Coercive polynomials and their Newton polytopes

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Abstract

Many interesting properties of polynomials are closely related to the geometry of their Newton polytopes. In this article we analyze the coercivity on $\mathbb{R}^n$ of multivariate polynomials $f \in \mathbb{R}[x]$ in terms of their Newton polytopes. In fact, we introduce the broad class of so-called gem regular polynomials and characterize their coercivity via conditions imposed on the vertex set of their Newton polytopes. These conditions solely contain information about the geometry of the vertex set of the Newton polytope, as well as sign conditions on the corresponding polynomial coefficients. For all other polynomials, the so-called gem irregular polynomials, we introduce sufficient conditions for coercivity based on those from the regular case. For some special cases of gem irregular polynomials we establish necessary conditions for coercivity, too. Using our techniques, the problem of deciding the coercivity of a polynomial can be reduced to the analysis of its Newton polytope. We relate our results to the context of the polynomial optimization theory and the existing literature therein, and we illustrate our results with several examples.

Keywords: Newton polytope, coercivity, polynomial optimization, non-compact semi-algebraic sets.


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1 Introduction

It is an interesting question in polynomial optimization theory whether a given multivariate polynomial \( f \) attains its infimum on \( \mathbb{R}^n \), or on some non-compact basic semi-algebraic set \( S \subseteq \mathbb{R}^n \). In fact, our subsequent studies are motivated by the following statement from [17, Sec. 7] which is also cited in [20, 22]:

‘This paper proposes a method for minimizing a multivariate polynomial \( f(x) \) over its gradient variety. We assume that the infimum \( f^* \) is attained. This assumption is non-trivial, and we do not address the (important and difficult) question of how to verify that a given polynomial \( f(x) \) has this property.’

Coercivity of a polynomial \( f \) on \( \mathbb{R}^n \) is a sufficient condition for \( f \) having this property. It is, thus, an interesting problem how to verify or disprove that a given polynomial \( f \) is coercive on \( \mathbb{R}^n \). This is the topic of the present article.

For \( f \in \mathbb{R}[x] = \mathbb{R}[x_1, \ldots, x_n] \), the ring of real polynomials in \( n \) variables, we write \( f(x) = \sum_{\alpha \in A_f} f_{\alpha} x^\alpha \) with \( A_f \subseteq \mathbb{N}_0^n \), \( f_{\alpha} \in \mathbb{R} \) for \( \alpha \in A_f \), and \( x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n} \) for \( \alpha \in \mathbb{N}_0^n \). We will assume that the set \( A_f \) is chosen minimally in the sense that \( A_f = \{ \alpha \in \mathbb{N}_0^n | f_{\alpha} \neq 0 \} \) holds. The degree of \( f \) is defined as \( \deg(f) = \max_{\alpha \in A_f} |\alpha| \) with \( |\alpha| = \sum_{i=1}^n \alpha_i \).

The function \( f \) is called coercive on \( \mathbb{R}^n \) if \( f(x) \to +\infty \) holds whenever \( \|x\| \to +\infty \), where \( \| \cdot \| \) denotes some norm on \( \mathbb{R}^n \). Since \( f \in \mathbb{R}[x] \) is (lower semi-) continuous on \( \mathbb{R}^n \), coercivity implies the existence of a globally minimal point of \( f \) on \( \mathbb{R}^n \) (as well as the existence of a globally minimal point of \( f \) on any nonempty basic closed semi-algebraic set \( S = \{ x \in \mathbb{R}^n | g_1(x) = 0, \ldots, g_l(x) = 0, h_1(x) \geq 0, \ldots, h_m(x) \geq 0 \} \) with polynomials \( g_1, \ldots, g_l, h_1, \ldots, h_m \in \mathbb{R}[x] \)).

Clearly, for the investigation of coercivity the value of \( f_0 \) is irrelevant. However, for our analysis it will turn out to be helpful to assume that this value is positive. Hence, after adding an appropriate constant to \( f \), without loss of generality we can use the following assumption throughout this article:

\[ \text{The polynomial } f \in \mathbb{R}[x] \text{ satisfies } f_0 > 0. \] \hspace{1cm} (A)

In this article we will relate coercivity of \( f \) with properties of the Newton polytope

\[ \text{New}(f) := \text{conv } A_f \]
of $f$, that is, the convex hull of $A_f$. Note that, due to the assumption $(\text{A})$, the sets $A_f$ as well as $\text{New}(f)$ contain the origin. This construction is sometimes also called ‘Newton polytope at infinity’ of $f$ (cf., e.g., \[4\]), without explicit reference to the assumption $(\text{A})$. If no confusion is possible we will abbreviate the Newton polytope as $P := \text{New}(f)$ and the set $A_f$ as $A$.

Various algebraic and analytic properties of polynomials are encoded in the properties of their Newton polytopes. To name some of them, for example the number of roots of $n$ polynomial equations in $n$ unknowns can be bounded by the (mixed) volumes of their Newton polytopes (cf., e.g., \[12, 13\]), absolute irreducibility of a polynomial is implied by the indecomposability of its Newton polytope in the sense of Minkowski sums of polytopes \[6\], and there are also some results dealing with Newton polytopes in elimination theory \[13\].

For polynomials to be bounded from below, necessary conditions imposed on vertices of their Newton polytopes and on the corresponding coefficients are identified in \[23\]. These are in fact identical with our conditions $(\text{C1})$ and $(\text{C2})$ below (cf. Th. \[23\]). This is not a coincidence due to the fact that every coercive polynomial is a polynomial bounded from below. Our additional condition $(\text{C3})$ can be viewed as a special condition for a polynomial being convenient (see, e.g., \[4, 23\] for the definition of convenient polynomials). In spite of these connections, we shall derive the conditions $(\text{C1}) - (\text{C3})$ with other proof techniques, mainly based on the application of theorems of the alternative, which allow us to develop also results in degenerate cases as well as sufficient conditions.

In \[11\], Sec. 3.2, the authors introduce a sufficient condition for coercivity on $\mathbb{R}^n$ of polynomials $f \in \mathbb{R}[x]$. On the one hand, this sufficient condition is computationally tractable, because it can be checked by solving a hierarchy of semi-definite programs. On the other hand, it is not satisfied by many coercive polynomials, as we shall show in Example 3.14. A simple reason for this effect is presented in \[2\], where we prove that the sufficient condition from \[11\] actually characterizes the stronger property of so-called stable coercivity of gem regular polynomials, a concept which we first introduce in \[2\].

The coercivity of polynomials in the convex setting is partially analyzed in \[11\], while the coercivity of a polynomial $f$ defined on a basic closed semi-algebraic set $S$ and its relation to the Fedoryuk and Malgrange conditions are examined in \[22\]. In \[22\], Th. 4.2 the authors prove that under the assumption of $f$ being bounded from below on $S$, the Malgrange, or the Fedoryuk conditions (\[22\], Defs. 4.2, 4.3), characterize the coercivity of $f$.
on $S$. We shall use this result in Corollary 3.12 below to prove that the our Newton polytope type sufficient conditions for coercivity imply the Fedoryuk and Malgrange conditions.

As a further consequence of coercivity, $f$ is bounded below on $\mathbb{R}^n$ by some $v \in \mathbb{R}$, so that $f - v \in \mathbb{R}[x]$ is positive semi-definite on $\mathbb{R}^n$. Since the coercivity of $f$ is equivalent to the boundedness of its lower level sets $\{x \in \mathbb{R}^n \mid f(x) \leq \alpha\}$ for all $\alpha \in \mathbb{R}$, appropriate coercivity conditions are also useful as a tool for analyzing the boundedness property of basic semi-algebraic sets.

This article is structured as follows. In Section 2 we derive necessary conditions for coercivity of an arbitrary polynomial $f \in \mathbb{R}[x]$ which solely contain information about the geometry of the vertex set $V$ of the Newton polytope $P$ and sign conditions on the corresponding polynomial coëfficients (Th. 2.8). Our technique of proof bases on the idea to evaluate $f$ only along certain curves, which may be traced back at least to Reznick (19). In Definition 2.18 we introduce the broad class of gem regular polynomials and show that our (i.e. Reznick’s) approach cannot yield necessary conditions in addition to those stated in Theorem 2.8 in the gem regular case. For a special class of gem irregular polynomials, however, Theorem 2.29 states further necessary conditions for coercivity in terms of so-called circuit numbers (cf. 7).

Section 3 deals with sufficient conditions for coercivity of $f$ in terms of its Newton polytope. In Proposition 3.1 we prove that for gem regular polynomials the necessary conditions from Theorem 2.8 are in fact sufficient for coercivity. This leads to our main result, the Characterization Theorem 3.2 of coercivity for gem regular polynomials.

In Section 3.2 we formulate two sufficient conditions for coercivity for gem irregular polynomials (Ths. 3.4 and 3.7) in the spirit of those from the gem regular case and, again, containing information about the corresponding circuit numbers. Section 3.3 presents a connection between our sufficient conditions for coercivity and the Fedoryuk and Malgrange conditions. In Section 3.4 we show that, in contrast to our conditions, the sufficient condition for coercivity from [11, Sec. 3.2] cannot be verified for many coercive polynomials. For the explanation of the simple reason we relate the latter condition to the context of stable coercivity, which we analyze in more detail in 4. The article closes with some final remarks in Section 4.

Throughout this article we provide various illustrative examples. The complete proofs of Proposition 2.24, Theorem 2.29, Theorem 3.7, along with the proof of a nonhomogeneous version of Motzkin’s transposition theorem, can be found in appendix of this article.
2 Necessary conditions for coercivity

2.1 Necessary sign conditions

Our derivation of necessary conditions for coercivity of $f$ bases on a similar technique as presented by Reznick in [19] for the investigation of positive semi-definiteness of multivariate polynomials, that is, on evaluations of $f$ along curves \( \{x_{y,\beta}(t) \mid t \in \mathbb{R}\} \) with

\[
x_{y,\beta}(t) := (y_1 e^{\beta_1 t}, \ldots, y_n e^{\beta_n t})
\]

and $y, \beta \in \mathbb{R}^n$ for $t \in \mathbb{R}$. We will often require that at least one entry of $\beta$ is positive, that is, with $\mathbb{H} = \{h \in \mathbb{R} \mid h \geq 0\}$ we assume $\beta \in B := (-\mathbb{H}^n)^c$. As the vector $\beta$ will act as a direction we could also restrict our attention to the case $\|\beta\| = 1$ but dispense with this for the ease of exposition. We abbreviate

\[
I := \{1, \ldots, n\}, \quad Y := \left\{ y \in \mathbb{R}^n \mid \prod_{i \in I} y_i \neq 0 \right\},
\]

as well as

\[
\Omega := Y \times B.
\]

**Lemma 2.1** Any $(y, \beta) \in \Omega$ satisfies $\lim_{t \to \infty} \|x_{y,\beta}(t)\| = +\infty$.

**Proof.** In the case that $\| \cdot \|$ coincides with the $\ell_\infty$ - norm $\| \cdot \|_\infty$ we obtain for any $(y, \beta) \in \Omega$

\[
\lim_{t \to \infty} \|x_{y,\beta}(t)\|_\infty = \lim_{t \to \infty} \max_{i \in I} |y_i| e^{\beta_i t} = +\infty.
\]

The equivalence of any norm $\| \cdot \|$ with $\| \cdot \|_\infty$ thus yields the assertion. \hfill \blacksquare

Next, for $f \in \mathbb{R}[x]$, $(y, \beta) \in \mathbb{R}^n \times \mathbb{R}^n$ and $t \in \mathbb{R}$ we define

\[
\pi_f(y, \beta, t) := f(x_{y,\beta}(t)) = \sum_{\alpha \in \mathcal{A}} f_\alpha y^\alpha e^{(\alpha,\beta)t},
\]

where $\langle \cdot, \cdot \rangle$ denotes the standard inner product on $\mathbb{R}^n$, as well as

\[
\Omega_f := \{(y, \beta) \in \mathbb{R}^n \times \mathbb{R}^n \mid \lim_{t \to \infty} \pi_f(y, \beta, t) = +\infty\}.
\]

Lemma 2.1 then immediately yields the following result.
**Lemma 2.2** The coercivity of $f \in \mathbb{R}[x]$ on $\mathbb{R}^n$ implies $\Omega \subseteq \Omega_f$.

For any $\beta \in \mathbb{R}^n$ let us consider the optimization problem to maximize $\langle \alpha, \beta \rangle$ over the set $A$, and denote the optimal value and the optimal point set of the latter problem by

$$d(\beta) := \max_{\alpha \in A} \langle \alpha, \beta \rangle$$

and

$$A(\beta) := \{ \alpha \in A \mid \langle \alpha, \beta \rangle = d(\beta) \},$$

respectively. Note that $d(\beta) \geq 0$ holds for all $\beta \in \mathbb{R}^n$ by assumption (A) and that, as the all ones vector $\mathbb{1} \in \mathbb{R}^n$ satisfies $\langle \alpha, \mathbb{1} \rangle = |\alpha|$, we may write $\deg(f) = d(\mathbb{1})$.

For $f \in \mathbb{R}[x]$ and $\beta \in \mathbb{R}^n$ we define the auxiliary polynomial

$$f^\beta(x) := \sum_{\alpha \in A(\beta)} f_\alpha x^\alpha$$

for $x \in \mathbb{R}^n$.

