A discussion on the conservatism of robust linear optimization problems

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A discussion on the conservatism of robust linear optimisation problems

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Abstract

In 2004, Bertsimas and Sim proposed a robust approach that can control the degree of conservatism by applying a limitation $\Gamma$ to the maximum number of parameters that are allowed to change. However, the robust approach can become extremely conservative even when $\Gamma$ is relatively small. In this paper, we provide a theoretical analysis to explain why this extreme conservatism occurs. We further point out that the robust approach does not reach an extremely conservative state when $\Gamma$ is less than $k$, where $k$ is the number of nonzero components of the optimal solution of the extremely conservative robust approach. This research also shows that care must be taken when adjusting the value of $\Gamma$ to control the degree of conservatism because the approach may result in greater conservative than was intended. We subsequently apply our conclusion to additive combinatorial optimization problems. Finally, we employ numerical simulation results to demonstrate our conclusion.

Keywords: robust approaches; conservatism; linear programming; combinatorial optimization
1. Introduction

The classical linear programming model is based on the assumption that the coefficients are precisely known and equal to some constants. However, the assumption completely ignores uncertainty of the coefficients, which are often estimated or predicted using history data and trends. As such, the optimal solution may be substantially suboptimal, or even infeasible. This observation has prompted the development of an approach that circumvents the assumption that all data are certain, and this approach has been denoted as robust approach.

In recent years, robust approaches have been widely used in various fields, and numerous models have been developed by Bertsimas and Sim [1], Soyster [2], Ben-Tal and Nemirovski [3], Jeyakumar and Li [4], and so on. However, the conservatism of robust approaches has been a controversial issue since they were first proposed. To reduce the degree of conservatism, a number of formulations have been developed by Bertsimas and Sim [5], Ben-Tal and Nemirovski [6,7], and El-Ghaoui et al. [8,9]. One of the most influential formulations is that proposed by Bertsimas and Sim [5], who considered the following nominal linear optimization problem.

\[
\begin{align*}
\max \quad & c'x \\
\text{subject to} \quad & \sum_{j \in J} a_{ij} x_j \leq b_i, \forall i \\
& -x \leq 0
\end{align*}
\]

Here, \( J \) is the set of the indices, i.e. \( J = \{1, 2, \ldots, n\} \), where \( n \) is the dimension of \( x \).

For convenience, we have expressed the above constraints as \(-x \leq 0\) rather than \(x \geq 0\). Each entry \( a_{ij} \) is modeled as a symmetric and bounded random variable, and \( \hat{a}_{ij} \) takes values in \([a_{ij} - \hat{a}_{ij}, a_{ij} + \hat{a}_{ij}]\) (see Bertsimas and Sim [5]), where \( \hat{a}_{ij} \geq 0 \). In addition, a limit \( \Gamma \) is applied to the number of coefficients that are allowed to change simultaneously. The model employed by Bertsimas and Sim is given by (1) as follows.
\begin{align*}
\max & \mathbf{c}' \mathbf{x} \\
\text{subject to} & \sum_{j \in J} a_{ij} x_j + \max_{i \in J, \tilde{a}_i \in \Gamma_i} \left( \sum_{j \in J} \tilde{a}_{ij} x_j + (\Gamma - [\tilde{\gamma_i}]) \tilde{a}_{ij} x_j \right) \leq b_i \quad \forall i \\
& -\mathbf{x} \leq \mathbf{0} \\
\end{align*}

(1)

Here, \( J \) is the set of the indices, i.e. \( J = \{1, 2, \ldots, n\} \), where \( n \) is the dimension of \( \mathbf{x} \).

When \( \Gamma \) is equal to \( n \), we obtain the extremely conservative robust model given by (2) as follows [2].

\begin{align*}
\max & \mathbf{c}' \mathbf{x} \\
\text{subject to} & \sum_{j \in J} (a_{ij} + \tilde{a}_i) x_j \leq b_i, \forall i \\
& -\mathbf{x} \leq \mathbf{0} \\
\end{align*}

(2)

For convenience, we refer to Model (2) as the Extremely-Robust model and Model(1) as the \( \Gamma \)-Robust model. The \( \Gamma \)-Robust model is said to be extremely conservative if its optimal solution is the same as the Extremely-Robust model. As is mentioned in [5], the \( \Gamma \)-Robust model gives full control of the degree of the conservatism associated with the constraints. When \( \Gamma \) progressively increases from zero to \( n \), the degree of the conservatism of the above approach is expected to increase gradually. However, we find that this does not hold. Let us consider an example:

\begin{align*}
\max & \quad x_1 + x_2 + x_3 \\
\text{subject to} & \quad x_1 + 2x_2 + 3x_3 \leq 5 \\
& \quad x_1, x_2, x_3 \geq 0 \\
\end{align*}

