Robust Investment Management with Uncertainty in Fund Managers' Asset Allocation

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April 2014, revised December 2014

Abstract

We consider a problem where an investment manager must allocate an available budget among a set of fund managers, whose asset class allocations are not precisely known to the investment manager. In this paper, we propose a robust framework that takes into account the uncertainty stemming from the fund managers' allocation, as well as the more traditional uncertainty due to uncertain asset class returns, in the context of manager selection and portfolio management. We assume that only bounds on the fund managers' holdings (expressed as fractions of the portfolio) are available, and fractions must sum to 1 for each fund manager. We define worst-case risk as the largest variance attainable by the investment manager's portfolio over that uncertainty set. We propose two exact approaches (of different complexity) and a heuristic one to solve the problem efficiently. Numerical experiments suggest that our robust model provides better protection against risk than the nominal model when the fund managers' allocations are not known precisely.

1 Motivation and Literature Review

Institutional investors, such as pension funds, university endowments and insurance companies, actively manage their portfolio by investing money in outside fund managers with the expectation of generating superior returns while keeping risk at an acceptable level. Fund managers might have quite different risk and return profiles regarding their strategy and investment process, and the investment manager (institutional investor) must decide how to allocate his budget among them given a high-level idea of each fund's strategy,

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as expressed in its prospectus, but without knowing the fund managers' allocation exactly. While a rich body of literature exists on portfolio management, starting with the pioneering work of Markowitz [27] on mean-variance allocation and while the issue of uncertainty in stocks' expected returns and covariances has received significant attention in the operations research community, the double uncertainty – from an institutional investor's perspective – stemming from both asset returns and managers' allocation, has to the best of our knowledge received no attention at all in terms of quantitative decision-making models. The purpose of this paper is to address such a gap.

1.1 Literature review

Performance attribution

The reader is referred to Maginn et al. [25] and Swensen [34] for introductions to the management of investment portfolios. Methods developed in finance to explain fund managers' performance versus a benchmark index such as the return of the S&P 500 are called fund performance attribution analysis. Previous research has focused on evaluating fund managers' skills by decomposing observed fund returns into several components. Brinson et al. [9] suggest a method to decompose the manager's added value into three parts: (1) asset allocation, (2) stock selection and (3) intersection between the two. This method has the following drawbacks: it does not incorporate the fact that over-weighting a portfolio in a negative market that has outperformed the overall benchmark should still have a positive effect, and it fails to distinguish between the static manager and the dynamic manager, who is trying to capture opportunity when the market is up in one sector and thus over-weighs his portfolio in that sector. Lo [24] and Hsu and Myers [20] both propose approaches to capture the static and dynamic contributions of a fund manager's performance. In their models, weights are considered to be a stochastic process as opposed to fixed parameters. The dynamic component is measured by the sum of the covariances between returns and portfolio weights. The fund return is split into an active management component (reflecting the fund manager's skill) and a passive management component (reflecting stock performance). Reed et al. [29] investigate downside risk management from institutional investors' perspectives and present a decomposition that separate the risk contributions of individual securities from that of investment decisions.

Fund returns can also be decomposed into systematic and unsystematic components. Such a decomposition was first presented in the Capital Asset Pricing Model, where the unsystematic component can be considered as the value added by the manager's performance. It was further developed in Treynor [36], Sharpe [31], Jensen [21] and Jensen [22], who also provide risk-adjusted performance measures such as the Sharpe Ratio, Treynor Ratio and Information Ratio to evaluate the fund manager's performance. These measures are all static measures, based on the characteristics of returns in a single time period. Treynor and Mazuy [38] propose a method to measure the fund managers's ability to capture the up market by introducing the quadratic term in the excess return $(R_{mt} - R_f)^2$. Arnott et al. [1] and Treynor [37] also consider the covariance between portfolio weights and returns but this is only discussed in the context of providing capitalization-indifferent equity market indexes that deliver superior mean-variance performance. Grinblatt and Titman [19] point out that the positive covariance between portfolio weights and returns should bring benefit to investors, and propose a measure that exhibits this property.

Decentralized investment management

A research area related to the present paper is decentralized investment management, pioneered by Sharpe [32]. Barry and Starks [2] focus on risk-sharing as a reason for the investor's decision to employ multiple managers and provide conditions under which a multi-manager allocation is optimal when managers' specialization and diversification are not motives for the use of multiple managers. Elton and Gruber [16] investigate how to set up a structure that would lead, in a decentralized investment situation, to the optimal portfolio for the centralized decision-maker. Specifically, the authors list the following four tasks for a centralized decision-maker and explain that their paper focuses on the first two: "(1) decide how much to invest in each portfolio, (2) give the outside managers instructions that will result in their making optimum security allocations from the point of view of the overall plan, (3) design incentive systems so that the managers will behave optimally, and (4) evaluate and select the portfolio managers." The authors derive conditions under which a centralized decision-maker can form an optimal overall portfolio by employing outside portfolio managers. Two key factors are whether the centralized manager assigns informative value to negative alphas when there are multiple active managers and whether short sales are allowed.

Van Binsbergen et al. [40] also study a problem where a centralized decision-maker employs multiple asset managers. In their setting, the decision-maker is uncertain about the portfolio managers' risk appetites. The authors show how a well-chosen unconditional linear performance benchmark can better align the incentive between the centralized decision-maker and the portfolio managers, when considering two asset classes (bonds and stocks) and three assets per class (for bonds: government-rated bonds, Baa-rated corporate bonds and Aaa-rated corporate bonds, and for stocks: growth stocks, intermediate and value stocks). Blake et al. [8] provide an empirical study in the pension fund industry, for which they document two important trends: the switch from generalist balanced managers to more specialized ones and, within asset classes, the switch from a single-manager situation to settings with multiple competing managers.

These approaches offer investors valuable ways to evaluate fund managers, but do not address the problem of creating a portfolio in presence of uncertainty on the fund managers' allocation.

Parameter uncertainty

Parameter uncertainty in classical mean-variance portfolio management has been studied extensively (although not specifically in a decentralized investment management context) since Chopra and Ziemba [11] documented the impact of mean and covariance estimation errors on mean-variance portfolios. Techniques suggested to mitigate the impact of such errors include stochastic optimization (Pflug and Wozabal [28]), robust statistic models (DeMiguel and Nogales [13], Garlappi et al. [17]), shrinkage (a method that transforms the sample covariance matrix by pulling the more extreme coefficients toward central values, see DeMiguel et al. [14] and Ledoit and Wolf [23]) and robust optimization, which models uncertain parameters using range forecasts and optimizes the worst-case objective, here, minimizes the worst-case variance, with the worst case being computed over that uncertainty set (Ben-Tal et al. [3], Ben-Tal et al. [4], Bertsimas et al. [6]). Applications of robust optimization to classical portfolio management with uncertain parameters have been presented in Ben-Tal et al. [3] and Goldfarb and Iyengar [18], among others. In particular, Goldfarb and Iyengar [18] investigate robust mean-variance portfolio selection problems under a specific uncertainty structure that leads to second-order cone problems and thus can be solved efficiently. Portfolio optimization with uncertainty over a set of distributions has also been studied, for instance in El-Ghaoui et al. [15], which assumes that only bounds on the mean and covariance matrix are available in the context of worst-case value-at-risk optimization, and in Delage and Ye [12], which incorporates ambiguity in both the distribution and its moments in a tractable formulation applied to portfolio selection.

