A Mathematical approach applied to train scheduling in Brazilian railways

Thiago Henrique Nogueira · Carlos Roberto Venâncio de Carvalho · Larissa Cristina de Camargo · Gabriel Pinheiro Alves Santos

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Abstract In this paper, a mathematical approach was used to schedule trains in order to minimize the weighted delay time. The single-track railway studied, is composed of a part of two joined railways in Brazil, which belongs to the VALE company and to Centro-Atlântica railway (FCA). This single line is shared by these railways (VALE and FCA) and the trains travel in both directions. These railways have a bottleneck caused by a heavy traffic. Therefore, mathematical equations were developed to determine the shortest time interval admissible between two trains, ensuring that they will not overlap. These equations enable us to simplify the mathematical representation of the problem and define it as a single machine scheduling problem. Later, two formulations of mathematical models were developed and analyzed; one based on “Arc-Time-Indexed” variables and the other, on “Binary” variables. Each formulation reflects a specific concept on how the variables and parameters are defined and require particular settings and definitions. Extensive computational experiments are performed considering various instances to capture several aspects of practical situations. Based on the results, recommendations are made for the best adaptation of the MIP formulation for the considered problem.

Keywords Logistic · Railway transportation · Single-track line · Interchange point · Scheduling · Mathematical modeling

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1 Introduction

Throughout the Brazilian railways, million tons of products are transported every day, in several types of trains. To meet the demand, they often travel on interchange tracks between railways. At this point, the railway transportation planning problems generally involve schedule decision, in accordance to some priority rules: client to be responded, transported product, yard capacity, among others.

This article investigates a single-track line, composed of a part of two joined railways in Brazil, which belongs to the VALE company and to Centro-Atlântica railway (FCA). This single line is shared by these railways (VALE and FCA) and the trains travel in both directions. There is a bottleneck in the railways caused by heavy traffic. Nowadays, due this criticality, the main decision about the train schedule over the railways is based on the interchange point between them. Furthermore, there is a daily flow of passengers, freight and ore in this interchange. Thus, the decision planning that determines the schedule of the trains that will travel over the line is complex. Moreover, this decision is essential because other operations will also be scheduled according to this plan, including trains that do not cross the interchange point.

Rail transportation planning problems can be classified into three levels, namely, operational, tactical and strategic. According to Ghoseiri et al. (2004) and Lusby et al. (2011), the problems related to the strategic, network and line planning problems involve, respectively, construction and/or modification of the existent infra-structure and the selection of a line set together with their frequencies of use. The tactical level includes timetabling and/or train scheduling problems, which consist in defining, after a set of trains and capacities/operational restrictions of their lines, the date at which each train goes through and leaves each station; rolling stock planning, which consists in connecting the cars and the locomotives, dedicated to each line, to form the train; and crew management, in which the crew distribution and allocation is determined. Finally, the operational level consists in real time management and dealing with problems, such as trains delays, line maintenance, broken trains, or accidents that interfere with the established planning.

This work aims to develop a tactical decision support for the train scheduling problem. This study considers the delay times of the trains after their formation date and their priorities. This problem is defined as $NP\text{-}hard$ (see Lenstra et al. (1977) and Lawler et al. (1993)). The delays cause some derangements, such as: increase of travelling time, packed yards and reduced efficiency in line use; and the priority increases the complexity of decision making. Thus, this work focuses on obtaining a train schedule, weighted by priority, which minimizes the time waiting for permission to travel. This problem is not simple, because it involves a lot of variables. As two or more trains cannot simultaneously run on the same track, the chance of delay is substantially increased. More details about real problems and models are presented in the next section. The literature about train scheduling problems is quite extensive. Among the various articles in the literature, we highlight some works that present
objective and multi-objective functions related with time and cost. These works are organized in chronological order in Tables 1 and 2.

Further information can be obtained from the following surveys: Assad (1980) retracts the railway existing models, proposes the collection and categorization of railway modeling efforts, besides positioning this literature in other contexts of transportation models; Cordeau et al. (1998) investigate and analyze the contribution of optimization problems when applied to railway transportation problem, mainly focusing in routing and scheduling; and Lusby et al. (2011) define a conjunct of models and methods proposed in the literature for issues of operational, tactical and strategic level, such as train timetabling, train dispatching, and train routing problems.


In the reviewed works, a few efforts develop solution techniques with exact methods, and heuristic algorithms development prevails. Furthermore, many analyzed works also consider a single-track railway with flow in both directions. However, there are some works that consider just one direction, such as Jovanović and Harker (1991), Carey and Lockwood (1995), Caprara et al. (2006) and Gholami and Sotskov (2012). In the reviewed literature no work treated an interchange point between railways with single-track and flow in both directions. Most train scheduling works focus on time objective functions, while a few of them are related with costs and/or multi-objective functions. We highlight the works Mladenović and Čangalović (2007), Lee and Chen (2009) and Shafia et al. (2012), in which the objective functions are similar to ours.
### Table 1: Previous specific research works for train scheduling problems - Part One

<table>
<thead>
<tr>
<th>Authors</th>
<th>Railway Type</th>
<th>Mathematical Model</th>
<th>Solving Method</th>
<th>Case Study</th>
<th>Objective Function</th>
</tr>
</thead>
<tbody>
<tr>
<td>Szpigel (1973)</td>
<td>Single track</td>
<td>Job-shop</td>
<td>Branch-and-Bound</td>
<td>Single track railroad in eastern Brazil</td>
<td>Minimize the sum of travel time</td>
</tr>
<tr>
<td>Jovanović and Harker (1991)</td>
<td>Single and double track</td>
<td>Flow-shop</td>
<td>Branch-and-Bound</td>
<td>Illustrative example</td>
<td>Maximize reliability</td>
</tr>
<tr>
<td>Cai and Goh (1994)</td>
<td>Single track</td>
<td>Integer programming</td>
<td>Greedy Heuristic</td>
<td>Illustrative example</td>
<td>Minimize the total cost due to stops and delays in the passing loops</td>
</tr>
<tr>
<td>Higgins et al. (1996)</td>
<td>Single track</td>
<td>Non-linear mixed integer program</td>
<td>Branch-and-Bound</td>
<td>British and Europe rail lines type</td>
<td>Minimize train delays and operating costs</td>
</tr>
<tr>
<td>Higgins et al. (1997)</td>
<td>Single-track Non-linear mixed integer program</td>
<td>Genetic Algorithms, Tabu Search and Two Hybrid Algorithms</td>
<td>Illustrative example</td>
<td>Minimize the total weighted travel time</td>
<td></td>
</tr>
<tr>
<td>Chiang et al. (1998)</td>
<td>North-south bound rail</td>
<td>Job-shop</td>
<td>Multi-objective: Maximizing the passengers service and minimizing the operation cost</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Adenso-Díaz et al. (1999)</td>
<td>Network</td>
<td>Stochastic analytical model</td>
<td>Iterative refinement algorithm</td>
<td>Minimize train delays</td>
<td></td>
</tr>
<tr>
<td>Salam (1999)</td>
<td>Single track</td>
<td>Mixed integer program</td>
<td>Heuristic algorithm</td>
<td>Spanish Railway</td>
<td>Minimize the number of passenger transported</td>
</tr>
<tr>
<td>De Oliveira et al. (1999)</td>
<td>Single track</td>
<td>Heuristic algorithm</td>
<td>Illustrative example</td>
<td>Minimize average delay</td>
<td></td>
</tr>
<tr>
<td>Adenso-Díaz et al. (2000)</td>
<td>Network</td>
<td>Heuristic algorithm</td>
<td>Illustrative example</td>
<td>Minimize the number of passenger transported</td>
<td></td>
</tr>
<tr>
<td>Dorheim and Molnár (2004)</td>
<td>Network</td>
<td>Discrete event model</td>
<td>Greedy Travel Advance Strategy</td>
<td>Illustrative example</td>
<td>Minimize the time to clear the line, the delay of all trains, and the number of delays</td>
</tr>
<tr>
<td>Chen et al. (2004)</td>
<td>Network</td>
<td>Heuristic algorithm</td>
<td>Illustrative example</td>
<td>Minimize the number of passenger transported</td>
<td></td>
</tr>
<tr>
<td>Medals et al. (2005)</td>
<td>Single track</td>
<td>Heuristic algorithm</td>
<td>Illustrative example</td>
<td>Minimize the number of passenger transported and train delays</td>
<td></td>
</tr>
<tr>
<td>Zhou and Zhao (2005)</td>
<td>Double track</td>
<td>Flow-shop</td>
<td>Modified B&amp;B algorithm</td>
<td>Heuristic Algorithm</td>
<td>Minimize the travel times and train delays</td>
</tr>
<tr>
<td>Caprara et al. (2006)</td>
<td>Single track</td>
<td>Traffic Management Problem (TCP)</td>
<td>Illustrative example</td>
<td>Minimize the points</td>
<td></td>
</tr>
<tr>
<td>Ghoseiri and Morshedsolouk (2006)</td>
<td>Single track</td>
<td>Heuristic algorithm</td>
<td>Illustrative example</td>
<td>Minimize the total train delays in the stations</td>
<td></td>
</tr>
</tbody>
</table>
### Table 2: Previous specific research works for train scheduling problems - Part Two