In this example, we assume that the coefficients have a maximum 10% deviation from their corresponding nominal values. The optimal solution of the nominal model is \( (5, 0, 0) \) achieving an optimal objective of 5. When \( \Gamma \) is 1, 2, or 3, the formulations of the robust approach from Bertsimas and Sim [5] are respectively given by the following.

\begin{align*}
\max & \quad x_1 + x_2 + x_3 \\
\text{subject to} & \quad x_1 + 2x_2 + 3x_3 + \max\{0.1x_1, 0.2x_2, 0.3x_3\} \leq 5 \\
& \quad x_1, x_2, x_3 \geq 0 \\
\end{align*}
The formulations can be linearized (Lemma 1) and thus solved efficiently for \( \Gamma = 1 \) and 2. For \( \Gamma = 3 \), the max in the constraint would disappear since there is only one term. Their optimal solutions are the same, i.e. \( x_1 = 5/1.1, x_2 = 0, x_3 = 0 \), with an objective of \( 5/1.1 \). In other words, the degree of the conservatism of the formulation above does not change when \( \Gamma \) increases from 1 to 3. Moreover, the formulation is extremely conservative even when \( \Gamma \) is equal to 1. Because the optimal robust solution has only one non-zero element here, the solution for the smallest non-zero \( \Gamma \) value is the same as, and thus as conservative as, the solution for the extremely conservative (highest \( \Gamma \) value) case (Theorem 1, 2). The example demonstrates that the \( \Gamma \)-Robust model may become extremely conservative even when \( \Gamma \) is far less than \( n \).

A related work of Thiele [10] focused on conditions where the cost coefficients of a linear programming problem were uncertain. Two examples were given establishing that ‘the decision variables adjust to the uncertain parameters and lead to a more conservative approach than what the manager had planned’. In the present paper, we consider conditions where the coefficients of the constraints of a linear programming problem are uncertain, and reach similar conclusions.

### 1.1. The contributions of the present work

In this paper, we provide a theoretical analysis to demonstrate that the \( \Gamma \)-Robust
model may become extremely conservative even when \( \Gamma \) is far less than \( n \).
Therefore, attempts to control the degree of the conservatism of the \( \Gamma \)-Robust model
by adjusting \( \Gamma \) to a certain value may result in a greater degree of conservatism than
that expected. In addition, we prove that, when \( \Gamma \) is less than \( k \), where \( k \) is the
number of nonzero components of the optimal solution of the Extremely-Robust model,
the \( \Gamma \)-Robust model does not reach the extremely conservative state. Finally, we
apply our conclusion to additive combinatorial optimization problems.

1.2. Structure of the paper

In Section 2, the conditions are given wherein the optimal solution of the \( \Gamma \)-Robust
model and the optimal solution of the Extremely-Robust model are equivalent. In
Section 3, the above idea is applied to robust additive combinatorial optimization
problems. Section 4 presents some simulation results that demonstrate our conclusion.
Section 5 presents concluding remarks.

2. The conditions yielding equivalent optimal solutions

All our conclusions will be under the assumption that the constraints are inequality
constraints. The assumption is without loss of generality since we can use ‘\( \alpha x \geq 0 \)’ and
‘\( -\alpha x \geq 0 \)’ instead of ‘\( \alpha x = 0 \)’.

Because the Extremely-Robust model is extremely conservative, our discussion
will be equivalent to evaluating the value at which, as \( \Gamma \) increases, the optimal
solution of the \( \Gamma \)-Robust model is equivalent to that of the Extremely-Robust model.

Note that the \( \Gamma \)-Robust model is not linear. Fortunately, Bertsimas and Sim
have reformulated the \( \Gamma \)-Robust model to that of an equivalent linear formulation.

Lemma 1. The \( \Gamma \)-Robust model is equivalent to the following linear formulation.
\[ \beta(\Gamma) = \max c'x \]
subject to
\[ \sum_{j \in J} a_{ij} x_j + z \Gamma + \sum_{j \in J} p_{ij} \leq b_i \quad \forall i \]
\[ -x_i \leq 0 \quad \forall j \]
\[ -z_i \leq 0 \quad \forall i \]
\[ -p_{ij} \leq 0 \quad \forall i, j \]

**Proof** See Bertsimas and Sim [5].

Suppose \( \bar{x} \) is a feasible solution of the \( \Gamma \)-Robust model. Below, we construct the corresponding feasible solution \( (\bar{x}, \bar{z}, \bar{p}) \) of Model (3). For any fixed \( i \), sort
\[ \{ \hat{a}_{ij}, \forall j \} \]
in a descending order. Let the sorted \( \{ \hat{a}_{ij}, \forall j \} \) be
\[ \{ \hat{a}_{i_{1j}}, \hat{a}_{i_{2j}}, \ldots, \hat{a}_{i_{t_{ij}}}, \ldots, \hat{a}_{i_{nj}} \} \]. Define \( (\bar{x}, \bar{z}, \bar{p}) \) according to the following.