A key technique we will use is delayed constraint generation to address the issue of scale in our robust formulations. Delayed constraint generation was first introduced in Benders [5]. The use of this technique in the context of tractable robust formulations is for instance presented in Thiele et al. [35] and Zeng [41]. Important variations on Benders' decomposition, still for robust optimization but in the context of inventory management with basestock levels, are described in Bienstock and Ozbay [7]. Further, Zhang [42] presents delayed constraint generation for multi-period pricing of perishable products under uncertainty.

1.2 Contributions

In this paper, our goal is to provide an optimization approach to construct a portfolio of funds for the investment manager while taking into account that the funds' tactical allocation is not precisely known. The methodology we will use to achieve that goal is robust optimization. We will investigate this methodology in the context of the well-known mean-variance framework, or Markowitz framework, where we seek to minimize worst-case variance of the investment manager's portfolio (with the worst case computed over the parameters' uncertainty set) subject to a constraint on the expected return. Short sales are not allowed in our model; they are usually prohibited in university endowment funds. Further, this allows the investment manager further insights into the fund managers he selects, as defined by non-negative fund investments. (In the presence of short sales, all fund managers are always selected but the allocation into their fund may be negative.) Uncertainty in finance has been well studied in the context of stock returns, but not regarding the asset allocation of fund managers. Our paper presents a new framework that extends the traditional mean-variance model to the problem faced by the investment (or fund of funds) manager. We also propose an efficient algorithm to solve the problem and provide insights into the fund managers who are chosen by the investment manager. Our numerical results indicate that uncertainty in managers' asset allocation does affect the investment manager's optimal strategy, and suggest that our robust model protects the investment manager against (fund managers') allocation risk.

Structure of the Paper

We present the general framework of the robust manager selection model in Section 2. In Section 3, we propose two algorithms for solving this problem. In Section 4, we apply the two proposed approaches to the robust model in numerical experiments and compare results with the nominal model. A heuristic method is also presented in order to solve the problem efficiently. Finally, Section 5 contains some concluding remarks.

2 Robust Fund Manager Selection

2.1 Problem Setup

We seek to minimize the worst-case portfolio risk (variance) of the investment manager, while guaranteeing that expected return achieves or beats a certain benchmark. Investment managers typically express their portfolio holdings in terms of broad asset classes rather than specific assets; this is therefore the practice that we will follow throughout this paper. From a mathematical standpoint, it has also the advantage of keeping the problem size smaller, since assets are grouped into a smaller number of asset classes. Each manager's allocation in each asset class is subject to ambiguity but is known to fall within a certain range. Non-negative allocation weights model that short sales are not allowed. The fund managers selected by the model will be those for whom the investment manager's allocation has strictly positive weights.

We will use the following notation:

Decision Variables

x_i: allocation in fund manager i

Parameters related to fund managers' allocations

w_{ij}: (uncertain) allocation of manager i in asset class j

w_{ij}^{+}: upper bound of allocation of manager i in asset class j

w_{ij}^{-}: lower bound of allocation of manager i to asset class j

$\overline{w}_{ij}$: nominal allocation of manager i to asset class j

Other parameters

n: number of fund managers

m: number of asset classes

$\bar{r}_j$: expected return from asset class j

cov(r_j, r_l) : covariance between the returns of asset class j and asset class l

τ: portfolio return benchmark.

2.2 Formulation

The problem without uncertainty on the fund managers' allocation, in the classical Markowitz framework, can be formulated as:

$$\begin{aligned} \min_x \quad & \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{n}\sum_{l=1}^{m} \overline{w}_{ij}\overline{w}_{kl}\,\text{cov}(r_j, r_l)\,x_i x_k \\ \text{s.t.} \quad & \sum_{i=1}^{n} x_i = 1 \\ & \sum_{i=1}^{n}\sum_{j=1}^{m} \overline{w}_{ij}\overline{r}_j\,x_i \geq \tau \\ & x_i \geq 0, \forall i \end{aligned} \tag{1}$$

When there is uncertainty on the fund managers' allocation, Problem (1) becomes:

$$
\begin{aligned}
\min_x \quad & \max_\omega \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{n}\sum_{l=1}^{m} w_{ij}w_{kl}\,\mathrm{cov}(r_j,r_l)\,x_ix_k \qquad (2)\\
& \text{s.t.} \;\sum_{j=1}^{m} w_{ij}=1,\forall i\\
& \qquad w_{ij}^{-}\le w_{ij}\le w_{ij}^{+},\forall i,j\\
\text{s.t.} \quad & \sum_{i=1}^{n} x_i = 1\\
& \sum_{i=1}^{n}\sum_{j=1}^{m}\overline{w}_{ij}\overline{r}_j\,x_i \ge \tau\\
& x_i \ge 0,\forall i
\end{aligned}
$$

The following section describes how to solve this problem efficiently. (Note that in order to compare our solution with that obtained by the nominal model, we do not incorporate parameter ambiguity into the benchmark constraint, so that our robust feasible solutions will also be feasible in the nominal model.)

The approach can easily accommodate additional linear or quadratic constraints in x: linear because the feasible set would remain a polyhedron as in the nominal problem and quadratic because the algorithm we will introduce to solve the robust problem involves solving a problem with a linear objective and linear or quadratic constraints. Hence, more linear or quadratic constraints will not change the type of problems to be solved. This allows the investment manager in particular to consider bounds constraints on each manager's risk to model the equivalent of risk-parity, risk-budgeting or marginal risk models (see Maillard et al. [26] and Roncalli [30] for insights into equally-weighted risk contribution portfolios and an introduction to risk parity and budgeting, respectively).