<table>
<thead>
<tr>
<th>Authors</th>
<th>Railway Type</th>
<th>Mathematical Model</th>
<th>Solving Method</th>
<th>Case Study</th>
<th>Objective Function</th>
</tr>
</thead>
<tbody>
<tr>
<td>D'Ariano et al. (2007)</td>
<td>Network</td>
<td>Job-shop</td>
<td>Branch and Bound</td>
<td>Dutch rail network</td>
<td>Minimize the maximum accumulated delay for all trains at all visited stations</td>
</tr>
<tr>
<td>Mazzarello and Ottaviani (2007)</td>
<td>Network</td>
<td>Alternative Graph</td>
<td>Local Heuristic Algorithm</td>
<td>Schiphol Bottleneck</td>
<td>Minimize delay costs</td>
</tr>
<tr>
<td>Mladenovic and Cangalovic (2007)</td>
<td>Single track</td>
<td>Job-shop</td>
<td>Modified Branch and Bound, Evaluation of the Representation (Germany)</td>
<td>Minimize the maximum tardiness; minimize the total tardiness; minimize the maximum slack of trains in stations; minimize the makespan</td>
<td></td>
</tr>
<tr>
<td>Rodriguez (2007)</td>
<td>Network</td>
<td>Job-shop</td>
<td>Branch and Bound</td>
<td>Pierrefitte-Gonesse (Paris) through the junction</td>
<td>Minimize the sum of delays</td>
</tr>
<tr>
<td>Törnquist (2007)</td>
<td>Network</td>
<td>HOAT</td>
<td>Heuristic Algorithm</td>
<td>Swedish Railway</td>
<td>Minimize total final delay; minimize the total accumulated delay; minimize the total delay cost</td>
</tr>
<tr>
<td>Törnquist and Persson (2007)</td>
<td>Network</td>
<td>Mixed Integer Linear Program</td>
<td>Four Heuristic Algorithm</td>
<td>Swedish Railway</td>
<td>Minimize the total final delay of the traffic; minimize the total cost of delays</td>
</tr>
<tr>
<td>Zhou and Zhong (2007)</td>
<td>Single track</td>
<td>Job-shop</td>
<td>Modified Branch and Bound algorithm</td>
<td>Laizhou to Shaowu (3 methods to reduce the solution space: Lagrangian relaxation based lower bound rule, exact lower bound rule, tight upper bound by a beam search method)</td>
<td>Minimize the total travel time</td>
</tr>
<tr>
<td>Abril et al. (2008)</td>
<td>Single track</td>
<td>Constraint Satisfaction</td>
<td>Meta-tree structure, Distributed Algorithm and an Intra-agent Search Algorithm</td>
<td>Spanish Railway</td>
<td>Minimize the satisfaction criteria</td>
</tr>
<tr>
<td>D'Armano et al. (2008)</td>
<td>Network</td>
<td>Single and Double-track</td>
<td>Linear Programming</td>
<td>Dutch Railway network</td>
<td>Minimize the maximum accumulated delay</td>
</tr>
<tr>
<td>Liu and Chen (2009)</td>
<td>Network</td>
<td>Single and Double-track</td>
<td>Linear Programming</td>
<td>Taiwan Railway</td>
<td>Minimize the sum of the difference between the services scheduled departure time and the target departure time, weighted by the priorities of the services</td>
</tr>
<tr>
<td>Liu and Zhong (2009)</td>
<td>Single track</td>
<td>Job-shop</td>
<td>Heuristic Procedure at each iteration of a Shifting Bottleneck Algorithm</td>
<td>Railway line between Isfahan and Tehran (Iran)</td>
<td>Minimize the total weighted travel distance</td>
</tr>
</tbody>
</table>
| Corman et al. (2010) | Network | Job-Shop Scheduling | Tabu Search | Hybrid Metaheuristic Algorithm | Minimize the makespan, maximum tardiness/tardiness, total (weighted) completion time, total (weighted) tardiness/tardiness.
This study aims to propose a specific mathematical representation model for specific decisions about railways concerning the train scheduling problem. Furthermore, an equation group is developed, which allows the mathematical representation of the problem in a simpler way. These equations perform a preprocessing of the problem data, which simplifies the problem. It is a particular case for the studied problem. This specific train scheduling problem is defined based on the single machine scheduling problem. Moreover, two single machine scheduling formulations are proposed and compared. The first is a new formulation proposition with “Arc-Time-Indexed” variables, based on the study conducted by Pessoa et al. (2010), while the latter is based on the traditional “Binary Variables Formulation” Manne (1959) and others. These formulations reflect a specific concept of how the variables and parameters are defined, requiring particular changes and definitions for the particular real problem. These proposed models are applied in real instances and their performances are compared, considering the optimality and lower/upper bounds generated with a commercial solver. With larger instances, the required computational time was a limiting factor to the solver to achieve optimal solutions.

The remainder of this study is organized as follows: Section 2 describes the railway problem decision process and its real context. Section 3 describes and formulates this specific train scheduling problem; the equations group, which allows the mathematical representation of the problem in a simpler way, with illustrative examples. These models are tested and analyzed in Section 4 in a real-life situation at the railways. Finally, Section 5 concludes and points out future directions for research.

2 Presentation of the problem

As already mentioned, the studied single-track line, has a heavy daily flow between the railways, whose importance can be ascribed to the constant ore traffic, general loads and passengers, besides fuel and service trains. The convoys that travel on this track have different shapes and sizes. Their travel is not always established, it can be changed daily, if necessary. The only fixed priority is set for the passengers trains, that travel daily (one in each direction). Their passage is preferential above any other kind of train (except in rare occasions). This studied single-track railway network is explained in Figure 1.

The studied single-track railway network is located in Minas Gerais Brazil between Contagem (Eldorado Station - EEL) and Barão de Cocais (Costa Lacerda yard - VCS). It belongs to the railways of VALE and FCA. They merge at the interchange point “I” highlighted in Figure 1. Considering the direction EEL to VCS, the railway traverses the cities (C1, C2 and C3), yards (VCS, P02, P03, P04, P05, P06, P07 and P08), stations (EEL, EHF and EGC) and mines (M1 and M2). In this study, these mines, stations and yards will be referred to as “railway points”. Starting the journey at EEL, the railway crosses the stations EHF and EGC. In the latter, there is a branch to Capitão Eduardo (C5), Corrego de Melo (C3) and Sabará (C4). Then, it goes through Corrego de Melo (C3), yards P02, P03, P04, P05, P06, P07 and P08 until it reaches the last yard (Costa Lacerda - VCS). The interchange point, “I”, between VALE and FCA is in the yard P07 as highlighted in Figure 1. Furthermore,
there are two mines at the railway line. The flow of minerals extracted from mine $M_1$ (Gongo Soco) is made by $P_05$, and from mine $M_2$ (Brucutu), by $P_07$ and $P_08$. Each of the yards $P_02$, $P_03$, $P_04$, $P_05$, $P_06$, $P_07$ and $P_08$ has a siding with an extension compatible with the size of the trains, i.e., if a train goes into a siding, the railway is released for the passage of another train. Finally, it must be emphasized that the yards of the extremities ($VCS$ and $EEL$) are venues of reception and maneuver for new train formations. In these yards the trains are formed only when there is expected release; thus, in this study, their capacity is not considered.

