledown}$ whenever there is an $F \in E(\pi, \mathcal{P}_q)$ such that two of the projections $p_i(F)$, $i = 1, 2, 3$, are these two cells and the third projection is in $\mathcal{P}_{q,\boxdot\diagdown}$. By the symmetry between p_1 and p_2 and the symmetry in the definition of $E(\pi, \mathcal{P}_q)$, for every $F \in E(\pi, \mathcal{P}_q)$ there exists an $F' \in E(\pi, \mathcal{P}_q)$ such that $p_1(F) = p_2(F')$, $p_2(F) = p_1(F')$, and $p_3(F) = p_3(F')$. Therefore, when considering an $F \in E(\pi, \mathcal{P}_q)$ with two projections in $I, J \in \mathcal{P}_{q,\triangle\triangledown}$ and a third projection $K \in \mathcal{P}_{q,\boxdot\diagdown}$, we can always assume that either $p_2(F) = K$ or $p_3(F) = K$.

Some faces in $\mathcal{E}_{\diagdown}$ are inherently also in $\mathcal{E}_{\boxdot}$. Figure 7 depicts how this can happen and also shows an edge in $\mathcal{E}_{\diagdown}$ that is not necessarily in $\mathcal{E}_{\boxdot}$. Thus, $\mathcal{E}_{\boxdot}$ alone is not sufficient to describe all the relations in the graph that we need to consider.

The functions π, π^1, π^2 have related slopes on faces that are connected in the graph.

Lemma 5.6. *Let $L \subseteq \mathbb{R}^2$ be a linear subspace. Let $I, J \in \mathcal{P}_{q,\triangle\triangledown}$ with $\{[I], [J]\} \in \mathcal{E}$. Suppose π^1, π^2 are valid functions with $\pi = \frac{1}{2}(\pi^1 + \pi^2)$. For $\theta = \pi, \pi^1$, or π^2, if θ is affine with respect to L over I, then θ is affine with respect to L over J as well.*

Proof. By Lemma 1.4 and Assumption 1, π^1, π^2 are minimal and continuous and $E(\pi, \mathcal{P}_q) \subseteq E(\theta, \mathcal{P}_q)$ for $\theta = \pi, \pi^1, \pi^2$.

Case (i). Suppose $\{[I],[J]\} \in \mathcal{E}_{\cdot}$. Then there exists $\mathbf{a} \in \frac{1}{q}\mathbb{Z}^2$ such that, setting $K = \{\mathbf{a}\} \in \mathcal{P}_{q,\cdot}$, there exists $F \in E(\pi, \mathcal{P}_q)$ such that either $(I, J, K) = (p_1(F), p_2(F), p_3(F))$, $(I, K, J) = (p_1(F), p_2(F), p_3(F))$, or $(J, K, I) = (p_1(F), p_2(F), p_3(F))$; these are Cases a, b, and c from Definition 5.5, respectively. We only consider the case $(I, J, K) = (p_1(F), p_2(F), p_3(F))$; the other cases are similar. Then $\theta|_I(\mathbf{x}) + \theta|_J(\mathbf{y}) = \theta|_K(\mathbf{a})$ for all $\mathbf{x} \in I$, $\mathbf{y} \in J$, $\mathbf{x} + \mathbf{y} = \mathbf{a}$. Consider any $\mathbf{y}^1, \mathbf{y}^2 \in J$ such that $\mathbf{y}^2 - \mathbf{y}^1 \in L$. Set $\mathbf{x}^i = \mathbf{a} - \mathbf{y}^i \in I$ for $i = 1, 2$. Thus, $\theta|_J(\mathbf{y}^2) - \theta|_J(\mathbf{y}^1) = \theta|_I(\mathbf{x}^1) - \theta|_I(\mathbf{x}^2)$ and $\mathbf{x}^1 - \mathbf{x}^2 = \mathbf{y}^2 - \mathbf{y}^1 \in L$. Since θ is affine with respect to L over I, θ is affine with respect to L over J.

Case (ii). Suppose $\{[I],[J]\} \in \mathcal{E}_{\diagdown}$. We show that for any $\mathbf{y} \in \operatorname{rel\,int}(J)$, and any $\mathbf{p} \in L$ there exists $\epsilon > 0$ such that θ is affine over $\{\, \mathbf{y} + \lambda\mathbf{p} \mid -\epsilon \le \lambda \le \epsilon \,\}$. Using Lemma 2.7, this will then imply that θ is affine with respect to L over J. We only consider the case when there exists $F \in E(\pi, \mathcal{P}_q)$ with $(I, J, K) = (p_1(F), p_2(F), p_3(F))$ (Case a from Definition 5.5); the Cases b and c are similar. Thus, $I, J \in \mathcal{P}_{q,\triangle}$ and $K \in \mathcal{P}_{q,\diagdown}$. Then $\theta|_I(\mathbf{x}) + \theta|_J(\mathbf{y}) = \theta|_K(\mathbf{a})$ for all $\mathbf{x} \in I$, $\mathbf{y} \in J$, $\mathbf{x} + \mathbf{y} = \mathbf{a} \in K$.

Let $\mathbf{y} \in \operatorname{rel\,int}(J)$. Using Lemma 2.9, there exists $\mathbf{x} \in \operatorname{rel\,int}(I)$ and $\mathbf{a} \in \operatorname{rel\,int}(K)$ such that $\mathbf{x} + \mathbf{y} = \mathbf{a}$. Since $\mathbf{y} \in \operatorname{rel\,int}(J)$, there exists $\epsilon > 0$ such that $\{\, \mathbf{x} + \lambda\mathbf{p} \mid -\epsilon \le \lambda \le \epsilon \,\} \subseteq I$ and $\{\, \mathbf{y} + \lambda\mathbf{p} \mid -\epsilon \le \lambda \le \epsilon \,\} \subseteq J$. Since θ is affine over $\{\, \mathbf{x} + \lambda\mathbf{p} \mid -\epsilon \le \lambda \le \epsilon \,\} \subseteq I$ and $\theta|_I(\mathbf{x}) + \theta|_J(\mathbf{y}) = \theta|_K(\mathbf{a})$ for all $\mathbf{x} \in I$, $\mathbf{y} \in J$, $\mathbf{x} + \mathbf{y} = \mathbf{a}$, a similar argument as Case (i) proves θ is affine over $\{\, \mathbf{y} + \lambda\mathbf{p} \mid -\epsilon \le \lambda \le \epsilon \,\}$. $\square$

With this in mind, for each $I \in \mathcal{P}_{q,\triangle}$, let $\mathcal{G}_I$ be the connected component of $\mathcal{G}$ containing $[I]$. We define the two sets of faces that contain complete connected components in the graph $\mathcal{G}$,

$$\mathcal{S}^1_{q,\triangle} = \{\, J \in \mathcal{P}_{q,\triangle} \mid [J] \in \mathcal{G}_I \text{ for some } I \in \mathcal{P}^1_{q,\triangle} \,\},$$
$$\mathcal{S}^2_{q,\triangle} = \{\, J \in \mathcal{P}_{q,\triangle} \mid [J] \in \mathcal{G}_I \text{ for some } I \in \mathcal{P}^2_{q,\triangle} \,\}.$$

Observation 5.7. *It follows from Lemma 5.6, Lemma 4.10 and the periodicity of π, π^1, and π^2 that π is affine imposing in $\mathcal{S}^2_{q,\triangle}$. Similarly, it follows from Lemma 5.6, Lemma 4.11 and the periodicity of π, π^1, and π^2 that π is diagonally affine imposing in $\mathcal{S}^1_{q,\triangle}$.*

Observation 5.8 (Geometrically adjacent triangles)**.** *From Lemma 4.13, it follows that if $I \in \mathcal{S}^2_{q,\triangle}$, $J \in \mathcal{S}^1_{q,\triangle}$ and $I \cap J \in \mathcal{P}_{q,\llcorner}$, then π is affine imposing in J. Furthermore, by periodicity of π, π^1, and π^2, π is affine imposing in all $J' \in [J]$.*

This observation motivates the following graph definition that is a super-graph of $\mathcal{G}$.

Definition 5.9. Let $\bar{\mathcal{G}} = \bar{\mathcal{G}}(\mathcal{P}_{q,\triangle}/\mathbb{Z}^2, \bar{\mathcal{E}})$ be the finite undirected graph with node set $\mathcal{P}_{q,\triangle}/\mathbb{Z}^2$ and edge set $\bar{\mathcal{E}} = \mathcal{E}_{\cdot} \cup \mathcal{E}_{\diagdown} \cup \mathcal{E}_{\llcorner}$ where $\mathcal{E}_{\cdot}$ and $\mathcal{E}_{\diagdown}$ are defined in Definition 5.5 and where $\{[I],[J]\} \in \mathcal{E}_{\llcorner}$ if and only if $[I] \neq [J]$ and for some $I' \in [I]$, $J' \in [J]$ we have $I', J' \in \mathcal{S}^1_{q,\triangle} \cup \mathcal{S}^2_{q,\triangle}$ and $I \cap J \in \mathcal{P}_{q,\llcorner}$.

In contrast to the graph $\mathcal{G}$ and Lemma 5.6, faces in $\bar{\mathcal{G}}$ connected by edges from $\mathcal{E}_{\llcorner}$ do not necessarily have related slopes, even if π is affine imposing on these faces.

For each $I \in \mathcal{P}_{q,\triangle}$, let $\bar{\mathcal{G}}_I$ be the connected component of $\bar{\mathcal{G}}$ containing $[I]$. Let

$$\bar{\mathcal{S}}^2_{q,\triangle} = \{\, K \in \mathcal{P}_{q,\triangle} \mid [K] \in \bar{\mathcal{G}}_I \text{ for some } I \in \mathcal{S}^2_{q,\triangle} \,\}. \tag{12}$$

Note that $\bar{\mathcal{S}}^2_{q,\triangle} \subseteq \mathcal{S}^1_{q,\triangle} \cup \mathcal{S}^2_{q,\triangle}$. Let

$$\bar{\mathcal{S}}^1_{q,\triangle} = \mathcal{S}^1_{q,\triangle} \setminus \bar{\mathcal{S}}^2_{q,\triangle}. \tag{13}$$

Theorem 5.10. *If $\bar{\mathcal{S}}^2_{q,\triangle} = \mathcal{P}_{q,\triangle}$, then π is affine imposing in $\mathcal{P}_{q,\triangle}$, and therefore θ is continuous piecewise linear over $\mathcal{P}_q$ for $\theta = \pi^1, \pi^2$ whenever we have that π^1, π^2 are valid functions and $\pi = \frac{1}{2}(\pi^1 + \pi^2)$.*

Proof. By Lemma 1.4, π^1, π^2 are minimal and continuous. Since they are minimal, they are also periodic. From Observation 5.7, π is affine imposing in $\mathcal{S}^2_{q,\triangle}$ and diagonally affine imposing in $\mathcal{S}^1_{q,\triangle}$. By Observation 5.8, π is affine imposing in any J' such that there exists a J with $J \in [J']$ and $I \cap J \in \mathcal{P}_{q,\llcorner}$ for some I such that π is affine imposing in I. In particular, this holds for all $I \in \mathcal{S}^2_{q,\triangle}$. Consider any $K \in \mathcal{S}^1_{q,\triangle}$

where $[K]$ is connected by a path to $[I]$ in the graph $\bar{\mathcal{G}}$. By induction on the number of edges in the path from $[K]$ to $[I]$ and using Lemma 5.6, π is affine imposing in K. Therefore, π is affine imposing in $\mathcal{S}^1_{q,\text{◣◥}} \cap \bar{\mathcal{S}}^2_{q,\text{◣◥}}$. Since $\mathcal{P}_{q,\text{◣◥}} = \bar{\mathcal{S}}^2_{q,\text{◣◥}} \subseteq \mathcal{S}^1_{q,\text{◣◥}} \cup \mathcal{S}^2_{q,\text{◣◥}} \subseteq \mathcal{P}_{q,\text{◣◥}}$, it follows that π is affine imposing in all of $\mathcal{P}_{q,\text{◣◥}}$. $\square$

5.3. **Non-extremality by two-dimensional equivariant perturbation.** In this and the following subsection, we will prove the following result.

Lemma 5.11. *Let π be a minimal, continuous piecewise linear function over $\mathcal{P}_q$ that is diagonally constrained. If $\bar{\mathcal{S}}^2_{q,\text{◣◥}} \neq \mathcal{P}_{q,\text{◣◥}}$, then π is not extreme.*

In the proof, we will need $\psi^m_{q,\bullet}$ and $\psi^m_{q,\diagdown}$, as constructed in subsection 5.1.