3 Solution Approach

The traditional approach to solve robust optimization problems, which are min-max problems, is to take the dual of the inner maximization problem, invoke strong duality and reinject the newly obtained inner minimization problem into the outer minimization problem to obtain a single minimization problem. However, in our case, the inner maximization problem is a non-convex problem, since we maximize a convex function. Hence, strong duality does not hold, but an optimal solution to the inner problem is achieved at a corner point of the feasible set. Let $\mathcal{S}$ be the set of corner points of that polyhedron, so that the optimal objective of the inner problem is $\max_{s\in\mathcal{S}}\sum_{i=1}^{n}x_i\sum_{j=1}^{m}w_{ij}^s\sum_{k=1}^{n}x_k\sum_{l=1}^{m}w_{kl}^s\,\mathrm{cov}(r_j,r_l)$. Then Problem

(2) becomes:

$$\begin{aligned} \min_{x,Z} \quad & Z \\ \text{s.t.} \quad & Z \geq \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{n}\sum_{l=1}^{m} w_{kl}^{s} w_{ij}^{s} \operatorname{cov}(r_j, r_l)\, x_i x_k,\ \forall s \in \mathcal{S} \\ & \sum_{i=1}^{n} x_i = 1 \\ & \sum_{i=1}^{n}\sum_{j=1}^{m} \overline{w}_{ij}\overline{r}_j x_i \geq \tau \\ & x_i \geq 0, \forall i \end{aligned} \tag{3}$$

Therefore, we will implement a delayed constraint generation algorithm where we only generate constraints of the type $Z \geq \sum_{i=1}^{n} x_i \sum_{j=1}^{m} w_{ij}^{s} \sum_{k=1}^{n} x_k \sum_{l=1}^{m} w_{kl}^{s} \operatorname{cov}(r_j, r_l)$ on an as-needed basis. A key step is to solve the inner problem efficiently so that we can add the constraints with the appropriate corner point of the feasible set of the inner problem, w^s.

We propose two algorithms to solve the inner problem, and then solve the outer problem via delayed constraint generation. The first approach to solve the inner problem is to use that the optimal solution of the inner problem will be at a corner point of the feasible set, eliminate the quadratic (binary) term in the objective function through linearization, and transform the problem into a mixed 0-1 linear program. The second approach is to implement Chen and Burer [10]'s algorithm to solve the nonconvex quadratic problem globally via completely positive programming.

3.1 Inner Problem

3.1.1 Approach 1

Consider

$$\begin{aligned} \max_{\omega} \quad & \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{n}\sum_{l=1}^{m} w_{ij} w_{kl} \operatorname{cov}(r_j, r_l)\, x_i x_k \\ \text{s.t.} \quad & \sum_{j=1}^{m} w_{ij} = 1, \forall i \\ & w_{ij}^{-} \leq w_{ij} \leq w_{ij}^{+}, \forall i, j \end{aligned}$$

We are maximizing a convex function over a bounded, non-empty polyhedron; hence, the optimal solution will be achieved at a corner point of the feasible set. For the specific feasible set we are considering, Tuy [39] shows that corner points are such that, for each manager i, there exists at most one w_{ij} that possibly does not reach its bound, and all the others are equal to either their upper or their lower bound. We write this in mathematical terms below.

Let J_i be the index of the w_{ij} not equal to either bound for manager i and denote $w_{ij} = w_{ij}^- + \triangle w_{ij} \cdot u_{ij}$, where $\triangle w_{ij} = w_{ij}^+ - w_{ij}^-$. Then for each manager i,

- for $j = J_i, w_{ij} \in (w_{ij}^-, w_{ij}^+)$, i.e. $w_{ij} = w_{ij}^- + \triangle w_{ij} \cdot u_{ij}$, where $0 \leq u_{ij} \leq 1$
- for $j \neq J_i, w_{ij}$ is w_{ij}^- or w_{ij}^+, i.e. $w_{ij} = w_{ij}^- + \triangle w_{ij} \cdot u_{ij}$, where $u_{ij} \in \{0, 1\}$

Therefore, we can enumerate values of J_i from 1 to m for each manager i and the original inner nonconvex problem can be reformulated into m^n mixed integer subproblems. This of course only remains tractable as long as m^n is relatively small. For a given subproblem, i.e., a specific value of J_i between 1 and m, we have to solve:

$$\begin{aligned} \max_\omega \quad & \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{n}\sum_{l=1}^{m} x_i x_k \mathrm{cov}(r_j, r_l)(w_{ij}^- + \triangle w_{ij} u_{ij})(w_{kl}^- + \triangle w_{kl} u_{kl}) \\ \text{s.t.} \quad & u_{ij} \in \{0,1\}, \text{for } j \neq J_i \\ & 0 \leq u_{ij} \leq 1, \text{for } j = J_i \end{aligned} \tag{4}$$

Note that this formulation allows w_{ij} for $j = J_i$ to also reach its bound in case of degeneracy.

Sherali and Adams [33] propose the following Reformulation-Linearisation Technique. The constraint $y_{ij} = z_i z_j$, together with $z_i \in \{0, 1\}$ and $z_j \in \{0, 1\}$ is rewritten as:

$$y_{ij} \geq 0, y_{ij} \leq z_i, y_{ij} \leq z_j, y_{ij} \geq z_i + z_j - 1.$$

Substituting $u_{ij}u_{kl}$ by v_{ijkl}, Problem (4) can therefore be reformulated as:

$$\begin{aligned} \max_{\omega} \quad & \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{n}\sum_{l=1}^{m} x_i x_k \text{cov}(r_j, r_l) \left(w_{ij}^{-} w_{kl}^{-} + 2w_{kl}^{-} \triangle w_{ij} \cdot u_{ij} + \triangle w_{ij} \triangle w_{kl} \cdot v_{ijkl} \right) \\ \text{s.t.} \quad & v_{ijkl} \leq u_{ij}, \forall i, j \\ & v_{ijkl} \leq u_{kl}, \forall k, l \\ & v_{ijkl} \geq u_{ij} + u_{kl} - 1, \forall i, j, k, l \\ & u_{ij} \in \{0, 1\}, \text{for } j \neq J_i, \ \forall i \\ & 0 \leq u_{ij} \leq 1, \text{for } j = J_i, \ \forall i. \end{aligned} \tag{5}$$

3.1.2 Approach 2

Chen and Burer [10] propose a method to solve nonconvex quadratic programming problems to global optimality via completely positive programming. Their approach consists in employing a finite branch-and bound ($B\&B$) scheme, in which branching is based on the first-order KKT conditions and polyhedral-semidominant relaxation are solved at each node of the ($B\&B$) tree. The relaxations are derived from completely positive and doubly nonnegative programming problems. The original quadratic programming problem is reformulated as a quadratic programming problem with linear equality, nonnegativity and complementarity constraints. Such a problem can be further reformulated as a completely positive programming problem and relaxed to a doubly nonnegative programming problem.

3.2 Algorithm

As explained above, we solve the outer problem using delayed constraint generation. For the Sth iteration, the outer problem can be formulated as:

$$\begin{aligned} \min_{x,Z} \quad & Z \\ \text{s.t.} \quad & Z \geq \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{n}\sum_{l=1}^{m} w_{ij}^{s} w_{kl}^{s} \ \text{cov}(r_j, r_l) \ x_i x_k, \forall s = 1, 2, ...S \\ & \sum_{i=1}^{n}\sum_{j=1}^{m} x_i \overline{w}_{ij} r_j \geq \tau, \\ & \sum_{i=1}^{n} x_i = 1 \\ & x_i \geq 0, \forall i. \end{aligned} \tag{6}$$

Therefore, we suggest the following algorithm to solve the investment manager's problem, which converges in a finite number of steps by definition of the master problem (3):

Algorithm 3.1 (Delayed constraint generation algorithm)**.**

Step 1 *Start with a feasible solution* $x \in \mathrm{X}$ *and set iteration number* $s := 0$.