As mentioned, the yards have enough size to be used by a given train so that the main line is released for the passage of another train. However, in the present model, the sidings are not used, due to the planning rules of VALE and FCA, mainly due to: (i) sidings are mainly intended to trains in need of repair; (ii) a stopped train is very costly, due to the increased travelling time and the high fuel expenditures to turn the train on and off. Therefore, a train must not be interrupted after it initiates its travel.

Due to the criticality of the decision that the railways need to make (see Figure 1), and the intense flow between them, the administrators of the railways have daily meetings to define this tactical decision. The train schedule between the railways must be defined considering all points between Eldorado Station - $EEL$ (VALE) and Costa Lacerda yard - $VCS$ (FCA).

Other decisions (trains that do not cross the interchange point), as mentioned above, will be made later. Considering that the fundamental decision of the administrators is guided by the train flow between the railways, they decided to prioritize the schedule of the trains that cross the interchange point ($I$). Therefore, the railways aim to avoid train congestions. The tactical decision is guided by the priority assigned to each train, and also by the date when they are available to travel. This decision is complex as it relates the priority of each train and delay time.

In summary, the tactical decision making consists in defining the schedule of trains that will travel between the railways (crossing the interchange point) aiming to minimize their delay times after formation date, weighted by priorities rules and respecting the established constraints. These trains priorities, the weights for each train type, are set in numerical scale. Nowadays, the railways of VALE and FCA manage their train scheduling problem by daily meetings of the teams of decision
makers. These meetings are held in the morning and the teams decide the schedule plan, which is, as already mentioned, guided by the interchange point, for the next two days. On the first day of the planning horizon the existing plan is refined. On the second day, a new plan is defined. After this planning, other operations and other trains (example: trains that do not pass through interchange point) are also scheduled.

The practitioners solve the problem through tactical knowledge. These teams define train schedule based on the train priorities, weights, and available time. Then, the defined schedule is simulated by a support software system. It considers each train departure date in order to avoid overlaps. The software system also provides the delay time after the formation date of each train. This is calculated by the difference between its formation date and the date of its release from the departure station. Based on this information and priorities, the team can easily calculate the sum of delays, the average delay and the priority weighted delay. For the railways, the latter metric is considered the most important, as it simultaneously takes into account delay time and priority. However, it must be emphasized that there is no support tool capable of evaluating if the outcome schedule is the most efficient or if it is at least among the most efficient ones. Therefore, the railway software system can only aid in the comparison of the solutions generated by the teams.

The railways demand agility in the preparation of train schedules, considering all the details of the problem. Due to time constraints, this solution must be achieved in less than 24 hours before its operationalization. Therefore, the railways demand the development of a tactical decision support system. This system must be able to generate a schedule that minimizes the delay time of trains that cross the interchange point, considering the established priorities.

3 Modeling the problem

As already mentioned, in analysis of Tables 1 and 2, there is a wide variety of problem representations for train scheduling works, each one with its own peculiarities. Each specific representation reflects on how the mathematical models are defined. In this work, the railway problem is to define the schedule that minimizes the delay time, considering the established priorities for the trains that travel between the railways. In this study, an equation group is developed for preprocessing the problem data (see more details in section 3.3.2). These equations define the shortest time interval admissible between two distinct trains crossing the interchange point, ensuring that they never overlap. With this preprocessing, the railway problem can be simplified and represented by the single machine scheduling environment. Similar approaches can be viewed in Ghoseiri and Morshedsolouk (2006). Considerations about this mathematical representation will be exposed in the next section.

In this effort, the train scheduling problem is modeled based on the machine scheduling problem and solved by a commercial solver. In the literature review on train scheduling, most models use mathematical formulations with binary variables. However, some surveys compare the existing mathematical formulations for scheduling problems, among which we highlight: Queyranne et al. (1994) accomplish a synthesis of polyhedral approaches for single machine and parallel machine problems.
The formulations are classified based on the choice of decision variables: “Time-Indexed”, “Linear Ordering”, “Assignment and Positional Date” and “Completion Time”. Assessments are made for the lower and upper bound quality, obtained by the different formulations. Khowala et al. (2005) compare the computational performance of the same formulations for the single machine problem, aiming to minimize the weighted delay sum. The experiments show that the most suitable formulation relies on the processing time sum. Rocha et al. (2008) compares the performance of formulations based on “Completion Time” and “Assignment and Positional Date” variables with a Branch and Bound algorithm, in search for optimum solutions to parallel machine problems. The “Completion Time” formulation presented better results than the “Assignment and Positional Date” formulation.

Keha et al. (2009) compare the performance of the same MIP formulations of Queyranne et al. (1994) for different objective functions of the single machine scheduling problem. The paper concludes that these formulations depend on the problems characters, and the “Time-Indexed” formulation is recommended for problems with a small time horizon or with arrival date. Based on the nomenclature of Queyranne et al. (1994), Unlu and Mason (2010) define and compare the four formulations for parallel machine scheduling problems. They analyze the formulations in extensive problems and verify how tight the bottom bounds are. The work concludes that the “Time-Indexed” formulation achieves the solution in a greater number of instances and with tighter lower bounds. However, it takes a larger computational time. Lastly, Adamu and Adewumi (2014) present a review of single machine scheduling to minimize the weighted number of tardy jobs. This paper provides an extensive review of authors, methods and techniques used in this problem; moreover, the study points out possible directions for future research works.

Although differently organized, other remarkable works provide further information about scheduling models, including Blazewicz et al. (1991), Allahverdi et al. (1999), Allahverdi et al. (2008) and Pinedo (2008). Pessoa et al. (2010) presents an innovation in terms of mathematical formulation. Unlike the works presented earlier, a new formulation called “Arc-Time-Indexed” is submitted, which, according to the authors, is at least as good as the “Time-Indexed” formulation, with tight bounds in a great number of instances consulted in the literature. Finally, Nogueira et al. (2014) analyse the existent MIP formulations for single machine environment with sequence dependent setup time. This work concludes that, for problems with setup times, release dates and objective function that aim to minimize the weighted delay (called Tardiness), the MIPs formulations based on “Arc-Time-Indexed” variables and on “Binary” variables present results with better performance.

With all the previously mentioned information in the hands, the train scheduling problem will be formulated as the single machine environment in two distinct forms. The first MIP will be defined with “Arc-Time-Indexed” variables and based on the study conducted by Pessoa et al. (2010), while the last MIP model will be defined with “Binary” variables, based on Manne (1959). However, these proposed MIP formulations need to reflect a specific concept on how the variables and parameters are defined, thus requiring particular changes and new definitions.
3.1 Problem Statement

As explained, a mathematical model can be determined, which enables to repre-
sent, within certain limits, the discussed problem. Considering that this specific train
scheduling problem can be modeled as a machine scheduling problem environment,
some timely considerations and adaptations in this present case are necessary and are
described in this section.

Summarizing the previous definitions about the studied single-track railway, it
consists of a set of railway points (mines, stations and yards). Track segments are
defined as the connection between two points in the single-track railway. In this study,
these track segments interconnect the railway points in the single-track.

The train route is then defined as a set of railway points (mines and/or stations
and/or yards) through which the train must pass. Thus, there is a set of track segments
located between these points. The problem decision is to define train scheduling so
as to ensure that the trains will not overlap during their travel.