Recall that $\Delta\pi(\mathbf{x},\mathbf{y}) := \pi(\mathbf{x}) + \pi(\mathbf{y}) - \pi(\mathbf{x}+\mathbf{y})$ and that when π is piecewise linear over $\mathcal{P}_q$, we have that $\Delta\pi$ is piecewise linear over $\Delta\mathcal{P}_q$, as explained in section 4.

Lemma 5.12 (Perturbation only on interior of triangles)**.** *Let π be a minimal, continuous piecewise linear function over $\mathcal{P}_q$ with $\mathbf{f} \in \operatorname{vert}(\mathcal{P}_q)$ that is diagonally constrained. Suppose there exists $I^* \in \mathcal{P}_{q,\text{◣◥}} \setminus (\mathcal{S}^2_{q,\text{◣◥}} \cup \mathcal{S}^1_{q,\text{◣◥}})$. Then π is not extreme.*

Furthermore, for any $m \in \mathbb{Z}_{\geq 3}$, there exist distinct minimal valid functions π^1, π^2 that are continuous piecewise linear over $\mathcal{P}_{mq}$ such that $\pi = \frac{1}{2}(\pi^1 + \pi^2)$.

Proof. Fix $m \in \mathbb{Z}_{\geq 3}$. Let $R = \bigcup\{\, J \mid [J] \in \mathcal{G}_{I^*} \,\}$. Let $\psi^m_{q,\bullet}\colon \mathbb{R}^2 \to \mathbb{R}$ be the function constructed in subsection 5.1. Let $\bar{\pi} = \delta_R \cdot \psi^m_{q,\bullet}$ where δ_R is the indicator function for the set R. Since $\{\mathbf{0}\}, \{\mathbf{f}\} \in \mathcal{P}_{q,\bullet}$, by Lemma 5.3 (i), we have $\psi^m_{q,\bullet}(\mathbf{0}) = 0$ and $\psi^m_{q,\bullet}(\mathbf{f}) = 0$. Hence, $\bar{\pi}(\mathbf{0}) = 0$ and $\bar{\pi}(\mathbf{f}) = 0$.

Since $\psi^m_{q,\bullet}\big|_I = 0$ for all $I \in \mathcal{P}_{q,|\,\_\,\diagdown}$ and R is a union of faces in $\mathcal{P}_{q,\text{◣◥}}$, we find that $\bar{\pi}$ is continuous. Since $\psi^m_{q,\bullet}$ is piecewise linear over $\mathcal{P}_{mq}$, $\bar{\pi}$ is also piecewise linear over $\mathcal{P}_{mq}$. Finally, notice that $\bar{\pi}$ is periodic modulo $\mathbb{Z}^2$ since $\psi^m_{q,\bullet}$ and δ_R are both periodic modulo $\mathbb{Z}^2$.

We will show that $E(\pi) \subseteq E(\bar{\pi})$. Since $I^* \in R$ and $\psi^m_{q,\bullet} \neq 0$ on $\operatorname{int}(I^*)$, we have that $\bar{\pi} \not\equiv 0$. Since $\bar{\pi}(\mathbf{f}) = 0$ and $\bar{\pi} \not\equiv 0$, by Theorem 3.13, this will show that π is not extreme. By Lemma 3.12, we only need to consider maximal faces in the complex $\Delta\mathcal{P}_q$. Let $F \in E_{\max}(\pi, \mathcal{P}_q)$. Define $\Delta\bar{\pi}(\mathbf{x},\mathbf{y}) := \bar{\pi}(\mathbf{x}) + \bar{\pi}(\mathbf{y}) - \bar{\pi}(\mathbf{x}+\mathbf{y})$. We will show that $\Delta\bar{\pi}|_F = 0$. Note that $\bar{\pi}$ is defined over the finer complex $\mathcal{P}_{mq}$. Therefore $\Delta\bar{\pi}$ is piecewise linear over $\Delta\mathcal{P}_{mq}$. Since $F \in \Delta\mathcal{P}_q$, the function $\Delta\bar{\pi}$ is not necessarily affine over F.

Let $I = p_1(F)$, $J = p_2(F)$, and $K = p_3(F)$. By Lemma 3.5, $F = F(I,J,K)$. Since π is diagonally constrained, we enumerate the possible cases for I, J, K as listed in Lemma 4.9 and show that $F = F(I,J,K) \subseteq E(\bar{\pi})$. Observe that we can write $\Delta\bar{\pi}|_F(\mathbf{x},\mathbf{y}) = \bar{\pi}|_I(\mathbf{x}) + \bar{\pi}|_J(\mathbf{y}) - \bar{\pi}|_K(\mathbf{x}+\mathbf{y})$ and that $F \subseteq E(\bar{\pi})$ if and only if $\Delta\bar{\pi}|_F = 0$.

(Type 1) $I, J, K \in \mathcal{P}_{q,\bullet\,\diagdown}$. By Lemma 5.3 (i), $\psi^m_{q,\bullet} = 0 = \bar{\pi}$ on the faces I, J, K and thus we have $\Delta\bar{\pi}|_F = 0$.

(Type 2) $I, J, K \in \mathcal{P}_{q,\text{◣◥}}$. By definition of $\mathcal{S}^2_{q,\text{◣◥}}$, we have $I, J, K \in \mathcal{S}^2_{q,\text{◣◥}}$. Therefore $I \cap R, J \cap R, K \cap R \in \mathcal{P}_{q,\bullet\,|\,\_\,\diagdown}$. By Lemma 5.3 (i), $\psi^m_{q,\bullet} = 0$ on $I \cap R$, $J \cap R$, $K \cap R$. Since $\delta_R = 0$ on $I \setminus R$, $J \setminus R$, $K \setminus R$, we have $\bar{\pi} = 0$ on I, J, K and thus $\Delta\bar{\pi}|_F = 0$.

(Type 3) One of I, J, K is in $\mathcal{P}_{q,\bullet}$, while the other two are in $\mathcal{P}_{q,\text{◣◥}}$. Label I, J, K as I', J', K' where $I' \in \mathcal{P}_{q,\bullet}$ and $J', K' \in \mathcal{P}_{q,\text{◣◥}}$. By Lemma 5.3 (i), $\psi^m_{q,\bullet} = 0 = \bar{\pi}$ on I'. We consider four cases.

- Case i. $[J'], [K'] \notin \mathcal{G}_{I^*}$. Then $J' \cap R, K' \cap R \in \mathcal{P}_{q,\bullet\,|\,\_\,\diagdown}$. By Lemma 5.3 (i), $\psi^m_{q,\bullet} = 0 = \bar{\pi}$ on $J' \cap R$ and $K' \cap R$. Furthermore, $\delta_R = 0$ on $J' \setminus R$ and $K' \setminus R$. Hence, $\bar{\pi} = 0$ on I', J', K' and hence $\Delta\pi|_F = 0$.
- Case ii. $[J'], [K'] \in \mathcal{G}_{I^*}$. By the relations in Lemma 5.3 (ii) and the fact that $\delta_R = 1$ on J', K', we have that $\Delta\bar{\pi}|_F = 0$.
- Case iii. $[J'] \in \mathcal{G}_{I^*}$, $[K'] \notin \mathcal{G}_{I^*}$. We show that this case cannot happen. Since $F \in E(\pi)$ and $I' \in \mathcal{P}_{q,\bullet}$, we have that $\{[J'], [K']\} \in \mathcal{E}_{\bullet}$. Therefore, $[K'] \in \mathcal{G}_{J'}$. Since $[J'] \in \mathcal{G}_{I^*}$, we have that $\mathcal{G}_{I^*} = \mathcal{G}_{J'}$, which is a contradiction because then $[K'] \in \mathcal{G}_{I^*}$.
- Case iv. $[K'] \in \mathcal{G}_{I^*}$, $[J'] \notin \mathcal{G}_{I^*}$. This is similar to the previous case.

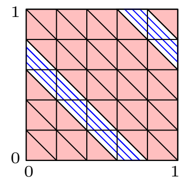


FIGURE 8. A case where $\mathcal{P}_{q,\triangle} = \mathcal{S}^2_{q,\triangle} \cup \mathcal{S}^1_{q,\triangle} = \bar{\mathcal{S}}^2_{q,\triangle} \cup \bar{\mathcal{S}}^1_{q,\triangle}$. Therefore, on every triangle, π is either affine imposing (*shaded triangles*), or only diagonally affine imposing (*striped triangles*). Observation 5.8 shows that a shaded triangle that is geometrically adjacent to a striped triangle along a vertical or horizontal face in $\mathcal{P}_{q,|\,-}$ forces the striped triangle to become shaded. Therefore, no striped triangle can be geometrically adjacent to a shaded triangle along a vertical or horizontal face in $\mathcal{P}_{q,|\,-}$. In this example, every striped triangle is connected in the graph $\bar{\mathcal{G}}$ by a path with edges in $\mathcal{E}_{|\,-}$. Therefore, all of these triangles form a connected component in the graph. We can choose any one of these triangles as I^* in Lemma 5.14 to perturb on this connected component.

(Type 4) One of I, J, K is in $\mathcal{P}_{q,\diagdown}$, while the other two are in $\mathcal{P}_{q,\triangle}$. In this case, by definition, the two triangles are in $\mathcal{S}^1_{q,\triangle}$. Since triangles in $\mathcal{S}^1_{q,\triangle}$ only intersect R on lower-dimensional faces $\mathcal{P}_{q,\cdot\,|\,-\,\diagdown}$, we have that $I \cap R, J \cap R, K \cap R \in \mathcal{P}_{q,\cdot\,|\,-\,\diagdown}$. By Lemma 5.3 (i), $\psi^m_{q,\cdot} = 0 = \bar{\pi}$ on $I \cap R$, $J \cap R$, and $K \cap R$. Since $\delta_R = 0$ on $I \setminus R$, $J \setminus R$ and $K \setminus R$, we have $\bar{\pi} = 0$ on I, J, K and we have $\Delta\bar{\pi}|_F = 0$.

We conclude that $E(\pi) \subseteq E(\bar{\pi})$, $\bar{\pi}(\mathbf{f}) = 0$, and π and $\bar{\pi}$ are both piecewise linear over $\mathcal{P}_{mq}$. Therefore, by Theorem 3.13, π is not extreme and there exist distinct minimal functions π^1, π^2 that are continuously piecewise linear over $\mathcal{P}_{mq}$. □

5.4. **Non-extremality by diagonal equivariant perturbation.** We continue to build machinery to prove Lemma 5.14. In this section, we will use the function $\psi^m_{q,\diagdown}$, as defined in subsection 5.1, as the basis for a perturbation function $\bar{\pi}$.

As in Lemma 5.12, we will allow the perturbation $\bar{\pi}$ to only apply to a subset of the triangles, this time corresponding to a connected component in the graph $\bar{\mathcal{G}}$. Since $\psi^m_{q,\diagdown}$ is non-zero on the vertical and horizontal faces $\mathcal{P}_{q,|\,-}$, we must be careful about geometrically adjacent triangles.

To handle the geometrically adjacent triangles easier, we consider the case where $\mathcal{P}_{q,\triangle} = \mathcal{S}^2_{q,\triangle} \cup \mathcal{S}^1_{q,\triangle}$.

Observation 5.13. *Suppose $\mathcal{P}_{q,\triangle} = \mathcal{S}^2_{q,\triangle} \cup \mathcal{S}^1_{q,\triangle}$ and let $I^* \in \bar{\mathcal{S}}^1_{q,\triangle}$. Let $J, K \in \mathcal{P}_{q,\triangle}$ such that $[J] \in \bar{\mathcal{G}}_{I^*}$ and $J \cap K \in \mathcal{P}_{q,|\,-}$. Then $[K] \in \bar{\mathcal{G}}_{I^*}$ as well. This is because $\mathcal{P}_{q,\triangle} = \mathcal{S}^2_{q,\triangle} \cup \mathcal{S}^1_{q,\triangle}$ and therefore $\{[J],[K]\} \in \mathcal{E}_{|\,-} \subseteq \bar{\mathcal{E}}$. See Figure 8.*

Lemma 5.14 (Diagonal perturbation touching vertical and horizontal boundaries of triangles)**.** *Suppose π is continuous piecewise linear over $\mathcal{P}_q$ with $\mathbf{f} \in \operatorname{vert}(\mathcal{P}_q)$ and is diagonally constrained. Suppose further that $\mathcal{P}_{q,\triangle} = \mathcal{S}^2_{q,\triangle} \cup \mathcal{S}^1_{q,\triangle}$ and there exists $I^* \in \bar{\mathcal{S}}^1_{q,\triangle}$. Then π is not extreme.*

Furthermore, for any $m \in \mathbb{Z}_{\geq 3}$, there exist distinct minimal valid functions π^1, π^2 that are continuous piecewise linear over $\mathcal{P}_{mq}$ such that $\pi = \frac{1}{2}(\pi^1 + \pi^2)$.