Step 2 *Solve the inner problem (in* w*) with candidate solution* x^s *as parameter using either Approach 1 or Approach 2 and obtain optimal solution* w*, denoted* w^{s+1}.

Step 3 *Solve the outer problem (6) (in* x*) with candidate weight* w^{s+1} *as parameter and obtain optimal solution* x*, denoted* x^{s+1}*. Set* $s := s + 1$.

Step 4 *Repeat Steps 2 and 3 until there is no new delayed constraint to generate, i.e., the optimal* w *solution outputted in Step 2 in step* s *has already been found in a previous step* $s' < s$ *and thus already appears in the constraints of Problem (6).*

4 Numerical Results

In this section, we present three experiments to illustrate our robust approach to the manager selection problem with uncertainty in the asset allocations. The first set of experiment is to compare the performance of the two approaches proposed in Section 3 to solve the inner problem and obtain corner points used in the delayed constraint generation algorithm. The second experiment is to compare our robust approach with the nominal approach from a risk standpoint. In the third experiment, we propose a heuristic algorithm that improves solution time by pre-selecting a subset of the fund managers from the large candidate pool into the robust manager selection model, and we compare the results of this heuristic method with the two approaches we proposed in Section 3. The data was provided by the Lehigh University's Investment Office, including bounds on real funds' allocations.

4.1 Comparison of the two approaches to solve the robust inner problem

In this set of experiments, we test the efficiency of the two approaches to solve the robust inner problem, which are used to generate constraints in the delayed algorithm for the master problem: the MIP approach and the Chen and Burer [10] approach. We test three instances: four managers with four asset classes, six managers with six asset classes, and twelve managers with six asset classes. We observe that the first approach (MIP approach) is more efficient than the second one when the problem size is small. However, as the problem size increases, the computational advantage of the second approach becomes more significant. When we consider the problem with twelve managers and six asset classes, the first approach needs to solve 6^{12} independent mixed integer problems and the computations exceed the allowed time limit, while the second approach can solve the problem in a reasonable time for this size. Tables 1-3 present the results as a function of the expected-return benchmark τ, increasing from 0.15 onwards. Running times are expressed in CPU seconds throughout. The worst variance of manager i is defined as the worst-case variance of his portfolio's return when his weights w_{ij} sum to 1 over all assets j and fall within the bounds $[w_{ij}^{-}, w_{ij}^{+}]$.

4.1.1 Four Managers with Four Asset Classes

Table 1: Four Managers with Four Asset Classes

	worst variance	nominal return	0.15	0.2	0.25	0.3	0.35	0.4	0.45
Manager 1	13.2566	0.0259	0	0	0	0	0	0	0
Manager 2	10.161	0.0227	0	0	0	0	0	0	0
Manager 3	11.2965	0.0457	0	0	0	0.17	0.4368	0.7	0.9632
Manager 4	7.645	0.0267	1	1	1	0.83	0.5632	0.3	0.0368
Variance (objective)			7.645	7.645	7.645	7.93	8.6456	9.6935	11.0762
Excess return			0.0117	0.0067	0.0017	0	0	0	0
Running Time	Approach 1	(CPU sec.)	8.092	7.222	7.294	9.96	16.394	13.791	15.745
Running Time	Approach 2	(CPU sec.)	13.274	12.95	15.055	25.6	27.134	26.487	44.493

We see from Table 1 that it is optimal for the investment manager to invest solely in the fund of Manager 4 for small values of τ (specifically, as long as the benchmark constraint is not tight). Once τ reaches 0.3, the investment manager begins to diversify into the fund of Manager 3, and increases his allocation into that fund until his allocation consists almost solely of Fund 3 for $\tau = 0.45$. The MIP approach (Approach 1) is initially about 39% faster than Approach 2, and for the largest value of τ considered, solves in about a third of the time it takes for Approach 2 to terminate.

4.1.2 Six Managers with Six Asset Classes

Table 2: Six Managers with Six Asset Classes

	worst var.	nominal return	0.15	0.25	0.3	0.35	0.4	0.45	0.5	0.55
Manager 1	7.1842	0.0461	0.4303	0.4303	0.4303	0.4303	0.4937	0.3524	0.1778	0.0158
Manager 2	6.4727	0.0341	0.5697	0.5697	0.5697	0.5697	0.5063	0.3593	0.2284	0.0915
Manager 3	7.5552	0.0432	0	0	0	0	0	0	0	0
Manager 4	9.6658	0.0506	0	0	0	0	0	0	0	0
Manager 5	7.8224	0.0573	0	0	0	0	0	0.2883	0.5937	0.8927
Manager 6	10.3974	0.0525	0	0	0	0	0	0	0	0
Variance (obj)			6.2028	6.2028	6.2028	6.2028	6.2145	6.4857	6.9147	7.4998
Excess return			0.0242	0.0142	0.0092	0.0042	0	0	0	0
Running time	Approach 1	(CPU sec.)	25768	26218	26423	26487	26407	26672	15859	28566
Running time	Approach 2	(CPU sec.)	6848	7152	7242	7258	6870	9594	13233	16236

Table 2 shows increased diversification as τ increases, from investing in the funds of Managers 1 and 2 only while the expected-return benchmark constraint is not tight, to investing in the funds of Managers 1, 2 and 5 with growing weight into Fund 5. We also observe that Approach 2 now takes less time to solve. For small values of τ, Approach 1 takes about four times longer to terminate. For large values of τ, it takes approximately 76% longer.

4.1.3 Twelve Managers with Six Asset Classes

The case with twelve managers and six asset classes is only solved using Approach 2, since Approach 1 runs into time limits. The investment manager initially allocates his budget between Managers 5, 10 and 12. As τ increases, Fund 4 is also chosen, and Fund 12 is taken out of the selection. Fund 6 is chosen for some values of τ as well. The fact that some managers are never chosen will motivate the design of the pre-selection heuristic we present in Section 4.3.

Table 3: Twelve Managers with Six Assets

	worst var.	nominal return	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5	0.55	0.6
Manager 1	8.921	0.0526	0	0	0	0	0	0	0	0	0	0
Manager 2	9.1132	0.0415	0	0	0	0	0	0	0	0	0	0
Manager 3	11.306	0.0461	0	0	0	0	0	0	0	0	0	0
Manager 4	8.9833	0.059	0	0	0	0	0	0.2253	0.431	0.622	0.8118	0.4923
Manager 5	7.6327	0.0316	0.6421	0.6421	0.6421	0.6421	0.6212	0.4865	0.3427	0.1858	0.0274	0
Manager 6	10.2603	0.0667	0	0	0	0	0	0	0.0005	0	0	0.4042
Manager 7	10.3677	0.029	0	0	0	0	0	0	0	0	0	0
Manager 8	8.8524	0.0201	0	0	0	0	0	0	0	0	0	0
Manager 9	9.9881	0.0145	0	0	0	0	0	0	0	0	0	0
Manager 10	10.031	0.0386	0.3468	0.3469	0.3468	0.3468	0.3051	0.2664	0.2258	0.1922	0.1608	0.1035
Manager 11	7.9499	0.031	0	0	0	0	0	0	0	0	0	0
Manager 12	8.885	0.0494	0.0111	0.011	0.0111	0.0111	0.0736	0.0218	0	0	0	0
Variance (obj)			6.8357	6.8357	6.8357	6.8357	6.8387	6.975	7.2184	7.6183	8.1685	8.9515
Excess return			0.0192	0.0142	0.0092	0.0042	0	0	0	0	0	0
Running time	(CPU	sec.)	30719	30599	29864	30553	31039	54754	55120	57464	48503	27397

4.2 Comparison between the nominal and the robust models

In this set of experiments, we compare our robust model with the nominal model where fund managers' allocations are assumed equal to their nominal value. We then test the performance of the two models under the nominal asset allocation scenario and the worst case asset allocation scenario. We also compare the difference in the manager selection policy under the two models. The cases of six managers with six asset classes and twelve managers with six asset classes are presented for illustration purposes.