**Proposition 2.3** The inclusion $\Omega \subseteq \Omega_f$ implies the following assertions:

a) For all $\beta \in B$ we have $d(\beta) > 0$.

b) For all $\beta \in B$ the polynomial $f^\beta$ is positive semi-definite on $\mathbb{R}^n$.

**Proof.** For the proof of part a) assume that $d(\beta) = 0$ holds for some $\beta \in B$. Then all $\alpha \in A$ satisfy $\langle \alpha, \beta \rangle \leq d(\beta) = 0$ so that

$$\pi_f(\mathbb{1}, \beta, t) = \sum_{\alpha \in A} f_\alpha e^{\langle \alpha, \beta \rangle t}$$

is, as a function of $t$, bounded for $t \to \infty$. On the other hand, we have $(\mathbb{1}, \beta) \in \Omega$, so that the assumption $\Omega \subseteq \Omega_f$ implies $\lim_{t \to \infty} \pi_f(\mathbb{1}, \beta, t) = +\infty$, a contradiction.

For the proof of part b) choose any $(y, \beta) \in \Omega$. Then the assumption $\Omega \subseteq \Omega_f$ yields $\lim_{t \to \infty} \pi_f(y, \beta, t) = +\infty$. This implies that the leading term

$$\sum_{\alpha \in A(\beta)} f_\alpha y^\alpha e^{d(\beta)t} = e^{d(\beta)t} f^\beta(y)$$

of $\pi_f(y, \beta, \cdot)$ cannot tend to $-\infty$ for $t \to +\infty$. However, in view of part a), the latter would happen in the case $f^\beta(y) < 0$, so that $f^\beta(y) \geq 0$ has to hold for all $y \in Y$. As the topological closure of $Y$ is $\mathbb{R}^n$, the continuity of $f^\beta$ yields the assertion.
2.2 Necessary conditions on the vertices of the Newton polytope

In the next step we will relate the assertions of Proposition 2.3 with statements about the Newton polytope $P = \text{New}(f) = \text{conv} A$ of $f$. In fact, let

$$V := \text{vert } P$$

denote the vertex set of $P$. Note that we have $V \subseteq A$ by, for example, [24, Prop. 2.2(ii)]. Moreover, the element $0 \in A$ (cf. ass. (3)) actually is among the vertices of $P$, since $A \subseteq \mathbb{H}^n$ implies $P \subseteq \mathbb{H}^n$ and, thus, $\alpha = 0$ is the unique maximal point of $\langle \alpha, -\mathbb{1} \rangle$ on $P$. The vertex theorem of linear programming, hence, implies $0 \in V$, and altogether we obtain

$$0 \in V \subseteq A.$$

With respect to the following lemma note that the above arguments entail that the element $\tilde{\alpha} = 0$ of $V$ coincides with the singleton set $A(-\mathbb{1})$, where $-\mathbb{1}$ is not an element of $B$.

**Lemma 2.4** For all $\tilde{\alpha} \in V \setminus \{0\}$ the following assertions hold:

a) There exists some $\beta \in B$ with $A(\beta) = \{\tilde{\alpha}\}$.

b) In the case $\Omega \subseteq \Omega_f$ we have $f_\alpha > 0$ and $\tilde{\alpha} \in 2\mathbb{N}_0^n$.

**Proof.** Let $\tilde{\alpha} \in V \setminus \{0\}$. Then, due to $A \subseteq P$, in particular the system

$$\sum_{\alpha \in A \setminus \{\tilde{\alpha}\}} \lambda_\alpha \begin{pmatrix} \alpha \\ 1 \end{pmatrix} = \begin{pmatrix} \tilde{\alpha} \\ 1 \end{pmatrix}, \quad \lambda_\alpha \geq 0 \quad \text{for all } \alpha \in A \setminus \{\tilde{\alpha}\}$$

is inconsistent. By the Farkas lemma, the latter is equivalent to the existence of some $\beta \in \mathbb{R}^n$ and $\gamma \in \mathbb{R}$ with

$$\langle \tilde{\alpha}, \beta \rangle + \gamma > 0, \quad \langle \alpha, \beta \rangle + \gamma \leq 0, \quad \alpha \in A \setminus \{\tilde{\alpha}\}. \quad (2.1)$$

Due to $0 \in A$ and $\tilde{\alpha} \neq 0$ we have $0 \in A \setminus \{\tilde{\alpha}\}$ and conclude

$$\langle \tilde{\alpha}, \beta \rangle > -\gamma \geq 0$$

from (2.1). For $\beta \in -\mathbb{H}^n$ this would contradict $\tilde{\alpha} \in \mathbb{H}^n$, so that $\beta$ must be an element of $(-\mathbb{H}^n)^c = B$. Moreover, (2.1) implies

$$\langle \tilde{\alpha}, \beta \rangle > \langle \alpha, \beta \rangle, \quad \alpha \in A \setminus \{\tilde{\alpha}\}$$
so that \( d(\beta) = \langle \bar{\alpha}, \beta \rangle \) and \( A(\beta) = \{ \bar{\alpha} \} \) hold, that is, the assertion of part a).

To see part b), first use part a) to choose some \( \beta \in B \) with \( A(\beta) = \{ \bar{\alpha} \} \). Proposition 2.3(b) then implies \( f_\alpha x^\alpha \geq 0 \) for all \( x \in \mathbb{R}^n \). The choice \( x := 1 \) and \( f_\alpha \neq 0 \) yield the first assertion of part b). Moreover, for any \( i \in I \) the choice \( x := 1 - 2e_i \) leads to \( f_\alpha(-1)^{\bar{\alpha}_i} \geq 0 \), so that \( f_\alpha > 0 \) implies \( \bar{\alpha}_i \in 2\mathbb{N}_0 \) and, thus, the second assertion of part b).

In the next lemma, cone \( A \) denotes the convex cone generated by \( A \).

**Lemma 2.5** The inclusion \( \Omega \subseteq \Omega_f \) implies the following assertions:

a) The set cone \( A \) contains all unit vectors \( e_i, i \in I \).

b) For all \( i \in I \) the set \( V \) contains vectors of the form \( 2k_i e_i \) with \( k_i \in \mathbb{N} \).

**Proof.** To see the assertion of part a), let \( i \in I \) and choose some \( \beta \in \mathbb{R}^n \) with \( \langle e_i, \beta \rangle > 0 \). Then we have \( \beta \in (-\mathbb{R}^n)^c = B \). By Proposition 2.3(a) the value \( d(\beta) \) thus is positive or, in other words, the system

\[
\langle e_i, \beta \rangle > 0, \quad \langle \alpha, \beta \rangle \leq 0, \quad \alpha \in A
\]

is inconsistent. By the Farkas lemma, the latter is equivalent to \( e_i \in \text{cone} A \).

For the proof of part b), given any \( i \in I \) we rewrite the fact \( e_i \in \text{cone} A \) from part a) as the existence of \( K \subseteq \mathbb{R}^n \) with \( \alpha_j > 0, \alpha \in K \), with \( e_i = \sum_{\alpha \in K} \lambda_{\alpha} \alpha \). In particular, for any \( j \in I \setminus \{ i \} \) we have

\[
0 = \sum_{\alpha \in K} \lambda_{\alpha} \alpha_j.
\]

Due to \( \alpha_j \geq 0 \) for all \( \alpha \in K \) this is only possible for \( \alpha_j = 0 \), that is, all elements of \( K \) must have the form \( \alpha = k_i e_i \) with some \( k_i \in \mathbb{N} \) and, in particular, there exists some element \( \alpha \in A \) of this form.

Next, let \( k^*_i := \max\{ k_i \in \mathbb{N} | k_i e_i \in A \} \) and \( \alpha^i := k^*_i e_i \). We will proceed to show \( \alpha^i \in V \). Note that \( \alpha^i \in P \) is clear from \( A \subseteq P \).

Assume that \( \alpha^i \) is not a vertex of \( P \). Then there exist \( L \subseteq A \setminus \{ \alpha^i \} \) and \( \lambda_a > 0, \alpha \in L, \) with \( \sum_{\alpha \in L} \lambda_a \alpha = \alpha^i \) and \( \sum_{\alpha \in L} \lambda_a = 1 \). With the same reasoning as above, all elements of \( L \) must have the form \( \alpha = k_i e_i \) with some \( k_i \in \mathbb{N} \). In view of \( \alpha^i \notin L \), this implies

\[
k^*_i = \alpha^i = \sum_{\alpha \in L} \lambda_a \alpha_i = \sum_{\alpha \in L} \lambda_a k_i < \sum_{\alpha \in L} \lambda_a k^*_i = k^*_i,
\]
a contradiction. Hence, we arrive at $k_i^* e_i \in V$. Lemma 2.4(b) finally entails that $k_i^*$ necessarily must be even.

**Remark 2.6** Using $A \subseteq \mathbb{H}^n$, it is not hard to see that the assertion of Lemma 2.4(a) is equivalent to the statement $\text{cone } A = \mathbb{H}^n$.

For later reference we observe that not only the condition $\Omega \subseteq \Omega_f$ (cf. Prop. 2.3(a)) but still its necessary condition from Lemma 2.5(b) implies $d(\beta) > 0$ for all $\beta \in B$:

**Lemma 2.7** For all $i \in I$ let the set $V$ contain a vector of the form $2k_i e_i$ with $k_i \in \mathbb{N}$. Then $d(\beta) > 0$ holds for all $\beta \in B$.

**Proof.** For any $\beta \in B$ there exists some $i \in I$ with $\beta_i > 0$, so that for the choice $\alpha = 2k_i e_i \in A$ we obtain

$$d(\beta) = \max_{a \in A} \langle \alpha, \beta \rangle \geq \langle 2k_i e_i, \beta \rangle = 2k_i \beta_i > 0.$$ 

We may now state our main necessary conditions for coercivity of a polynomial involving the vertex set of $P$.

**Theorem 2.8** Let $f \in \mathbb{R}[x]$ be coercive on $\mathbb{R}^n$ and let assumption (A) be satisfied. Then the following three conditions hold:

1. $V \subseteq 2\mathbb{N}_0^n$. \hfill (C1)
2. All $\alpha \in V$ satisfy $f_\alpha > 0$. \hfill (C2)
3. For all $i \in I$ the set $V$ contains vectors of the form $2k_i e_i$ with $k_i \in \mathbb{N}$. \hfill (C3)

**Proof.** First note that the vertex $0 \in V$ obviously satisfies $0 \in 2\mathbb{N}_0^n$ and that, by assumption (A), we have $f_0 > 0$. This shows the conditions (C1) and (C2) for $\alpha = 0$. Lemmata 2.2, 2.4(b) and 2.5(b) yield all other assertions.

**Remark 2.9** For later reference we remark that the assumption of a coercive polynomial $f$ in Theorem 2.8 may be replaced by the assumption $\Omega \subseteq \Omega_f$. 

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Example 2.10 Assume that the function
\[ f(x) = f_{4,2}x_1^4x_2^2 + f_{3,3}x_1^3x_2^3 + f_{2,3}x_1^2x_2^4 + f_{1,3}x_1x_2^5 + f_{0,4}x_1^4 + f_{0,3}x_1^3 + f_{2,0}x_2^4 + f_{0,0} \]
is coercive on \( \mathbb{R}^2 \). In the following we shall use Theorem 2.8 to derive necessary conditions on the coefficients \( f_\alpha, \alpha \in A \), in \( f(x) = \sum_{\alpha \in A} f_\alpha x^\alpha \) with \( A \subseteq \{(4,2), (3,3), (2,3), (1,3), (0,4), (0,3), (2,0), (0,0)\} \) (see Fig. 1). To satisfy assumption (A) we assume \( f_{0,0} > 0 \), so that \( A \) has to contain the point \((0,0)\).

Due to (C1) the point \((3,3)\) cannot be contained in any choice of \( A \), as it would be a vertex of \( P \), while \((3,3) \notin 2\mathbb{N}_0^2 \). Hence, \( f_{3,3} \) has to vanish.

Due to (C3) the point \((2,0)\) must be contained in any choice of \( A \), and by (C2) we necessarily have \( f_{2,0} > 0 \).

Due to (C3) also the point \((0,4)\) must be contained in any choice of \( A \), as the alternative point \((0,3)\) would violate the evenness condition of (C3). By (C2) we also have \( f_{0,4} > 0 \).

If the point \((4,2)\) is not contained in \( A \), neither \((2,3)\) nor \((1,3)\) can be elements of \( A \), since \((2,3)\) would be a vertex of \( P \) while \((2,3) \notin 2\mathbb{N}_0^2 \) and, for the hence necessary case \((2,3) \notin A \) the point \((1,3)\) would be a vertex of \( P \), in contradiction to (C1). In this case we arrive at \( \{(0,4), (2,0), (0,0)\} \subseteq A \subseteq \{(0,4), (0,3), (2,0), (0,0)\} \) with \( f_{0,4}, f_{2,0} > 0 \) and \( f_{0,3} \in \mathbb{R} \).

If, on the other hand, \((4,2)\) is contained in \( A \), then it is a vertex of \( P \) and we conclude \( f_{4,2} > 0 \) from (C2). We arrive at \( \{(4,2), (0,4), (2,0), (0,0)\} \subseteq A \subseteq \{(4,2), (2,3), (1,3), (0,4), (0,3), (2,0), (0,0)\} \) with \( f_{4,2}, f_{0,4}, f_{2,0} > 0 \) and \( f_{2,3}, f_{1,3}, f_{0,3} \in \mathbb{R} \).