Proof. Let $R = \bigcup\{\, J \in \mathcal{P}_{q,\triangle} \mid [J] \in \bar{\mathcal{G}}_{I^*} \,\}$. Note that $\bar{\mathcal{G}}_{I^*} \subseteq \bar{\mathcal{S}}^1_{q,\triangle}$ and recall that $\bar{\mathcal{S}}^1_{q,\triangle} \cap \bar{\mathcal{S}}^2_{q,\triangle} = \emptyset$. Furthermore, $\bar{\mathcal{S}}^1_{q,\triangle} \cap \mathcal{S}^2_{q,\triangle} = \emptyset$. Let $\psi^m_{q,\diagdown} \colon \mathbb{R}^2 \to \mathbb{R}$ be the function constructed in subsection 5.1.

Let $\bar{\pi} = \delta_R(\mathbf{x}) \cdot \psi^m_{q,\diagdown}(\mathbf{x})$. First recognize that $\bar{\pi}$ is a continuous function. To see this, note that $\psi^m_{q,\diagdown}(\mathbf{x})$ is continuous and δ_R is continuous on R and $\mathbb{R}^2 \setminus R$. By Observation 5.13, it follows that $\partial R \subseteq \bigcup\{\, I \mid I \in \mathcal{P}_{q,\diagdown} \,\}$. By Lemma 5.4 (i), $\psi^m_{q,\diagdown}$ vanishes on ∂R, that is $\psi^m_{q,\diagdown} = 0$ on ∂R. These together imply that $\bar{\pi}$ is continuous.

Since $\psi^m_{q,\diagdown}$ is piecewise linear over $\mathcal{P}_{mq}$, $\bar{\pi}$ is also piecewise linear over $\mathcal{P}_{mq}$. Also, since $\psi^m_{q,\diagdown}\big|_I = 0$ for all $I \in \mathcal{P}_{q,\diagdown}$, we find $\bar{\pi}$ is also continuous. Finally, notice that $\bar{\pi}$ is periodic modulo $\mathbb{Z}^2$ since $\psi^m_{q,\diagdown}$ and δ_R are both periodic modulo $\mathbb{Z}^2$.

We will show that $E(\pi) \subseteq E(\bar{\pi})$. Since $I^* \in R$ and $\psi^m_{q,\diagdown} \not\equiv 0$ on $\operatorname{int}(I^*)$, we have $\bar{\pi} \not\equiv 0$. Since $\bar{\pi}(\mathbf{f}) = 0$ and $\bar{\pi} \not\equiv 0$, by Theorem 3.13, this will show that π is not extreme. By Lemma 3.12, we only need to consider maximal faces in the complex $\Delta\mathcal{P}_q$. Let $F \in E_{\max}(\pi, \mathcal{P}_q)$.

Define $\Delta\bar{\pi}(\mathbf{x}, \mathbf{y}) := \bar{\pi}(\mathbf{x}) + \bar{\pi}(\mathbf{y}) - \bar{\pi}(\mathbf{x} + \mathbf{y})$. We will show that $\Delta\bar{\pi}|_F = 0$. Note that $\bar{\pi}$ is defined over the finer complex $\mathcal{P}_{mq}$. Therefore $\Delta\bar{\pi}$ is piecewise linear over $\Delta\mathcal{P}_{mq}$. Since $F \in \Delta\mathcal{P}_q$, the function $\Delta\bar{\pi}$ is not necessarily affine over F.

Let $I = p_1(F)$, $J = p_2(F)$, and $K = p_3(F)$. By Lemma 3.5, $F = F(I, J, K)$. Since π is diagonally constrained, we enumerate the possible cases for I, J, K as listed in Lemma 4.9 and show that $F = F(I, J, K) \subseteq E(\bar{\pi})$. Observe that we can write $\Delta\bar{\pi}|_F(\mathbf{x}, \mathbf{y}) = \bar{\pi}|_I(\mathbf{x}) + \bar{\pi}|_J(\mathbf{y}) - \bar{\pi}|_K(\mathbf{x} + \mathbf{y})$ and that $F \subseteq E(\bar{\pi})$ if and only if $\Delta\bar{\pi}|_F = 0$.

(Type 1) $I, J, K \in \mathcal{P}_{q,\bullet\square\diagdown}$. By Lemma 5.4 (i), $\psi^m_{q,\diagdown} = 0 = \bar{\pi}$ on all faces I, J, K and thus we have $\Delta\bar{\pi}|_F = 0$.

(Type 2) $I, J, K \in \mathcal{P}_{q,\blacktriangle\blacktriangledown}$. By definition of $\bar{\mathcal{S}}^2_{q,\blacktriangle\blacktriangledown}$, we have $I, J, K \in \bar{\mathcal{S}}^2_{q,\blacktriangle\blacktriangledown}$. By Observation 5.13, we must have $I \cap R, J \cap R, K \cap R \in \mathcal{P}_{q,\bullet\square\diagdown}$ and hence $\psi^m_{q,\diagdown} = 0$ on $I \cap R, J \cap R, K \cap R$ by Lemma 5.4 (i). Therefore, $\bar{\pi} = 0$ on I, J, K and we have $\Delta\bar{\pi}|_F = 0$.

(Types 3 & 4) One of I, J, K is in $\mathcal{P}_{q,\bullet\square\diagdown}$, while the other two are in $\mathcal{P}_{q,\blacktriangle\blacktriangledown}$. Label I, J, K as I', J', K' where $I' \in \mathcal{P}_{q,\bullet\square\diagdown}$ and $J', K' \in \mathcal{P}_{q,\blacktriangle\blacktriangledown}$. By Lemma 5.4 (i), $\psi^m_{q,\diagdown} = 0$ on I'. We consider four cases.

- Case i. $[J'], [K'] \notin \bar{\mathcal{G}}_{I^*}$. By Observation 5.13, we must have $J' \cap R, K' \cap R \in \mathcal{P}_{q,\bullet\square\diagdown}$. Therefore, by Lemma 5.4 (i), $\psi^m_{q,\diagdown} = 0$ on $J' \cap R, K' \cap R$, while $\delta_R = 0$ on $J' \setminus R$, and $K' \setminus R$. Therefore $\bar{\pi} = 0$ on I, J, K and hence $\Delta\pi_F = 0$.
- Case ii. $[J'], [K'] \in \bar{\mathcal{G}}_{I^*}$. By Lemma 5.4 (ii) and the fact that $\delta_R = 1$ on J', K', we have that $\Delta\bar{\pi}|_F = 0$.
- Case iii. $[J'] \in \bar{\mathcal{G}}_{I^*}$, $[K'] \notin \bar{\mathcal{G}}_{I^*}$. We show that this case cannot happen. Since $F \subseteq E(\pi)$ and $I' \in \mathcal{P}_{q,\bullet\square\diagdown}$, we have that $\{[J'], [K']\} \in \mathcal{E} \subseteq \bar{\mathcal{E}}$. Therefore, $[K'] \in \bar{\mathcal{G}}_{J'}$. Since $[J'] \in \bar{\mathcal{G}}_{I^*}$, we have that $\bar{\mathcal{G}}_{I^*} = \bar{\mathcal{G}}_{J'}$ which is a contradiction because then $[K'] \in \bar{\mathcal{G}}_{I^*}$.
- Case iv. $[K'] \in \bar{\mathcal{G}}_{I^*}$, $[J'] \notin \bar{\mathcal{G}}_{I^*}$. This is similar to the previous case.

We conclude that $E(\pi) \subseteq E(\bar{\pi})$, $\bar{\pi}(\mathbf{f}) = 0$, and π and $\bar{\pi}$ are both continuous piecewise linear over $\mathcal{P}_{mq}$. Therefore, by Theorem 3.13, π is not extreme and there exist distinct minimal functions π^1, π^2 that are continuously piecewise linear over $\mathcal{P}_{mq}$. □

Proof of Lemma 5.11. This follows directly from Lemmas 5.12 and 5.14. □

The specific form of our perturbations as continuous piecewise linear functions over $\mathcal{P}_{mq}$ implies the following corollary.

Corollary 5.15. *Fix $m \in \mathbb{Z}_{\geq 3}$. Suppose π is a continuous piecewise linear function over $\mathcal{P}_q$ and is diagonally constrained. If π is not affine imposing over $\mathcal{P}_{q,\blacktriangle\blacktriangledown}$, then there exist distinct minimal π^1, π^2 that are continuous piecewise linear over $\mathcal{P}_{mq}$ such that $\pi = \frac{1}{2}(\pi^1 + \pi^2)$.*

5.5. **Extremality and non-extremality by linear algebra.** In this section we suppose π is a minimal continuous piecewise linear function over $\mathcal{P}_q$ that is affine imposing in $\mathcal{P}_{q,\blacktriangle\blacktriangledown}$. Therefore, by Lemma 1.4 and Definition 5.2, π^1 and π^2 must also be minimal continuous piecewise linear functions over $\mathcal{P}_q$. Recall from Lemma 1.4 that $E(\pi) \subseteq E(\pi^1), E(\pi^2)$.

We now set up a system of linear equations that π satisfies and that π_1 and π_2 must also satisfy. Let $\varphi \colon \frac{1}{q}\mathbb{Z}^2 \to \mathbb{R}$ be a periodic function modulo $\mathbb{Z}^2$. Suppose φ satisfies the following system of linear equations:

$$\begin{cases} \varphi(\mathbf{0}) = 0, \\ \varphi(\mathbf{f}) = 1, \\ \varphi(\mathbf{u}) + \varphi(\mathbf{v}) = \varphi(\mathbf{u} + \mathbf{v}) & \text{for all } \mathbf{u}, \mathbf{v} \in \frac{1}{q}\mathbb{Z}^2 \text{ with } \pi(\mathbf{u}) + \pi(\mathbf{v}) = \pi(\mathbf{u} + \mathbf{v}). \end{cases} \qquad (\mathrm{E}_q(\pi))$$

Since π exists and satisfies ($\mathrm{E}_q(\pi)$), we know that the system has a solution. Since φ and π are periodic, we can identify variables $\varphi(\mathbf{x})$ and $\varphi(\mathbf{x}+\mathbf{t})$ for $\mathbf{x} \in \frac{1}{q}\mathbb{Z}^2$ and $\mathbf{t} \in \mathbb{Z}^2$, and thus the system can be represented with finitely many variables and finitely many equations.

Theorem 5.16. *Let $\pi\colon \mathbb{R}^2 \to \mathbb{R}$ be a continuous piecewise linear valid function over $\mathcal{P}_q$.*

(i) *If the system ($\mathrm{E}_q(\pi)$) does not have a unique solution, then π is not extreme.*
(ii) *Suppose π is minimal and affine imposing in $\mathcal{P}_{q,\blacktriangle\triangledown}$. Then π is extreme if and only if the system of equations ($\mathrm{E}_q(\pi)$) has a unique solution.*

The proof is similar to one in [4].

Proof. Part (i). Suppose ($\mathrm{E}_q(\pi)$) does not have a unique solution. Let $\bar\varphi\colon \frac{1}{q}\mathbb{Z}^2 \to \mathbb{R}$ be a non-trivial element in the kernel of the system above. Then for any ϵ, $\pi\big|_{\frac{1}{q}\mathbb{Z}^2} + \epsilon\bar\varphi$ also satisfies the system of equations. Let $\bar\pi\colon \mathbb{R}^2 \to \mathbb{R}$ be the continuous piecewise linear extension of $\bar\varphi$ over $\mathcal{P}_q$. Therefore $\bar\pi(\mathbf{f}) = 0$ and $\bar\pi \not\equiv 0$. Let $\mathbf{u}, \mathbf{v} \in \frac{1}{q}\mathbb{Z}^2$. If $\Delta\pi(\mathbf{u}, \mathbf{v}) = 0$, then $\Delta\varphi(\mathbf{u}, \mathbf{v}) = 0$, as implied by the system of equations. Since $\operatorname{vert}(\Delta\mathcal{P}_q) \subseteq \frac{1}{q}\mathbb{Z}^2$, this shows that for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$, $\Delta\pi(\mathbf{x}, \mathbf{y}) = 0$ implies that $\Delta\bar\pi(\mathbf{x}, \mathbf{y}) = 0$. Therefore $E(\pi) \subseteq E(\bar\pi)$. Therefore, by Theorem 3.13, π is not extreme.