\[
\begin{align*}
\bar{x} &= \bar{x} \\
\bar{z_i} &= \max \{ \hat{a}_{ij}, j \in S_i \} \\
\bar{p}_{ij} &= \begin{cases} 0 & \text{if } j \in S_i \\ \hat{a}_{ij} \bar{x}_j - \bar{z}_i & \text{if } j \notin S_i \end{cases}
\end{align*}
\]

Here, \( S_i = \{ j_1, j_2, \ldots, j_{t_{ij}} \} \), \( S_i = J \setminus S_i \).

**Lemma 2.** \( (\bar{x}, \bar{z}, \bar{p}) \) is a feasible solution of Model (3).

**Proof** Firstly, \( (\bar{x}, \bar{z}, \bar{p}) \) satisfies nonnegative constraints. Secondly, it is obvious that \( (\bar{x}, \bar{z}, \bar{p}) \) satisfies the constraints \( \hat{a}_{ij} \bar{x}_j - \bar{z}_i - \bar{p}_{ij} \leq 0 \). Thirdly, we have the following.

\[
\sum_{j \in J} a_{ij} \bar{x}_j + \bar{z}_i \Gamma + \sum_{j \in J} \bar{p}_{ij} = \sum_{j \in J} a_{ij} \bar{x}_j + \bar{z}_i \Gamma + \sum_{j \in J} (\hat{a}_{ij} \bar{x}_j - \bar{z}_i) = \sum_{j \in J} a_{ij} \bar{x}_j + \bar{z}_i (\Gamma - \lceil \Gamma \rceil) + \sum_{j \in J} \hat{a}_{ij} \bar{x}_j
\]
\[
\leq \sum_{j \in J} a_{ij} \bar{x}_j + \max_{\{ s \leq \Gamma \} \cup J \setminus \{ \Gamma \} } \left\{ \sum_{j \in J} \hat{a}_{ij} \bar{x}_j + (\Gamma - \lfloor \Gamma \rfloor) \hat{a}_{ij} \bar{x}_j \right\} \leq b_i
\]

Let \( S_i \) be \( \tilde{S}_i \) and \( \hat{a}_{ij} \bar{x}_j \) be \( \bar{z}_i \), and the first non-strict inequality ‘\( \leq \)’ follows, and we conclude that \( (\bar{x}, \bar{z}, \bar{p}) \) is a feasible solution of Model (3).

For any linear programming problem, we give the following definition.
maximize \( w^T x \)
subject to \( H x \leq q \)
\[- I x \leq 0\]  

(5)

Here, \( I \) is the identity matrix.

Definition 1: for a constraint \( h_i x \leq q_i \), the constraint is called active at \( x \) if \( h_i x = q_i \).

For example, consider the following linear programming problem.

maximize \( 2x_1 + 3x_2 \)
subject to \( x_1 + x_2 \leq 5 \)
\( x_1 + 2x_2 \leq 7 \)
\(-x_1 \leq 0, -x_2 \leq 0 \)

In the above definition, the matrix constraints are \( x_1 + x_2 \leq 5, x_1 + 2x_2 \leq 7 \) and the sign constraints are \(-x_1 \leq 0, -x_2 \leq 0 \). For the solution \((5,0)\), the active constraints are \( x_1 + x_2 \leq 5, -x_2 \leq 0 \). The coefficient matrix of the active constraints is \( \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \).

Let \( \bar{x} \) be a feasible solution of Formulation (5). Let \( h_i \) be the \( i \)th row of \( H \), \( e_i \) be the \( i \)th row of \( I \), and \( q_i \) be the \( i \)th element of \( q \). Suppose there are \( p \) rows and \((r - p)\) rows in ‘\( H \bar{x} \leq q \)’ and ‘\(-I \bar{x} \leq 0 \)’, respectively, that the left side is equal to the right. Define \( H_{\bar{x}}, I_{\bar{x}}, \) and \( D_{\bar{x}} \) as follows:

\[
H_{\bar{x}} = \begin{bmatrix} h_1 \\ \vdots \\ h_p \end{bmatrix}, \quad I_{\bar{x}} = \begin{bmatrix} e_{i_p} \\ \vdots \\ e_{i_r} \end{bmatrix} \quad \text{and} \quad D_{\bar{x}} = \begin{bmatrix} H_{\bar{x}} \\ -I_{\bar{x}} \end{bmatrix},
\]

where \[
\begin{align*}
&h_i \bar{x} = q_i \quad \forall m \in \{1, \ldots, p\} \\
e_i \bar{x} = 0 \quad \forall m \in \{p + 1, \ldots, r\}.
\end{align*}
\]

In fact, \( D_{\bar{x}} \) is the coefficient matrix of the active constraints at \( \bar{x} \) of Formulation (5).