Example 2.11 By Theorem 2.8 the so-called Motzkin form \( m(x) = x_1^4x_2^2 + x_1^2x_2^4 + x_3^5 - 3x_1^2x_2^2x_3^2 \) is not coercive on \( \mathbb{R}^2 \), since the polynomial \( m + 1 \) violates (C3) (while (C1) and (C2) are satisfied).

2.3 A nondegeneracy notion for coercive polynomials

As a motivation for our further discussion note that the conditions (C1) and (C4) from Theorem 2.8 concern vertices of \( P \) and that these are, in view of Lemma 2.8(b), singleton sets \( A(\beta) \) for some \( \beta \in B \). Proposition 2.9(b), however, may provide additional necessary conditions in cases where \( A(\beta) \)
Figure 1: Illustration of Example 2.10. On the left: the exponent $(4, 2)$ is not contained in $A$. On the right: the exponent $(4, 2)$ is contained in $A$. The filled circles describe the vertices of the Newton polytope $\text{New}(f)$, which in both pictures corresponds to the shaded area. The shaded circles describe other possible exponents of $f$ with arbitrary real coefficients. The void circles describe exponents of $f$ with zero coefficients.

is not a singleton, especially if $A(\beta)$ contains some $\alpha \in V^c := A \setminus V$. In fact, for the special case $f(x) = x_1^4x_2^2 + x_1^2x_2^3 + x_1x_2^3 + x_2^4 + x_2^3 + x_1^2 + 1$ of the function from Example 2.10 we obtain $A((1, 2)) = \{(4, 2), (2, 3), (0, 4)\}$ with $(2, 3) \in V^c$. On the other hand, the latter situation is degenerate in the sense that the elements of $A((1, 2))$ are not in general position, where we say that finitely many points from $\mathbb{R}^n$ are in general position if for any $k \in \{2, \ldots, n+1\}$ no $k$ of them lie in a common affine subspace of dimension $k - 2$.

Remark 2.12 We emphasize that a perturbation analysis under this notion of general position would not be straightforward, as the points in our application are elements of $\mathbb{N}_0^n$, rather than $\mathbb{R}^n$.

In the following we shall first identify an appropriate nondegeneracy condition for coercive polynomials (Def. 2.13), then see that we cannot derive necessary conditions in addition to those from Theorem 2.8 for the nondegenerate case with our techniques (Lem. 2.25) and, in Section 2.3, move on to treat a degenerate case.

To develop the nondegeneracy notion, in the following we shall take a closer look at the face structure of $P$ and its relation to points in $A$. Recall that $F$ is a nonempty (closed) face of $P$ if and only if $F = \{\alpha \in P | \langle \alpha, \beta \rangle = \max_{\alpha \in P} \langle \alpha, \beta \rangle\}$ holds for some $\beta \in \mathbb{R}^n$. 

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Lemma 2.13 For all $\beta \in \mathbb{R}^n$ we have $\max_{\alpha \in P} \langle \alpha, \beta \rangle = d(\beta)$.

Proof. Let $\beta \in \mathbb{R}^n$. From $A \subseteq P$ the relation
\[ d(\beta) = \max_{\alpha \in A} \langle \alpha, \beta \rangle \leq \max_{\alpha \in P} \langle \alpha, \beta \rangle \]
is clear. To see the reverse inequality, choose some arbitrary point $\bar{\alpha} \in P$. Then there exist $K \subseteq A$ and $\lambda_\alpha > 0$, $\alpha \in K$, with $\sum_{\alpha \in K} \lambda_\alpha \bar{\alpha} = \bar{\alpha}$ and $\sum_{\alpha \in K} \lambda_\alpha = 1$. This implies
\[ \langle \bar{\alpha}, \beta \rangle = \sum_{\alpha \in K} \lambda_\alpha \langle \alpha, \beta \rangle \leq \sum_{\alpha \in K} \lambda_\alpha d(\beta) = d(\beta) \]
and, thus, $\max_{\alpha \in P} \langle \bar{\alpha}, \beta \rangle \leq d(\beta)$.

In view of Lemma 2.13, the nonempty faces of $P$ are given by the sets
\[ P(\beta) := \{ \alpha \in P \mid \langle \alpha, \beta \rangle = d(\beta) \} \]
with $\beta \in \mathbb{R}^n$. Since we are primarily interested in vectors $\beta \in B$, the next result clarifies which faces of $P$ are singled out by this choice, and how they are related to the sets $A(\beta)$. In fact, let us define
\[ \mathcal{F} := \{ F \subseteq \mathbb{R}^n \mid F \neq \emptyset \text{ is a face of } P \text{ with } 0 \notin F \} \]
as well as the gem of $f$,
\[ \text{Gem}(f) := \bigcup_{F \in \mathcal{F}} F. \]

Remark 2.14 The set $\text{Gem}(f)$ has widely been used in the literature on Newton polytopes of polynomials under different names. For example, in [18, 21] it is called ‘Newton boundary at infinity’. Our terminology is motivated by Definition 2.18 below.

Lemma 2.15 Under condition (C3) the following assertions hold:

a) $F \in \mathcal{F}$ holds if and only if there exists some $\beta \in B$ with $F = P(\beta)$.

b) $A_F = A \cap F$ holds with $F \in \mathcal{F}$ if and only if there exists some $\beta \in B$ with $A_F = A(\beta)$. 
Proof. For the proof of part a) choose $F \in \mathcal{F}$. As $F$ is a nonempty face of $P$, we have $F = P(\beta)$ with some $\beta \in \mathbb{R}^n$. Assume that this holds with $\beta \in -\mathbb{H}^n$. Then, due to $P \subseteq \mathbb{H}^n$, all $\alpha \in P$ satisfy $\langle \alpha, \beta \rangle \leq 0$, and the latter upper bound is attained for $0 \in P$. This implies $d(\beta) = 0$ and $0 \in P(\beta) = F$, a contradiction. Hence, we arrive at $F = P(\beta)$ with $\beta \in (\mathbb{H}^n)^c = B$.

To see the reverse inclusion, let $P(\beta)$ with $\beta \in B$ be given. Then $P(\beta)$ is a nonempty face of $P$, and all $\alpha \in P(\beta)$ satisfy $\langle \alpha, \beta \rangle = d(\beta) > 0$ by (2.3) and Lemma 2.7. This excludes that $P(\beta)$ contains the origin, that is, we have $P(\beta) \in \mathcal{F}$.

The assertion of part b) immediately follows from part a) and the identity $A \cap P(\beta) = A(\beta)$ for any $\beta \in B$.

In the following let $V_F$ denote the vertex set $\text{vert} F$ for any of the polytopes $F \in \mathcal{F}$. From, e.g., [24, Prop. 2.3(i)] we know the identity $V_F = V \cap F$, so that $V \subseteq A$ immediately implies the next result.

**Lemma 2.16** Each $F \in \mathcal{F}$ satisfies $V_F \subseteq A \cap F$.

Before we continue the motivation of our nondegeneracy condition, we briefly present the following result as a side effect of Lemma 2.16. Note that it may also be proven by different techniques, but that the presented approach sheds some additional light on the problem structure.

**Proposition 2.17** Let $f \in \mathbb{R}[x]$ be coercive on $\mathbb{R}^n$ and let assumption (A) be satisfied. Then the degree $\deg(f)$ of $f$ is even.

**Proof.** Recall that we may write $\deg(f) = d(\mathbb{1})$. Due to $\mathbb{1} \in B$, Lemma 2.2 and Theorem 2.8, the face $F = P(\mathbb{1})$ lies in $\mathcal{F}$ and, by Lemma 2.16, it satisfies $V_F \subseteq A \cap F = A(\mathbb{1})$. Consequently, $A(\mathbb{1})$ contains some vertex $\bar{\alpha} \in V$, and we arrive at $\deg(f) = d(\mathbb{1}) = \langle \bar{\alpha}, \mathbb{1} \rangle$. As all entries of $\bar{\alpha}$ are even by condition (C1) in Theorem 2.8, this yields the assertion.

The announced nondegeneracy notion just states equality in the assertion of Lemma 2.16. Note that $V \subseteq A$ and the definition $V^c = A \setminus V$ entail

$$A \cap F = (V \cup V^c) \cap F = V_F \cup (V^c \cap F)$$

so that the identity $V_F = A \cap F$ is equivalent to $V^c \cap F = \emptyset$. 

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Definition 2.18  
(Gem degenerate exponents and gem regular polynomials)  
  
a) An exponent vector \( \alpha \in A \) is called gem degenerate if \( \alpha \in V^c \cap F \) holds for some \( F \in \mathcal{F} \). We denote the set of all gem degenerate points \( \alpha \in A \) by \( D \).  

b) The polynomial \( f \in \mathbb{R}[x] \) is called gem regular if the set \( D \) is empty, otherwise it is called gem irregular.  

Clearly, gem regularity of \( f \in \mathbb{R}[x] \) is equivalent to \( V^c \cap F = \emptyset \) for all \( F \in \mathcal{F} \). Furthermore, the definition of \( D \) gives rise to a partitioning of \( V^c \) into \( D \) and a set of ‘remaining exponents’ \( R := V^c \setminus D \), so that we may write  
\[
A = V \cup D \cup R. 
\]  

Example 2.19 For the polynomial \( f(x) = x_1^4x_2^2 + x_1^2x_3^2 + x_1^4 + x_2^3 + x_1^2 + 1 \) we obtain \( V = \{(4, 2), (0, 4), (2, 0), (0, 0)\} \), \( D = \emptyset \), and \( R = \{(1, 3), (0, 3)\} \), so that \( f \) is gem regular (see Fig. 2). Note that for the face \( F = P((-1, 0)) \) we have \((0, 3) \in V^c \cap F \), but that due to \( F \not\in \mathcal{F} \) this does not mean gem degeneracy of the exponent vector \((0, 3)\).  

Example 2.20 The polynomial \( f(x) = x_1^4x_2^2 + x_1^2x_3^2 + x_1x_3^3 + x_4^3 + x_2^3 + x_1^2 + 1 \) satisfies \( V = \{(4, 2), (0, 4), (2, 0), (0, 0)\} \), \( D = \{(2, 3)\} \), and \( R = \{(1, 3), (0, 3)\} \), so that \( f \) is gem irregular (see Fig. 3).  

Example 2.21 The Motzkin form \( m(x) = x_1^4x_2^2 + x_1^2x_3^2 + x_1x_3^3 + x_4^3 - 3x_1^2x_2^3x_3^2 \) is a gem irregular polynomial with \( V = \{(4, 2, 0), (2, 4, 0), (0, 0, 6)\} \), \( D = \{(2, 2, 2)\} \), and \( R = \emptyset \).  

To term the condition from Definition 2.18b) a regularity condition is justified by the fact that it is related to requiring general position of certain elements of \( A \):  

Lemma 2.22 If for \( f \in \mathbb{R}[x] \) and each \( F \in \mathcal{F} \) the elements of \( A \cap F \) are in general position, then \( f \) is gem regular.
Figure 2: On the left: illustration of Example 2.19. On the right: illustration of Example 2.20. The filled circles describe the vertex set $V$ of the Newton polytope $\text{New}(f)$, which in both pictures corresponds to the shaded area. The shaded circles describe the set $R$ corresponding to $f$. The shaded square in the right picture describes the (singleton) set $D$ corresponding to $f$.

**Proof.** For each $F \in \mathcal{F}$ let the elements of $A \cap F$ be in general position and assume that $V^c \cap F \neq \emptyset$ holds for some $F \in \mathcal{F}$. Then, by Lemma 2.16 and (2.2), we have $|V_F| < |A \cap F|$. On the other hand, $\dim(F) + 1 \leq |V_F|$ holds as $F$ is a polytope, where $\dim(F)$ denotes the dimension of the affine hull $\text{aff}(F)$ of $F$. Hence, $A \cap F$ contains at least $\dim(F) + 2$ elements, while at the same time $A \cap F$ lies in the subspace $\text{aff}(F)$ of dimension $\dim(F)$. This contradicts the assumption that the elements of $A \cap F$ are in general position. 

**Remark 2.23** The polynomial $f(x) = x_1^2 + x_2^2 + x_3^2 + x_1^2 x_2^2 + x_1^2 x_3^2 + x_2^2 x_3^2 + x_1^2 x_2 x_3 + 1$ shows that gem regularity is strictly weaker than the type of general position assumed in Lemma 2.22. In fact, $\text{New}(f)$ is a cube and $D$ is void, while for any facet $F \in \mathcal{F}$ the set $A \cap F$ is not in general position.

The following characterization of the set $D$ will be crucial in Section 3. It states that $D$ contains exactly the exponent vectors in $A$ which cannot be written as a convex combination of elements from $V$ with $0 \in V$ entering with a positive weight. The proof is given in Section A.2, prepared by the proof of a nonhomogeneous version of Motzkin’s transposition theorem (L. A.1) in Section A.1.
Proposition 2.24  Under condition (C3) the following are equivalent:

a) $\alpha^* \in D$,

b) $\alpha^* \in V^c$, and any choice of coefficients $\lambda_\alpha$, $\alpha \in V$, with

$$\alpha^* = \sum_{\alpha \in V} \lambda_\alpha \alpha, \quad \sum_{\alpha \in V} \lambda_\alpha = 1, \quad \lambda_\alpha \geq 0, \quad \alpha \in V;$$

satisfies $\lambda_0 = 0$.

The following lemma clarifies in which cases the assertion of Proposition 2.3(b) may contain additional information on necessary conditions for coercivity, given the assertions of Theorem 2.8.