During the data collecting time, many kinds of trains were considered. Therefore,
in order to be included, they were separated into groups with similar formation and
average travelling time. It is important to highlight that: (i) a lot of trains have similar
travel time (even when they have different formation), once the maximum railway
speed is limited by track; (ii) for all of the trains, the speed varies according to the
travelling direction, considering that the route profile changes with the direction; (iii)
the travel time between two railway points varies due to the composition of distinct
trains or due to their inability to reach the speed limit. By grouping them, a reduced
number of distinct trains was obtained. Thus, in the planning horizon, there was an
average of around 30 types of different trains running in the studied railway.

Throughout the article, in the previous sections, we presented several considera-
tions and characteristics about the studied single-track railway. These considerations
and characteristics are fundamental to define the mathematical model. They are now
summarized as the main assumptions adopted: (i) the railway problem is to define the
schedule of the trains that will travel between them, crossing the interchange point,
without any kind of interruption; (ii) railways aim to minimize the trains delay times
weighted by priorities rules; (iii) a group of trains must travel through a set of mines
and/or stations and/or yards. As already mentioned, they are referred to as “railway
points” in this study; (iv) train departures and arrivals railway points, and the moment
at which they are formed, are known; (v) the travelling time between railway points
can be similar or different for distinct trains, but all of these data are known; (vi) a
set of track segments interconnects the railway points in the railway; (vii) the studied
railway has a single-track, hence, only one train passes through a track at each time;
(viii) in the studied line the trains travel in both directions; (ix) once the train starts
the travel, it must end it; (x) the studied line has sidings, but they will not be used
as possible route; (xi) the reception capacity in the railway extreme points, EEL and
VCS, is not considered; (xii) the delay time is defined as the deviation from the train
formation date, i.e., the time difference between the train release date from the depar-
ture station and its formation date, or the time difference between the date at which
the train crosses the interchange point and its planned time (formation date plus travel
time); (xiii) the weight priority is defined as the priority measure assigned for each type of train.

With the information already summarized, the sets and parameters that will be needed by the mathematical models can be defined. In a few words, the following notation describes the used sets and parameters in the mathematical formulations. The “sets”: (i) $H = \{1, 2, \ldots, h\}$: discretized planning time horizon with size $h$, this value will be defined with the mathematical model definitions (3.3); (ii) $J = \{1, 2, \ldots, n\}$: set of trains $j$; (iii) $K = \{1, 2, \ldots, k\}$: sequence of railway points (mines and/or stations and/or yards) interlinked by single-tracks. The index relative to each one increases from left to right, accordingly to the convention established by the railway. The “Parameters”: (i) $r_j$: train $j \in J$ formation date, which means, the date after which $j \in J$ is ready to start the travel ($r_j \in \mathbb{R}_+^*$); (ii) $a_j$: train $j \in J$ departure point ($a_j \in K$); (iii) $b_j$: train $j \in J$ arrival point ($b_j \in K$); (iv) $k_j$: interchange point $k_j \in K$, settled so that it can assume as value the departure point, the arrival one or any other between them, regardless of the route; (v) $L_j$: requested time interval so that the train $j \in J$ travels a distance equal to its own size, which is the requested time to evacuate the branch after reaching $b_j \in K$ ($L_j \in \mathbb{R}_+^*$); (vi) $p_{ij}$: is the asymmetric data matrix that represents the shortest admissible time interval between train $i \in J$ and its successor $j \in J$ travel through $k_j \in K$, so they never overlap. This value is calculated with the problem data and it will be defined after the data presentation ($p_{ij} \in \mathbb{R}_+^*$); (vii) $p'_{ij}$: train $j \in J$ travelling time from $a_j \in K$ to $k_j \in K$, travelling direction apart ($p'_{ij} \in \mathbb{R}_+^*$); (viii) $p^{\alpha,j}_{ij}$: train $i \in J$ travelling time from $k_j \in K$ to $b_j \in K$. Note that, if $i = j$, than $p^{\alpha,j}_{ij}$ is the train $j \in J$ requested travel time from $k_j \in K$ to $b_j \in K$ ($p^{\alpha,j}_{ij} \in \mathbb{R}_+^*$); (ix) $D_{ij}$: train $j \in J$ travelling time from $a_j \in K$ to $b_i \in K$ ($D_{ij} \in \mathbb{R}_+^*$); (x) $w_j$: train $j \in J$ weight priority ($w_j \in \mathbb{N}^*$).

Since the train route is defined by a set of railway points that must be crossed by the train, it can be seen as set of operations to be performed. The estimated travelling time of the train between departure and arrival points is known. Therefore, the parameter $p_{ij}$ can be established and thus the release time between two distinct trains, such that they never overlap, is known. Hence, in this study, each train travel between the departure and arrival points of the railway, can be seen as an operation. These operations must be scheduled at the interchange point between the railways. Thus, the interchange point is seen as a machine. In this case, this interchange between railways stays held by the train $i$ for any train $j$ for $p_{ij}$ unit times, i.e., the setup time in scheduling machines. Thus, no train can be released from its departure point if the interchange point stays held for a time longer than the travel time to there, $p'_{ij}$. Finally, the parameter $r_j$, train formation date, for the train $j$ is known and can be defined as the release date.

Based on these previous information and the collected data, one may specifically define the single machine scheduling problem that better represents the studied problem. The scheduling problems can be classified as the proposed assortment made by Graham et al. (1979) and also exposed in Blazewicz et al. (1991) and Pinedo (2008). This classification is described by a triplet $\alpha|\beta|\gamma$ of fields. The field $\alpha$ describes the machine environment, the field $\beta$ provides details of processing characteristics and constraints and the field $\gamma$ describes the objective to be minimized.
In this case, the specific train scheduling problem studied can be defined as \(1| r_j + p_j, d_j, p_{ij} \sum w_j \delta_j\), where 1 defines the single machine environment; \(r_j + p_j\) is similar to the job \(j \in J\) release date in scheduling problems; \(d_j\) is the job \(j \in J\) due date which is defined as \(r_j + p_j\) in our particular case; \(p_{ij}\) is similar to the machine setup time of \(i \in J\) to \(j \in J\); \(\delta_j\) is the job \(j \in J\) delay time; and \(w_j\) is the job \(j \in J\) weight priority. The last field \(\sum w_j \delta_j\) represents the objective function that minimizes the weighted delay sum. The problem is defined as NP-hard, as mentioned by Lenstra et al. (1977) and Lawler et al. (1993). The proposed mathematical models will be presented after the \(p_{ij}\) statement.

3.2 \(p_{ij}\) statement

The definition of the parameter \(p_{ij}\), as previously commented, simplifies train scheduling problems. The value of this parameter is not previously available. Thus, it must be calculated based on other available data, before the mathematical model resolution. The estimated train travelling times between distinct railway points and its departure and arrival railway points are known. Therefore, the development of equations that determinate the \(p_{ij}\) value is done according to these data. After defining \(p_{ij}\), it is settled that only one train can fill a space at time. Thus, even if \(p_{ij}\) attends the time interval within which the distinct trains pass through the interchange point, its formulation must ensure that they will not meet or overlap in any point of the railway.

The equations to be defined refer to the moment when the trains go through the interchange point. However, they can be applied to any point of the line, including the departure ones. These equations were deducted from the analysis of trains travelling in the same or in opposite directions. Each one of these equations will be presented as a problem definition and illustrative examples will be used to validate them.

Let \(i \in J\) and \(j \in J\) be two trains travelling on the same single-track line. The train \(i\) leaves from railway point \(a_i \in K\) to railway point \(b_i \in K\) and the train \(j\) leaves from railway point \(a_j \in K\) to railway point \(b_j \in K\). The interchange point is denominated \(k' \in K\). The Figures 2a and 2b, represent, respectively, the trains \(i\) and \(j\) travelling in the same direction, definition 1, and in opposite directions, definition 2.