Part (ii). Suppose there exist distinct, valid functions π^1, π^2 such that $\pi = \frac{1}{2}(\pi^1 + \pi^2)$. Since π is minimal and affine imposing in $\mathcal{P}_{q,\blacktriangle\triangledown}$, π^1, π^2 are minimal continuous piecewise linear functions over $\mathcal{P}_q$. Furthermore, $\pi\big|_{\frac{1}{q}\mathbb{Z}^2}$ and, also $\pi^1\big|_{\frac{1}{q}\mathbb{Z}^2}, \pi^2\big|_{\frac{1}{q}\mathbb{Z}^2}$ satisfy the system of equations ($\mathrm{E}_q(\pi)$). If this system has a unique solution, then $\pi = \pi^1 = \pi^2$, which is a contradiction since π^1, π^2 were assumed distinct. Therefore π is extreme.

On the other hand, if the system ($\mathrm{E}_q(\pi)$) does not have a unique solution, then by Part (i), π is not extreme. ☐

5.6. **Connection to a finite group problem.**

Theorem 5.17. *Fix $m \in \mathbb{Z}_{\geq 3}$. Let π be a minimal continuous piecewise linear function over $\mathcal{P}_q$ that is diagonally constrained. Then π is extreme if and only if the system of equations ($\mathrm{E}_{mq}(\pi)$) with $\frac{1}{mq}\mathbb{Z}^2$ has a unique solution.*

Proof. Since π is piecewise linear over $\mathcal{P}_q$, it is also piecewise linear over $\mathcal{P}_{mq}$. The forward direction is the contrapositive of Theorem 5.16 (i), applied when we view π piecewise linear over $\mathcal{P}_{mq}$. For the reverse direction, observe that if the system of equations ($\mathrm{E}_{mq}(\pi)$) has a unique solution, then there cannot exist distinct minimal π^1, π^2 that are continuous piecewise linear over $\mathcal{P}_{mq}$ such that $\pi = \frac{1}{2}(\pi^1 + \pi^2)$. By the contrapositive of Corollary 5.15, π is affine imposing in $\mathcal{P}_{q,\blacktriangle\triangledown}$. Then π is also affine imposing on $\mathcal{P}_{mq,\blacktriangle\triangledown}$ since it is a finer set. By Theorem 5.16 (ii), since π is affine imposing in $\mathcal{P}_{mq,\blacktriangle\triangledown}$ and the system of equations ($\mathrm{E}_{mq}(\pi)$) on $\mathcal{P}_{mq}$ has a unique solution, π is extreme. ☐

Theorem 1.8 and Theorem 1.9 are direct consequences of Theorem 5.17.

Appendix A. Reflection groups and equivariant perturbations

We provide a general framework to motivate the definition of the functions $\psi^m_{q,\square}$ and $\psi^m_{q,\diagdown}$ from subsection 5.1. We describe the construction at a more abstract level with the vision that it could be a useful tool to analyze infinite group problems in higher dimensions.

We follow the lead of [4] where the relevant arithmetics of the one-dimensional problem is captured by studying sets of additivity relations of the form $\pi(t^i) + \pi(y) = \pi(t^i + y)$ and $\pi(x) + \pi(r^i - x) = \pi(r^i)$, where the points t^i and r^i are breakpoints of a one-dimensional minimal valid function π. This is an important departure from the previous literature, which only uses additivity relations over non-degenerate intervals. The arithmetic nature of the problem comes into focus when one realizes that isolated additivity relations over single points are also important for studying extremality. These isolated additivity relations give rise to a subgroup of the group $\operatorname{Aff}(\mathbb{R}^k)$ of invertible affine linear transformations of $\mathbb{R}^k$ as follows.

A.1. **Reflection groups and their fundamental domains.** For a point $\mathbf{r} \in \mathbb{R}^k$, define the *reflection* $\rho_{\mathbf{r}} \colon \mathbb{R}^k \to \mathbb{R}^k$, $\mathbf{x} \mapsto \mathbf{r} - \mathbf{x}$. For a vector $\mathbf{t} \in \mathbb{R}^k$, define the *translation* $\tau_{\mathbf{t}} \colon \mathbb{R}^k \to \mathbb{R}$, $\mathbf{x} \mapsto \mathbf{x} + \mathbf{t}$. We consider the reflections $\rho_{\mathbf{r}}$ and translations $\tau_{\mathbf{t}}$ as elements of the group $\operatorname{Aff}(\mathbb{R}^k)$.

Given a set R of points in $\mathbb{R}^k$ and a set T of vectors in $\mathbb{R}^k$, we define the *reflection group* $\Gamma = \Gamma(R, T) = \langle\, \rho_{\mathbf{r}}, \tau_{\mathbf{t}} \mid \mathbf{r} \in R,\, \mathbf{t} \in T \,\rangle$. A *group character* of Γ is a group homomorphism $\chi \colon \Gamma \to \mathbb{C}^\times$. The *orbit* of a point $\mathbf{x} \in \mathbb{R}^k$ under the group $\Gamma \subseteq \operatorname{Aff}(\mathbb{R}^k)$ is the set $\Gamma(\mathbf{x}) = \{\, \gamma(\mathbf{x}) \mid \gamma \in \Gamma \,\}$. We extend this notation to subsets of $\mathbb{R}^k$: for a subset $X \subseteq \mathbb{R}^k$, $\Gamma(X) = \bigcup_{\mathbf{x} \in X} \Gamma(\mathbf{x})$.

In the following, we assume that $R \neq \emptyset$, i.e., at least one of the generators is a reflection. The structure of the group Γ is easy to describe completely. The following lemma, which appeared in [4] for $k = 1$, summarizes the structure of this group and generalizes easily from [4].

Lemma A.1. *Let $\mathbf{r}_1 \in R$. Then the group $\Gamma = \Gamma(R, T) = \langle\, \rho_{\mathbf{r}}, \tau_{\mathbf{t}} \mid \mathbf{r} \in R, \mathbf{t} \in T \,\rangle$ is the semidirect product $\Gamma^+ \rtimes \langle \rho_{\mathbf{r}_1} \rangle$, where the (normal) subgroup of translations is of index 2 in Γ and can be written as*

$$\Gamma^+ = \{\, \tau_{\mathbf{t}} \mid \mathbf{t} \in Y \,\}, \tag{14}$$

where Y is the additive subgroup of $\mathbb{R}^k$

$$Y = \langle\, \mathbf{r} - \mathbf{r}_1, \mathbf{t} \mid \mathbf{r} \in R, \mathbf{t} \in T \,\rangle_{\mathbb{Z}} \subseteq \mathbb{R}^k. \tag{15}$$

There is a unique group character $\chi \colon \Gamma \to \{\pm 1\} \subset \mathbb{C}^\times$ with $\chi(\rho) = -1$ for every reflection $\rho \in \Gamma$ and $\chi(\tau) = +1$ for every translation $\tau \in \Gamma$.

Definition A.2. A function $\psi \colon \mathbb{R}^k \to \mathbb{R}$ is called Γ-*equivariant* if it satisfies the *equivariance formula*

$$\psi(\gamma(\mathbf{x})) = \chi(\gamma)\psi(\mathbf{x}) \quad \text{for } \mathbf{x} \in \mathbb{R}^k \text{ and } \gamma \in \Gamma. \tag{16}$$

We will use formula (16) to give an alternative derivation of the functions $\psi^m_{q,\boxdot}$ and $\psi^m_{q,\boxbslash}$ defined in subsection 5.1. These functions provide the perturbation functions when Theorem 3.13 is invoked in subsections 5.3 and 5.4.

Observation A.3. *Let $\Gamma = \Gamma(R, T)$ be a reflection group with $R \cap T \neq \emptyset$ and let ψ be any Γ-equivariant function. Then, $\rho_{\mathbf{0}} \in \Gamma$ and $\psi(\mathbf{0}) = 0$.*

Proof. Let $\mathbf{r} \in R \cap T$; then $\rho_{\mathbf{0}} = \rho_{\mathbf{r}} \circ \tau_{\mathbf{r}}$. Also, we have $\psi(\mathbf{0}) = \psi(\rho_{\mathbf{0}}(\mathbf{0})) = \chi(\rho_{\mathbf{0}})\psi(\mathbf{0}) = -\psi(\mathbf{0})$; hence, $\psi(\mathbf{0}) = 0$. □

It follows from Observation A.3 that when $R \cap T \neq \emptyset$ and ψ is Γ-equivariant, we have $\psi \equiv 0$ on all of $\Gamma(\mathbf{0})$. If we restrict ourselves to continuous Γ-equivariant functions and Y defined in (15) is dense in $\mathbb{R}^k$, then $\psi \equiv 0$ is the unique Γ-equivariant function. On the other hand, when Y has inherent discreteness properties, which we make precise in the following discussion, we can construct many non-trivial continuous Γ-equivariant functions. To do so, we only need to construct a function on a subset of $\mathbb{R}^k$.

Definition A.4. A *fundamental domain* of a reflection group Γ is a subset of $\mathbb{R}^k$ that is a system of representatives of the orbits.

Given a reflection group Γ for $k = 1$, if the group Y from (15) in Lemma A.1 is discrete, a fundamental domain of Γ can be chosen as a certain closed interval. In higher dimensions, when Y is discrete, the fundamental domain is no longer a closed set. Even so, it is easy to describe the closure of a fundamental domain. This is made concrete in the following discussion and Lemma A.5.

A well known fact is that for any discrete subgroup Λ of $\mathbb{R}^k$ there exists a finite set of vectors $\mathbf{t}^1, \dots, \mathbf{t}^\ell \in \mathbb{R}^k$ such that $\Lambda = \langle \mathbf{t}^1, \dots, \mathbf{t}^\ell \rangle_{\mathbb{Z}}$. These vectors are called the *basis* of Λ. We say that Λ is a *lattice* of the linear subspace $\langle \mathbf{t}^1, \dots, \mathbf{t}^\ell \rangle_{\mathbb{R}}$. The set $V_\Lambda = \{\, \sum_{i=1}^{\ell} \lambda_i \mathbf{t}^i \mid 0 \leq \lambda_i \leq 1 \,\}$ is called the *closed fundamental parallelepiped* of Λ with respect to the basis $\mathbf{t}^1, \dots, \mathbf{t}^\ell$. Define $\mathbf{t} = \sum_{i=1}^{\ell} \mathbf{t}^i$ and set $M := \max\{\, \mathbf{t} \cdot \mathbf{x} \mid \mathbf{x} \in V \,\} = \mathbf{t} \cdot \mathbf{t}$. Define $V_\Lambda^+ = \{\, \mathbf{x} \in V \mid \mathbf{t} \cdot \mathbf{x} \leq \frac{M}{2} \,\}$ and $V_\Lambda^- = \{\, \mathbf{x} \in V \mid \mathbf{t} \cdot \mathbf{x} \geq \frac{M}{2} \,\}$. (These definitions are of course with respect to the particular basis $\{\mathbf{t}^1, \dots, \mathbf{t}^\ell\}$; the basis will usually be fixed in a particular context).

A *mixed-lattice* is a subgroup $Y \subseteq \mathbb{R}^k$ such that $Y = \Lambda + L$ where Λ is a lattice of a linear subspace L' of $\mathbb{R}^k$, L is a linear subspace of $\mathbb{R}^k$, and L' and L are complementary subspaces, i.e., $\mathbb{R}^k = L' + L$ and $L \cap L' = \{\mathbf{0}\}$.

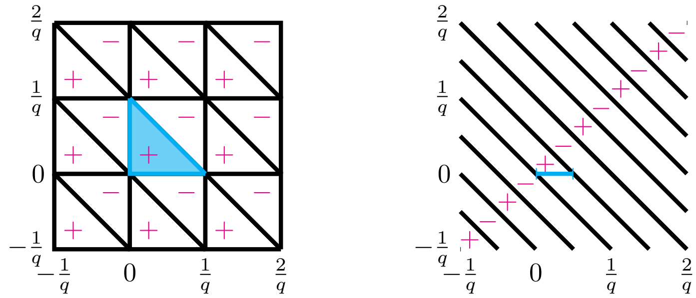


FIGURE 9. Reflection groups $\Gamma_{q,\boxdot}$ and $\Gamma_{q,\diagdown}$ and the closures of their fundamental domains (*blue*) for $m = 3$. *Left*, the case $\Gamma_{q,\boxdot}$. Translating the closure of the fundamental domain, $V^+_{q,\boxdot} = {}_{\mathbf{0}}\!\diagdown$ (*blue triangle*), by $\tau_{\mathbf{t}}$ for $\mathbf{t} \in \Lambda_{q,\boxdot} = \frac{1}{q}\mathbb{Z}^2$ gives the triangles labeled with $+$. Reflections by $\rho_{\mathbf{r}}$ for $\mathbf{r} \in \frac{1}{q}\mathbb{Z}^2$ take these triangles to the triangles labeled with $-$. *Right*, the case $\Gamma_{q,\diagdown}$. Translating the closure of the fundamental domain, $V^+_{q,\diagdown}$ (*blue line segment*), by $\tau_{\mathbf{t}}$ for $\mathbf{t} \in L_{\diagdown}$ gives the diagonal strip, labeled $+$, containing the fundamental domain. Further translations by $\tau_{\mathbf{t}}$ for $\mathbf{t} \in \Lambda_{q,\diagdown}$ give the remaining diagonal strips labeled $+$. The reflections $\rho_{\mathbf{r}}$ in $\Gamma_{q,\diagdown}$ then take these strips to the strips labeled $-$.