4.2.1 Six managers with six asset classes

Figures 1 and 2 compare the optimal manager selection policy under the nominal and robust models, respectively. In the nominal model, Manager 1 is always chosen for all values of the benchmark requirement (parameter τ), but with a decreasing weight as the benchmark return increases and Manager 5 is selected with an increasing weight in the investment manager's portfolio as the benchmark return exceed 0.045; however, in the robust model, Manager 1 has much less weight in the portfolio compared to his weight in the nominal model. Manager 2, who is never selected in the nominal model, has an initially large weight in the robust manager selection model, which decreases as τ increases.

Figure 3 presents the risk under the nominal model, robust model and the case where the nominal manager allocation is applied when the worst-case manager's allocation occurs. As expected, the nominal model always gives the lowest risk for the fund managers' nominal asset allocation, while the robust model gives the lowest risk for the fund managers' robust asset allocation. In the scenario that the worst case manager's asset allocation occurs, the nominal manager selection policy consistently results in a higher risk than the risk under the robust manager selection policy. From Figure 3, we see that the robust model provides a good protection for the worst case scenario, with a decrease in risk of about 8% in the worst case.

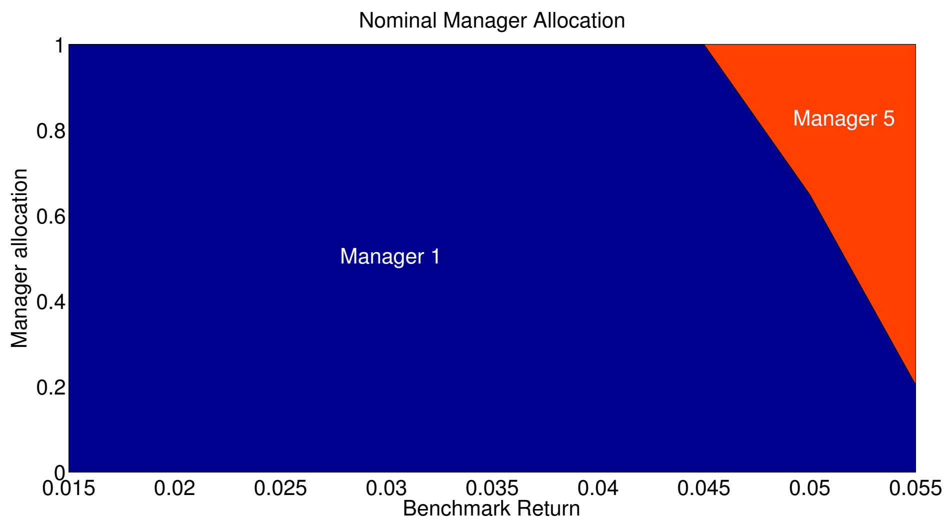

Figure 1: Nominal Manager Allocation: Six Managers with Six Asset Classes

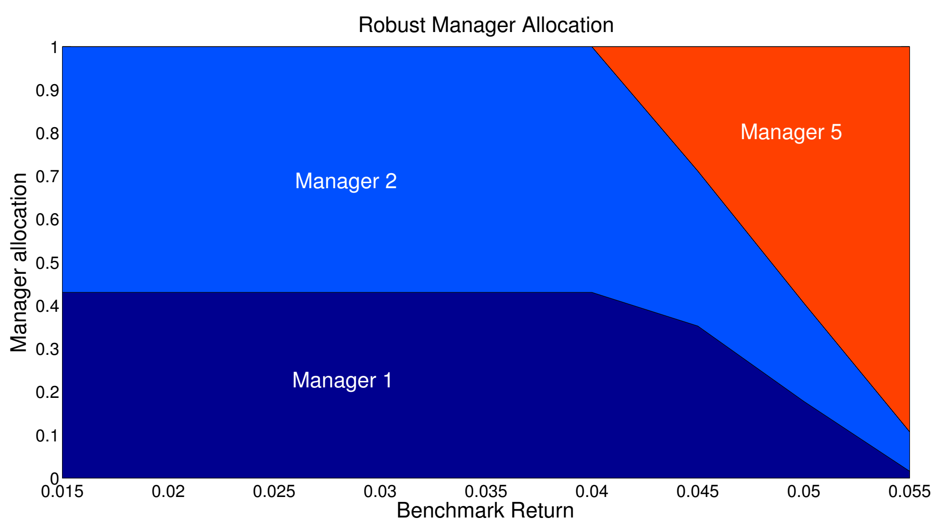

Figure 2: Robust Manager Allocation: Six Managers with Six Asset Classes

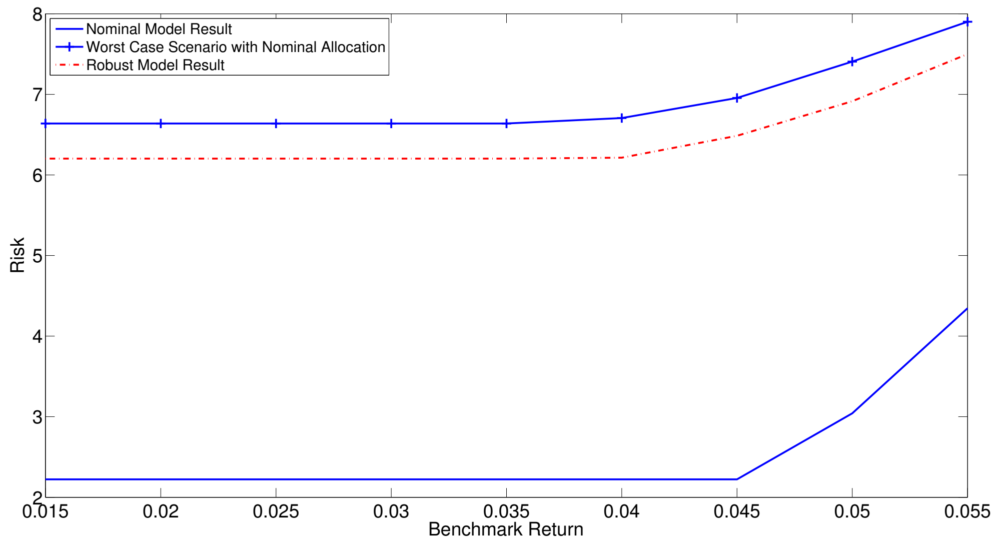

Figure 3: Nominal Model vs Robust Model: Six Managers with Six Asset Classes

4.2.2 Twelve managers with six asset classes

Here as well, the nominal and robust models result in very different manager selection policies. Managers 4, 10 and 12, who receive a large weight under the robust manager selection policy, are never chosen in the nominal case. Meanwhile, Managers 1 and 8 are never selected under the robust manager selection policy. Detailed manager allocation information are shown in Figures 4 and 5 for the nominal and robust models, respectively.