Before giving the main result, the following lemma is needed.
Lemma 3. Assume \( w \neq 0 \). Let \( \tilde{x} \) be a feasible solution of Formulation (5). The necessary and sufficient condition that \( \tilde{x} \) is the optimal solution of Formulation (5) is that \( w \in D_\tilde{x} \), where \( D_\tilde{x} \) is the coefficient matrix of the active constraints at \( \tilde{x} \) and
\[
D_\tilde{x} = \{ w | w = D_\tilde{x} \hat{\lambda}, \lambda \geq 0 \}.
\]

Proof. Firstly, we will prove that \( \tilde{x} \) is the optimal solution of model (5) if \( w \in D_\tilde{x} \). For any \( x \) that is a feasible solution of Formulation (5), it satisfies \( D_x x \leq q_d \), where \( q_d \) is a subvector of \( \begin{bmatrix} q \\ 0 \end{bmatrix} \) corresponding to \( D_x \). According to the definition of \( D_x \), \( q_d = D \tilde{x} \), and, due to \( w \in D_\tilde{x} \), there exists a \( \hat{\lambda} \) satisfying \( w = D_x \hat{\lambda}, \hat{\lambda} \geq 0 \). Then, we have
\[
D_x x \leq q_d \Rightarrow D_x x \leq D_x \tilde{x} \Rightarrow \hat{\lambda}' D_x x \leq \hat{\lambda}' D_x \tilde{x} \Rightarrow w' x \leq w' \tilde{x}.
\]
The statement follows immediately.

Secondly, it is proved that \( w \in D_\tilde{x} \) if \( \tilde{x} \) is the optimal solution of Formulation (5). The dual of Formulation (5) is as follows.
\[
\min \quad q'y
\text{subject to} \quad H'y \geq w

\quad -y \leq 0
\]

(6)

Let \( y \) be the optimal solution of (6). According to complementary slackness, if \( h_x \tilde{x} \leq q_i \), \( \bar{y}_i = 0 \). Let \( \bar{w} = H'y \). Then, we have \( H_x' \bar{y}_x = H'y = \bar{w} \geq w \), where \( \bar{y}_x \) is a subvector of \( \bar{y} \) corresponding to \( H_x \). Let \( \bar{w}_i \) and \( w_i \) be the ith element of \( \bar{w} \) and \( w \), respectively. If \( \bar{w}_i = w_i \), then \( x_i = 0 \) (complementary slackness), i.e. \( e_i \) is a row of \( I_x \). According to the definition of \( D_x \), we have
\[
\mathbf{w} = \begin{bmatrix} H'_{x} & -I'_{x} \end{bmatrix} \begin{bmatrix} \tilde{y}_{x} \\ \tilde{w}_{i_{p+1}} - w_{i_{p+1}} \\ \vdots \\ \tilde{w}_{i_{k}} - w_{i_{k}} \end{bmatrix} = D'_{x} \begin{bmatrix} \tilde{y}_{x} \\ \tilde{w}_{i_{p+1}} - w_{i_{p+1}} \\ \vdots \\ \tilde{w}_{i_{k}} - w_{i_{k}} \end{bmatrix}.
\]

That is, there exists a \( \tilde{\lambda} \geq 0 \) that satisfies \( \mathbf{w} = D'_{x} \tilde{\lambda} \). The statement follows immediately.

Hence, \( \tilde{x} \) is the optimal solution of Formulation (5) \( \leftrightarrow \mathbf{w} \in D_{x} \). \( \blacksquare \)

Remark 1 \( D_{x} \) is an empty set when \( \tilde{x} \) is an inner point.

Remark 2 Lemma 3 can be extended to the case that includes the equality constraints. We will not discuss this case here.

Suppose \( x^{*} \) is the optimal solution of the Extremely-Robust model with \( \mathbf{c} \) as the target vector, and \( (x^{*},z^{*},p^{*}) \) is the corresponding feasible solution of Model (3). Let 
\( D_{x}(1) \) be the coefficient matrix of the active constraints of the Extremely-Robust model at \( x^{*} \), and 
\( D_{x}(0) = \{ c | c = D_{x}(0) \tilde{\lambda}, \tilde{\lambda} \geq 0 \} \). Let \( D_{x}(2)(\Gamma) \) be the coefficient matrix of the active constraints of Model (3) at \( (x^{*},z^{*},p^{*}) \), and
\( D_{x}(2)(\Gamma) = \{ c | (c,0,0) = D_{x}(2)(\Gamma) \tilde{\lambda}, \tilde{\lambda} \geq 0 \} \).