3.2.1 \(p_{ij}\) statement - Trains Travelling in the Same Direction

**Definition 1** For trains travelling in the same direction, the shortest admissible time interval between the first train \(i \in J\) and the second train \(j \in J\) travelling through \(k' \in K\), so that they never overlap, is defined as:

\[
p_{ij} = \max(L_i, p_{ib}^h + L_j - p_{jb}^h) \quad \forall i, j \in J, \quad i \neq j,
\]

whereby \(b \in K\) refers the nearest arrival point (\(b_i \in K\) or \(b_j \in K\)) of the interchange point \(k' \in K\).

If the train \(i \in J\) goes through the interchange point before than the train \(j \in J\) and train \(i\) has a “higher speed” than train \(j\), two situations may occur: (i) \(b_i\) is closer to \(k'\) than \(b_j\); (ii) \(b_j\) is closer to \(k'\) than \(b_i\)
Regardless of the situation, the train $i$ has a “higher speed” than train $j$, thus the train $j$ will never reach the other train. As $p_{ij}$ is the shortest admissible time interval between two trains, $i, j \in J$, in the interchange point, the train $j$ should wait only during the time that the train $i$ spends to pass all its extension through the interchange point. Thus, $p_{ij} = L_i$.

If the train $i \in J$ goes through the interchange point before than the train $j \in J$ and train $i$ has a speed “lower or equal” to train $j$, then the same situations described above may occur. In the situation (i), the shortest admissible time interval between two trains at the interchange point must be large enough so that the train $i$ arrives at its arrival point before being reached by train $j$. Therefore, besides considering $L_i$, it must also be considered the difference between the travel times, $p''_{ij} - p''_{ij}$. Moreover, in the situation (ii), besides considering $L_i$, it must also be considered the difference between the travel times, $p''_{ij} - p''_{ij}$.

If train $j \in J$ goes through the interchange point before than the train $i \in J$ all situations previously described are valid and present the same logic and conclusions. Therefore, the $p_{ij}$ value in the general Equation 1 is valid for all mentioned situations for trains in the same direction, regardless of the departure and arrival points and independently of train speeds.

### 3.2.2 $p_{ij}$ statement - Trains Travelling in Opposite Directions

**Definition 2** For trains travelling in opposite directions, the shortest admissible time interval between the first train $i \in J$ and the second train $j \in J$ travelling through $k' \in K$, so that they never overlap, is defined as:

$$p_{ij} = L_1 + p''_j + p''_{ij} - D_{jc} \quad \forall i, j \in J, \; i \neq j,$$

where $1 \in J$ refers to the first train that goes through $k' \in K$ and $2 \in J$ is its posterior index. The $c \in K$ index refers to the railway point closest to $k' \in K$ (between $a_2 \in K$ and $b_1 \in K$).
Let \( i \in J \) and \( j \in J \) be two trains travelling in opposite directions. All possible directions for each train are: (i) The railway point \( a_i \) is further from \( k' \) than \( b_j \) and \( b_j \) is nearer \( k' \) than \( a_i \). The train \( i \) goes through \( k' \) before the train \( j \); (ii) The railway point \( a_i \) is further from \( k' \) than \( b_j \) and \( b_j \) is nearer \( k' \) than \( a_i \). The train \( j \) goes through \( k' \) before the train \( i \); (iii) The railway point \( a_i \) is further from \( k' \) than \( b_j \) and \( a_i \) is nearer \( k' \) than \( b_j \). The train \( i \) goes through \( k' \) before the train \( j \); (iv) The railway point \( a_i \) is further from \( k' \) than \( b_j \) and \( a_i \) is nearer to \( k' \) than \( b_j \). The train \( j \) goes through \( k' \) before the train \( i \).

In the situation (i), to ensure that the trains will never overlap, the time that the train \( i \in J \) needs to release the railway, \( t_i \) (\( r_i + \delta_i + p_i^t + p_i^m \)), must be shorter or equal to the time that the train \( j \in J \) takes to reach this same railway point (\( r_j + \delta_j + D_{ji} \)). Therefore,

\[
 r_i + \delta_i + p_i^t + p_i^m \leq r_j + \delta_j + D_{ji}.
\]

As the time when any train \( i \in J \) goes through the interchange point is \( r_i + \delta_i + p_i^t \), it can be defined as \( t_i \). The equation (3) can be rewritten as:

\[
 t_i + p_i^m \leq t_j - p_j^t + D_{ji} \text{ or } t_j - t_i \geq L_i + p_i^m + p_j^t - D_{ji}.
\]

Therefore, as the smallest value for \( t_j - t_i \) is \( p_{ij} \), it is defined as \( L_i + p_i^m + p_j^t - D_{ji} \).

In the situation (ii), if the steps of the former case are adopted, it is obtained \( t_i - t_j \geq L_j + p_j^a + p_j^t - D_{ij} \) and thus \( p_{ij} = L_j + p_j^a + p_j^t - D_{ij} \). For the situation (iii), following the same logic, \( t_j - t_i \geq L_i + p_i^a + p_i^t \) and thus \( p_{ji} = L_i + p_i^a + p_i^t \). Finally, in the situation (iv) \( t_i - t_j \geq L_j + p_j^a + p_j^t - D_{ij} \) and \( p_{ji} = L_j + p_j^a + p_j^t - D_{ij} \) are obtained. Therefore, it is possible observe that the \( p_{ij} \) value in the general Equation (2) is valid for all mentioned situations for trains in opposite directions. Furthermore, the \( p_{ij} \) value does not depend on the relative positions of the departure and arrival points and train speeds.

### 3.3 Mathematical Formulation

The railways decision problem is to define the schedule of the trains that will travel between the railways and cross the interchange point, without any kind of interruption. As already described, this specific problem can be represented by the single machine scheduling problem environment. The problem is then defined as \( 1|p_j^t,d_j,p_{ij}|\sum w_j \delta_j \) which aims to minimize the weighted delay sum with release dates \( (r_j + p_j^t) \) and time dependent on the sequence \( (p_{ij}) \). As already mentioned, this objective is important because train priority and delay are considered.

such as Mladenović and Čangalović (2007), Lee and Chen (2009) and Shafia et al. (2012).

As mentioned, these proposed mathematical models are based on Pessoa et al. (2010) for “Arc-Time-Indexed” variables and on Manne (1959) for “Binary” variables. However, in the proposed mathematical models, the factor time dependent of sequence is not considered. Therefore, in these proposed mathematical models, is necessary to change how variables and parameters are defined, and particular changes and definitions are also required for the particular real problem.

3.3.1 Arc-Time-Indexed Formulation

This formulation is based on Tanaka and Araki (2008), Sourd (2009) and Pessoa et al. (2010). It is important to mention the works of Fox (1973) and Fox et al. (1980) on the time-dependent travelling salesman problem (TDTSP) and the adaptation for single machine scheduling problems by Bigras et al. (2008). The model with “Arc-Time-Indexed” variables uses a discretized time horizon $H$. Its value size $h$ is based on Unlu and Mason (2010), which is defined as:

$$h = \max_{j \in J'} (r_j + p_j^t) + \sum_{j \in J'} \max_{i \in J'} (p_{ji}),$$

where the set $J'$ is defined as $J' = J \cup \{0\}$. In this paper all time data are multiple of 5 minutes. Thus the time horizon is discretized in 5-minute intervals. Therefore, the total $h$ in minutes can be divided by 5. Furthermore, the set “$\{0\}$” is named as fictitious train, and refers to the beginning and the end of the train schedule at $k' \in K$ (the values are null in every fictitious train parameter).

The binary Arc-Time-Indexed variable ($x_{ij}$) defines that if the train $j \in J'$ goes through $k' \in K$ during the time $t \in H$ with the train $i \in J'$ immediately before $j \in J'$ its value is 1, otherwise 0. However, if there is no train $j \in J'$, $j \neq i$ to succeed $i \in J'$ immediately after its $p_{ij}$ then the interchange point will be idle. The $k' \in K$ idleness is determined when there is a schedule for a train $i \in J'$ to itself, $x_{ii}^t = 1$, with $p_{ii} = 1$ for all $i \in J$. In summary, the variable is defined as (6):

$$x_{ij}^t \in \{0, 1\} \ \forall i, j \in J' \text{ with } i \neq 0 \text{ or } j \neq 0, \quad t \in \{\phi, \ldots, h\} \subset H,$$

where $\phi = \max\{r_i + p_i^t + p_{ij}, r_j + p_j^t\}$.