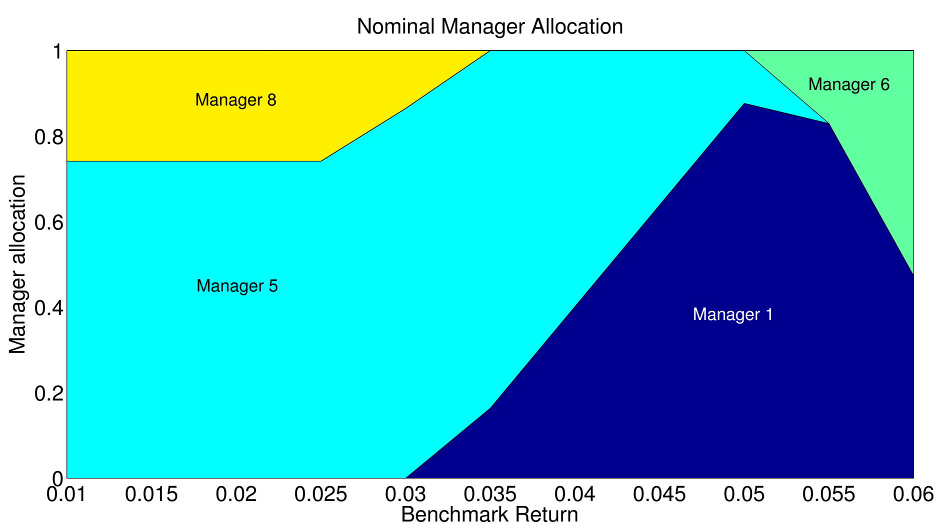

Figure 4: Nominal Manager Allocation: Twelve Managers with Six Asset Classes

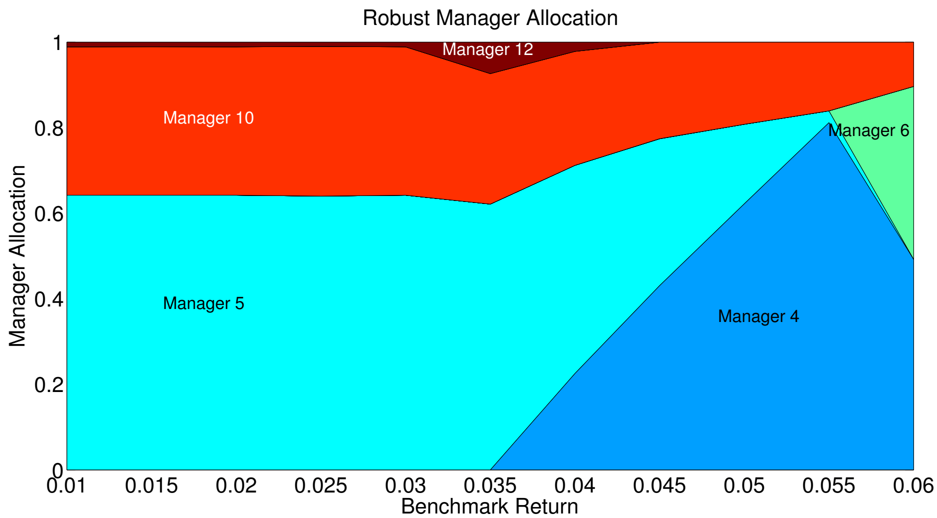

Figure 5: Robust Manager Allocation: Twelve Managers with Six Asset Classes

Figure 6 shows that the robust model better protects the investment manager against allocation risk given the worst-case asset allocation scenario, achieving a decrease in risk in the worst case of about 9%. The large differences in fund manager allocation combined with the increased protection against risk suggests that it is important for the investment manager to implement such a robust approach if fund managers' allocations are not precisely known.

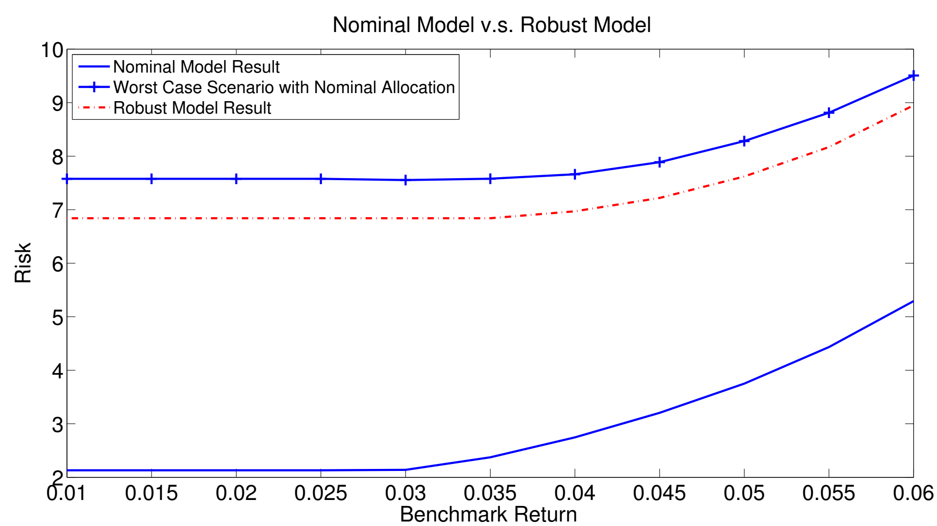

Figure 6: Nominal Model vs Robust Model: Twelve Managers with Six Asset Classes

4.3 A Heuristic

In this section, we are interested in studying a heuristic to pre-select fund managers, thus decreasing computational time (perhaps enough to make the MIP approach – Approach 1 – a viable alternative to Approach 2). We define “the worst-case efficient managers” as the managers whose worst-case risk and nominal return pairs lie on the boundary of the convex hull of all the managers’ worst-case risk vs nominal return pairs and are non-dominated. (A worst-case risk and nominal return pair for a given manager is said to be dominated when one can find another manager with a smaller-or-equal worst-case risk and a higher-or-equal nominal return. On the graphs, dominated pairs correspond to managers’ markers for which there exists another manager’s marker above and to the left.) These worst-case efficient managers will be the only managers selected in the heuristic, which we then test in the numerical experiments presented above.