Lemma 2.25  For $f \in \mathbb{R}[x]$ the following assertions hold:

a) If the conditions (C1)--(C3) from Theorem 2.8 hold and $f$ is gem regular, then for all $\beta \in B$ the polynomial $f^\beta$ is positive semi-definite on $\mathbb{R}^n$.

b) If $\Omega \subseteq \Omega_f$ holds, then for all $F \in \mathcal{F}$ with $D \cap F \neq \emptyset$ we have

$$\sum_{\alpha \in V_F} f_\alpha x^\alpha \geq - \sum_{\alpha \in D \cap F} f_\alpha x^\alpha \quad (2.4)$$

for all $x \in \mathbb{R}^n$.

Proof. Let $\beta \in B$ and any $x \in \mathbb{R}^n$ be given. By (C3) and Lemma 2.15(b) there is some $F \in \mathcal{F}$ with $A(\beta) = A \cap F$ so that

$$f^\beta(x) = \sum_{\alpha \in A \cap F} f_\alpha x^\alpha \quad (2.5)$$

holds. Under the assumption of part a) equations (2.4) and (2.5) yield

$$f^\beta(x) = \sum_{\alpha \in V_F} f_\alpha x^\alpha,$$

so that $V_F \subseteq V$, (C1) and (C2) imply the assertion of part a).

To see the assertion of part b), let $F \in \mathcal{F}$ with $D \cap F \neq \emptyset$ be given. By Lemma 2.5(b) the inclusion $\Omega \subseteq \Omega_f$ implies (C3), so that Lemma 2.15(b)
guarantees the existence of some $\beta \in B$ with $A \cap F = A(\beta)$ and \((2.5)\). Hence, the inclusion $\Omega \subseteq \Omega_f$, Proposition \((2.3b)\) and \((2.5)\) imply

$$0 \leq f^\beta(x) = \sum_{\alpha \in A \cap F} f_\alpha x^\alpha = \sum_{\alpha \in V_F} f_\alpha x^\alpha + \sum_{\alpha \in D \cap F} f_\alpha x^\alpha$$

for all $x \in \mathbb{R}^n$. This shows the assertion of part b).

Lemma \((2.25a)\) expresses that Proposition \((2.3b)\) and, thus, the approach used in Section 2, cannot provide necessary conditions for coercivity of gem regular polynomials $f$ in addition to the conditions \((C1)-(C3)\) stated in Theorem \((2.8)\).

In particular, although \((C1)-(C3)\) where derived using only the special case of singleton sets $A(\beta)$ (cf., e.g., Lem. \((2.4)\)), the consideration of $\beta \in B$ with more general sets $A(\beta)$ in Proposition \((2.3b)\) is superfluous.

For gem irregular polynomials $f$, however, further necessary conditions for coercivity may be derived from the assertion of Lemma \((2.25b)\). The proof of the according statement directly follows from Lemma \((2.2)\) and Lemma \((2.25b)\).

**Proposition 2.26** Let $f \in \mathbb{R}[x]$ be coercive on $\mathbb{R}^n$. Then for all $F \in \mathcal{F}$ with $D \cap F \neq \emptyset$ the inequality

$$\sum_{\alpha \in V_F} f_\alpha x^\alpha \geq - \sum_{\alpha \in D \cap F} f_\alpha x^\alpha$$

holds for all $x \in \mathbb{R}^n$.

For the following we observe that, under condition \((2.11)\), the unique correspondence between the sets $A(\beta)$, $\beta \in B$, and $A \cap F$, $F \in \mathcal{F}$, stated in Lemma \((2.4)\) allows us to interchange the notation $f^\beta$ with $f^F$ so that, for example, equation \((2.5)\) reads

$$f^F(x) = \sum_{\alpha \in A \cap F} f_\alpha x^\alpha.$$

In \([23]\) the polynomials $f^F$ are called quasi-homogeneous components of $f$.

### 2.4 Necessary conditions in a degenerate case

Lemma \((2.2)\), Proposition \((2.3b)\), Lemma \((2.5b)\), and Lemma \((2.6b)\) obviously allow to state a multitude of inequalities on the coefficients $f_\alpha$, $\alpha \in A$, of a
coercive polynomial, just by evaluating \( f^\beta \) at special vectors \( x \) for all \( \beta \in B \) (or, equivalently, for all \( F \in \mathcal{F} \)). For example, the choice \( x := 1 \) yields
\[
\sum_{\alpha \in A \cap F} f_\alpha \geq 0
\]
for all \( F \in \mathcal{F} \), and the choice \( x = -1 \) leads to
\[
\sum_{\alpha \in A \cap F} f_\alpha \geq \sum_{\alpha \in A \cap F} f_\alpha
\]
for all \( F \in \mathcal{F} \).

While, in view of Lemma 2.25 a), many of these inequalities may not contain any information improving the conditions (C1)–(C3) from Theorem 2.8 due to \( |D \cap F| = 0 \), in the case of \( F \in \mathcal{F} \) with \( |D \cap F| > 0 \) Proposition 2.26 provides a systematic way to gain further relations on the coefficients \( f_\alpha, \alpha \in A \). Our main result in the present section will state bounds on these coefficients in the case \( |D \cap F| = 1 \), under the additional assumption that \( F \) is a simplex, that is, the convex hull of affinely independent points. Note that in [11] the corresponding polynomial \( f^F(x) = \sum_{A \cap F} f_\alpha x^\alpha \) is termed a circuit polynomial. The following examples illustrate this case.

**Example 2.27** Consider the polynomial
\[
f(x) = f_{1,2}x_1^4x_2^2 + f_{2,3}x_1^2x_2^3 + f_{1,3}x_1x_2^3 + f_{0,4}x_2^4 + f_{0,3}x_2^3 + f_{2,0}x_1^2 + 1
\]
with \( f_{1,2} \neq 0 \), whose coercivity on \( \mathbb{R}^2 \) implies \( f_{1,2}, f_{0,4}, f_{2,0} > 0 \) as well as \( f_{2,3}, f_{1,3}, f_{0,3} \in \mathbb{R} \), as we saw in Example 2.10. For \( f_{2,3} \neq 0 \) the face \( F = P((1, 2)) \) lies in \( \mathcal{F} \), is a simplex, and satisfies \( |D \cap F| = |\{(2, 3)\}| = 1 \). In particular, the function \( f^F(x) = f_{1,2}x_1^4x_2^2 + f_{2,3}x_1^2x_2^3 + f_{0,4}x_2^4 \) is a circuit polynomial.

**Example 2.28** The Newton polytope of the Motzkin form \( m(x) = x_1^4x_2^2 + x_1^2x_2^4 + x_3^6 - 3x_1^2x_2^2x_3^2 \) from Example 2.10 is a simplex and satisfies \( |D \cap \text{New}(m)| = |\{(2, 2, 2)\}| = 1 \). Thus, \( m \) is a circuit polynomial.

Recall that, for any simplex \( F \) and \( \alpha^* \in F \), the coefficients \( \lambda_\alpha, \alpha \in V_F \), with
\[
\sum_{\alpha \in V_F} \lambda_\alpha \binom{\alpha}{1} = \binom{\alpha^*}{1}, \quad \lambda_\alpha \geq 0, \ \alpha \in V_F,
\]
are unique. Using the natural convention \(0^0 := 1\) in the polynomial setting (to cover the case of vanishing coefficients \(\lambda_\alpha\)), we may define the circuit number (cf. [7])

\[
\Theta(f, V_F, \alpha^*) := \prod_{\alpha \in V_F} \left(\frac{f_\alpha}{\lambda_\alpha}\right)^{\lambda_\alpha}
\]

(2.7)

of \(\alpha^*\) with respect to \(f_F\). Note that the arithmetic-geometric mean inequality immediately yields that for any \(\alpha^* \in F\) the circuit number \(\Theta(f, V_F, \alpha^*)\) bounds the sum of coefficients \(\sum_{\alpha \in V_F} f_\alpha\) from below.

The following assertion has a similar structure as [7, Th. 1.1]. Given the slightly different context, we provide a self-contained proof of this result in Section A.3.

**Theorem 2.29** Let \(f \in \mathbb{R}[x]\) be coercive on \(\mathbb{R}^n\) and let assumption (A) hold. Then the conditions (C1)–(C3) from Theorem 2.8 are satisfied, and for any \(\alpha^* \in D\) such that there exist a simplicial face \(F \in \mathcal{F}\) with \(\alpha^* \in F\) and \(D \cap F = \{\alpha^*\}\), the following assertions hold.

a) We have

\[
f_{\alpha^*} \geq -\Theta(f, V_F, \alpha^*). \tag{2.8}
\]

b) For \(\alpha^* \notin 2\mathbb{N}_0^n\) we also have

\[
f_{\alpha^*} \leq \Theta(f, V_F, \alpha^*). \tag{2.9}
\]

**Example 2.30** Consider the polynomial

\[
f(x) = f_{4,2}x_1^4x_2^2 + f_{2,3}x_1^2x_2^3 + f_{1,3}x_1x_2^3 + f_{0,4}x_2^4 + f_{0,3}x_2^2 + f_{2,0}x_1^2 + 1
\]

with \(f_{4,2} \neq 0\), whose coercivity on \(\mathbb{R}^2\) implies \(f_{4,2}, f_{0,4}, f_{2,0} > 0\) as well as \(f_{2,3}, f_{1,3}, f_{0,3} \in \mathbb{R}\), as we saw in Example 2.10 and, for \(f_{2,3} \neq 0\), the exponent \(\alpha^* = (2, 3)\) lies in \(D\) and satisfies the assumptions of Theorem 2.29, as we saw in Example 2.27. In fact, we have \(V_F = \{(4, 2), (0, 4)\}\) and \(\lambda_{4,2} = \lambda_{0,4} = 1/2\). Hence, by Theorem 2.29(a) and b) the coercivity of \(f\) implies

\[-2\sqrt{f_{4,2}f_{0,4}} \leq f_{2,3} \leq 2\sqrt{f_{4,2}f_{0,4}}.
\]

**Example 2.31** Let us modify the Motzkin form from Example 2.11 such that the resulting polynomial does not violate the condition (C3), for example to

\[
\tilde{m}(x) = x_1^4x_2^2 + x_1^2x_2^4 + x_0^6 + \tilde{m}_{2,2,2}x_1^4x_2^2x_3^2 + x_1^2 + x_2^2 + x_3^2 + 1.
\]

For \(\tilde{m}_{2,2,2} \neq 0\)
the exponent \( \alpha^* = (2, 2, 2) \) lies in \( D \) and satisfies the assumptions of Theorem 2.29 with the face \( F = P(1) \), as we saw in Example 2.21. In fact, we have \( V_F = \{ (4, 2, 0), (2, 4, 0), (0, 0, 6) \} \) and \( \lambda_{4,2,0} = \lambda_{2,4,0} = \lambda_{0,0,6} = 1/3 \). By Theorem 2.29(a) the coercivity of \( m \) hence implies \( m_{2,2,2} \geq -3 \) which shows that the choice of the coefficient \( m_{2,2,2} \) in the original Motzkin form is, in this sense, a critical one.

Example 2.32 In [10, Ex. 3.2] the coercivity of \( f(x) = x_1^6 + x_2^5 + f_{3,3}x_1^3x_2^3 + x_1^4 - x_2 + 1 \) on \( \mathbb{R}^2 \) is shown for the choice \( f_{3,3} = -1 \). The conditions (C1)–(C3) are clearly satisfied for any choice \( f_{3,3} \in \mathbb{R} \). Moreover, the face \( F = P(1) \in \mathcal{F} \) is a simplex with \( |D \cap F| = |\{(3,3)\}| = 1 \) and, thus, \( \alpha^* = (3,3) \) satisfies the assumptions of Theorem 2.29 with \( V_F = \{ (6,0),(0,6) \} \) and \( \lambda_{6,0} = \lambda_{0,6} = 1/2 \). The coercivity of \( f \) hence implies \( f_{3,3} \in [-2,2] \).

Remark 2.33 The assumptions of Theorem 2.29 obviously exclude situations with \( |D \cap F| > 1 \) for \( F \in \mathcal{F} \). While this makes our analysis incomplete, note that already the case \( |D \cap F| > 0 \) is degenerate in the sense that \( f \) then cannot be gem regular, and the elements of \( A \) then cannot be in general position. In this sense, cases with \( |D \cap F| > 1 \) are even more degenerate.

Remark 2.34 The assumptions of Theorem 2.29 also exclude cases in which no face \( F \in \mathcal{F} \) with \( \alpha^* \in F \) is a simplex. While such situations may be covered by our notion of gem regularity, they still are degenerate in the more restrictive sense that the vertices of each such \( F \) then cannot be in general position.

We believe, however, that it should be possible to generalize the assertion of Theorem 2.29 to non-simplicial faces of \( P \) by replacing the complete vertex set \( V_F \) of a face \( F \) corresponding to \( \alpha^* \in D \) by any affinely independent subset \( V^* \subseteq V_F \) with \( \alpha^* \in \text{conv } V^* \), and by using the according circuit number \( \Theta(f, V^*, \alpha^*) \) in the estimates for \( f_{\alpha^*} \). Note that at least one such set \( V^* \) exists by Carathéodory’s theorem, but as there may be several possible choices for \( V^* \), we would obtain several necessary inclusions for the coefficient \( f_{\alpha^*} \) by the technique from Theorem 2.29, and the tightest inclusions would form the necessary conditions. Unfortunately, we do not see how such results may be inferred from Proposition 2.26, as its assertion only covers complete sets \( A \cap F \). Hence, we expect that these results cannot directly be deduced from our (i.e., Reznick’s) approach taken in Section 4.
3 Sufficient conditions for coercivity

We start by treating sufficient coercivity conditions for gem regular polynomials in Section 3.1 which actually lead to a coercivity characterization, before we move on to the degenerate case in Section 3.2.