If $x_{ij}^t = 1$, one may admit that the train $j \in J'$ is the first to be scheduled in the interchange point at date $a \in H$. When $x_{ij}^t = 1$, one may admit that the train $j \in J'$ was the last to be scheduled in the interchange point at date $b \in H$. As mentioned, considering that the train $i \in J'$ may pass through $k' \in K$ and that there is idleness between the passage of $i \in J'$ and the next train $j \in J'$, $j \neq i$, one may define that, scheduling the train from $i \in J'$ to $i \in J'$ means idleness. In this case, the interchange point will be idle from $t \in H$ to $t + p_{ii} \in H$ ($p_{ii}$ is defined as 1) and the last train was $i \in J'$ ($x_{ii}^t = 1$).
The constraint set (7) establishes that all the trains \( j \in J' \) must go through the interchange point \( k' \in K \) only once, in time horizon \( H \).

\[
\sum_{i \in J'} \sum_{t \in \phi} x_{ij}^t = 1 \quad \forall j \in J',
\]

where \( \phi = \max\{r_i + p_i^j + p_{ij}, r_j + p_j^i\} \).

The constraint set (8) designates the trains schedule in the interchange point. In these constraints the train \( j \in J' \) can be in the interchange point immediately after the train \( i \in J \), if, and only if, the date that the train \( j \) crosses this point is equal to: (i) the date \( t \) that the train \( i \) crosses this point \((x_{ij}^t = 1\) with \( l \in J' \) and \( l \neq i \neq j \) plus \( p_{ij} \)) or; (ii) the date \( t \) when the interchange point was idle with train \( i \), its last scheduled train \((x_{ij}^t = 1, \) plus \( p_{ij} \)).

These conditions are defined by \( x_{ij}^t - x_{ij}^{t+p_{ij}} = 0 \) or \( x_{ij}^t = x_{ij}^{t+p_{ij}} \) for a given \( i \in J \), \( j, l \in J', l \neq j \) and \( t \in H \). Therefore, \( x_{ij}^t = 1 \) if, and only if, \( x_{ij}^{t+p_{ij}} = 1 \).

\[
\sum_{i \in J'} \sum_{t \in \phi} x_{ij}^t - \sum_{j \in J'} \sum_{t \in \phi} x_{ij}^{t+p_{ij}} = 0 \quad \forall i \in J, \ t \in \{r_i + p_i^j, \ldots, h\} \subset H.
\]

The objective function is minimize \( \sum_j w_j \delta_j \). The delay time \((\delta_j)\) of train \( j \in J' \) can be obtained by the difference between the date \( t \) that the train \( j \in J' \) crosses \( k' \in K \) \((t \in H) \) and its due date \((r_j + p_j^i)\). The binary variable \( x_{ij}^t \), as already defined, is 1 if the train \( j \in J' \) goes through \( k' \in K \) during the time \( t \in H \) and 0 otherwise. Thus, \( \delta_j \) can be defined by \( x_{ij}^t(t - r_j - p_j^i) \) for a given train \( j \in J' \) on a given time period \( t \in H \), which will be \( t - r_j - p_j^i \) if the train \( j \in J' \) goes through \( k' \in K \) during the time \( t \in H \) and 0 otherwise. Therefore, the equation (9) defines the objective function that minimizes the weighted delay sum on trains release.

\[
\text{minimize } \sum_j \sum_{i \in J'} \sum_{t \in \phi} w_j \left| x_{ij}^t(t - r_j - p_j^i) \right|
\]

where \( \phi = \max\{r_i + p_i^j + p_{ij}, r_j + p_j^i\} \).

All of the “Time-Indexed” models present some known limitations. The first limitation of the presented model is treating the time, continuous by definition, discretized. In this study, the time is discretized in 5 minutes. Thus, all decision variables only present values between 5 minutes. This was possible because all parameter time values are multiple of 5 minutes in this study.

If any time parameter value requires greater precision, then the time will have to be discretized in smaller periods and the time horizon length will be extended. Therefore, the “Time-Indexed” models have planning horizon size, \( h \), affected by the parameter length value and by its configuration. An illustrative example: if there are \( n - 1 \) trains with time parameters multiple of 5, and there is only one train with this parameter with value 12, then the time horizon must be discretized in values
$H = \{0, 5, 10, 12, 15, 17, 20, 22, 25, 27, \ldots, h\}$, so that all the time possibilities are considered.

Besides that, the variables and the described constraint sets, it can be said that the formulation size is $O(n^2h)$ columns (variables) and $O(nh)$ rows (constraint sets), with $(h \gg n)$. Therefore, for the same number of trains, if the time horizon increases in one unit then the variables number increases by $n^2$ and the constraints number by $n$. Thereby, since $h \gg n$ the number of variables and constraints increases rapidly by the $h$ horizon increase. Therefore, as mentioned by Pessoa et al. (2010), it cannot be loaded and solved by a commercial solver in a reasonable computational time due to its pseudo-polynomially large number of variables. Moreover, Pessoa et al. (2010) defines that the “Time-Indexed” bound may still leave a significant duality gap and all exact algorithms based on it sometimes need to explore large enumeration trees. Thus, in their experiments, results from these approaches are only promising for small instances of the scheduling problems.

### 3.3.2 Binary Formulation

This formulation was originally proposed by Manne (1959) for the Job-shop scheduling problem $J||C_{\text{max}}$. The adapted formulation is based on previous works by Manne (1959); Queyranne (1993); Queyranne et al. (1994); Ballicu et al. (2002); Khowala et al. (2005); Eren and Guner (2006); Balas et al. (2008); Rocha et al. (2008); Keha et al. (2009) and Unlu and Mason (2010).

In this formulation, the binary variables $\gamma_{ij}$ (11), are equal to 1 if train $i \in J$ crosses the interchange point $k' \in K$ before train $j \in J$ and equal to 0 otherwise. When $\gamma_{ij} = 1$, the train $i$ is not necessarily crossing the interchange point immediately before train $j$. Moreover, the integer decision variable $C_j$ (10), is necessary. This variable defines the date in which the train $j \in J$ crosses the interchange point. These variables are summarized below:

$$C_j \geq 0 \ \forall j \in J,$$

$$\gamma_{ij} \in \{0, 1\} \ \forall i, j \in J, i \neq j.$$

Furthermore, in this formulation, the big-$M$ constant is necessary. This formulation is going to be directly tested with a commercial solver. Therefore, the big-$M$ value selection is essential for the mathematical model performance. The big-$M$ constant is defined with its value size based on Unlu and Mason (2010) and it is defined as:

$$M = \max_{j \in J}(r_j + p'_j) + \sum_{j \in J} \max_{i \in J}(p_{ji}) + \max_{j \in J}(p_{ij}).$$

Note that instead of a global “big-$M$”, an individual “big-$M_{ij}$” may be defined for each inequality in (14), satisfying

$$M_{ij} = M - r_j - p'_j + p_{ij},$$
so that the constraint set (16) is respected.