The investment manager can also consider further pre-selecting managers if the set of worst-case efficient managers is too large. Possible pre-selection schemes include selecting three worst-case efficient managers, one with the lowest worst-case risk, one with the largest worst-case risk (because he will also have the largest expected return, since he is efficient by definition), and one “in the middle”, following a criterion to be specified by the investment manager (e.g., the fund manager with the smallest worst-case standard deviation that is above half the worst-case standard deviation of the other two managers who have just been identified), and proceeding iteratively to either change one manager, keeping the other two constant, or adding a fourth manager. The proper procedure must be analyzed with care, not only in the details of its definition but also in a comparison with the nominal benchmark, and is therefore outside the scope of this paper.

In the case of four managers with four asset classes, Managers 3 and 4 dominate the other managers from the viewpoint of worst case risk vs nominal return, as shown in Figure 7. Because it was in fact optimal to only select Managers 3 and 4, we thus recover the optimal solution. In the case of six managers with six asset classes, Managers 1, 2 and 5 are on the boundary of the convex hull of worst case risk vs. nominal return pairs and it was also optimal to select those managers (and only them) in the exact algorithm. In the case of twelve managers with six asset classes, Managers 4, 5 and 6 are “the worst-case efficient managers” according to our definition above. Manager 10 (marked as a circle) and Manager 12 (marked as a cross) were selected in the optimal robust allocation but are not ”worst case efficient managers” (their risk-return pairs do not lie on the boundary of the convex hull) and thus are not selected in the heuristic method.

For the cases of four managers with four assets, and six managers with six assets, respectively, Tables 4 and 5 compare the result of the optimal manager selection policy and the running time from the heuristic method with those in the two approaches proposed in Section 4. The heuristic method yields the same optimal result as the other two approaches (since the set of worst-case efficient managers was the set of managers selected in the exact approaches at optimality), but requires significantly less time to solve the problem due to the smaller number of candidate managers, especially in the instance with six managers and six asset classes. The heuristic solves 290-610 times faster than Approach 1 and 160-190 times faster than Approach 2. The efficient frontier obtained from the heuristic method, defined as the benchmark expected return τ (vertical axis) for the optimal standard deviation or square root of the objective for the investment manager (horizontal axis), overlaps with the exact efficient frontier obtained with the other approaches as shown in Figures 7 and 8. “With Four Managers” refers to the problem solved with the full fund-manager pool of four managers. “With Two Most Efficient Managers” refers to the problem solved when only the two most efficient managers, as defined above, are considered for possible funding, i.e., the fund-manager pool is reduced according to the heuristic selection.

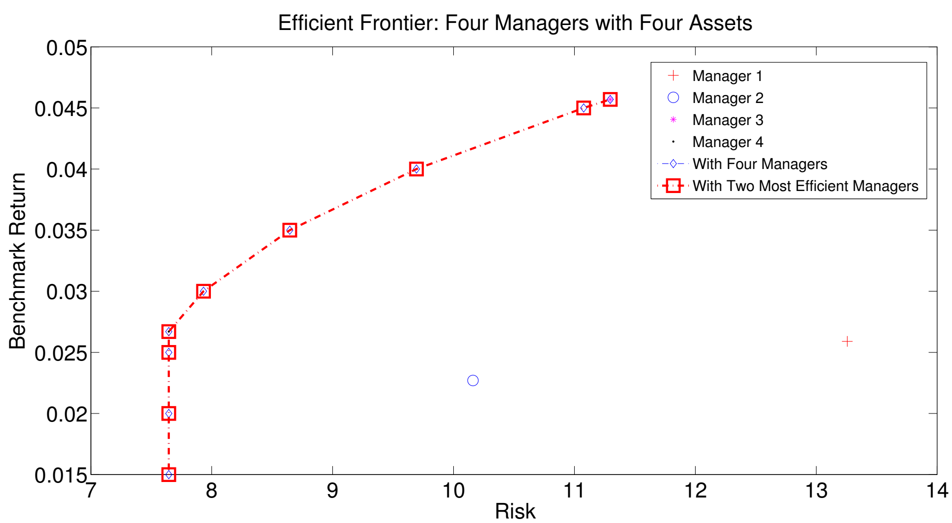

Figure 7: Efficient Frontier: Four Managers with Four Asset Classes

Table 4: Heuristic Method vs Other Methods: Four Managers with Four Asset Classes

	worst variance	nominal return	0.15	0.2	0.25	0.3	0.35	0.4	0.45
Manager 3	11.2965	0.0457	0	0	0	0.1737	0.4368	0.7	0.9632
Manager 4	7.645	0.0267	1	1	1	0.8263	0.5632	0.3	0.0368
Variance (objective)			7.645	7.645	7.645	7.9324	8.6456	9.6935	11.0762
Running Time			9.36	10.43	11.07	12.01	12.02	12.82	13.37
						With Four Managers			
Variance (objective)			7.645	7.645	7.645	7.9324	8.6456	9.6935	11.0762
Running time	(Approach 1)		8.092	7.222	7.294	9.957	16.394	13.791	15.745
Running time	(Approach 2)		13.274	12.95	15.055	25.636	27.134	26.487	44.493

In Figure 8, "With Six Managers" refers to the problem solved with the full fund-manager pool of six managers. "With Two Most Efficient Managers" refers to the problem solved when only the three most efficient managers, as defined above, are considered for possible funding, i.e., the fund-manager pool is reduced according to the heuristic selection.

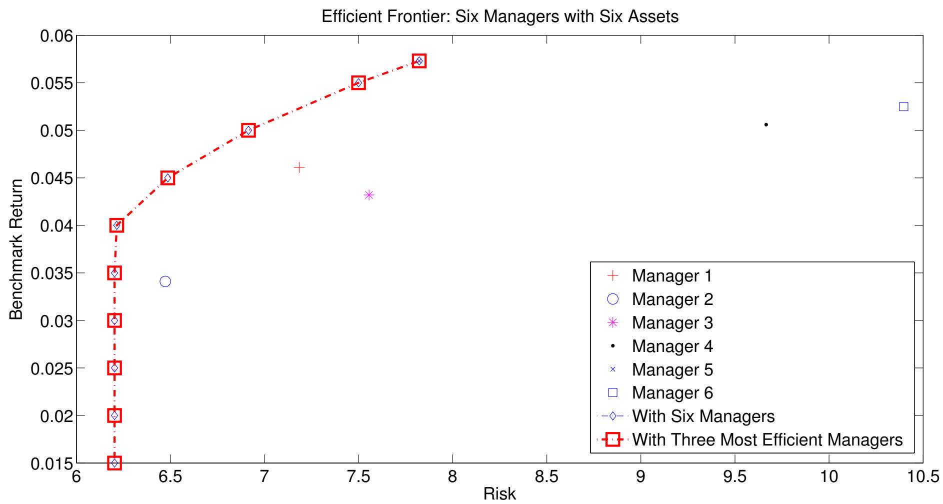

Figure 8: Efficient Frontier: Six Managers with Six Asset Classes

Table 5: Heuristic Method vs Other Methods: Six Managers with Six Asset Classes

	worst variance	nominal return	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5	0.55
Manager 1	7.1842	0.0461	0.4303	0.4303	0.4303	0.4303	0.4303	0.4937	0.3524	0.1778	0.0158
Manager 2	6.4727	0.0341	0.5697	0.5697	0.5697	0.5697	0.5697	0.5063	0.3593	0.2284	0.0915
Manager 5	7.8224	0.0573	0	0	0	0	0	0	0.2883	0.5937	0.8927
Running time			42	100	38	58	36	44	36	65	99
							With Six Managers				
Variance (obj)			6.2028	6.2028	6.2028	6.2028	6.2028	6.2145	6.4857	6.9147	7.4998
Running time	(Approach1)		25768	26467	26218	26423	26487	26407	26672	15859	28566
Running time	(Approach2)		6848	7167	7152	7242	7258	6870	9594	13233	16236