In the following we will give the necessary and sufficient condition that \( x^{*} \) is the optimal solution of \( \Gamma \)-Robust model and point out that \( D_{x}(2)(\Gamma) \) increases monotonically.

**THEOREM 1:**

(a) \( \mathbf{c} \in D_{x}(1) \);

(b) If \( \Gamma \geq k \), \( x^{*} \) is the optimal solution of the \( \Gamma \)-Robust model iff
\[
\mathbf{c} \in D_{x}(2)(\Gamma).
\] (7)

(c) \( D_{x}(2)(\Gamma) \subset D_{x}(2)(\Gamma + 1) \)

(d) \( D_{x}(2)(n) \) is equivalent to \( D_{x}(0) \).

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Proof

(a) By Lemma 3, the statement follows immediately.

(b) By Lemma 3, the statement follows immediately.

(c) Suppose there exists a $\varepsilon$ that belongs to $\mathcal{D}_\varepsilon^{(2)}(\Gamma)$ but not $\mathcal{D}_\varepsilon^{(2)}(\Gamma+1)$. When the target vector is $\varepsilon$, by Lemma 3, $(x',z',p')$ is the optimal solution of Formulation $\beta(\Gamma)$ (see Model (3)), but not the optimal solution of Formulation $\beta(\Gamma+1)$. However, the feasible region of Formulation $\beta(\Gamma+1)$ is a subset of the feasible region of Formulation $\beta(\Gamma)$ and $(x',z',p')$ is the feasible solution of both. Therefore, if $(x',z',p')$ is the optimal solution of Formulation $\beta(\Gamma)$, it must be the optimal solution of Formulation $\beta(\Gamma+1)$. The assumption fails.

(d) $\beta(n)$ is equivalent to the Extremely-Robust model. The statement follows immediately.

Remark 3 According to Theorem 1 (c) and (d), $\mathcal{D}_\varepsilon^{(2)}(\Gamma)$ is a subset of $\mathcal{D}_\varepsilon^{(3)}$, and, by Theorem 1(a) and (b), when $\varepsilon$ belongs to $\mathcal{D}_\varepsilon^{(2)}(\Gamma) \cap \mathcal{D}_\varepsilon^{(3)}$, the optimal solution of Formulation $\beta(\Gamma)$ is the same as that of the Extremely-Robust model. This explains why the $\Gamma$-Robust model may become extremely conservative even when $\Gamma$ is far less than $n$. Moreover, only if $\varepsilon$ belongs to $\mathcal{D}_\varepsilon^{(3)}(n) \setminus \mathcal{D}_\varepsilon^{(3)}(n-1)$, the $\Gamma$-Robust model does not reach the extremely conservative state unless $\Gamma$ is equal to $n$.

We give the necessary and sufficient condition that $x'$ is the optimal solution of $\Gamma$-Robust model above. Bellow we will give the condition that $x'$ is not the optimal solution of $\Gamma$-Robust model.
THEOREM 2 Let $k$ be the number of nonzero components in $x^*$. If

$$\sum_{j=1}^k |c_j x_j^*| > 0, \quad k \geq 1 \text{ and } 1 \leq k \leq k-1, \quad x^* \text{ is not the optimal solution of the } \Gamma \text{-Robust model.}$$

Proof

1. If $\Gamma$ is not an integer, then the following holds.

$$\sum_{j \in J \setminus (S \cup \{t_i\})} (a_{ij} + \hat{a}_{ij}) x_j^* + \sum_{j \in S} (a_{ij} + \hat{a}_{ij}) x_j^* + \sum_{j \in S \setminus \{t_i\}} \hat{a}_{ij} x_j^* = \sum_{j \in J \setminus (S \cup \{t_i\})} (a_{ij} + \hat{a}_{ij}) x_j^* + \sum_{j \in S \setminus \{t_i\}} \hat{a}_{ij} x_j^* \quad (8)$$

