A POLYHEDRAL INVESTIGATION OF STAR COLORINGS

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Abstract. Given a weighted undirected graph G and a nonnegative integer k, the maximum k-star colorable subgraph problem consists of finding an induced subgraph of G which has maximum weight and can be star colored with at most k colors; a star coloring does not color adjacent nodes with the same color and avoids coloring any 4-path with exactly two colors. In this article, we investigate the polyhedral properties of this problem. In particular, we characterize cases in which the inequalities that appear in a natural integer programming formulation define facets. Moreover, we identify graph classes for which these base inequalities give a complete linear description. We then study path graphs in more detail and provide a complete linear description for an alternative polytope for $k = 2$. Finally, we derive complete balanced bipartite subgraph inequalities and present some computational results.

1. Introduction

For an undirected graph $G = (V, E)$, a coloring of G is an assignment of colors to the nodes of G such that no two adjacent nodes receive the same color. A *star coloring* of G is a coloring such that no four consecutive nodes on a path are colored with exactly two colors. Star colorings were introduced by Grünbaum [15] and have applications in the computation of sparse Hessians, see, e.g., Coleman and Moré [6] and Section 2 for more details.

The goal of this paper is to investigate the polyhedral properties of star colorings; to the best of our knowledge this has not been considered before. We consider a positive integer k and the corresponding polytope $\mathrm{P}^{\star}_{k}(G)$, which is the convex hull of incidence vectors of star colorings of subgraphs of G using at most k colors.

An integer programming (IP) formulation of star colorings, with binary variables indicating whether a node receives a given color, contains three nontrivial families of inequalities: *packing inequalities*, indicating that a node receives at most one color, *edge inequalities* forbidding adjacent nodes with the same color, and *star inequalities* ensuring the star condition mentioned above. We show that packing inequalities always define facets if $k \geq 2$. Edge inequalities are (possibly) dominated by clique inequalities, which define facets if and only if the cliques are maximal. This is similar to the stable set polytope, see, e.g., Chvátal [5] and Padberg [30]. A characterization for star inequalities to define facets is more complex, depending on the structure of the neighborhood of the path.

Afterwards, we focus on complete linear descriptions of $\mathrm{P}^{\star}_{k}(G)$. We first characterize graphs for which the above three families of inequalities and nonnegativity constraints completely describe the polytope. It turns out that

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the star inequalities for the corresponding graphs are in fact redundant. This highlights the surprising fact that the mentioned inequalities never suffice to describe the polytope for cases in which the star coloring constraints are "active" (non-redundant).

A base case for which the investigation of the structure of the polytope $\mathrm{P}_k^\star(G)$ seems to be particularly interesting are path graphs. However, it turns out that $\mathrm{P}_k^\star(G)$ is rather complicated for $k = 2$ even for path graphs. We therefore consider a projected formulation that indicates whether a node is colored or not, but does not distinguish the two colors. We then obtain a complete description using only trivial inequalities plus "projected star inequalities"; in fact, we prove that this description is totally dual integral (TDI). The projection, however, is only valid for $k = 2$.

Furthermore, we investigate a generalization of star inequalities to complete bipartite graphs. We prove that the corresponding inequalities define facets of $\mathrm{P}_k^\star(G)$, if G is a balanced complete bipartite graph. These inequalities are then used in an implementation, for which we present computational experiments.

As mentioned above, one application of the star coloring problem is the efficient computation of sparse Hessians via automatic differentiation, see, e.g., Coleman and Moré [6], Gebremedhin et al. [12, 13, 14]. However, star colorings have been investigated in the literature mainly with respect to lower and upper bounds, i.e., the computation of star colorings with the least number of used colors (*star chromatic number*) and lower bounds on the number of colors needed, see, e.g., Albertson et al. [2], Fertin et al. [9], and Lyons [25]. Lyons [25] identifies graph classes for which the number of colors needed coincides with the number of colors in an ordinary coloring. Moreover, Lyons [26] investigates classes of graphs for which the star chromatic number equals the so-called acyclic chromatic number (acyclic colorings also arise in the context of sparse Hessians).

The polyhedral properties of ordinary colorings have been investigated, for instance, in Mendez-Díaz and Zabala [27, 28] and Coll et al. [7]. The problem of finding subgraphs of a given graph that can be colored with a specific number of colors is studied by Narasimhan [29]. A polyhedral analysis of this problem and an investigation of its algorithmic treatment as well as the handling of color and graph symmetries can be found in [19, 20].