3.1 A characterization of coercivity for gem regular polynomials

Proposition 3.1 Let $f$ be a gem regular polynomial satisfying assumption (A) as well as the conditions (C1)–(C3) from Theorem 2.8. Then $f$ is coercive on $\mathbb{R}^n$.

**Proof.** Let $(x^k)_{k \in \mathbb{N}}$ be any sequence in $\mathbb{R}^n$ with $\lim_{k \to \infty} \|x^k\| = +\infty$. We have to show $\lim_{k \to \infty} f(x^k) = +\infty$.

With the definition $f^W(x) = \sum_{\alpha \in W} f_{\alpha} x^\alpha$ for $W \subseteq A$ and (2.3) we have $f = f^V + f^R$, as $D$ is void by the assumption of gem regularity. The conditions (C1)–(C3) immediately imply the coercivity of $f^V$ on $\mathbb{R}^n$, so that $\lim_{k \to \infty} f^V(x^k) = +\infty$ holds. In particular, we have $f^V(x^k) > 0$ for almost all $k \in \mathbb{N}$.

The proof will be complete if we can show the existence of some $\varepsilon > 0$ with

$$f^R(x^k) \geq (\varepsilon - 1) f^V(x^k) \quad \text{for almost all } k \in \mathbb{N},$$

as this implies

$$f(x^k) = f^V(x^k) + f^R(x^k) \geq \varepsilon f^V(x^k) \quad \text{for almost all } k \in \mathbb{N}$$

and, thus, $\lim_{k \to \infty} f(x^k) = +\infty$.

In fact, by Proposition 2.24 for any $\alpha^* \in R$ there exist coefficients $\lambda_\alpha$, $\alpha \in V$, with

$$\alpha^* = \sum_{\alpha \in V} \lambda_\alpha \alpha, \quad \sum_{\alpha \in V} \lambda_\alpha = 1, \quad \lambda_\alpha \geq 0, \quad \alpha \in V \setminus \{0\}, \quad \lambda_0 > 0.$$
Hence, using (C1), the convention $0^0 = 1$ as well as (A.12) we may write
\[ f_{\alpha^*} (x^k)^{\alpha^*} \geq -|f_{\alpha^*}| \left| (x^k)^{\alpha^*} \right| = -|f_{\alpha^*}| \left| x^k \right|^{\alpha} \prod_{\alpha \in V \setminus \{0\}} ((x^k)^{\alpha})^{\lambda_{\alpha}} \]
\[ \geq -|f_{\alpha^*}| \prod_{\alpha \in V \setminus \{0\}} \left( \max_{\alpha \in V \setminus \{0\}} (x^k)^{\alpha} \right)^{\lambda_{\alpha}} \]
\[ = -|f_{\alpha^*}| \left( \max_{\alpha \in V \setminus \{0\}} (x^k)^{\alpha} \right)^{1-\lambda_0}. \]

In the following we denote, for $k \in \mathbb{N}$, by $(x_k)$ some $x \in V \setminus \{0\}$ with $(x^k)^{\alpha(k)} = \max_{\alpha \in V \setminus \{0\}} (x^k)^{\alpha}$. (C1) and (C2) imply
\[ f_{\alpha} (x^k)^{\alpha} \geq f_{\alpha(k)} (x^k)^{\alpha(k)} \geq \left( \min_{\alpha \in V \setminus \{0\}} f_{\alpha} \right) (x^k)^{\alpha(k)}, \]
so that, again by (C2),
\[ f_{\alpha^*} (x^k)^{\alpha^*} \geq -|f_{\alpha^*}| \left( (x^k)^{\alpha(k)} \right)^{1-\lambda_0} \]
\[ \geq - \left( \min_{\alpha \in V \setminus \{0\}} f_{\alpha} \right)^{-1} |f_{\alpha^*}| \left( (x^k)^{\alpha(k)} \right)^{-\lambda_0} f_{V}(x^k). \] (3.2)

Next we shall show $\lim_{k \to \infty} (x^k)^{\alpha(k)} = +\infty$. On the contrary, assume that some subsequence $((x_{k_\ell})^{\alpha(k_\ell)})_{\ell \in \mathbb{N}}$ is bounded above by some $M \in \mathbb{R}$. Then the definition of $\alpha(k_\ell)$ yields
\[ f_{\alpha} (x_{k_\ell})^{\alpha} \leq \sum_{\alpha \in V} f_{\alpha} (x_{k_\ell})^{\alpha(k_\ell)} \leq M \sum_{\alpha \in V} f_{\alpha} \]
for all $\ell \in \mathbb{N}$. On the other hand, as a subsequence of $(x^k)_{k \in \mathbb{N}}$ the sequence $(x_{k_\ell})_{\ell \in \mathbb{N}}$ satisfies $\lim_{\ell \to \infty} \|x_{k_\ell}\| = +\infty$, so that the coercivity of $f_{V}$ implies $\lim_{\ell \to \infty} f_{V}(x_{k_\ell}) = +\infty$, a contradiction.

The positivity of $\lambda_0$, thus, implies
\[ \lim_{k \to \infty} (x^k)^{\alpha(k)}^{-\lambda_0} = 0 \]
and we arrive at $\lim_{k \to \infty} \gamma_k(\alpha^*) = 0$ for the term
\[ \gamma_k(\alpha^*) := \left( \min_{\alpha \in V \setminus \{0\}} f_{\alpha} \right)^{-1} \left( |f_{\alpha^*}| \left( (x^k)^{\alpha(k)} \right)^{-\lambda_0} \right) \]
from (3.2). This implies
\[- \sum_{\alpha^* \in R} \gamma_k(\alpha^*) \geq -\frac{1}{2} \]
for almost all \(k \in \mathbb{N}\), so that summing up the inequalities (3.2) over all \(\alpha^* \in R\) yields
\[
f^R(x^k) \geq \left( - \sum_{\alpha^* \in R} \gamma_k(\alpha^*) \right) f^V(x^k) \geq -\frac{1}{2} f^V(x^k) \tag{3.3}\]
for almost all \(k \in \mathbb{N}\), and (3.1) holds with \(\varepsilon := \frac{1}{2}\). \(\blacksquare\)

**Theorem 3.2 (Characterizations of Coercivity)**

For a gem regular polynomial \(f \in \mathbb{R}[x]\) satisfying assumption (A), the following three assertions are equivalent:

a) \(f\) is coercive on \(\mathbb{R}^n\).

b) \(\Omega \subseteq \Omega_f\) holds.

c) The conditions (C1)–(C3) from Theorem 2.8 hold.

**Proof.** Lemma 2.2 states that assertion a) implies b), in view of Remark 2.9 assertion b) implies c), and by Proposition 3.1 assertion c) implies a). \(\blacksquare\)

**Remark 3.3** While the equivalence of assertions a) and c) in Theorem 3.2 definitely is the important one from the application point of view, we emphasize that the equivalence of assertions a) and b) also is interesting in the following sense: it shows that Reznick’s approach from [19], namely the analysis of polynomials merely along certain curves, is sufficiently strong to yield a characterization of an important property of polynomials, at least in the gem regular case.

### 3.2 Sufficient conditions in the degenerate case

By Carathéodory’s theorem, for any degenerate multiplier \(\alpha^* \in D\) there exists a set of affinely independent points \(V^* \subseteq V\) with \(\alpha^* \in \text{conv} \ V^*\). In the
case that a simplicial face $F \subseteq \mathcal{F}$ contains $\alpha^*$, the set $V^*$ can be chosen as the vertex set $V_F$ of $F$. For non-simplicial faces $F$, however, there may exist several possibilities to choose $V^* \subseteq V_F$.

For any set of affinely independent points $V^*$ with $\alpha^* \in \text{conv } V^*$, the solution $\lambda$ of

$$\sum_{\alpha \in V^*} \lambda_\alpha \left( \begin{array}{c} \alpha \\ 1 \end{array} \right) = \left( \begin{array}{c} \alpha^* \\ 1 \end{array} \right), \quad \lambda_\alpha \geq 0, \ \alpha \in V^*$$

is unique, and again we may consider the circuit number

$$\Theta(f, V^*, \alpha^*) = \prod_{\alpha \in V^*} \left( \frac{f_\alpha}{\lambda_\alpha} \right)^{\lambda_\alpha}.$$

If, in addition, $V^*$ is chosen minimally in the sense that the presence of all points in $V^*$ is necessary for $\alpha^* \in \text{conv } V^*$ to hold, then we also have $\lambda_\alpha > 0$ for all $\alpha \in V^*$.

While we were not able to use this approach in the derivation of necessary conditions in the degenerate case (cf. Rem. 2.3), it will be fruitful for the following.

**Theorem 3.4** Let $f \in \mathbb{R}[x]$ be a polynomial satisfying assumption (A) as well as the conditions (C1)–(C3) from Theorem 2.8. Furthermore for each $\alpha^* \in D$ let $V^* \subseteq V$ denote a minimal affinely independent set with $\alpha^* \in \text{conv } V^*$, let $w(\alpha^*) > 0$, $\alpha^* \in D$, denote weights with $\sum_{\alpha^* \in D} w(\alpha^*) \leq 1$, and let

$$f_{\alpha^*} > -w(\alpha^*) \Theta(f, V^*, \alpha^*) \quad \text{if} \quad \alpha^* \in 2\mathbb{N}_0^n$$

and

$$|f_{\alpha^*}| < w(\alpha^*) \Theta(f, V^*, \alpha^*) \quad \text{else.}$$

Then $f$ is coercive on $\mathbb{R}^n$.

**Proof.** As in the proof of Proposition 3.1, let $(x^k)_{k \in \mathbb{N}}$ be any sequence in $\mathbb{R}^n$ with $\lim_{k \to \infty} x^k = +\infty$. In view of (2.3) we have $f = f^V + f^D + f^R$, where the conditions (C1)–(C3) imply $\lim_{k \to \infty} f^V(x^k) = +\infty$ and, thus, $f^V(x^k) > 0$ for almost all $k \in \mathbb{N}$. The proof will be complete if we can show the existence of some $\varepsilon > 0$ with

$$f^D(x^k) + f^R(x^k) \geq (\varepsilon - 1) f^V(x^k) \quad \text{for almost all } k \in \mathbb{N}, \quad (3.5)$$

as this implies

$$f(x^k) = f^V(x^k) + f^D(x^k) + f^R(x^k) \geq \varepsilon f^V(x^k) \quad \text{for almost all } k \in \mathbb{N}.$$
and, thus, \( \lim_{k \to \infty} f(x^k) = +\infty \).

In fact, the proof is based upon the estimate
\[
f^{V^*}(x^k) \geq \Theta(f, V^*, \alpha^*) |x^k|^{\alpha^*} \tag{3.6}
\]
for any \( k \in \mathbb{N} \) and \( \alpha^* \in D \), where \( \Theta(f, V^*, \alpha^*) \) is defined via the unique multipliers \( \lambda_\alpha, \alpha \in V^* \), from (3.4).

To see (3.6), we distinguish similar cases as in Remark 3.2 and define the index sets
\[
I_0(x^k) := \{ i \in I \mid x_i^k = 0 \} \quad \text{and} \quad I_0(\alpha^*) = \{ i \in I \mid \alpha_i^* = 0 \}.
\]
In the case \( I_0(x^k) \not\subseteq I_0(\alpha^*) \) there exists some \( i \in I \) with \( x_i^k = 0 \) and \( \alpha_i^* \neq 0 \), so that \( (x^k)^{\alpha_i^*} = 0 \) and, thus, \( |x^k|^{\alpha^*} = 0 \) holds. The relation (3.6) then collapses to the nonnegativity of \( f^{V^*}(x^k) \) which clearly holds in view of (3.4) and (3.5).