Lemma A.5. *Let $\Gamma = \Gamma(R,T)$ be a reflection group with $\emptyset \subsetneq R \subseteq T$ such that the corresponding Y from* (15) *is a mixed-lattice, i.e., $Y = \Lambda + L$ and let $\mathbf{t}^1, \dots, \mathbf{t}^\ell$ be a basis of Λ. Let $L' = \langle \mathbf{t}^1, \dots, \mathbf{t}^\ell \rangle_{\mathbb{R}}$. Let V_Λ^+ be defined with respect to this basis. Then there exists a fundamental domain $\tilde V$ for Γ such that $\operatorname{int}_{L'}(V_\Lambda^+) \subseteq \tilde V \subseteq V_\Lambda^+$.*

Proof. We first show that V_Λ^+ contains a representative for every point $\mathbf{x}$ in $\mathbb{R}^k$. Let $\mathbf{x} = \sum_{i=1}^{\ell} \lambda_i \mathbf{t}^i + \mathbf{p}$ for some $0 \le \lambda_i \le 1$, $\mathbf{p} \in Y$. We will show that $\gamma(\mathbf{x}) \in V_\Lambda^+$ for some $\gamma \in \Gamma$. Let $\mathbf{x}' = \sum_{i=1}^{\ell} \lambda_i \mathbf{t}^i$ and let $\mathbf{t} = \sum_{i=1}^{k} \mathbf{t}^i$. If $\mathbf{x}' \in V_\Lambda^+$, then we are done by taking $\gamma = \tau_{-\mathbf{p}}$. Otherwise, $\mathbf{t} \cdot \mathbf{x}' > \frac{M}{2}$. Consider $\tau_{\mathbf{t}} \circ \rho_{\mathbf{0}}(\mathbf{x}') = \mathbf{t} - \mathbf{x}' = \sum_{i=1}^{\ell} (1-\lambda_i)\mathbf{t}^i$. By Observation A.3, $\rho_{\mathbf{0}} \in \Gamma$ and so $\gamma = \tau_{\mathbf{t}} \circ \rho_{\mathbf{0}} \circ \tau_{-\mathbf{p}} \in \Gamma$. Further, $\mathbf{t} \cdot (\mathbf{t} - \mathbf{x}') = M - \mathbf{t} \cdot \mathbf{x}' < \frac{M}{2}$, and hence $\gamma(\mathbf{x}) = \mathbf{t} - \mathbf{x}' \in V_\Lambda^+$. Hence, V_Λ^+ contains a representative for every point in $\mathbb{R}^k$.

Next we show that every point $\mathbf{x} \in \operatorname{int}_{L'}(V_\Lambda^+)$ is a unique representative in V_Λ^+ because for any non-trivial $\tau_{\mathbf{t}} \in \Gamma^+$, $\tau_{\mathbf{t}}(\mathbf{x}) \notin V_\Lambda$, and for any $\mathbf{r} \in R \subseteq T$, $\rho_{\mathbf{r}}(\mathbf{x}) = \tau_{\mathbf{r}} \circ \rho_{\mathbf{0}}(\mathbf{x})$ lies in $\Gamma^+(\operatorname{int}_{L'}(V_\Lambda^-))$, which does not intersect V_Λ^+ (recall that Γ^+ is the subgroup defined in (14) for Γ). ☐

The following lemma explains how to construct Γ-equivariant functions using the fundamental domain.

Lemma A.6 (Construction of Γ-equivariant functions)**.** *Let $\Gamma = \Gamma(R,T)$ be a reflection group with $\emptyset \subsetneq R \subseteq T$ such that the corresponding Y from* (15) *is a mixed-lattice, i.e., $Y = \Lambda + L$ and let $\mathbf{t}^1, \dots, \mathbf{t}^\ell$ be a basis of Λ. Let $L' = \langle \mathbf{t}^1, \dots, \mathbf{t}^\ell \rangle_{\mathbb{R}}$. Let V_Λ^+ be defined with respect to this basis. Let $\psi\colon V_\Lambda^+ \to \mathbb{R}$ be any function such that $\psi\big|_{\partial_{L'}(V_\Lambda^+)} = 0$, where $\partial_{L'}(V_\Lambda^+)$ denotes the boundary of V_Λ^+ with respect to the linear subspace L'. Then the* equivariance formula (16) *gives a well-defined extension of ψ to all of $\mathbb{R}^k$.*

Figures 6 and 9 illustrate this construction.

Proof. By Lemma A.5, V_Λ^+ contains a fundamental domain. Since $\operatorname{int}_{L'}(V_\Lambda^+)$ has unique representatives for the orbits of Γ and $\psi = 0$ on the boundary $\partial_{L'}(V_\Lambda^+)$, the extension is well-defined. ☐

A.2. **Deriving the perturbation functions $\psi^m_{q,\boxdot}, \psi^m_{q,\diagdown}$ using equivariance formulas.** In [4], the authors use $\Gamma = \langle \rho_g, \tau_g \mid g \in \frac{1}{q}\mathbb{Z} \rangle$, where $Y = \Lambda = \frac{1}{q}\mathbb{Z}$. Using the lattice basis $\{t^1 = \frac{1}{q}\}$, we obtain the fundamental parallelepiped $V_\Lambda = [0, \frac{1}{q}]$ and hence $V_\Lambda^+ = [0, \frac{1}{2q}]$. In this one-dimensional case, V_Λ^+ is actually a fundamental domain for Γ.

We proceed similarly with two different reflection groups in dimension two. We first consider the reflection group $\Gamma_{q,\boxdot} = \langle\, \rho_{\mathbf{g}}, \tau_{\mathbf{g}} \mid \mathbf{g} \in \frac{1}{q}\mathbb{Z}^2 \,\rangle$ generated by reflections and translations corresponding to all possible vertices of $\mathcal{P}_q$; see Figure 9 (left). The corresponding lattice $Y_{q,\boxdot} = \Lambda_{q,\boxdot} = \frac{1}{q}\mathbb{Z}^2$. Using the lattice basis $\{\mathbf{t}^1 = \binom{1/q}{0}, \mathbf{t}^2 = \binom{0}{1/q}\}$, we obtain the fundamental parallelepiped $V_{q,\boxdot} = [0, \frac{1}{q}]^2$ from which we obtain $V^+_{q,\boxdot} = {}_{\mathbf{0}}\triangle = \frac{1}{q}\operatorname{conv}(\{\binom{0}{0}, \binom{1}{0}, \binom{0}{1}\})$. We make this particular choice of fundamental domain in part because $V^+_{q,\boxdot} \in \mathcal{P}_{q,\triangle\triangledown}$ and $\Gamma_{q,\boxdot}(V^+_{q,\boxdot}) \subseteq \mathcal{P}_{q,\triangle\triangledown}$. (Note that we have simplified the notation, e.g., $V_{\Lambda_{q,\boxdot}}$ is now denoted by $V_{q,\boxdot}$.)

For any $m \in \mathbb{Z}_{\geq 3}$, we may now interpret the function $\psi^m_{q,\boxdot}\colon \mathbb{R}^2 \to \mathbb{R}$ defined in subsection 5.1 in the following way: at all vertices of $\mathcal{P}_{mq}$ that lie on the boundary of ${}_{\mathbf{0}}\triangle$, let $\psi^m_{q,\boxdot}$ take the value 0, and at all vertices of $\mathcal{P}_{mq}$ that lie on the interior of of ${}_{\mathbf{0}}\triangle$, we assign $\psi^m_{q,\boxdot}$ to have the value 1. Interpolate these values to define $\psi^m_{q,\boxdot}$ on ${}_{\mathbf{0}}\triangle$. By Lemma A.6, the extension of $\psi^m_{q,\boxdot}$ to $\mathbb{R}^2$ via the equivariance formula (16) is well-defined. This is an alternative description for the function $\psi^m_{q,\boxdot}$ defined in subsection 5.1; refer back to Figure 6 (left) for an illustration.

One possible choice of a fundamental domain for $\Gamma_{q,\boxdot}$ is

$$\tilde{V}_{q,\boxdot} = \operatorname{int}({}_{\mathbf{0}}\triangle) \cup \big[\tbinom{0}{0}, \tbinom{0}{1/2q}\big] \cup \big[\tbinom{0}{0}, \tbinom{1/2q}{0}\big] \cup \big[\tbinom{1/2q}{1/2q}, \tbinom{1/2q}{0}\big),$$

where $[\mathbf{x}, \mathbf{y}]$ and $[\mathbf{x}, \mathbf{y})$ denote the closed and half open line segments, respectively, between $\mathbf{x}$ and $\mathbf{y}$. For our construction, only its closure, $V^+_{q,\boxdot} = {}_{\mathbf{0}}\triangle$, matters.

Lemma A.7. *The function $\psi^m_{q,\boxdot}\colon \mathbb{R}^2 \to \mathbb{R}$ has the following properties:*

(i) $\psi^m_{q,\boxdot}\big|_I = 0$ *on all edges and vertices* $I \in \mathcal{P}_{q,\boxdot\,\vert\,\underline{\square}\,\diagdown}$.
(ii) $\psi^m_{q,\boxdot}(\mathbf{x}) = -\psi^m_{q,\boxdot}(\rho_{\mathbf{g}}(\mathbf{x})) = -\psi^m_{q,\boxdot}(\mathbf{g} - \mathbf{x})$ *for all* $\mathbf{x} \in \mathbb{R}^2$ *and* $\mathbf{g} \in \frac{1}{q}\mathbb{Z}^2$.
(iii) $\psi^m_{q,\boxdot}(\mathbf{x}) = \psi^m_{q,\boxdot}(\tau_{\mathbf{g}}(\mathbf{x})) = \psi^m_{q,\boxdot}(\mathbf{g} + \mathbf{x})$ *for all* $\mathbf{x} \in \mathbb{R}^2$ *and* $\mathbf{g} \in \frac{1}{q}\mathbb{Z}^2$.
(iv) *Let* $i = 1, 2$ *or* 3, *and let* $F \in \Delta\mathcal{P}_q$ *be such that* $p_i(F) \in \mathcal{P}_{q,\boxdot}$. *Then,* $F \subseteq E(\psi^m_{q,\boxdot})$.
(v) $\psi^m_{q,\boxdot}$ *is continuous piecewise linear over* $\mathcal{P}_{mq}$.

Proof. Properties (i), (ii), (iii) follow directly from the equivariance formula (16). The function is continuous because it is continuous on the interior of each $I \in \mathcal{P}_{q,\triangle\triangledown}$ by construction and because $\psi^m_{q,\boxdot}\big|_I = 0$ on all edges $I \in \mathcal{P}_{q,\vert\,\underline{\square}\,\diagdown}$. Property (iv) follows from properties (i), (ii), and (iii) and the fact that $\operatorname{vert}(\mathcal{P}_q) = \frac{1}{q}\mathbb{Z}^2$. Finally, the function is continuous piecewise linear by construction as well. ☐

We next analyze $\psi^m_{q,\diagdown}$ from subsection 5.1. Let $\Gamma_{q,\diagdown} = \langle\, \rho_{\mathbf{y}}, \tau_{\mathbf{y}} \mid \mathbf{y} \in \mathbb{R}^2, \mathbf{1}\cdot\mathbf{y} \equiv 0 \pmod{\frac{1}{q}} \,\rangle \supseteq \Gamma_{q,\boxdot}$ be the group generated by reflections and translations corresponding to all points on diagonal edges of $\mathcal{P}_q$; see Figure 9 (right). In this case, $Y_{q,\diagdown} = \Lambda_{q,\diagdown} + L_{\diagdown}$ where $\Lambda_{q,\diagdown} = \frac{1}{q}\mathbb{Z} \times \{\mathbf{0}\}$ and $L_{\diagdown}$ is as defined in Definition 4.12. We choose the lattice basis $\{\mathbf{t}^1 = \binom{1/q}{0}\}$, which has the fundamental parallelepiped $V_{q,\diagdown} = \big[\binom{0}{0}, \binom{1/q}{0}\big]$ and hence $V^+_{q,\diagdown} = \big[\binom{0}{0}, \binom{1/2q}{0}\big]$. (Note that we have simplified the notation, e.g., $V_{\Lambda_{q,\diagdown}}$ is now denoted by $V_{q,\diagdown}$.)