Because there are $k$ components in $x^*$ that are nonzero and $|S| \cup \{t_i\} = \lceil \Gamma \rceil \leq k-1$, there exists at least one $j \in J \setminus (S \cup \{t_i\})$ that satisfies $x_j > 0$, i.e.,

$$\sum_{j \in J \setminus (S \cup \{t_i\})} \hat{a}_{ij} x_j^* > 0. \text{ Then, the following hold.}$$

$$\left( \sum_{j \in S} (a_{ij} + \hat{a}_{ij}) x_j^* + [a_{ij} + (\Gamma - \lceil \Gamma \rceil) \hat{a}_{ij}] x_j^* + \sum_{j \in J \setminus (S \cup \{t_i\})} \hat{a}_{ij} x_j^* \right) < \sum_{j \in J} (a_{ij} + \hat{a}_{ij}) x_j^* \quad (9)$$

$$\leq b_i, \quad \forall S_i \subseteq J, |S_i| = \lceil \Gamma \rceil, t_i \in J \setminus S_i, \forall i \quad (10)$$

$$\sum_{j \in S} (a_{ij} + \hat{a}_{ij}) x_j^* + [a_{ij} + (\Gamma - \lceil \Gamma \rceil) \hat{a}_{ij}] x_j^* + \sum_{j \in J \setminus (S \cup \{t_i\})} \hat{a}_{ij} x_j^* < b_i, \quad \forall S_i \subseteq J, |S_i| = \lceil \Gamma \rceil, t_i \in J \setminus S_i, \forall i \quad (11)$$

$$\sum_{j \in J \setminus \{t_i\}} \hat{a}_{ij} x_j^* + \max_{|S_i| \leq \lceil \Gamma \rceil, t_i \in J \setminus S_i} \left( \sum_{j \in S_i} (a_{ij} + \hat{a}_{ij}) x_j^* + (\Gamma - \lceil \Gamma \rceil) \hat{a}_{ij} x_j^* \right) < b_i, \forall i \quad (12)$$

Inequality (9) follows from Inequality (8) and $\sum_{j \in J \setminus (S \cup \{t_i\})} \hat{a}_{ij} x_j^* > 0$. Because $x^*$ is the optimal solution of the Extremely-Robust model, Inequality (10) follows. Inequality (11) follows from Inequality (9) and Inequality (10). Inequality (12) follows from Inequality (11).

Due to $\sum_{j=1}^k |c_j x_j^*| > 0$, there exists at least one $j$ to make both $x_j^*$ and $c_j$ nonzero.

Suppose $j$ is equal to 1 and $|c|$ is a number that is sufficiently small to satisfy
$c, \varepsilon > 0$. Set $\tilde{x} = x^* + (\varepsilon, 0 \cdots 0)^T$. For $|\varepsilon|$ is sufficiently small and $x^*$ satisfies the strict Inequality (12), $\tilde{x}$ satisfies Inequality (12) and is a feasible solution of Model (1). Then we have $c\tilde{x} = cx^* + c, \varepsilon > cx^*$.

(2) If $\Gamma$ is an integer, the only difference is that there exists no $(\Gamma - \bar{\alpha})a_s x_s$ and the proof is similar to the proof given above. We omit it here. Hence, $x^*$ is not the optimal solution of the $\Gamma$-Robust model.

3. Application on additive combinatorial optimization problems

In this section, we will apply our conclusion to combinatorial optimization problems. Firstly, a definition of absolute robust additive combinatorial optimization problems is given.

Let $T$ be a finite set, $|T| = M$, and $E \subset 2^T$ be a set of feasible subsets of $T$. $w_t$ is the weight assigned to $t \in T$ and $w_t \in [a_i - \hat{a}, a_i + \hat{a}]$ where $a_i$ is a real number and $\hat{a}$ is a positive real number. A vector $s = \{w_t, t \in T\}$ is called a scenario (as described by Averbakh [11]). Let $S$ be the set of all scenarios. For a feasible subset $X \in E$ and a scenario $s$, define

$$F(s, X) = \sum_{t \in X} w_t^s.$$  

A class of absolute robust additive combinatorial optimization problems can be defined as follows:

ARO Problem OPT: $\min_{x} \max_{s \in S} F(s, X)$.

Since all the parameters are allowed to change, ARO Problem is extremely conservative.

A scenario $\hat{s}$ is dominant if it satisfies

$$F(\hat{s}, X) \geq F(s, X) \quad \forall s \in S, X \in E.$$  

Lemma 4: $s_a = \{a_i + \hat{a}, t \in T\}$ is a dominant scenario.
Proof  For \( w' \in [a_i - \hat{a}_i, a_i + \hat{a}_i] \), \( w' \leq a_i + \hat{a}_i \), and, for \( s \in S \), \( X \in E \),

\[
F(s, X) = \sum_{n \in X} (a_i + \hat{a}_i) \geq \sum_{n \in X} w'_n = F(s, X), \text{ i.e., } F(s, X) \geq F(s, X), \forall s \in S, X \in E.
\]

The statement of the lemma follows immediately.