To study the second case, \( I_0(x^k) \subseteq I_0(\alpha^*) \), let us first discuss its special subcase \( I_0(x^k) = \emptyset \). Then we have \( |x^k|^{\alpha} > 0 \) for any \( \alpha \in V^* \), so that the arithmetic-geometric mean inequality, together with (3.4) and (3.5), yields
\[
f^{V^*}(x^k) = \sum_{\alpha \in V^*} f_\alpha (x^k)^{\alpha} = \sum_{\alpha \in V^*} f_\alpha |x^k|^{\alpha} \geq \prod_{\alpha \in V^*} \left( \frac{f_\alpha |x^k|^{\alpha}}{\lambda_\alpha} \right)^{\lambda_\alpha}
= \prod_{\alpha \in V^*} \left( \frac{f_\alpha}{\lambda_\alpha} \right)^{\lambda_\alpha} \prod_{\alpha \in V^*} (|x^k|^{\alpha})^{\lambda_\alpha} = \Theta(f, V^*, \alpha^*) |x^k|^{\alpha^*},
\]
that is, (3.6). Finally, for \( \emptyset \neq I_0(x^k) \subseteq I_0(\alpha^*) \) each \( i \in I_0(x^k) \) satisfies \( (x^k)^{\alpha_i^*} = 0 \) and, thus, \( |x^k|^{\alpha^*} = \prod_{i \in I \setminus I_0(x^k)} |x_i^k|^{\alpha_i} \). Moreover, for each \( i \in I_0(\alpha^*) \) we find
\[
0 = \alpha_i^* = \sum_{\alpha \in V^*} \lambda_\alpha \alpha_i,
\]
so that the positivity of all \( \lambda_\alpha, \alpha \in V^* \), implies \( \alpha_i = 0 \) for all \( \alpha \in V^* \). Hence, for any \( \alpha \in V^* \) and \( i \in I_0(x^k) \subseteq I_0(\alpha^*) \) we also have \( (x_i^k)^{\alpha_i} = 0 \) and, thus, \( |x^k|^{\alpha} = \prod_{i \in I \setminus I_0(x^k)} |x_i^k|^{\alpha_i} \), so that we may write
\[
f^{V^*}(x^k) = \sum_{\alpha \in V^*} f_\alpha |x^k|^{\alpha} \prod_{i \in I \setminus I_0(x^k)} |x_i^k|^{\alpha_i}.
\]
Since \( |x_i^k| > 0 \) holds for all \( i \in I \setminus I_0(x^k) \), we may apply the arithmetic-geometric mean inequality to this term, as above in the case \( I_0(x^k) = \emptyset \), and arrive at
\[
f^{V^*}(x^k) \geq \Theta(f, V^*, \alpha^*) \prod_{i \in I \setminus I_0(x^k)} |x_i^k|^{\alpha_i^*} = \Theta(f, V^*, \alpha^*) |x^k|^{\alpha^*}.
\]
Hence, we have shown the estimate \((3.6)\) in any case.

In view of
\[
\begin{align*}
\alpha^* \left( x^k \right) & \begin{cases} 
= f_{\alpha^*} |x^k|^{\alpha^*} & \text{for } \alpha^* \in 2\mathbb{N}_0^n, \\
\geq -|f_{\alpha^*}| |x^k|^{\alpha^*} & \text{else},
\end{cases}
\end{align*}
\]
under the assumptions of the theorem there exists some \(\delta(\alpha^*) > 0\) with
\[
\begin{align*}
f_{\alpha^*} \left( x^k \right)^{\alpha^*} & \geq (\delta(\alpha^*) - w(\alpha^*) \Theta(f, V^*, \alpha^*)) |x^k|^{\alpha^*} \\
& = (\delta(\alpha^*) \Theta^{-1}(f, V^*, \alpha^*) - w(\alpha^*)) \Theta(f, V^*, \alpha^*) |x^k|^{\alpha^*} \\
& \geq (\delta(\alpha^*) \Theta^{-1}(f, V^*, \alpha^*) - w(\alpha^*)) f^{V^*}(x^k) \quad (3.7) \\
& \geq (\delta(\alpha^*) \Theta^{-1}(f, V^*, \alpha^*) - w(\alpha^*)) f^V(x^k), \quad (3.8)
\end{align*}
\]
where \((3.7)\) holds due to \((3.6)\) for a sufficiently small choice of \(\delta(\alpha^*)\), and \((3.8)\) due to \((3.1)\) and \((3.2)\).

Thus, with the notation from the proof of Proposition 3.1 for \(\alpha^* \in R\) and \((3.3)\), we arrive at
\[
\begin{align*}
f^D(x^k) + f^R(x^k) & \geq \left( \sum_{\alpha^* \in D} (\delta(\alpha^*) \Theta^{-1}(f, V^*, \alpha^*) - w(\alpha^*)) - \sum_{\alpha^* \in R} \gamma_k(\alpha^*) \right) f^{V^*}(x^k) \\
& \geq \left( \sum_{\alpha^* \in D} \delta(\alpha^*) \Theta^{-1}(f, V^*, \alpha^*) - \sum_{\alpha^* \in R} \gamma_k(\alpha^*) - 1 \right) f^{V^*}(x^k)
\end{align*}
\]
and, due to
\[
\lim_{k \to \infty} \sum_{\alpha^* \in R} \gamma_k(\alpha^*) = 0,
\]
may choose
\[
\varepsilon := \frac{1}{2} \sum_{\alpha^* \in D} \delta(\alpha^*) \Theta^{-1}(f, V^*, \alpha^*)
\]
in \((3.6)\).

**Remark 3.5** We emphasize that, in contrast to our necessary condition for the degenerate case from Theorem 2.24, the sufficient condition from Theorem 3.4 holds for general polynomials \(f \in \mathbb{R}[x]\), and does not make any assumptions on the structure of faces related to degenerate exponent vectors.

**Remark 3.6** For the special case of a gem irregular polynomial \(f \in \mathbb{R}[x]\) (satisfying \((3)\)) with a singleton set \(D = \{\alpha^*\}\) such that the minimal face
$F \in \mathcal{F}$ with $\alpha^* \in F$ is simplicial, the gap between the necessary condition from Theorem 2.29 and the sufficient condition from Theorem 3.4 reduces to the strictness of an inequality: the necessary condition states that \((\text{C1})-(\text{C3})\) as well as
\[
f_{\alpha^*} \geq -\Theta(f, V_F, \alpha^*) \quad \text{if} \quad \alpha^* \in 2\mathbb{N}_0^n
\]
and
\[
|f_{\alpha^*}| \leq \Theta(f, V_F, \alpha^*) \quad \text{else}
\]
hold, and the sufficient condition just replaces the nonstrict by strict inequalities in either case. Note that the choice $V^* = V_F$ is mandatory for a minimal simplicial face $F$.

Other than in the special degenerate case from Remark 3.6, the gap between necessary and sufficient conditions is significantly larger, so that we expect that the necessary (cf. also Rem. 2.34) as well as the sufficient condition can be improved further. In fact, already for the case $D = \{(\alpha^*)^1, (\alpha^*)^2\}$ such that the minimal faces $F_i \in \mathcal{F}$ with $(\alpha^*)^i \in F_i$ are simplicial and not identical, the need to choose weights $w((\alpha^*)^1)$ and $w((\alpha^*)^2)$ in Theorem 3.4 leads to a larger discrepancy to the necessary conditions from Theorem 2.29 than just the strictness of inequalities.

In the following we will show how Theorem 3.4 can be modified to improve the sufficient conditions in this respect. The price to pay is, unfortunately, that we need to require an extra condition on the polynomial $f \in \mathbb{R}[x]$ (cf. Rem. 3.5). For the statement of this condition, for any $\alpha^* \in D$ choose a minimal affinely independent set $V^*(\alpha^*) \subseteq V$ with $\alpha^* \in \text{conv } V^*(\alpha^*)$ and define the set $\mathcal{V} := \{V^*(\alpha^*)| \alpha^* \in D\}$. In particular, if two exponent vectors $(\alpha^*)^1$ and $(\alpha^*)^2$ satisfy $V^*((\alpha^*)^1) = V^*((\alpha^*)^2)$, then this set is only listed once in $\mathcal{V}$. We will need to require that the sets in $\mathcal{V}$ can be chosen to be mutually disjoint. The necessary modifications of the proof of Theorem 3.4 to show the following result are given in Section A.3.

**Theorem 3.7** Let $f \in \mathbb{R}[x]$ be a polynomial satisfying assumption (A) as well as the conditions \((\text{C1})-(\text{C3})\) from Theorem 2.8. Furthermore for each $\alpha^* \in D$ let $V^*(\alpha^*) \subseteq V$ denote a minimal affinely independent set with $\alpha^* \in \text{conv } V^*(\alpha^*)$ such that the sets in $\mathcal{V} = \{V^*(\alpha^*)| \alpha^* \in D\}$ are mutually disjoint, let $w(\alpha^*) > 0$, $\alpha^* \in D$, denote weights with $\sum_{\alpha^* \in D \cap V^*} w(\alpha^*) \leq 1$ for each $V^* \in \mathcal{V}$, and let
\[
f_{\alpha^*} > -w(\alpha^*) \Theta(f, V^*, \alpha^*) \quad \text{if} \quad \alpha^* \in 2\mathbb{N}_0^n
\]
and

$$|f_{\alpha^*}| < w(\alpha^*) \Theta(f, V^*, \alpha^*) \text{ else.}$$

Then $f$ is coercive on $\mathbb{R}^n$.

As an application of Theorem 3.7 recall the above mentioned situation $D = \{(\alpha^*)^1, (\alpha^*)^2\}$ such that the minimal faces $F_i \in F$ with $(\alpha^*)^i \in F_i$ are simplicial and not identical. If, in addition, $F_1$ and $F_2$ are actually disjoint, then Theorem 3.7 may be applied, and the resulting sufficient conditions for coercivity differ from the necessary conditions of Theorem 2.29 again just by the strictness of inequalities.

**Example 3.8** Examples 2.30, 2.31, and 2.32 all satisfy the special condition discussed in Remark 3.6. In particular, the coercivity of the polynomial $f(x) = x_1^6 + x_2^6 + f_{3,3}x_1^3x_2^3 + x_1^4 - x_2 + 1$ on $\mathbb{R}^2$ may not only be guaranteed for $f_{3,3} = -1$, as stated in [10], but by Theorem 3.4 even for any $f_{3,3} \in (-2, 2)$.

**Example 3.9** Minimal examples for polynomials satisfying the special condition from Remark 3.6, but being critical in the sense that only the necessary conditions from Theorem 3.4 hold, but not the sufficient ones from Theorem 3.7, are $f^+(x) = x_1^2 + 2x_1x_2 + x_2^2 + 1$. Direct inspection immediately reveals that neither $f^+$ nor $f^-$ are coercive.

Note that Theorem 3.4 presents our most general sufficient conditions for coercivity, while Theorems 3.2 and 3.7 refine them under more special assumptions.

As any coercive and lower semi-continuous function on $\mathbb{R}^n$ attains its infimum, an obvious first application of Theorem 3.4 is that any polynomial $f \in \mathbb{R}[x]$ satisfying the assumptions of Theorem 3.4 attains its infimum $v$ over $\mathbb{R}^n$. In particular, $f$ is then bounded below, and $f - v$ is positive semi-definite on $\mathbb{R}^n$.

Moreover, as all lower level sets of any coercive function are bounded, a basic closed semi-algebraic set

$$S = \{x \in \mathbb{R}^n | g_1(x) = 0, \ldots, g_l(x) = 0, h_1(x) \geq 0, \ldots, h_m(x) \geq 0\}$$

with polynomials $g_1, \ldots, g_l, h_1, \ldots, h_m \in \mathbb{R}[x]$ is bounded if at least one of the functions $g_i$, $i = 1, \ldots, l$, $-g_i$, $i = 1, \ldots, l$, $-h_j$, $j = 1, \ldots, m$, satisfies the assumptions of Theorem 3.4. In particular, the zero set of any polynomial $f \in \mathbb{R}[x]$ satisfying the assumptions of Theorem 3.4 is bounded.

A less obvious application is given in the next section.
3.3 The Malgrange and Fedoryuk conditions

In the following, using results from [22], we will show that the assumptions from Theorem 3.4 imposed on \( f \in \mathbb{R}[x] \) directly imply that \( f \) fulfills the so-called Malgrange and Fedoryuk conditions on \( \mathbb{R}^n \). Before doing so, we shortly recall their definitions.

**Definition 3.10 (Malgrange condition, see [15], [22])**

For \( f \in \mathbb{R}[x] \) let

\[
K_1(f, \mathbb{R}^n) := \{ y \in \mathbb{R} | \exists \text{ sequence } (x^k)_{k \in \mathbb{N}} \subseteq \mathbb{R}^n, \|x^k\| \to \infty \text{ with } f(x^k) \to y
\]

be the set of asymptotic critical values at infinity of \( f \) on \( \mathbb{R}^n \). A polynomial \( f \in \mathbb{R}[x] \) is said to satisfy the Malgrange condition on \( \mathbb{R}^n \) if \( K_1(f, \mathbb{R}^n) = \emptyset \).

**Definition 3.11 (Fedoryuk condition, see [5], [22])**

A polynomial \( f \in \mathbb{R}[x] \) is said to satisfy the Fedoryuk condition on \( \mathbb{R}^n \) if there are positive constants \( \delta \) and \( R \) such that

\[
\|\nabla f(x)\| \geq \delta \quad \text{for all } x \in \mathbb{R}^n \text{ with } \|x\| \geq R.
\]

The Fedoryuk and Malgrange conditions arise in the context of analyzing the bifurcation sets and generalized critical values of polynomials \( f : \mathbb{K}^n \to \mathbb{K} \) with \( \mathbb{K} = \mathbb{C} \) or \( \mathbb{K} = \mathbb{R} \). For more detail see, e.g., [5, 8, 9, 15, 22].

**Corollary 3.12** Let \( f \in \mathbb{R}[x] \) satisfy the assumptions of Theorem 3.4. Then \( f \) also satisfies the Fedoryuk and Malgrange conditions on \( \mathbb{R}^n \).

**Proof.** By Theorem 3.4, the polynomial \( f \) is coercive and thus \( \inf_{x \in \mathbb{R}^n} f(x) > -\infty \) holds. By setting \( S := \mathbb{R}^n \) in [22, Th. 4.2] the assertion directly follows.

3.4 A growth condition

While Example 3.8 shows that, in particular, the sufficient condition for coercivity from [10] can be improved with respect to possible values of polynomial coefficients, in the following we will show that our sufficient condition from
Theorem 3.4 covers whole classes of polynomials which cannot be treated at all by the approach from [10].