We consider an alternative description for the function $\psi^m_{q,\diagdown}$, $m \geq 3$. This is done by setting $\psi^m_{q,\diagdown}(\binom{0}{0}) = 0$, $\psi^m_{q,\diagdown}(\binom{1/2q}{0}) = 0$, and for integer i with $1 \leq i < \frac{m}{2}$ we set $\psi^m_{q,\diagdown}(\binom{i/mq}{0}) = 1$. Then the function is interpolated over the vertices of $\mathcal{P}_{mq}$ that lie in $V^+_{q,\diagdown}$. We extend the function to all of $\mathbb{R}^2$ by applying the equivariance formula (16) (the extension is well-defined by Lemma A.6). This results in the continuous piecewise linear function $\psi^m_{q,\diagdown}$ defined in subsection 5.1; refer back to Figure 6 (right) for an illustration.

Lemma A.8. *The function $\psi^m_{q,\diagdown}\colon \mathbb{R}^2 \to \mathbb{R}$ has the following properties:*

(i) $\psi^m_{q,\diagdown}\big|_I = 0$ *on all edges and vertices* $I \in \mathcal{P}_{q,\boxdot\,\diagdown}$.
(ii) $\psi^m_{q,\diagdown}(\mathbf{x}) = -\psi^m_{q,\diagdown}(\rho_{\mathbf{y}}(\mathbf{x})) = -\psi^m_{q,\diagdown}(\mathbf{y} - \mathbf{x})$ *for all* $\mathbf{x} \in \mathbb{R}^2$ *and* $\mathbf{y} \in \mathbb{R}^2$ *such that* $\mathbf{1}\cdot\mathbf{y} \equiv 0 \pmod{\frac{1}{q}}$.
(iii) $\psi^m_{q,\diagdown}(\mathbf{x}) = \psi^m_{q,\diagdown}(\tau_{\mathbf{y}}(\mathbf{x})) = \psi^m_{q,\diagdown}(\mathbf{y} + \mathbf{x})$ *for all* $\mathbf{x} \in \mathbb{R}^2$ *and* $\mathbf{y} \in \mathbb{R}^2$ *such that* $\mathbf{1}\cdot\mathbf{y} \equiv 0 \pmod{\frac{1}{q}}$.

(iv) *Let $i = 1, 2,$ or 3 and let $F \in \Delta\mathcal{P}_q$ be such that $p_i(F) \in \mathcal{P}_{q,\boxdot\boxbslash}$. Then, $F \subseteq E(\psi^m_{q,\boxbslash})$.*
(v) *$\psi^m_{q,\boxbslash}$ is continuous piecewise linear over $\mathcal{P}_{mq}$.*

Proof. Properties (i), (ii), (iii) follow directly from the equivariance formula (16). Property (iv) follows from properties (i), (ii), and (iii) and the fact that all faces of $\mathcal{P}_{q,\boxdot\boxbslash}$ are contained in the set $\{\, \mathbf{y} \in \mathbb{R}^2 \mid \mathbf{1} \cdot \mathbf{y} \equiv 0 \pmod{\frac{1}{q}} \,\}$. The function is continuous because the restriction to $V^+_{q,\boxbslash}$ is continuous and the function vanishes on the relative boundary of $V^+_{q,\boxbslash}$. Finally, the function is piecewise linear by construction as well. ☐

Appendix B. Genuinely k-dimensional functions

B.1. **Preliminaries.** In this section, we prove useful properties of a special class of functions called *genuinely k-dimensional* functions. In the process, we motivate our assumption in Theorem 1.8 and Theorem 1.9 that $\mathbf{f} \in \operatorname{vert}(\mathcal{P}_q)$.

Definition B.1. A function $\theta\colon \mathbb{R}^k \to \mathbb{R}$ is *genuinely k-dimensional* if there does not exist a function $\varphi\colon \mathbb{R}^{k-1} \to \mathbb{R}$ and a linear map $T\colon \mathbb{R}^k \to \mathbb{R}^{k-1}$ such that $\theta = \varphi \circ T$.

Genuinely k-dimensional functions were studied in [6]. We will show that $\mathbf{f}$ must be a vertex of the complex $\mathcal{P}$ whenever π is a minimal piecewise linear function over $\mathcal{P}$ that is genuinely k-dimensional. We will then show that it suffices to consider only genuinely k-dimensional functions. This is because if the function is not genuinely k-dimensional we can study the function in a lower dimension by instead studying its restriction to a linear subspace of $\mathbb{R}^k$.

We will need the following lemma, which is implied by Lemma 13 in [2] and is a consequence of Dirichlet's Approximation Theorem for the reals.

Lemma B.2 ([6]). *Let $\mathbf{y} \in \mathbb{R}^k$ be any point and $\mathbf{r} \in \mathbb{R}^k \setminus \{\mathbf{0}\}$ be any direction. Then for every $\epsilon > 0$ and $\bar\lambda \geq 0$, there exists $\mathbf{w} \in \mathbb{Z}^k$ such that $\mathbf{y} + \mathbf{w}$ is at distance less than ϵ from the half line $\{\, \mathbf{y} + \lambda\mathbf{r} \mid \lambda \geq \bar\lambda \,\}$.*

The proof of the next lemma is adapted from the proof of Claim 2 in [2]. For any linear subspace M of $\mathbb{R}^k$, $\operatorname{proj}_M(\cdot)$ will denote orthogonal projection onto M. Also $M^\perp$ will denote the orthogonal complement of M.

Lemma B.3. *Let L be any linear subspace of $\mathbb{R}^k$. Then $\operatorname{proj}_L(\mathbb{Z}^k)$ has the following form: there exists a linear subspace $L' \subseteq L$ (we allow the possibility $L' = \{\mathbf{0}\}$) such that $\operatorname{proj}_L(\mathbb{Z}^k) = \Lambda + D$, where Λ is a lattice that spans $L'^\perp \cap L$ and D is a dense subset of L'.*

Proof. Let $\Lambda' = \operatorname{proj}_L(\mathbb{Z}^k)$. Let V_ϵ be the linear subspace of L spanned by the points in $\{\, \mathbf{y} \in \Lambda' \mid \|\mathbf{y}\| < \epsilon \,\}$. Notice that, given $\epsilon' > \epsilon'' > 0$, then $V_{\epsilon'} \supseteq V_{\epsilon''} \supseteq \{\mathbf{0}\}$. Since $\dim(V_\epsilon)$ changes discretely as $\epsilon \to 0$, there exists $\epsilon_0 > 0$ such that $V_\epsilon = V_{\epsilon_0}$ for every $0 < \epsilon < \epsilon_0$. Let $L' = V_{\epsilon_0}$. Observe that $\Lambda' \cap L'$ is dense in L' and $\Lambda = \operatorname{proj}_{L'^\perp \cap L}(\Lambda')$ is discrete (i.e., $B(\mathbf{0}, \epsilon_0) \cap \Lambda = \{\mathbf{0}\}$). Since Λ is the projection of a subgroup of $\mathbb{R}^k$, it is also a subgroup and therefore it is a discrete subgroup, i.e., a lattice. We thus have the result using $D = \Lambda' \cap L'$. ☐

The following lemma can be found within the proof of Lemma 2.10 in [6] for the case where L is a one-dimensional linear space.

Lemma B.4. *Suppose $\theta\colon \mathbb{R}^k \to \mathbb{R}$ is a subadditive function such that $\theta = 0$ on a linear space L. For any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^k$ such that $\mathbf{x} - \mathbf{y} \in L$, we have $\theta(\mathbf{x}) = \theta(\mathbf{y})$.*

Proof. Since $\mathbf{x} - \mathbf{y} \in L$, $\theta(\mathbf{x} - \mathbf{y}) = 0$. By subadditivity, $\theta(\mathbf{y}) + \theta(\mathbf{x} - \mathbf{y}) \geq \theta(\mathbf{x})$, which implies $\theta(\mathbf{y}) \geq \theta(\mathbf{x})$. Similarly, $\theta(\mathbf{x}) \geq \theta(\mathbf{y})$, and hence we have equality. ☐

The following lemma is modified version of Lemma 2.10 from [6] to give detail about when we can choose a linear map T that can be represented as a rational matrix. We allow for Lipschitz continuity because this continuity is implicit in continuous piecewise linear functions.

Lemma B.5. *Let $\theta\colon \mathbb{R}^k \to \mathbb{R}$ be nonnegative, Lipschitz continuous, subadditive and periodic modulo the lattice $\mathbb{Z}^k$. Suppose there exist $\mathbf{r} \in \mathbb{R}^k \setminus \{\mathbf{0}\}$ and $\bar\lambda > 0$ such that $\theta(\lambda\mathbf{r}) = 0$ for all $0 \le \lambda \le \bar\lambda$. Then θ is not genuinely k-dimensional, i.e., there exists a linear map $T\colon \mathbb{R}^k \to \mathbb{R}^{k-1}$ and a function $\varphi\colon \mathbb{R}^{k-1} \to \mathbb{R}$ such that $\pi = \varphi \circ T$. Furthermore, if $\mathbf{r} \in \mathbb{Q}^k$, then T can be represented by a rational matrix.*

Proof. Let the Lipschitz constant for θ be K, that is, $|\theta(\mathbf{x}) - \theta(\mathbf{y})| \le K\|\mathbf{x} - \mathbf{y}\|$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^k$.

We will begin by showing that $\theta(\lambda\mathbf{r}) = 0$ for all $\lambda \in \mathbb{R}$. Let $\lambda' \in \mathbb{R}$.

Suppose that $\lambda' > \bar\lambda$ and let $M \in \mathbb{Z}_+$ such that $0 \le \lambda'/M \le \bar\lambda$. From the hypothesis, we have that $\theta(\frac{\lambda'}{M}\mathbf{r}) = 0$. By nonnegativity and subadditivity of θ we see $0 \le \theta(\lambda'\mathbf{r}) \le M\theta(\frac{\lambda'}{M}\mathbf{r}) = 0$, and therefore, $\theta(\lambda'\mathbf{r}) = 0$. This shows that $\theta(\lambda\mathbf{r}) = 0$ for all $\lambda \ge 0$.

Next suppose $\lambda' < 0$. By Lemma B.2, for all $\epsilon > 0$ there exists a $\mathbf{w} \in \mathbb{Z}^k$ such that $\lambda'\mathbf{r} + \mathbf{w}$ is at distance less than ϵ from the half line $\{\, \lambda'\mathbf{r} + \lambda\mathbf{r} \mid \lambda \ge -\lambda' \,\} = \{\lambda\mathbf{r} \mid \lambda \ge 0\}$. That is, there exists a $\tilde\lambda \ge 0$ such that $\|\lambda'\mathbf{r} + \mathbf{w} - \tilde\lambda\mathbf{r}\| \le \epsilon$. Since $\theta(\tilde\lambda\mathbf{r}) = 0$, by periodicity and then Lipschitz continuity, we see that $0 \le \theta(\lambda'\mathbf{r}) = \theta(\lambda'\mathbf{r} + \mathbf{w}) = \theta(\lambda'\mathbf{r} + \mathbf{w}) - \theta(\tilde\lambda\mathbf{r}) \le K\epsilon$. This holds for every $\epsilon > 0$ and therefore $\theta(\lambda'\mathbf{r}) = 0$. Thus, we have shown that $\theta(\lambda\mathbf{r}) = 0$ for all $\lambda \in \mathbb{R}$.

Let $L = \{\, \lambda\mathbf{r} \mid \lambda \in \mathbb{R} \,\}$. By Lemma B.4, for any $\mathbf{x}, \mathbf{y}$ such that $\mathbf{x} - \mathbf{y} \in L$, we have $\theta(\mathbf{x}) = \theta(\mathbf{y})$.

We conclude that $\theta = \varphi \circ \operatorname{proj}_{L^\perp}$ for some function $\varphi\colon \mathbb{R}^{k-1} \to \mathbb{R}$ and therefore θ is not genuinely k-dimensional. Finally, if $\mathbf{r} \in \mathbb{Q}^k$, then $\operatorname{proj}_{L^\perp}$ can be represented by a rational matrix. □

B.2. **Dimension reduction for functions that are not genuinely *k*-dimensional.** We now show that it suffices to consider only genuinely k-dimensional functions for testing extremality of continuous piecewise linear functions.