Similarly, to control the degree of conservatism, we apply a limitation \( \Gamma \) to the number of elements that are assigned the upper bounds. Suppose \( \Gamma \) is a positive integer. \( S(\Gamma) \) is a set of scenarios

\[
S(\Gamma) = \{ s \in S | \text{ there are at most } \Gamma \text{ elements } t \in T \text{ that assigned the upper bounds } a_i + \hat{a}_i \}.
\]

The \( \Gamma \) absolute robust additive combinatorial optimization problems can be defined as follows:

\[
\Gamma \text{-ARO Problem OPT: } \min_{X \in E} \max_{s \in S(\Gamma)} F(s, X).
\]

For \( \Gamma \) -ARO Problem, there are at most \( \Gamma \) elements that are allowed to change. When \( \Gamma \) progressively increases, the degree of the conservativeness of \( \Gamma \) -ARO Problem is expected to increase gradually. However, this does not hold. We will give the value at which, as \( \Gamma \) increases, the optimal solution of ARO Problem is the same as the \( \Gamma \) -ARO Problem. The \( \Gamma \) -ARO Problem is said to be extremely conservative if

\[
\min_{X \in E} \max_{s \in S(\Gamma)} F(s, X) = \min_{X \in E} \max_{s \in S} F(s, X).
\]

To motive the following result, we give a simple example. Consider for example the complete undirected graph on three nodes \( G(E, V) \), i.e. \( V = \{ 1, 2, 3 \} \), \( E = \{ (1, 2), (1, 3), (2, 3) \} \). Let \( c_{ij} \) be the nominal weight on edge \((i, j)\). Consider \( c_{12} = 2, c_{13} = 3, c_{23} = 4 \).

Assume that the weights have a maximum 10% deviation from their corresponding nominal values. The optimal solution of the nominal minimum spanning tree problem on \( G(E, V) \) is \( \{ (1, 2), (1, 3) \} \) and the sum of weights is 5. By Lemma 4, we can compute that the optimal solution of ARO minimum spanning tree problem on \( G(E, V) \) is \( \{ (1, 2), (1, 3) \} \) and the sum of weights is 5.5. By the definition above, we
have \( S(1) = \{(2.2,3,4),(2,3.3,4),(2,3,4.4)\} \). For every tree compute the sum of weights on all \( s \in S(1) \) and we can get that the optimal solution of 1-ARO minimum spanning tree problem is \( \{(1,2),(1,3)\} \) and the sum of weights is 5.2. Similarly, we can get that the optimal solution of 2-ARO minimum spanning tree problem on \( G(E,V) \) is \( \{(1,2),(1,3)\} \) and the sum of weights is 5.5, which is the same as ARO minimum spanning tree problem. Below we will give the reason why the optimal solution of 2-ARO Problem is the same as the ARO Problem.

**THEOREM 3** Suppose \( X^* \) is the optimal solution of an ARO problem. Let \(|X|\) be the number of the elements in \( X \) and \( \Gamma_i = \max_{X \in \mathcal{E}} |X| \).

(a) \( X^* \) is the optimal solution of \( \Gamma_i \)-ARO problems,

(b) \( \min_{X \in \mathcal{E}} \max_{s \in \mathcal{S}(\Gamma_i)} F(s,X) = \min_{X \in \mathcal{E}} \max_{s \in \mathcal{S}(\Gamma_i)} F(s,X) \)

**Proof**

(a) \[ \min_{X \in \mathcal{E}} \max_{s \in \mathcal{S}(\Gamma_i)} F(s,X) = \min_{X \in \mathcal{E}} \sum_{s \in \mathcal{S}(\Gamma_i)} (a_i + \hat{a}_i) = \sum_{s \in \mathcal{S}(\Gamma_i)} (a_i + \hat{a}_i) \quad (13) \]

The first equal sign follows from Lemma 4. For any fixed \( X \), we have

\[ \max_{s \in \mathcal{S}(\Gamma_i)} F(s,X) = \max_{s \in \mathcal{S}(\Gamma_i)} \sum_{e \in X} w'_e \leq \sum_{e \in X} (a_i + \hat{a}_i) \quad (14) \]

We construct scenario \( s_x \) as follows.

Assign \( a_i \) as weights to all elements \( e \in E \setminus X \) and assign \( a_i + \hat{a}_i \) as weights to all elements \( e \in X \).

Obviously \( s_x \in S(\Gamma_i) \). Then, we have

\[ \max_{s \in \mathcal{S}(\Gamma_i)} \sum_{e \in X} w'_e \geq \sum_{e \in X} w'_{s_x} = \sum_{e \in X} (a_i + \hat{a}_i) \quad (15) \]

\[ \max_{s \in \mathcal{S}(\Gamma_i)} F(s,X) = \sum_{e \in X} (a_i + \hat{a}_i) \quad (16) \]
Equality (16) follows from Inequality (14) and Inequality (15). The first equal sign of Equality (17) follows from Equality (16), and the second follows Equality (13). The statement of the theorem follows immediately.