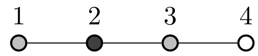
The structure of this paper is as follows: In Section 2 we describe the application of star colorings in the computation of sparse Hessians, and in Section 3 we introduce some basic notation. A polyhedral description for the maximum k-star colorable subgraph problem is developed in Section 4, and in Section 5 we provide graph classes for that we can find a complete linear description. In Section 6 we derive a projected polyhedral model for $k = 2$, and we study polyhedral properties of the new model and connections to the previously defined polytope. Section 7 applies the results from the previous section to path graphs, whereas Section 8 provides further facet inducing inequalities for $\mathrm{P}_k^\star(G)$. Computational results on the maximum k-star colorable subgraph problem are presented in Section 9, and we conclude this paper in Section 10.

$$H = \begin{pmatrix} \star & \star & 0 & 0 \\ \star & \star & \star & 0 \\ 0 & \star & \star & \star \\ 0 & 0 & \star & \star \end{pmatrix}$$

(a) nonzero pattern of Hessian

$$\begin{pmatrix} \star & \star & 0 & 0 \\ \star & \star & \star & 0 \\ 0 & \star & \star & \star \\ 0 & 0 & \star & \star \end{pmatrix}$$

(b) column partition



(c) adjacency graph

FIGURE 1. Example for the computation of sparse Hessians.

2. COMPUTATION OF SPARSE HESSIANS

Since the above mentioned application to the computation of sparse Hessians is important, we will briefly describe the connection to star colorings in this section. The presentation is based on Gebremedhin et al. [12, Section 4].

Suppose we are given a twice continuously differentiable function $f : \mathbb{R}^n \to \mathbb{R}$ and the goal is to efficiently compute its Hessian H, e.g., in a (globalized) Newton or SQP method (see, e.g., Bonnans et al. [4]). Consider, for example, a Hessian with the sparsity pattern as shown in Figure 1(a). The first two components of the directional derivative of ∇f along $e_1 + e_4$ equal the nonzeros of the first column of H, and its last two components are the nonzero entries in the fourth column of H; here, e_i is the ith unit vector. This is true since the nonzeros of the first and fourth column are disjoint. The final Hessian can then be completed by taking the derivative of ∇f along e_2 and e_3, respectively. In contrast, the naïve approach would require four derivatives.

This argument can be easily extended to the general case by partitioning the columns such that for each row there is at most one column in each part which has a nonzero in this row. Then for each part π, the components of the derivative of ∇f along the sum of unit vectors for each index in π give the nonzeros of the corresponding columns in H. The efficiency of this scheme increases if the number of parts decreases. Furthermore, we can obtain a speed-up by exploiting the symmetry of H, since it suffices to find a partition of the columns of H such that we are able to recover H_{ij} or H_{ji}.

Coleman and Moré [6] show that a column partition of H (assuming that the diagonal entries are nonzero) allow such a symmetric reconstruction if and only if it induces a star coloring of the adjacency graph of H. The *adjacency graph* is defined by introducing a node for each column of H; there is an edge between two distinct nodes i and j if and only if $H_{ij} = H_{ji} \neq 0$.

Consider again the example in Figure 1. Taking the derivative d of ∇f along $e_1 + e_3$ allows to compute H_{11}, H_{33}, and H_{43}, but only $H_{21} + H_{23}$. If we take the derivative along $e_2 + e_4$, we cannot recover H_{32} and H_{34}. Thus, taking just two derivatives (parts) does not suffice to compute H, since columns $\{1, 3\}$ would receive one color and $\{2, 4\}$ a second one. Hence, the corresponding coloring would not be a star coloring (with two colors) of the adjacency graph. Taking the partitioning in Figure 1(b), however, we can additionally reconstruct $H_{32} = H_{23}$ and H_{34}. Entry H_{21} can then be recovered as $d_2 - H_{32}$.

3. Notation, basic definitions and complexity

Throughout this article we use the following notation.

Let $G = (V, E)$ be an undirected graph with node set V and edge set E. We write $n \coloneqq |V|$ for the number of nodes. For any subset $S \subseteq V$ we denote with $G[S]$ the subgraph of G that is induced by S, i.e., the subgraph of G with node set S and edge set $E[S] \coloneqq \{\{u, v\} \in E \,:\, u, v \in S\}$. A graph $G' = (V', E')$ is a *subgraph* of G if $V' \subseteq V$ and $E[V'] \subseteq E'$. For $n \in \mathbb{N} = \{1, 2, \dots\}$, we define $[n] \coloneqq \{1, \dots, n\}$.