Remark B.6. Given a piecewise linear continuous valid function $\zeta\colon \mathbb{R} \to \mathbb{R}$ for the one-dimensional infinite group problem $R_\mathbf{f}(\mathbb{R}, \mathbb{Z})$, Dey–Richard [10, Construction 6.1] consider the function $\kappa\colon \mathbb{R}^2 \to \mathbb{R}$, $\kappa(\mathbf{x}) = \zeta(\mathbf{1}\cdot\mathbf{x})$, where $\mathbf{1} = \binom{1}{1}$, and show that κ is minimal and extreme if and only if ζ is minimal and extreme, respectively. If ζ has rational breakpoints in $\frac{1}{q}\mathbb{Z}$ with $q \in \mathbb{Z}_+$, then κ belongs to our class of diagonally constrained continuous piecewise linear functions over $\mathcal{P}_q$. However, these functions are not genuinely 2-dimensional, and as Dey–Richard point out, we can study the one-dimensional function ζ instead of the 2-dimensional function κ. We call the function κ a *diagonal embedding* of ζ.

The following two theorems can be found in [10] for the special case of diagonal embeddings. We also refer the interested reader to [11] where the authors exhibit a sequential merge procedure, creating extreme functions in higher dimensions from extreme functions in lower dimensions and vice versa.

Lemma B.7. *Let $T\colon \mathbb{R}^k \to \mathbb{R}^\ell$ be a linear map. Suppose $\pi\colon \mathbb{R}^k \to \mathbb{R}$ and $\varphi\colon T\mathbb{R}^k \to \mathbb{R}$ satisfy $\pi = \varphi \circ T$. Then π is minimal for $R_\mathbf{f}(\mathbb{R}^k, \mathbb{Z}^k)$ if and only if φ is minimal for $R_{T\mathbf{f}}(T\mathbb{R}^k, T\mathbb{Z}^k)$.*

Proof. ($\Longleftarrow$) Suppose φ is minimal for $R_{T\mathbf{f}}(T\mathbb{R}^k, T\mathbb{Z}^k)$. We demonstrate that π satisfies the criterion from Theorem 1.1 to be minimal.

(1) For any $\mathbf{z} \in \mathbb{Z}^k$, $0 = \varphi(T\mathbf{z}) = \pi(\mathbf{z})$.
(2) For any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^k$ we have

$$\pi(\mathbf{x}) + \pi(\mathbf{y}) - \pi(\mathbf{x} + \mathbf{y}) = \varphi(T\mathbf{x}) + \varphi(T\mathbf{y}) - \varphi(T(\mathbf{x} + \mathbf{y})) = \varphi(T\mathbf{x}) + \varphi(T\mathbf{y}) - \varphi(T\mathbf{x} + T\mathbf{y}) \ge 0.$$

(3) For any $\mathbf{x} \in \mathbb{R}^k$, we have

$$\pi(\mathbf{x}) + \pi(\mathbf{f} - \mathbf{x}) = \varphi(T\mathbf{x}) + \varphi(T(\mathbf{f} - \mathbf{x})) = \varphi(T\mathbf{x}) + \varphi(T\mathbf{f} - T\mathbf{x}) = 1.$$

Therefore π is minimal by Theorem 1.1.

($\Longrightarrow$) Suppose π is minimal for $R_\mathbf{f}(\mathbb{R}^k, \mathbb{Z}^k)$. We demonstrate that φ satisfies the criterion from Theorem 1.1 to be minimal.

(1) For any $\mathbf{z} \in \mathbb{Z}^k$, $0 = \pi(\mathbf{z}) = \varphi(T\mathbf{z})$.
(2) For any $\mathbf{x}, \mathbf{y} \in T\mathbb{R}^k$, let $\hat{\mathbf{x}} \in T^{-1}\mathbf{x}$, $\hat{\mathbf{y}} \in T^{-1}\mathbf{y}$. Then

$$0 \le \pi(\hat{\mathbf{x}}) + \pi(\hat{\mathbf{y}}) - \pi(\hat{\mathbf{x}} + \hat{\mathbf{y}}) = \varphi(\mathbf{x}) + \varphi(\mathbf{y}) - \varphi(\mathbf{x} + \mathbf{y}).$$

(3) Similarly, for any $\mathbf{x} \in T\mathbb{R}^k$, let $\hat{\mathbf{x}} \in T^{-1}\mathbf{x}$. Then

$$1 = \pi(\hat{\mathbf{x}}) + \pi(\mathbf{f} - \hat{\mathbf{x}}) = \varphi(\mathbf{x}) + \varphi(T\mathbf{f} - \mathbf{x}).$$

Therefore φ is minimal by Theorem 1.1. □

Lemma B.8. *Let $\pi\colon \mathbb{R}^k \to \mathbb{R}$ be a minimal valid function. Let $T\colon \mathbb{R}^k \to \mathbb{R}^\ell$ be a linear map and let $\varphi\colon T\mathbb{R}^k \to \mathbb{R}$ such that $\pi = \varphi \circ T$. Then π is extreme for $R_{\mathbf{f}}(\mathbb{R}^k, \mathbb{Z}^k)$ if and only if φ is extreme for $R_{T\mathbf{f}}(T\mathbb{R}^k, T\mathbb{Z}^k)$.*

Proof. ($\Longrightarrow$) We prove the contrapositive. Suppose φ is not extreme for $R_{T\mathbf{f}}(T\mathbb{R}^k, T\mathbb{Z}^k)$. Then, by Lemma 1.4, there exist distinct minimal valid functions φ^1, φ^2 for $R_{T\mathbf{f}}(T\mathbb{R}^k, T\mathbb{Z}^k)$ such that $\varphi = \frac{1}{2}(\varphi^1 + \varphi^2)$. But then $\pi^1 = \varphi^1 \circ T$ and $\pi^2 = \varphi^2 \circ T$ are distinct functions, and $\pi = \frac{1}{2}(\pi^1 + \pi^2)$. By Lemma B.7, π^1, π^2 are minimal for $R_{\mathbf{f}}(\mathbb{R}^k, \mathbb{Z}^k)$. Therefore π is not extreme.

($\Longleftarrow$) We again prove the contrapositive. Suppose that π is not extreme for $R_{\mathbf{f}}(\mathbb{R}^k, \mathbb{Z}^k)$. Then there exist distinct minimal valid functions π^1, π^2 for $R_{\mathbf{f}}(\mathbb{R}^k, \mathbb{Z}^k)$ such that $\pi = \frac{1}{2}(\pi^1 + \pi^2)$. Since π, π^1, π^2 are minimal by Lemma 1.4, $\pi(\mathbf{0}) = \pi^1(\mathbf{0}) = \pi^2(\mathbf{0}) = 0$. Since $E(\pi) \subseteq E(\pi^1), E(\pi^2)$ by Lemma 1.4, and $0 = \pi(\mathbf{x}) + \pi(-\mathbf{x}) - \pi(\mathbf{0}) = \Delta\pi(\mathbf{x}, -\mathbf{x})$ for all $\mathbf{x} \in T^{-1}(\mathbf{0})$, it follows that $\pi^i(\mathbf{x}) = -\pi^i(-\mathbf{x})$ for $i = 1, 2$. Since π^i are valid functions, $\pi^i \geq 0$, therefore we must have $\pi^i(\mathbf{x}) = 0$ for all $\mathbf{x} \in T^{-1}(\mathbf{0})$. By Lemma B.4, $\pi^i(\mathbf{x}) = \pi^i(\mathbf{y})$ whenever $\mathbf{x} - \mathbf{y} \in T^{-1}(\mathbf{0})$. Therefore, we must have φ^1, φ^2 such that $\pi^1 = \varphi^1 \circ T$ and $\pi^2 = \varphi^2 \circ T$. Since π^1, π^2 are distinct, the functions φ^1, φ^2 are distinct as well. Also since $\pi = \frac{1}{2}(\pi^1 + \pi^2)$, we have $\varphi = \frac{1}{2}(\varphi^1 + \varphi^2)$. By Lemma B.7, φ^1, φ^2 are minimal for $R_{T\mathbf{f}}(T\mathbb{R}^k, T\mathbb{Z}^k)$. Therefore φ is not extreme. □

Given any family of polyhedra $\mathcal{F}$ (not necessarily a polyhedral complex), we say a polyhedral complex $\mathcal{P}$ is a *refinement* of $\mathcal{F}$ if every polyhedron of $\mathcal{F}$ is a union of polyhedra from $\mathcal{P}$.

Proposition B.9 (Dimension reduction)**.** *Let $\mathcal{P}$ be a pure and complete polyhedral complex in $\mathbb{R}^k$ that is periodic modulo $\mathbb{Z}^k$. Let $\pi\colon \mathbb{R}^k \to \mathbb{R}$ be a piecewise linear function over $\mathcal{P}$, such that π is nonnegative, subadditive, periodic modulo $\mathbb{Z}^k$ and $\pi(\mathbf{0}) = 0$. If π is not genuinely k-dimensional, then there exists a natural number $0 \leq \ell < k$, a pure and complete polyhedral complex $\mathcal{X}$ in $\mathbb{R}^\ell$ that is periodic modulo $\mathbb{Z}^\ell$, a nonnegative and subadditive function $\phi\colon \mathbb{R}^\ell \to \mathbb{R}$ that is piecewise linear over $\mathcal{X}$, and a point $\mathbf{f}' \in \mathbb{R}^\ell \setminus \mathbb{Z}^\ell$ with the following properties:*

(1) π is minimal for $R_{\mathbf{f}}(\mathbb{R}^k, \mathbb{Z}^k)$ if and only if ϕ is minimal for $R_{\mathbf{f}'}(\mathbb{R}^\ell, \mathbb{Z}^\ell)$.
(2) π is extreme for $R_{\mathbf{f}}(\mathbb{R}^k, \mathbb{Z}^k)$ if and only if ϕ is extreme for $R_{\mathbf{f}'}(\mathbb{R}^\ell, \mathbb{Z}^\ell)$.

Proof. Since π is not genuinely k-dimensional, it follows by iteratively applying the definition of genuinely k-dimensional functions that there exist a number $0 \leq \ell < k$, a function $\varphi\colon \mathbb{R}^\ell \to \mathbb{R}$, and a linear map $T\colon \mathbb{R}^k \to \mathbb{R}^\ell$ such that $\varphi\colon \mathbb{R}^\ell \to \mathbb{R}$ is genuinely ℓ-dimensional and $\pi = \varphi \circ T$. Since π is nonnegative, φ must also be nonnegative. Since π is subadditive and T is additive, φ must be subadditive.

Claim 1. $T\mathbb{Z}^k$ is a lattice that spans $\mathbb{R}^\ell$.

Since every linear map is a projection composed with an isomorphism, Lemma B.3 implies that there exists a linear subspace $L \subseteq \mathbb{R}^\ell$ such that $T\mathbb{Z}^k = \Lambda + D$, where Λ is a lattice spanning $L^\perp$ and D is dense in L. If $L = \{\mathbf{0}\}$ then we are done. So we assume $\dim(L) \geq 1$. Since π is continuous (it is piecewise linear over a locally finite polyhedral complex), and T is linear map, it follows that φ is continuous. Also, since π vanishes over $\mathbb{Z}^k$, φ vanishes over $T\mathbb{Z}^k$. But this implies that φ vanishes over D, and thus over L. By Lemma B.4, φ is constant on the affine subspaces parallel to L. This contradicts the assumption that φ is genuinely ℓ-dimensional. This concludes the proof of Claim 1.

Let $\mathcal{U} = \bigcup_{I \in \mathcal{P}} \{I \cap [0,1]^n\}$. Since π is piecewise linear over $\mathcal{P}$, π is also piecewise linear over a refinement of $\mathcal{P}$, in particular, over the polyhedral complex $\bigcup_{I \in \mathcal{U}, \mathbf{w} \in \mathbb{Z}^k} \{I + \mathbf{w}\}$, that is periodic modulo $\mathbb{Z}^k$. Since $T\mathbb{Z}^k$ is a lattice and for every $I \in \mathcal{U}$, TI is a polytope (it is the projection of the polytope I), we can find a refinement of the family of polytopes $\bigcup_{I \in \mathcal{U}, \mathbf{w} \in T\mathbb{Z}^k} \{TI + \mathbf{w}\}$; we denote this refinement by $\mathcal{P}'$, which is a pure and complete polyhedral complex of $\mathbb{R}^\ell$. We observe that φ is piecewise linear over $\mathcal{P}'$ and $\mathcal{P}'$ is a polyhedral complex that is periodic modulo $T\mathbb{Z}^k$.