To see this, we start by repeating the result from [10] explicitly (where the choice of the norm is, again, irrelevant).

Lemma 3.13 ([10, Lemma 3.1])
Decompose \( f \in \mathbb{R}[x] \) with \( \text{deg}(f) \in 2\mathbb{N} \) into a sum of polynomials, \( f = f_0 + \cdots + f_{\text{deg}(f)} \), where \( f_i \) is homogenous of degree \( i \) for \( i = 0, \ldots, \text{deg}(f) \). If the growth condition
\[
\exists \delta > 0 \; \forall x \in \mathbb{R}^n: \quad f_{\text{deg}(f)}(x) \geq \delta \|x\|^{\text{deg}(f)} \tag{G}
\]
is satisfied, then \( f \) is coercive on \( \mathbb{R}^n \).

The following example presents a polynomial which is coercive on \( \mathbb{R}^n \) while violating the growth condition (G).

Example 3.14 Consider the gem regular polynomial \( f(x) := x_1^4 + x_2^3 + x_1^2 x_2^2 + 1 \) which clearly fulfills the assumption (A) and conditions (C1)–(C3). By our Characterization Theorem 3.2 the polynomial \( f \) is coercive on \( \mathbb{R}^2 \), but this cannot be verified using the sufficiency criterion (C). In fact, we have \( \text{deg}(f) = 4 \), \( f_4 = x_1^2 x_2^2 \), and choosing the Euclidean norm we obtain for every positive constant \( \delta \)
\[
0 = f_4(0, 1) < \delta \|(0, 1)\|_2^4 = \delta.
\]
The sufficiency criterion (C) is, hence, violated although \( f \) is coercive. Many different examples having this property can be constructed easily in the same way. One only has to find a coercive polynomial \( f \in \mathbb{R}[x] \) (e.g. using Ths. 3.3, 7.7 or 7.9) and a point \( \bar{x} \neq 0 \) such that \( f_{\text{deg}(f)}(\bar{x}) = 0 \).

In [2] we show that, for gem regular polynomials of even degree and satisfying assumption (A), the growth condition (G) actually implies our sufficient conditions (C1)–(C4) for coercivity and is then, in view of Example 3.14, strictly stronger than our conditions. In fact, in [2] it turns out that, under the above assumptions, the growth condition (G) characterizes the stronger property of so-called stable coercivity of gem regular polynomials. The latter refers to the condition that coercivity prevails under certain sufficiently small perturbations of the polynomial coefficients (cf. [2] for details). An alternative characterization of stable coercivity is possible by conditions (C1)–(C3) and an extra condition (C4) from [2], again in terms of the Newton polytope, so that in the gem regular case the even degree of the polynomial together with condition (G) may be characterized by (C1)–(C4).
4 Final remarks

In the univariate case, that is, for \( n = 1 \) our results collapse to trivial statements. In fact, then we have \( \text{New}(f) = [0, \deg(f)] \) for any polynomial \( f \) satisfying assumption (A) so that, in particular, each polynomial \( f \) is gem regular. The characterization of coercivity from Theorem 3.2 by conditions (C1)–(C4) then simply states that the leading term of \( f \) has even degree and a positive coefficient.

For \( n > 1 \) a natural and more interesting question arising throughout this article is whether gem regularity, the conditions (C1)–(C3), and the remaining conditions introduced in Theorems 2.29, 3.4, and 3.7 can be verified algorithmically. To this end, in particular one needs to compute all vertices and faces of the polytope \( \text{New}(f) \). This could be done, for example, by using vertex and facet enumeration algorithms (cf., e.g., [1, 3]), but is beyond the scope of the present article.

In some applications stronger notions of coercivity are needed, like stable coercivity of a polynomial (cf. [2]), superlinear coercivity of a function \( f : \mathbb{R}^n \to \mathbb{R} \) which is satisfied when \( f(x)/\|x\| \to +\infty \) holds for \( \|x\| \to +\infty \), or locally uniform coercivity of a parametric function \( f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R} \) which is satisfied at \( \bar{t} \in \mathbb{R}^r \) when \( f(t, x) \to +\infty \) holds for \( t \to \bar{t} \) and \( \|x\| \to +\infty \). The application of our techniques to the latter two concepts in the case of polynomial functions \( f \) is subject of future research.

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References


A Appendix

A.1 A nonhomogeneous Motzkin transposition theorem

In the proof of Proposition 2.24 we will use the following nonhomogeneous version of Motzkin’s transposition theorem.
Lemma A.1 For matrices and vectors of appropriate dimensions, the system
\[ Ax = a, \quad Bx \geq 0, \quad Cx > 0 \quad \text{(A.1)} \]
is inconsistent if and only if at least one of the systems
\[ A^T \rho + B^T \sigma + C^T \tau = 0, \quad a^T \rho > 0, \quad \sigma, \tau \geq 0 \quad \text{(A.2)} \]
and
\[ A^T \rho + B^T \sigma + C^T \tau = 0, \quad a^T \rho = 0, \quad \sigma, \tau \geq 0, \quad \tau \neq 0 \quad \text{(A.3)} \]
is consistent.

Proof. The system (A.1) is inconsistent if and only if the homogeneous system
\[ (A, -a) \begin{pmatrix} x \\ y \end{pmatrix} = 0, \quad (B, 0) \begin{pmatrix} x \\ y \end{pmatrix} \geq 0, \quad \begin{pmatrix} C & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} > 0 \quad \text{(A.4)} \]
is inconsistent, as for any solution \( x \) of (A.1) the vector \( (x, 1) \) solves (A.4), and for any solution \( (x, y) \) of (A.4) we have \( y > 0 \), and \( x/y \) solves (A.1). By Motzkin’s (homogeneous) transposition theorem, the system (A.4) is inconsistent if and only if the system
\[ \begin{pmatrix} A^T \\ -a^T \end{pmatrix} \rho + \begin{pmatrix} B^T \\ 0^T \end{pmatrix} \sigma + \begin{pmatrix} C^T & 0 \\ 0^T & 1 \end{pmatrix} \begin{pmatrix} \tau \\ \mu \end{pmatrix} = 0, \quad \sigma, \tau, \mu \geq 0, \quad (\tau, \mu) \neq 0 \]
is consistent. Rewriting this fact for the two cases \( \mu > 0 \) and \( \mu = 0 \) yields the assertion. \hfill \bullet

A.2 Proof of Proposition 2.24

For any \( \alpha^* \in \mathbb{R}^n \), the fact that any choice of coefficients \( \lambda_\alpha, \alpha \in V \), with
\[ \alpha^* = \sum_{\alpha \in V} \lambda_\alpha \alpha, \sum_{\alpha \in V} \lambda_\alpha = 1, \quad \lambda_\alpha \geq 0, \quad \alpha \in V, \]
satisfies \( \lambda_0 = 0 \) is equivalent to the inconsistency of the system
\[ \sum_{\alpha \in V} \lambda_\alpha \begin{pmatrix} \alpha \\ 1 \end{pmatrix} = \begin{pmatrix} \alpha^* \\ 1 \end{pmatrix}, \quad \lambda_\alpha \geq 0, \quad \alpha \in V \setminus \{0\}, \quad \lambda_0 > 0. \quad \text{(A.5)} \]
For the application of Lemma A.1 we define

$$A := \begin{pmatrix} \cdots & \alpha & \cdots & 0 \\ \cdots & 1 & \cdots & 1 \end{pmatrix}, \quad B := \begin{pmatrix} 1 & & & 0 \\ \end{pmatrix},$$

where $\alpha$ runs through the set $V$ and where we use the convention that $0 \in V$ corresponds to its last entry, as well as

$$a := \begin{pmatrix} \alpha^* \\ 1 \end{pmatrix}, \quad C := c^T := (0, \ldots, 0, 1).$$

By Lemma A.1 the inconsistency of (A.5) is equivalent to the consistency of at least one of the systems

$$A^T \rho + B^T \sigma + \tau c = 0, \quad \langle a, \rho \rangle > 0, \quad \sigma, \tau \geq 0 \quad (A.6)$$

and

$$A^T \rho + B^T \sigma + \tau c = 0, \quad \langle a, \rho \rangle = 0, \quad \sigma \geq 0, \quad \tau > 0, \quad (A.7)$$

where we have used that $\tau$ is a scalar. Setting $\rho = (\beta, \gamma)$ with $\gamma \in \mathbb{R}$ yields that the consistency of (A.6) is equivalent to the consistency of

$$\langle \alpha, \beta \rangle \leq \tau, \quad \alpha \in V, \quad \langle \alpha^*, \beta \rangle > \tau, \quad \tau \geq 0, \quad (A.8)$$

and that the consistency of (A.7) is equivalent to the consistency of

$$\langle \alpha, \beta \rangle \leq \tau, \quad \alpha \in V, \quad \langle \alpha^*, \beta \rangle = \tau, \quad \tau > 0. \quad (A.9)$$

Note that in both systems we used that the inequalities corresponding to the choice $\alpha = 0 \in V$ are consistent in view of the nonnegativity of $\tau$.

So far we have shown that part b) of the assertion can be reformulated as $\alpha^* \in V^c$ and the consistency of at least one of the systems (A.8) and (A.9). Next we shall prove that for any $\alpha^* \in V^c$ the system (A.8) must be inconsistent. In fact, as $\alpha^*$ possesses some description

$$\alpha^* = \sum_{\alpha \in V} \lambda_{\alpha} \alpha \quad \text{with} \quad \sum_{\alpha \in V} \lambda_{\alpha} = 1, \quad \lambda_{\alpha} \geq 0, \quad \alpha \in V,$$

the consistency of both (A.8) or (A.9) implies the existence of some $\beta \in \mathbb{R}^n$ and $\tau \geq 0$ with

$$\langle \alpha^*, \beta \rangle = \sum_{\alpha \in V} \lambda_{\alpha} \langle \alpha, \beta \rangle \leq \sum_{\alpha \in V} \lambda_{\alpha} \tau = \tau. \quad (A.10)$$
However, the consistency of (A.8) would also imply $\langle \alpha^*, \beta \rangle > \tau$, a contradiction. Hence, for any $\alpha^* \in V^c$ the inconsistency of (A.5) is equivalent to the consistency of (A.9).

In the next step we will show that the consistency of (A.9) is equivalent to the existence of some $\beta \in B$ with $\alpha^* \in A(\beta)$. In fact, (A.10) shows that the consistency of (A.9) implies the optimality of the point $\alpha^*$ for the maximization of $\langle \alpha, \beta \rangle$ over $A$, that is, $\alpha^* \in A(\beta)$ with some $\beta \in \mathbb{R}^n$. More precisely, $\alpha^* \geq 0$ and $\langle \alpha^*, \beta \rangle = \tau > 0$ yield that the consistency of (A.9) entails $\beta \in (-\mathbb{H}^n)^c = B$. For the reverse implication, note that $\alpha^* \in A(\beta)$ for some $\beta \in B$ means $\langle \alpha, \beta \rangle \leq \langle \alpha^*, \beta \rangle$ for all $\alpha \in V$ and some $\beta \in B$. Moreover, by (C3) and Lemma 2.7 we have $d(\beta) = \langle \alpha^*, \beta \rangle > 0$, so that the choice $\tau := d(\beta)$ proves the consistency of (A.9).

Altogether, this shows that part b) can be reformulated as $\alpha^* \in V^c$ and the existence of some $\beta \in B$ with $\alpha^* \in A(\beta)$. In view of (C3) and Lemma 2.15, the latter is equivalent to $\alpha^* \in V^c$ and the existence of some $F \in \mathcal{F}$ with $\alpha^* \in A \setminus F$, that is, to $\alpha^* \in V^c \setminus F$ for some $F \in \mathcal{F}$. This is, finally, just the definition for $\alpha^*$ to lie in $D$, that is, part a) of the assertion.

\section*{A.3 Proof of Theorem 2.29}

First, by Theorem 2.8, the conditions (C1)–(C3) are satisfied. Furthermore, under the stated assumptions Lemma 2.2 and Proposition 2.26 yield

$$\sum_{\alpha \in V_F} f_\alpha x^\alpha \geq -f_{\alpha^*} x^{\alpha^*} \text{ for all } x \in \mathbb{R}^n. \quad (A.11)$$

As a first step, we will rewrite this condition in terms of absolute values of $x$, where we put

$$|x|^\alpha := \prod_{i \in I} |x_i|^{\alpha_i} \quad (A.12)$$

for any $\alpha \in \mathbb{N}_0^n$. Due to conditions (C1) and (C2), in the left hand side of (A.11) we may replace $x^\alpha$ by $|x|^\alpha$ for any $\alpha \in V_F$. In the right hand side we replace $x^{\alpha^*}$ by $\text{sign}(x^{\alpha^*})|x^{\alpha^*}| = \text{sign}(x^{\alpha^*})|x|^{\alpha^*}$.