The first constraint set, (14), ensures that the date at which the train \( j \in J \) crosses the interchange point happens only after the date that the train \( i \in J \) crosses this point plus the release time between \( i \) and \( j \), \( p_{ij} \).

\[
C_j \geq C_i + p_{ij} - M_{ij}(1 - \gamma_{ij}) \quad \forall i, j \in J, i \neq j. \tag{14}
\]

The second constraint set, (15), imposes that either the train \( i \) crosses the interchange point before than the train \( j \) or otherwise.

\[
\gamma_{ij} + \gamma_{ji} = 1 \quad \forall i, j \in J, i < j. \tag{15}
\]

The third constraint set, (16), ensures that the date that the train \( j \) crosses the interchange point is later than or equal to the train formation date plus its travelling time to the interchange point.

\[
C_j \geq r_j + p_j' \quad \forall j \in J. \tag{16}
\]

Similarly to the “Arc-Time-Index” formulation, the objective function is minimize \( \sum_j w_j \delta_j \). Likewise, the delay time \( (\delta_j) \) of the train \( j \in J' \) is obtained by the difference between the date \( t \) that the train \( j \in J' \) crosses \( k' \in K \) \( (t \in H) \) and the planned date \( (r_j + p_j') \). As the integer variable \( C_j \) defines the date that the train \( j \) crosses the interchange point, \( \delta_j \) can be defined by \( C_j - r_j - p_j' \). Therefore, the equation (17) defines the objective function that minimizes the weighted delay sum on trains release.

\[
\text{minimize} \sum_{i \in J} w_j(C_j - r_j - p_j'). \tag{17}
\]

In the “Binary” formulation, it can be said, about the variables and the described constraint sets, that the formulation size is \( O(n^2) \) columns (variables) and \( O(n^2) \) rows (constraint sets). Therefore, as \( h \gg n \) the “Binary” formulation is smaller than the “Arc-Time-Indexed” formulation. Thus, in the “Binary” formulation, its LP relaxation can be solved faster and a larger number of nodes can be explored in a fixed amount of time. However, as mentioned in Keha et al. (2009); Unlu and Mason (2010); Nogueira et al. (2014), the initial gap in B&B algorithms is higher than in “Time-Indexed” formulations because the LP relaxation gap is not as tight as the “Time-Indexed” formulations.

4 Computational Results

An extensive computational experiment is performed to identify the strength and the weaknesses of each proposed formulation as a support decision tool. The practical case requires an average schedule of 30 trains by the planning team. To analyze the model validity as a solution tool, its performance will be tested in a typical planning horizon (an average of 30 trains, ranging from 15 to 60). Furthermore, in order to better highlight the limitations of this model, other parameters will be varied: time discretization, density of traffic in a prediction horizon and the infrastructure layout (setting of the interchange point). A specific benchmark including these different features and characteristics was created for this purpose.
4.1 Data

Eight different classes of instances are created based on real instances. All parameters of the instances are randomly generated from a probability distribution and their minimal and maximal values are based on specific scale parameters. The instance classes and their scale parameters are listed in Table 3.

Table 3: Distribution values of the instances

<table>
<thead>
<tr>
<th>Input data</th>
<th>Distribution value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Infrastructure Layout Position ($k' \in K$)</td>
<td>Left, Right</td>
</tr>
<tr>
<td>Train Travelling Direction ($D_t$)</td>
<td>Be(Right, 0.65)</td>
</tr>
<tr>
<td>Train Travelling Time until Interchange Point in hours ($p'_j$)</td>
<td>$U(1, \alpha_2)P_f$</td>
</tr>
<tr>
<td>Train Formation Date in hours ($r_j$)</td>
<td>$U(1, \alpha_1)$</td>
</tr>
<tr>
<td>Priority ($w_j$)</td>
<td>$U(1, n - 2)$</td>
</tr>
</tbody>
</table>

The “Infrastructure Layout Position” defines the setting of the interchange point. The studied railways have an interchange point in the right position. However, the effects of the left position are studied. In this context is defined the position factor $P_f$. The position factor defines a factor for the travelling time until reaching the interchange point. If the train departure station and Infrastructure Layout Position are in the same side of the railway, then the $P_f$ is 0.5, otherwise is 1.5. “Train Travelling Direction” defines Bernoulli’s discrete distribution based on real data. Finally, the trains priorities distribution is defined with uniform values from 1 to $n - 2$. The last two trains are defined as passengers trains with weight 1000, (priority above any other train).

The train travels from right to left (right departure) in the railway with the probability value equals to 0.65 and its value is 0.35 if in the opposite direction. The scale parameters $\alpha_1$ and $\alpha_2$ define the distribution scenario of “Travelling Time” and “Formation date” respectively. The parameter $\alpha_1 \in \{1, 2\}$ modifies the travelling time extent and therefore the time horizon length; the $\alpha_2 \in \{1, 5\}$ defines the congestion level. Other non mentioned model parameters are maintained in the standard railway values. These values are omitted due the confidentiality of the information. In each condition (1 to 4) there is a change in one scale parameter.

These conditions are generated for the interchange point on the right side and on the left side, totaling eight instance classes. The created conditions are, namely: “Condition 1” - all scale parameters have minimum values; “Condition 2” - $\alpha_1$ has the maximum value (2) and other scale parameters have minimum values; “Condition 3” - $\alpha_2$ has the maximum value (6) and other scale parameters have minimum values; “Condition 4” - all scale parameters have maximum values.

Each class and each side of the interchange point, right and left, present special characteristics. “Condition 1” is our base railway system. “Condition 2” considers a long planning horizon affected by a longer travel time. “Condition 3” defines a system with high congestion level, reduces its formation date values for the beginning of time horizon. “Condition 4” determines a scheduling system with emphasized conditions. The later defines a complex system and presents long planning horizons and a signif-
icant congestion level. For each class, 10 independent instances are considered with size $n \in \{15, 20, 25, 30, 35, 40, 45, 50, 55, 60\}$. Thus, 800 instances are randomly and independently generated. The “Binary formulation” requires a $p_{ij}$ data that satisfy the triangle inequality. All instances are slightly modified to satisfy this ($p_{ij} \leq p_{il} + p_{lj}$, where $i, j$ and $l \in J$ and $i \neq j \neq k$).

4.2 Results

The mathematical formulations were modeled and solved using AMPL and CPLEX 12.6 with default settings. The experiments were run on a Linux Debian Edition with a single 1.8 GHz processor and 8 GB memory. The runs were concluded after one hour of CPU time (3600 seconds). To analyse the differences between the formulations it was carried out a comparison of the optimality gap within 3600 seconds, the linear programming relaxation gap, CPU times and their sizes. The linear programming relaxation gap is defined as the relative difference between the best integer solution found for each instance and the LP (linear programming) relaxation value. The average results of the experiments are presented in Table 5.

Table 5 shows the average GAP results for MIP and LP relaxation for “Arc-Time-Indexed” and “Binary” formulations in all instance classes proposed. It must be highlighted that in several occasions these formulations were unable to load the whole problem into the solver. In those cases the GAP was defined as 100% and its computational time was defined as 3600 seconds.

A small real instance, provided by the railways, is used to validate the result obtained by the mathematical model, Table 4. The instance provided by them was carefully obtained to reflect the single-track railway under normal operation conditions. The instances are described as a set of 12 trains. This set is used to compare the solution provided by railways and by the proposed mathematical models. The “Solution of the Railway Decision Makers” is initially defined by the schedule of the trains in the interchange point, after that the delay of each train is calculated, with the support software, and then, the weighted delay is calculated. The “Model Solutions” are directly obtained by both mathematical models. The weight (priorities) data for each train type ranges between 1 and 10, except for passengers trains. Their values are defined as 1000, because they have priority above any other type.

Railway planning decision is based on priority preference, weights, formation date and the search of a viable solution. The comparison of the solutions obtained in this instance, the solution of the railways and the mathematical model solutions, show that the mathematical models present a delay sum that is about 45% of the value obtained by the railway solution. Furthermore, the objective function value of the mathematical models (minimize $\sum \delta_j$) is about 71% of that obtained by the decision planning team. When the average delay by train is compared, the mathematical models present a time of 2.9 hours while the planning team solution presents 6.5 hours.