A graph (V, E) is called a path of length $k \in \mathbb{N}$, or *k-path*, if for $V = \{v_1, \dots, v_k\}$, $|V| = k$, we have $E = \big\{\{v_i, v_{i+1}\} \,:\, i \in [k-1]\big\}$. Furthermore, a graph (V, E) is called a *k-cycle* if for $V = \{v_1, \dots, v_k\}$, $|V| = k$, $E = \big\{\{v_i, v_{i+1}\} \,:\, i \in [k-1]\big\} \cup \{\{v_k, v_1\}\}$. We define

$$\mathcal{P}_4(G) \coloneqq \{P \subseteq V \,:\, |P| = 4,\ G[P] \text{ contains a 4-path as a subgraph}\},$$

and we write $\mathcal{P}_4$ instead of $\mathcal{P}_4(G)$ when the underlying graph is obvious. Note that the paths in $\mathcal{P}_4$ are not necessarily induced. Similarly, we define

$$\mathcal{Q}_4 = \{Q \subseteq V \,:\, |Q| = 4,\ G[Q] \text{ is either a 4-path or a 4-cycle}\}.$$

A *star* is a connected bipartite graph that has one partition set with cardinality 1. The node forming the partition set of cardinality one is called the *hub* of the star, and the other nodes are called *spokes*.

For $k \in \mathbb{N}$, $\phi : V \to [k]$ is a *k-coloring* of G if each two adjacent nodes assume different values of ϕ, i.e., $\phi(u) \neq \phi(v)$ for all $\{u, v\} \in E$; the elements of $[k]$ are called *colors*. Moreover, ϕ is a *k-star coloring* if ϕ is a k-coloring such that no $P \in \mathcal{P}_4$ is colored with exactly two colors, i.e., no 4-path subgraph is colored with exactly two colors. If there exists a k-star coloring, G is called k-star colorable. A coloring of a graph with two colors is called a *bicoloring*.

The term star coloring becomes clear if we concentrate on the subgraphs that are induced by the nodes of two color classes: The connected components are stars, see for instance Gebremedhin et al. [14].

Definition 1. Let $G = (V, E)$ be an undirected graph with a positive weight vector $w \in \mathbb{R}^V$, and let $k \in \mathbb{N}$. The *maximum k-star colorable subgraph problem* is to find a set $S \subseteq V$ with maximum weight $\sum_{v \in S} w_v$ such that the induced subgraph $G[S]$ is k-star colorable.

The choice of positive weight vectors is not a restriction of the problem, because nodes with negative weight will never be a part of an optimal solution. Similarly, nodes of weight 0 can be removed from an optimal solution without changing the objective. Hence, we can delete nodes with nonpositive weight.

Furthermore, we can restrict the problem to simple and loopless graphs: Multiple edges between two nodes can be replaced by a single edge, and loops enforce that a node can never be colored. For this reason, we assume throughout this article all graphs to be simple. Additionally, we assume that the graphs are connected. Otherwise, every connected component can be treated separately. Finally, we assume $k < n$, because for $k \geq n$ we can color all nodes by assigning each node a different color. Since the solution set S of the maximum k-star colorable subgraph problems induces a *sub*graph

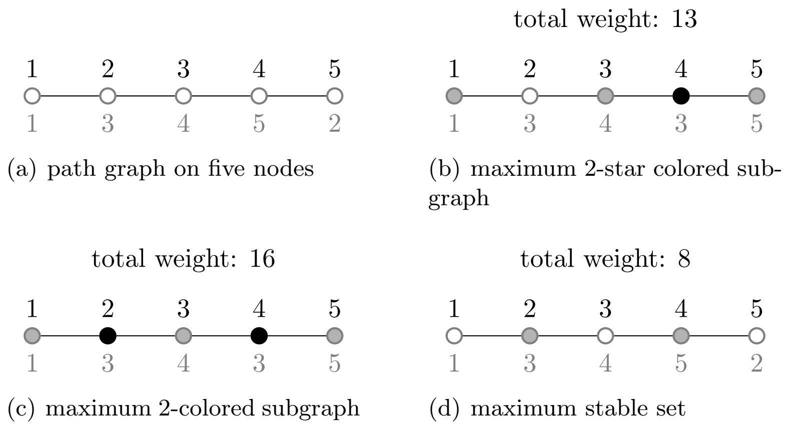


FIGURE 2. The three different problems on a path graph: weights are shown in gray, labels are displayed above the corresponding node. Uncolored nodes are shown in white.

of G which can be star colored with at most k colors by a coloring ϕ, we call the function that coincides with ϕ on S and which is 0 on $V \setminus S$ a *k-star subcoloring* of G or a *star subcoloring* if the number of colors is clear from the context.