In the following we focus on the case

$$x \in X := \left\{ x \in \mathbb{R}^n \mid \prod_{i \in I} x_i \neq 0 \right\}$$

(see Rem. A.2 for a discussion of the case $x \notin X$). Then we have $|x|^{\alpha^*} > 0$, so that (A.11) implies

$$\sum_{\alpha \in V_F} f_\alpha |x|^{\alpha - \alpha^*} \geq -f_{\alpha^*} \text{sign}(x^{\alpha^*}) \text{ for all } x \in X. \quad (A.13)$$
With the two sets $X^\pm := \{ x \in X \mid \text{sign}(x^{\alpha^*}) = \pm 1 \}$ we arrive at the two separate conditions

$$\inf_{x \in X^+} \sum_{\alpha \in V_F} f_\alpha |x|^{\alpha - \alpha^*} \geq -f_{\alpha^*}, \tag{A.14}$$

and

$$\inf_{x \in X^-} \sum_{\alpha \in V_F} f_\alpha |x|^{\alpha - \alpha^*} \geq f_{\alpha^*}. \tag{A.15}$$

Note that $X^+$ is nonempty for any $\alpha^* \in \mathbb{N}_0^n$, whereas $X^-$ is nonempty if and only if $\alpha^* \notin 2\mathbb{N}_0^n$. This explains why the assertion of this theorem is split into parts a) and b).

In fact, let $X^*$ either denote the set $X^+$ or a nonempty set $X^-$. We will show that the infimum appearing in conditions (A.14) and (A.15), that is, the infimum $v_Q$ of the problem

$$Q : \min_{x \in \mathbb{R}^n} \sum_{\alpha \in V_F} f_\alpha |x|^{\alpha - \alpha^*} \quad \text{s.t.} \quad x \in X^*$$

is bounded above by the infimum $v_S$ of the problem

$$S : \min_{s \in \mathbb{R}^{|V_F|}} \sum_{\alpha \in V_F} f_\alpha e_s^{\alpha} \quad \text{s.t.} \quad \sum_{\alpha \in V_F} \lambda_\alpha s_\alpha = 0,$$

where $\lambda_\alpha, \alpha \in V_F$, denote the unique coefficients from (2.6) (in fact, both infima even coincide, see Rem. A.3). As the objective function of $Q$ is a posynomial, we will borrow some standard techniques from geometric programming for our further analysis.

We will use that for any $\bar{s} \in M_S$ the system of equations

$$\langle \alpha - \alpha^*, z \rangle = \bar{s}_\alpha, \quad \alpha \in V_F, \tag{A.16}$$

possesses a solution $\bar{z}$. In fact, as the vectors $\alpha \in V_F$ are affinely independent as vertices of a simplex, the vectors $\left(\alpha - \alpha^* \atop 1\right), \alpha \in V_F$, are linearly independent, and the system

$$\left(\left(\alpha - \alpha^* \atop 1\right), \left(z \atop \zeta\right)\right) = \bar{s}_\alpha, \quad \alpha \in V_F,$$

possesses a solution $(\bar{z}, \bar{\zeta})$. Moreover, the feasibility of $\bar{s}$ implies

$$0 = \sum_{\alpha \in V_F} \lambda_\alpha \bar{s}_\alpha = \sum_{\alpha \in V_F} \lambda_\alpha \left(\langle \alpha - \alpha^*, \bar{z} \rangle + \bar{\zeta}\right) = \left(\sum_{\alpha \in V_F} \lambda_\alpha \langle \alpha - \alpha^*, \bar{z} \rangle\right) + \bar{\zeta} = \bar{\zeta}.$$
so that $\bar{z}$ solves (A.10).

Next, from any solution $\bar{z} \in \mathbb{R}^n$ of (A.10) we can construct an element of $X^*$. In fact, for $X^* = X^+$ the point $x$ defined by $\bar{x}_i := e^{\bar{z}_i}$, $i \in I$, lies in $X^+$. On the other hand, if $X^* = X^-$ holds with a nonempty set $X^-$, then $\alpha^*$ possesses at least one odd entry $\alpha^*_j$. The point $x$ defined by $\bar{x}_j := -e^{\bar{z}_j}$ and $\bar{x}_i := e^{\bar{z}_i}$, $i \in I \setminus \{j\}$ then lies in $X^-$. Hence, in any of the two cases we arrive at $\bar{x} \in X^*$ which implies

$$v_Q \leq \sum_{\alpha \in V_F} f_\alpha |\bar{x}|^{\alpha - \alpha^*}.$$  

Furthermore, the latter right hand side satisfies

$$\sum_{\alpha \in V_F} f_\alpha |\bar{x}|^{\alpha - \alpha^*} = \sum_{\alpha \in V_F} f_\alpha \prod_{i \in I} e^{(\alpha_i - \alpha^*_i)\bar{z}_i} \prod_{\alpha \in V_F} f_\alpha e^{(\alpha - \alpha^*, \bar{z})} = \sum_{\alpha \in V_F} f_\alpha e^{\bar{z}_\alpha}$$

so that, as $\bar{s} \in M_S$ was chosen arbitrarily, we arrive at $v_Q \leq v_S$.

Finally, let us explicitly compute $v_S$. Since $S$ is a convex optimization problem with polyhedral feasible set, the globally minimal points of $S$ coincide with its Karush-Kuhn-Tucker points. In fact, $s$ is a Karush-Kuhn-Tucker point of $S$ if there exists some $\mu \in \mathbb{R}$ with

$$f_\alpha e^{s_\alpha} = \mu \lambda_\alpha, \quad \alpha \in V_F.$$  

(A.17)

The feasibility of $s$ and (A.17) entail

$$1 = e \left( \sum_{\alpha \in V_F} \lambda_\alpha s_\alpha \right) = \prod_{\alpha \in V_F} (e^{s_\alpha})^{\lambda_\alpha} = \prod_{\alpha \in V_F} \left( \mu \frac{\lambda_\alpha}{f_\alpha} \right)^{\lambda_\alpha} = \mu \prod_{\alpha \in V_F} \left( \frac{\lambda_\alpha}{f_\alpha} \right)^{\lambda_\alpha}$$

so that $\mu$ as well as (by (A.17)) $s$ are uniquely determined, and $s$ coincides with the unique minimal point of $S$. The value $v_S$ is the corresponding minimal value of $S$ which, in view of (A.17), may be written as

$$\sum_{\alpha \in V_F} f_\alpha e^{s_\alpha} = \sum_{\alpha \in V_F} \mu \lambda_\alpha = \mu.$$  

and, thus, the infimum of $S$ is

$$v_S = \mu = \prod_{\alpha \in V_F} \left( \frac{f_\alpha}{\lambda_\alpha} \right)^{\lambda_\alpha} = \Theta(f, V_F, \alpha^*).$$  

(A.18)

As the infimum of $Q$ is bounded above by $v_S$, the choice $X^* = X^+$ in $Q$ and (A.14) yields part a) of the assertion. Under the additional assumption of part b) the set $X^-$ is nonempty, so that the choice $X^* = X^-$ in $Q$ together with (A.15) shows the assertion of part b).
Remark A.2 In the above proof of Theorem 2.29, no additional conditions can be derived from (A.11) in the case \( x \not\in X \). To see this, let us define the index sets \( I_0(x) = \{ i \mid x_i = 0 \} \) and \( I_0(\alpha^*) = \{ i \mid \alpha^*_i = 0 \} \). Clearly, the condition \( x \not\in X \) is equivalent to \( I_0(x) \neq \emptyset \). In the case \( I_0(x) \subseteq I_0(\alpha^*) \) there exists some \( i \in I \) with \( x_i = 0 \) and \( \alpha^*_i > 0 \) which implies \( x_i^{\alpha^*_i} = 0 \) and \( x^{\alpha^*} = 0 \).

The condition resulting from (A.11) then contains no additional information as, in view of conditions (A.1) and (A.2), it holds anyway. Note that, in view of \( I_0(x) \neq \emptyset \), this case includes the case \( I_0(\alpha^*) = \emptyset \), that is, \( \alpha^* \in \mathbb{N}^n \).

On the other hand, in the case \( I_0(x) \subseteq I_0(\alpha^*) \) all \( i \in I_0(x) \) satisfy \( x_i^{\alpha^*_i} = 0^0 = 1 \). Moreover, due to \( \alpha^* \in \text{conv} V_F \) we necessarily have \( \alpha_i = 0 \) and, thus, \( x_i^{\alpha_i} = 0^0 = 1 \) for all \( \alpha \in V_F \). Removing the variables \( x_i, i \in I_0(x) \), and the exponent vector entries \( \alpha_i, i \in I_0(x) \), from condition (A.11) reduces it to a condition in a lower dimensional space of dimension \( \tilde{n} = n - |I_0(x)| \) with \( \tilde{n} \geq 1 \) (as \( I_0(x) = I \) is impossible due to \( \alpha^* \neq 0 \)). Since the lower dimensional variables \( \tilde{x} \) possess no vanishing entries, the argument from the proof of Theorem 2.29 for the case \( x \in X \) could be repeated, but as the resulting estimate of \( f_{\alpha^*} \) by the circuit number is independent of the dimension \( n \) of the decision variable \( Q \), we would not obtain new necessary conditions.

Summarizing, the condition (A.11) is not interesting for the case \( x \not\in X \).

Remark A.3 The bounds on \( f_\alpha \) stated in Theorem 2.29 actually are best possible in the sense that no better bounds can be derived from conditions (A.11) and (A.13). This is due to the fact that not only the estimate \( v_Q \leq v_S \) holds, but even equality. In fact, the reverse inequality \( v_Q \geq v_S \) readily follows from the arithmetic-geometric mean inequality: for any \( \lambda_\alpha \geq 0, \alpha \in V_F \), with \( \sum_{\alpha \in V_F} \lambda_\alpha = 1 \) it yields for any \( x \in \mathbb{R}^n \)

\[
\sum_{\alpha \in V_F} f_\alpha \left| x \right|^{\alpha - \alpha^*} \geq \prod_{\alpha \in V_F} \left( \frac{f_\alpha \left| x \right|^{\alpha - \alpha^*}}{\lambda_\alpha} \right)^{\lambda_\alpha} = \left| x \right|^2 \sum_{\alpha \in V_F} \lambda_\alpha (\alpha - \alpha^*) \prod_{\alpha \in V_F} \left( \frac{f_\alpha}{\lambda_\alpha} \right)^{\lambda_\alpha}
\]

where, again, the convention \( 0^0 = 1 \) is used. If the \( \lambda_\alpha, \alpha \in V_F \), are additionally chosen such that \( \alpha^* = \sum_{\alpha \in V_F} \lambda_\alpha \alpha \), we obtain

\[
\sum_{\alpha \in V_F} f_\alpha \left| x \right|^{\alpha - \alpha^*} \geq \prod_{\alpha \in V_F} \left( \frac{f_\alpha}{\lambda_\alpha} \right)^{\lambda_\alpha} \tag{A.19}
\]

for all such \( \lambda \) as well as all \( x \in \mathbb{R}^n \). While these inequalities give rise to the duality theory of geometric programming, we do not make use of it, as under the assumptions of Theorem 2.29 there only exists a single vector \( \lambda \) with the
required specifications, and the right hand side in (A.19) may be replaced by the circuit number $\Theta(f, V_F, \alpha^*)$. By (A.18) the circuit number coincides with $v_S$, so that the infimum of the left hand side in (A.19) taken over any set $X^* \subseteq \mathbb{R}^n$ is bounded below by $v_S$. As $v_Q$ is such an infimum, the relation $v_Q \geq v_S$ is shown.

A.4 Proof of Theorem 3.7

This proof is identical to the proof of Theorem 3.4 until the estimate (3.7), from which we do not deduce the coarser estimate (3.8), but proceed as follows. To bound $f_D(x^k)$ from below, first we group the sum over all $\alpha^* \in D$ which share the same set $V^* \in V$ and write

$$f_D(x^k) = \sum_{V^* \in V} \sum_{\alpha^* \in D \cap V^*} f_{\alpha^*}(x^k)^{\alpha^*}. $$

For any $V^* \in V$ the inner sum satisfies

$$\sum_{\alpha^* \in D \cap V^*} f_{\alpha^*}(x^k)^{\alpha^*} \geq \sum_{\alpha^* \in D \cap V^*} \left( (\delta(\alpha^*)\Theta^{-1}(f, V^*, \alpha^*) - w(\alpha^*)) f^{V^*}(x^k) \right) $$

$$\geq \left( \sum_{\alpha^* \in D \cap V^*} \delta(\alpha^*)\Theta^{-1}(f, V^*, \alpha^*) - 1 \right) f^{V^*}(x^k) $$

$$\geq \min_{V^* \in V} \left( \sum_{\alpha^* \in D \cap V^*} \delta(\alpha^*)\Theta^{-1}(f, V^*, \alpha^*) - 1 \right) f^{V^*}(x^k). $$

As the sets $V^* \in V$ are mutually disjoint, for sufficiently small choices of $\delta(\alpha^*), \alpha^* \in D$, the conditions (A.11) and (A.12) imply

$$f_D(x^k) = \sum_{V^* \in V} \sum_{\alpha^* \in D \cap V^*} f_{\alpha^*}(x^k)^{\alpha^*} $$

$$\geq \min_{V^* \in V} \left( \sum_{\alpha^* \in D \cap V^*} \delta(\alpha^*)\Theta^{-1}(f, V^*, \alpha^*) - 1 \right) \sum_{V^* \in V} f^{V^*}(x^k) $$

$$\geq \left( \min_{V^* \in V} \sum_{\alpha^* \in D \cap V^*} \delta(\alpha^*)\Theta^{-1}(f, V^*, \alpha^*) - 1 \right) f^V(x^k). $$

From here, the proof may be continued as the proof of Theorem 3.4, with the choice

$$\varepsilon := \frac{1}{2} \min_{V^* \in V} \sum_{\alpha^* \in D \cap V^*} \delta(\alpha^*)\Theta^{-1}(f, V^*, \alpha^*). $$