The analysis of the LP relaxation, shown in Table 5, reveals that the “Arc-Time-Indexed” formulation generally presents lower GAPs with higher CPU time values every time that the problem can be loaded into the solver, due to its size. On the
Table 4: Results for Validating the Mathematical Model Solution with Small Real Instances. “Planned (hr)” indicates the time established by the railways, when each train will cross the interchange point (train formation date plus travel time from departure station to interchange point); and “Weight” is the weight priority defined by the team of decision makers. “Delay (hr)” is the delay time defined for each train as the difference between the date at which the train crosses the interchange point and the time planned by the team of decision makers (formation date plus travel time) and by the mathematical model solutions; and “Weighted” means the delays multiplied by its weight. The solution provided by the FCA and VALE planning team is denominated “Solution of the Railway Decision Makers” and the solutions obtained by the mathematical models are denominated “Model Solutions”

<table>
<thead>
<tr>
<th>Trains</th>
<th>Railways Information</th>
<th>Solution of the Railway Decision Makers</th>
<th>Model Solutions</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Planned (hr)</td>
<td>Weight</td>
<td>Delay (hr)</td>
</tr>
<tr>
<td>T1</td>
<td>9.5</td>
<td>1000.0</td>
<td>0.0</td>
</tr>
<tr>
<td>T2</td>
<td>10.0</td>
<td>1.0</td>
<td>4.0</td>
</tr>
<tr>
<td>T3</td>
<td>14.0</td>
<td>3.0</td>
<td>0.0</td>
</tr>
<tr>
<td>T4</td>
<td>15.5</td>
<td>3.0</td>
<td>1.0</td>
</tr>
<tr>
<td>T5</td>
<td>17.5</td>
<td>1.0</td>
<td>10.0</td>
</tr>
<tr>
<td>T6</td>
<td>18.5</td>
<td>5.0</td>
<td>0.5</td>
</tr>
<tr>
<td>T7</td>
<td>19.0</td>
<td>3.0</td>
<td>5.3</td>
</tr>
<tr>
<td>T8</td>
<td>19.5</td>
<td>3.0</td>
<td>6.5</td>
</tr>
<tr>
<td>T9</td>
<td>21.0</td>
<td>2.0</td>
<td>12.5</td>
</tr>
<tr>
<td>T10</td>
<td>21.5</td>
<td>1.0</td>
<td>14.5</td>
</tr>
<tr>
<td>T11</td>
<td>23.0</td>
<td>2.0</td>
<td>15.5</td>
</tr>
<tr>
<td>T12</td>
<td>23.5</td>
<td>3.0</td>
<td>17.8</td>
</tr>
</tbody>
</table>

| Sum of Delay | 77.5 | 34.5 |

Objective Function - $\sum w_j \delta_j = 168.5$ (120.1)

other hand, the “Binary” formulation can load all instances, but it has lower computational time values and it produces poorer lower bounds. The “Arc-Time-Indexed” formulation presents better GAP results for the instance classes 1, 2 and 5, and poorer results for classes with considerable congestion level, 3, 4, 7, 8. While the “Binary” formulation presents poorer results in all instance classes.

The analysis of the average results for the MIP formulations, Table 5, show that all formulations have difficulty as the number of trains increases. It is possible to notice that the “Arc-Time-Indexed” formulation managed to find the optimum solution for some instances (small GAP value), but as the number of variables and constraints increases, the problems become rapidly unmanageable by the commercial solver. Nevertheless, the “Binary” formulation can generate viable solutions for all instances. As it can be seen, the “Arc-Time-Indexed” formulation is able to solve instances of up to 40 trains, depending on the class. This formulation presents better GAP results for instance classes 1, 5 and 6, and poorer results for classes 7 and 8, considerable congestion level and interchange point in the left side. The “Binary” formulation presents poorer results in instance classes with considerable congestion level (3, 4, 7, 8).

The mathematical techniques proposed, MIP models, are able to solve the trains scheduling problem in some variations of the operation days. The “Arc-Time-Indexed” formulation, although not able to optimally solve some instance groups, present a small GAP when it can load the instances in the commercial software system. Moreover, its MIP relaxation has solutions with a smaller GAP, when it load the problem in the software with computational shorter time, compared to its MIP problem. In the “Binary” formulation, all instance classes can be loaded, but these instances, generally present a considerable GAP. Furthermore, its MIP relaxation has poorer solutions, but its relaxation can be solved in a few seconds.
### Table 5: Average GAP Results for Mathematical Models Proposed for All Classes in All Sizes.

<table>
<thead>
<tr>
<th>Class Number</th>
<th>LP Relaxation</th>
<th>MIP Problem</th>
<th>SD(Gap)</th>
<th>SD(Time)</th>
</tr>
</thead>
<tbody>
<tr>
<td>8</td>
<td>50</td>
<td>0.03</td>
<td>100.00%</td>
<td>0.00%</td>
</tr>
<tr>
<td>3</td>
<td>45</td>
<td>0.02</td>
<td>100.00%</td>
<td>0.00%</td>
</tr>
</tbody>
</table>

**Average** GAP = 0.02, 0.02, 100.00% 0.00%

**Std Dev** SD(Gap) = 0.01, 0.01 0.00% 0.00%

**Std Dev** SD(Time) = 0.01, 0.01 0.00% 0.00%

**Average** SD(Gap) = 0.01, 0.01 0.00% 0.00%

**Average** SD(Time) = 0.01, 0.01 0.00% 0.00%
5 Concluding remarks and further research

The train scheduling problem investigated is characterized by a single-track line, an interchange point between railways and trains travelling in both directions. This study aimed to find a schedule that minimizes train release delay, given the priorities/weights established. Furthermore, as already mentioned, the railways (VALE and FCA) manage this problem based on tacit knowledge. There is no support tool capable of evaluating if the obtained solution is the best.

In this effort, a set of equations was presented. These equations allow determining the smallest admissible time interval between two trains, also ensuring that they will not overlap. Supported by these equations, the problem was defined as a single machine scheduling problem. This problem was mathematically modeled by two formulations and solved by the CPLEX software system. The experiments realized were based on real data and some variations were presented. The mathematical models were capable of generating good results, at a reasonable time, to support decision-making in some instance classes. However, when the number of trains increased or the problems became more complex (instance classes) the mathematical formulations were unable to solve the problem in the commercial solver. Nevertheless, the “Binary” formulation has shown viable solutions in all instances. This formulation showed to be a viable alternative for the railway problem.

However, as demonstrated by instance class results obtained by the “Binary” and “Arc-Time-Indexed” models, the railway scenarios affect train scheduling complexity factors, such as the distance of each train travel, a large quantity of trains with different priorities (weights), the density of traffic in a prediction horizon and the infrastructure layout (setting of the interchange point). Moreover, time length of the horizon only affects the “Arc-Time-Indexed” formulation. All formulations present difficulty to solve the instance class with considerable congestion level. In addition, a slight performance difference was observed with the change of the side of the interchange point.

This study was affected by the fact that, due to the confidentiality of the company’s information, besides the instances used to the validation, no data from applied solutions were available. It prevented the precise comparison of the method proposed by this effort and the one adopted by the railway carriers through the information related to reduced delay and/or financial impact.

The limitations of this effort, which should be addressed in the future, are the determinations of operational safety margins, imposed by regulation, legislation or by railway criterion. For instance, operational limits of the trains speed at the rail or the minimum time interval between two trains at the interchange station. This regulation can affect the generated solution, perhaps making it unfeasible, due to time increase of the passage of two distinct trains in the interchange station. It would also be interesting to measure the insertion of an impact on the parameters, such as a penalty for the occurrence of events external to the context that cause delay, including rail accidents or meteorological factors.

Hybrid heuristics are an important perspective for future research and should be studied for better results with low computational times (see surveys Alba (2005), Blum and Roli (2008), Blum et al. (2011), Jourdan et al. (2009), Puchinger and Raidl
These hybrid heuristics can be composed by the relax-and-fix with other metaheuristic, such as a local search procedure. Furthermore, the train scheduling problem can be considered in order to minimize fuel consumption. When it is required to wait for release, each train expenditure can be accounted, and it could be highlighted how train fuel costs are sizable. This is a remarkable study for railways, since fuel is one of the greatest railways expenditures.

References