For the case of twelve managers with six asset classes, Managers 4, 5 and 6 are selected as the worst-case efficient managers in the heuristic algorithm. The number of candidate managers is reduced from twelve to three, and running time decreases significantly. However, because up to 5 managers can be selected at optimality of the exact method, the heuristic method yields a larger worst-case variance compared to the other two approaches. The difference in variance decreases from 11% to 0 as the benchmark return increases, as shown in Figure 10.

Table 6: Heuristic Method v.s. Other Methods: Twelve Managers with Six Asset Classes

	worst variance	nominal return	0.015	0.03	0.035	0.04	0.045	0.05	0.055	0.06	0.065
Manager 4	8.9833	0.059	0.8692	0.8693	0.8698	0.6927	0.5109	0.3291	0.1473	0	0
Manager 5	7.6327	0.0316	0.1308	0.1302	0.3073	0.4891	0.6709	0.8527	0.875	0.2171	
			0.1307								
Manager 6	10.2603	0.0667	0	0	0	0	0	0	0	0.125	0.7829
Variance (objective)			7.5945	7.5945	7.5945	7.6518	7.8306	8.1309	8.5538	9.1266	9.9577
Running time			162	92	90	117	90	152	186	29	17
						With Twelve Managers					
Variance (objective)			6.8357	6.8357	6.8387	6.975	7.2184	7.6183	8.1685	8.9515	9.8855
Running time	(Approach2)		30719	30553	31039	54754	55120	57464	48503	27397	1660

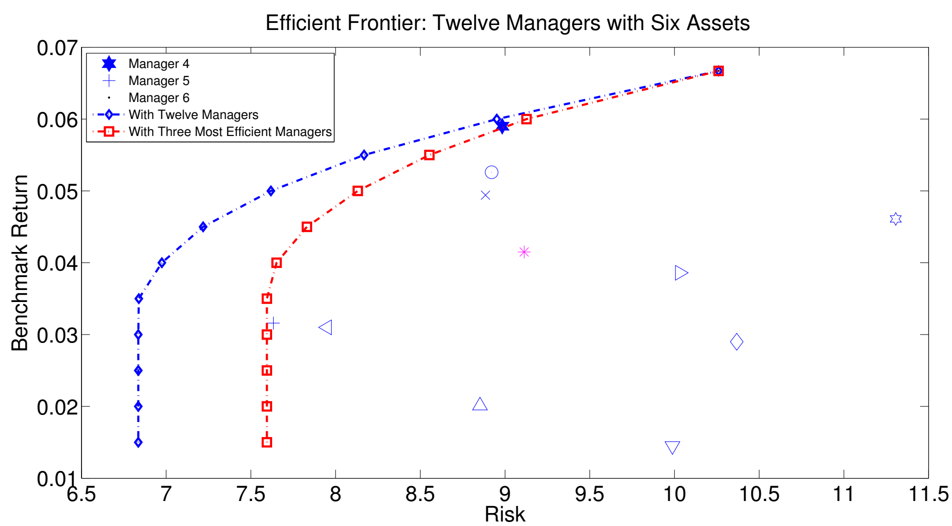

Figure 9: Efficient Frontier: Twelve Managers with Six Asset Classes

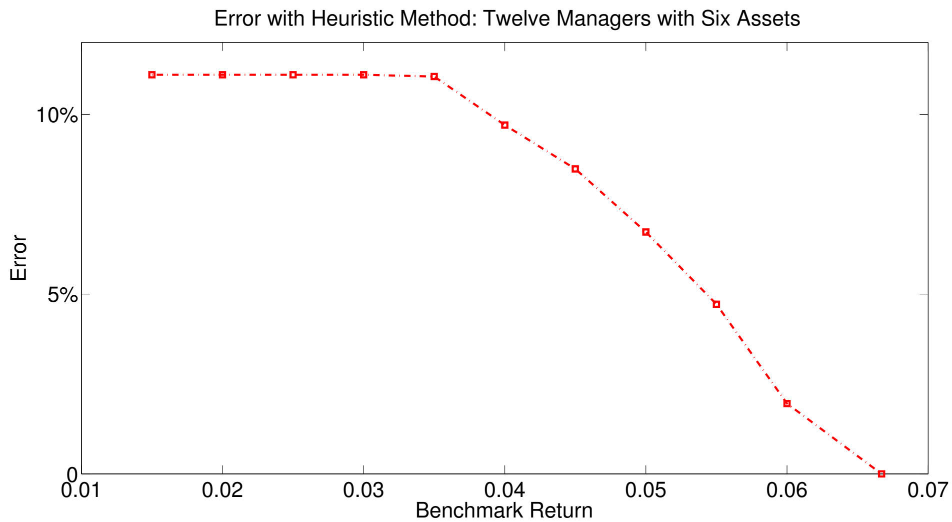

Figure 10: Error with Heuristic Method: Twelve Managers with Six Asset Classes

5 Conclusions

In this paper, we have proposed a robust mean-variance framework to the investment manager's portfolio problem that takes into account the uncertainty stemming from the asset allocation of fund managers, in the context of manager selection and portfolio management. We have also proposed two exact approaches (one that is tractable for small size instances where numerical experiments suggest it also solves the problem faster, and one that is tractable for all problem sizes but can take longer to solve) and a heuristic one preselecting fund managers in order to solve the problem efficiently. Our results indicate that our robust model provides the investment manager with better protection against fund allocation ambiguity in terms of worst-case variance. Future research directions include refining the heuristics for large numbers of fund managers or asset classes if both exact approaches and the heuristic based on worst-case efficient managers become computationally intractable. These heuristics would involve further preselection of fund managers among the worst-case efficient managers, possibly with iterative updating to identify the optimal (in a sense to be defined) subset of fund managers to be considered by the investment manager.

Acknowledgments

The authors would like to thank Lehigh University's Endowment Office for their help in making this paper possible. The authors are also grateful to two anonymous reviewers for their comments, which have significantly improved the clarity of this paper.

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