Now simply find an invertible linear transformation $A\colon T\mathbb{Z}^k \to \mathbb{Z}^\ell$ and let $\phi := \varphi \circ A^{-1}$ be the piecewise linear function defined over the pure and complete polyhedral complex $\mathcal{X} := A\mathcal{P}'$ and let $\mathbf{f}' := AT\mathbf{f}$. Then $\mathbf{f}' \notin \mathbb{Z}^\ell$, since $1 = \pi(\mathbf{f}) = \phi(\mathbf{f}')$ and $\phi(\mathbb{Z}^\ell) = \pi(\mathbb{Z}^k) = 0$. The two properties now follow from Lemmas B.7 and B.8. □

Remark B.10 (Dimension reduction)**.** Using Proposition B.9, the extremality/minimality question for π that is not genuinely k-dimensional can be reduced to the same question for a lower-dimensional genuinely ℓ-dimensional function (so $\ell < k$.) When $\mathcal{P}$ is a rational polyhedral complex, this reduction can be done algorithmically. The question of making this effective for the irrational case is beyond the scope of this paper.

B.3. **The assumption of $\mathbf{f} \in \mathrm{vert}(\mathcal{P})$.** We will show that $\mathbf{f}$ is a vertex for any minimal continuous piecewise linear function that is genuinely k-dimensional.

Theorem B.11. *Let $\mathcal{P}$ be a pure and complete polyhedral complex in $\mathbb{R}^k$ that is periodic modulo $\mathbb{Z}^k$. Let $\theta\colon \mathbb{R}^k \to \mathbb{R}$ be minimal, piecewise linear function over $\mathcal{P}$ that is genuinely k-dimensional. Then $\mathbf{f} \in \mathrm{vert}(\mathcal{P})$.*

Proof. For the sake of contradiction, suppose $\mathbf{f} \notin \mathrm{vert}(\mathcal{P})$. Therefore, there exists some $I \in \mathcal{P}$ with $\mathbf{f} \in \mathrm{rel\,int}(I)$ and the dimension of I is at least one. Since π is minimal, $0 \leq \pi \leq 1$. Since $\pi(\mathbf{f}) = 1$, $\pi \leq 1$, π is affine on I and $\mathbf{f} \in \mathrm{rel\,int}(I)$, we have that $\pi(\mathbf{x}) = 1$ for all $\mathbf{x} \in I$. Now consider π on $\mathbf{f} - I$ and note that $\mathbf{0} \in \mathbf{f} - I$. By symmetry, $\pi(\mathbf{x}) = 0$ for all $\mathbf{x} \in \mathbf{f} - I$. Since the dimension of I is at least one, there exists $\mathbf{r} \in (\mathbf{f} - I) \setminus \{\mathbf{0}\}$. But then $\pi(\lambda\mathbf{r}) = 0$ for all $\lambda \in [0, 1]$. Since π is continuous piecewise linear over $\mathcal{P}$, by Lemma 1.4, it satisfies the hypotheses of Lemma B.5. Therefore, π is not genuinely k-dimensional, which is a contradiction. Therefore, we must have $\mathbf{f} \in \mathrm{vert}(\mathcal{P})$. □

Remark B.12. Using Proposition B.9 and Theorem B.11, we can achieve dimension reduction when $\mathbf{f} \notin \mathrm{vert}(\mathcal{P})$. Thus, although the results presented in this paper assume that $\mathbf{f} \in \mathrm{vert}(\mathcal{P})$, this assumption is actually not very restrictive.

B.4. **Boundedness of cells for genuinely k-dimensional functions.** In this subsection, we show that for genuinely k-dimensional minimal valid functions that are piecewise linear over a pure and complete polyhedral complex $\mathcal{P}$ in $\mathbb{R}^k$ that is periodic modulo $\mathbb{Z}^k$, the cells of $\mathcal{P}$ are full-dimensional bounded polytopes (so they cannot be unbounded polyhedra).

Lemma B.13. *Let $\mathbf{r} \in \mathbb{R}^k$ be any vector and let $L = \mathbf{r}^\perp$ be the orthogonal complement of the subspace spanned by $\mathbf{r}$. Let U be a compact convex set with nonempty interior in $\mathbb{R}^k$. Then $\mathrm{proj}_L(U + \mathbb{Z}^k)$ is a closed set.*

Proof. Since orthogonal projections onto linear subspaces are linear operators, $\mathrm{proj}_L(U + \mathbb{Z}^k) = \mathrm{proj}_L(U) + \mathrm{proj}_L(\mathbb{Z}^k)$. Observe that $\mathrm{proj}_L(U)$ is also a compact convex set with nonempty interior with respect to L. By Lemma B.3, there exists a linear subspace $L' \subseteq L$ such that $\mathrm{proj}_L(\mathbb{Z}^k) = \Lambda + D$, where Λ is a lattice that spans $L'^\perp \cap L$ and D is a dense subset of L'. Since $\mathrm{proj}_L(U)$ is convex with nonempty interior, $\mathrm{proj}_L(U) + D = \mathrm{proj}_L(U) + L'$. Let U' be the orthogonal projection of $\mathrm{proj}_L(U)$ onto $L'^\perp \cap L$; so U' is compact convex set. Thus, we have

$$\begin{aligned}
\mathrm{proj}_L(U + \mathbb{Z}^k) &= \mathrm{proj}_L(U) + \mathrm{proj}_L(\mathbb{Z}^k) \\
&= \mathrm{proj}_L(U) + \Lambda + D \\
&= \mathrm{proj}_L(U) + L' + \Lambda \\
&= U' + L' + \Lambda.
\end{aligned}$$

Since U' is a compact set and Λ is a closed set, $U' + \Lambda$ is closed (see, e.g., [1] Lemma 5.3 (4)). Moreover, $U' + \Lambda \subseteq L'^\perp \cap L$. Therefore, $U' + \Lambda + L'$ is closed. □

Let $H := [0, 1]^k$ denote the unit hypercube.

Lemma B.14. *Let $\mathcal{P}$ be a locally finite polyhedral complex that is periodic modulo $\mathbb{Z}^k$. Then for any full-dimensional polyhedron $I \in \mathcal{P}$, the set $I + \mathbb{Z}^k$ is a finite union of the form $\bigcup_{j\in J}(I_j + \mathbb{Z}^k)$ where J is a finite index set and each I_j is a full-dimensional polytope contained in H.*

Proof. We can take I_j to be all full-dimensional polytopes contained in $(I + \mathbb{Z}^k) \cap H$. There are only finitely many of these polytopes by the locally finite property of $\mathcal{P}$ (see Definition 3.1 (iv)). □

Lemma B.15. *Let $\theta\colon \mathbb{R}^k \to \mathbb{R}$ be a piecewise linear minimal valid function over a polyhedral complex $\mathcal{P}$ that is pure, complete and periodic modulo the lattice $\mathbb{Z}^k$. If θ is genuinely k-dimensional, then the cells of $\mathcal{P}$ and $\Delta\mathcal{P}$ are full-dimensional polytopes.*

Proof. Suppose to the contrary that a cell I^* has a recession direction $\mathbf{r}$. Let L be the linear subspace orthogonal to $\mathbf{r}$, i.e., $L = \langle \mathbf{r}\rangle^\perp$. Let $U = \bigcup\{\, I \in \mathcal{P} \mid \mathbf{r} \text{ is a recession direction for } I \,\}$. Define $S = \operatorname{proj}_L(U)$.

Claim 1. $S = L$.

First, notice that $H \cap \mathcal{P}$ contains finitely many full-dimensional polytopes by the local finiteness of $\mathcal{P}$. Combining this observation with Lemma B.14, we can express $U = \bigcup_{j\in J}(I_j + \mathbb{Z}^k)$ where J is a finite index set and each I_j is a full-dimensional polytope. Therefore, $S = \operatorname{proj}_L(U) = \bigcup_{j\in J} \operatorname{proj}_L(I_j + \mathbb{Z}^k)$, which is a finite union of closed sets by Lemma B.13. Therefore, S is closed. The set S is nonempty because I^* has recession direction $\mathbf{r}$. If $S \neq L$, then there exists a boundary point $\mathbf{x}$ of S (considered as a subset of L). Thus, there exist $Q_0 \in \mathcal{P}$ and $\mathbf{y} \in Q_0$ such that $\mathbf{x} = \operatorname{proj}_L(\mathbf{y})$ and Q_0 has $\mathbf{r}$ as a recession direction. Moreover, we can choose $\mathbf{y}$ so that $\mathbf{y}$ is in the relative interior of a face $F_0 \subseteq Q_0$ where F_0 also has $\mathbf{r}$ as a recession direction. Let $Q_1, \dots, Q_p \in \mathcal{P}$ be the cells that also have F_0 as their face (using the local finiteness of $\mathcal{P}$). We set $p = 0$ if $F_0 = Q_0$. Since $\mathcal{P}$ is complete and $\mathbf{y} \in \operatorname{rel\,int}(F_0)$, we can choose $\delta > 0$ such that $B(\mathbf{y}, \delta) \subseteq Q_0 \cup Q_1 \cup \dots \cup Q_p$. Since F_0 is a face of each of these polyhedra, $\mathbf{r}$ is a recession direction for each $Q_0, Q_1, \dots, Q_p$. Thus, $B(\mathbf{y}, \delta) \subseteq U$ and thus, $\operatorname{proj}_L(B(\mathbf{y}, \delta)) \subseteq S$. But $\operatorname{proj}_L(\mathbf{y}) = \mathbf{x}$ and $\mathbf{x}$ is a boundary point of S. This is a contradiction. Therefore, $S = L$. This concludes the proof of Claim 1.

Fix $\mathbf{x} \in L = S$. Let $Q \in \mathcal{P}$ be the cell such that $\mathbf{x} \in \operatorname{proj}_L(Q)$ and $\mathbf{r}$ is a recession direction of Q. Thus, there exists a constant $\lambda(\mathbf{x}) \in \mathbb{R}$ such that $\mathbf{x} + \mu\mathbf{r} \in Q$ for all $\mu \geq \lambda(\mathbf{x})$. Since θ is bounded and affine over Q, θ must be constant on the half-line $\mathbf{x} + \mu\mathbf{r}$, $\mu \geq \lambda(\mathbf{x})$. Thus, there exists a constant $C(\mathbf{x})$ such that $\theta(\mathbf{x} + \mu\mathbf{r}) = C(\mathbf{x})$ for all $\mu \geq \lambda(\mathbf{x})$. We now show that $\theta(\mathbf{x} + \mu\mathbf{r}) = C(\mathbf{x})$ for all $\mu \in \mathbb{R}$. Let $\mu' < \lambda(\mathbf{x})$ and let $\mathbf{y} = \mathbf{x} + \mu'\mathbf{r}$. By Lemma B.2, for all $\epsilon > 0$ there exists $\mathbf{w} \in \mathbb{Z}^k$ such that $\mathbf{y} + \mathbf{w}$ is at distance less than ϵ from the half line $\{\, \mathbf{y} + \mu\mathbf{r} \mid \mu \geq \lambda(\mathbf{x}) - \mu' \,\} = \{\, \mathbf{x} + \lambda\mathbf{r} \mid \lambda \geq \lambda(\mathbf{x}) \,\}$. That is, there exists $\tilde\lambda \geq \lambda(\mathbf{x})$ such that $\|\mathbf{y} + \mathbf{w} - (\mathbf{x} + \tilde\lambda\mathbf{r})\| \leq \epsilon$. Since $\theta(\mathbf{x} + \tilde\lambda\mathbf{r}) = \theta(\mathbf{x} + \lambda(\mathbf{x})\mathbf{r})$, by periodicity and then Lipschitz continuity, we see that $|\theta(\mathbf{x} + \lambda(\mathbf{x})\mathbf{r}) - \theta(\mathbf{y})| = |\theta(\mathbf{x} + \tilde\lambda\mathbf{r}) - \theta(\mathbf{y} + \mathbf{w})| \leq K\epsilon$. This holds for every $\epsilon > 0$ and therefore $\theta(\mathbf{y}) = \theta(\mathbf{x} + \lambda(\mathbf{x})\mathbf{r})$. Thus, we have shown that for any $\mathbf{x} \in L$, θ is constant on the line $\mathbf{x} + \mu\mathbf{r}$, $\mu \in \mathbb{R}$. But this contradicts the fact that θ is genuinely k-dimensional.

Finally, if all cells of $\mathcal{P}$ are polytopes, then this property also holds for $\Delta\mathcal{P}$. □

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