(b) By Equality (13) and (17), the statement follows immediately.

Remark 4 Suppose \( G(V, E) \) is a graph, \( V \) is the set of vertices, \( E \) is the set of edges, and \( N = |V| \). With Theorem 3, we can draw the following conclusions.

(i) When \( \Gamma = N - 1 \), the optimal solution of an ARO minimum spanning tree problem is the optimal solution of \( \Gamma \) ARO minimum spanning tree problems.

(ii) When \( \Gamma = \frac{N}{2} \), the optimal solution of an ARO minimum matching problem is the optimal solution of \( \Gamma \) ARO minimum matching problems.

(iii) When \( \Gamma = N - 1 \), the optimal solution of an ARO shortest path problem is the optimal solution of \( \Gamma \) ARO shortest path problems because the number of edges that each path includes is not more than \( N - 1 \).

We illustrate several examples here. These examples show that the optimal solution of an ARO problem may be the optimal solution of \( \Gamma \) ARO problems even when \( \Gamma \ll |T| \). In other words, the optimal solution of the case that all the edges are assigned the upper bounds may be the same as that of the case when only \( \Gamma \) edges are assigned the upper bounds. This indicates that \( \Gamma \) ARO problems may become extremely conservative even when \( \Gamma \ll |T| \).

4. Simulation results

In this section, we provide simulation results that demonstrate our conclusions. Prior to
giving the results, we will first restate the definitions of the relative parameters:

\[ m: \text{the number of rows of the constraint matrix } A; \]

\[ n: \text{the number of columns of the constraint matrix } A; \]

\[ k_1: \text{the minimum number for which the optimal solution of the } \Gamma\text{-Robust model is equivalent to that of the Extremely-Robust model when } \Gamma = k_1. \]

In our study, we assume that the coefficients of the constraint matrix \( A \) have a 10% deviation from their corresponding nominal values, and \( n/m \) varies from 1 to 5. For every fixed \( n/m \), we run 5000 simulations of random yields. The simulation results are listed in Table 1.

Table 1 presents the distribution of \( k_1 \) from the 5000 simulations. For example, the first column and forth row of Table 1 show that there are 210 simulations whose corresponding \( k_1/n \) belongs to \((0.2, 0.3]\) among the 5000 simulations under the condition that \( n/m = 1 \).

Table 1 shows that nearly all the \( k_1/n \) simulations are less than or equal to 0.5. This indicates that, when \( \Gamma = k_1 \), which is much less than \( n \) for most cases, the optimal solution of the \( \Gamma\text{-Robust model is equivalent to that of the Extremely-Robust model. Furthermore, Table 1 also shows that, generally speaking, the value of } n/m \text{ decreases with increasing } k_1/n. \) For example, among the 5000 simulations, when \( n/m = 5 \), there are 4514 simulations whose corresponding values of \( k_1/n \) are less than or equal to 0.2. In other words, if the value of \( n/m \) is relatively large, the \( \Gamma\text{-Robust model is more likely to become extremely conservative. All the results above suggest that we should be cautious when seeking to control the degree of conservatism of a robust approach by adjusting the value of } \Gamma, \text{ for we may obtain a much more conservative approach than anticipated.} \]
5. Concluding remarks

The discussion indicates that we must be cautious when seeking to control the degree of conservatism by limiting the maximum number of parameters that are allowed to take their worst-case value. We have demonstrated in theory that both robust linear optimization problems and absolute robust additive combinatorial optimization problems may become extremely conservative even when the maximum number of parameters that are allowed to take their worst-case value are relatively small. We have also provided simulation results of robust linear optimization problems that demonstrate our theoretical conclusions, and the results are in accordance with our conclusions, particularly when the number of rows of the constraint matrix is far less than the number of columns.

References


Table 1. Results of the distribution $k_1$ with varying $n/m$.

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<tr>
<th>$n/m$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
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<tr>
<td>$k_1/n$</td>
<td>0</td>
<td>105</td>
<td>158</td>
<td>822</td>
<td>1039</td>
</tr>
<tr>
<td></td>
<td>(0.0,0.1]</td>
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<td>9</td>
<td>22</td>
<td>50</td>
</tr>
<tr>
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<td>110</td>
<td>656</td>
<td>640</td>
</tr>
<tr>
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<td>927</td>
<td>2845</td>
</tr>
<tr>
<td></td>
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<td>2991</td>
<td>2364</td>
<td>280</td>
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<tr>
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<td>1245</td>
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<tr>
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<td>9</td>
<td>39</td>
<td>1</td>
</tr>
<tr>
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<td>3</td>
</tr>
<tr>
<td></td>
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<td>1</td>
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<tr>
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