We furthermore need the following notation: The *neighborhood* of a node $v \in V$ is the set $\Gamma(v) := \{u \in V : \{u, v\} \in E\}$. Analogously, we define the neighborhood of a set $S \subseteq V$ as the set of all nodes not in S that are adjacent to some node in S, i.e.,

$$\Gamma(S) := \Big(\bigcup_{v \in S} \Gamma(v)\Big) \setminus S.$$

Related problems. The maximum k-star colorable subgraph problem is related to the following problems, see Figure 2:

- The *star coloring problem* consists of finding the smallest k such that the graph is k-star colorable. In this case, k is the smallest number such that G is k-star colorable, but *not* $(k-1)$-star colorable. Thus, by a binary search over k and checking whether all nodes can be k-star colored via the maximum k-star colorable subgraph problem, we can solve the star coloring problem. The latter has been proved to be $\mathcal{NP}$-hard by Coleman and Moré, see [6], and Gebremedhin et al. [13] could show that it is even hard to approximate. Furthermore, Coleman and Moré have shown that it is $\mathcal{NP}$-complete to decide whether a graph is k-star colorable for fixed $k \geq 3$. Consequently, the maximum k-star colorable subgraph problem is $\mathcal{NP}$-hard for any fixed $k \geq 3$.
- The *graph coloring problem* determines the smallest k such that G is k-colorable. This number k is called the *chromatic number* of G. For a graph in which every 4-path has a chord that does not close a cycle, the graph coloring problem coincides with the star coloring problem, see Section 5.

- Analogously to the maximum k-star colorable subgraph problem, we can define the *maximum k-colorable subgraph problem* that consists of finding an induced subgraph with maximum weight that is k-colorable, see [20].
- In the case $k = 1$, the maximum k(-star) colorable subgraph problem coincides with the (weighted) *stable set problem*.

Figure 2 shows an example for $k = 2$ and clarifies the differences between the maximum k-star colorable subgraph problem, the maximum k-colorable subgraph problem, and the stable set problem. We use a path graph on five nodes with positive weights, which are shown in gray. The labels are displayed above the corresponding node.

4. The maximum k-star colorable subgraph polytope

The maximum k-star colorable subgraph problem can be formulated using the following integer program:

$$(\mathrm{IP}^\star_k(G)) \quad \max \sum_{v\in V}\sum_{r\in[k]} w_v\, x_{vr} \tag{1}$$

$$\sum_{r\in[k]} x_{vr} \leq 1 \qquad \forall\, v \in V, \tag{2}$$

$$x_{ur} + x_{vr} \leq 1 \qquad \forall\, \{u,v\} \in E,\ \forall\, r \in [k], \tag{3}$$

$$\sum_{v\in P}(x_{vr} + x_{vs}) \leq 3 \qquad \forall\, P \in \mathcal{P}_4,\ \forall\, r,s \in [k], r < s, \tag{4}$$

$$x_{vr} \in \{0,1\} \qquad \forall\, v \in V,\ \forall\, r \in [k]. \tag{5}$$

The *packing inequalities* (2) guarantee that at most one color is assigned to each node. The *edge inequalities* (3) ensure that adjacent nodes get different colors (if they are colored). The *star inequalities* (4) imply that there is no 4-path subgraph that is colored with two colors: Assume there exists a $P \in \mathcal{P}_4$ that is completely colored with distinct colors $r, s \in [k]$. Then either $x_{vr} = 1$ or $x_{vs} = 1$ for each $v \in P$, which violates (4). Finally, for a solution $\hat{x}$ of $(\mathrm{IP}^\star_k(G))$, a node is colored if and only if $\hat{x}_{v1} + \cdots + \hat{x}_{vk} = 1$, cf. Definition 1. Thus, the objective function (1) computes the total weight of the colored nodes.

Remark 2. Note that the star inequalities (4) are redundant if $P \in \mathcal{P}_4$ is not an induced 4-path or an induced 4-hole. In these cases, P contains a chord, which avoids a bicoloring, see Proposition 11 below. Thus, we could replace $\mathcal{P}_4$ by $\mathcal{Q}_4$ in (4), avoiding all paths that contain a chord.

In the following, we will investigate the polytope corresponding to $(\mathrm{IP}^\star_k(G))$

$$\mathrm{P}^\star_k(G) \coloneqq \operatorname{conv}\left\{x \in \{0,1\}^{V\times[k]} \,:\, x \text{ fulfills (2)–(4)}\right\},$$

the *maximum k-star colorable subgraph polytope*.

For a detailed analysis of this polytope, we introduce the following notation: For $v \in V$ and $r \in [k]$ we define $E^{vr} \in \{0,1\}^{V\times[k]}$ to be the (v,r)-*unit matrix*, i.e., E^{vr} has a 1 in entry (v,r) and 0 in all other entries.

Remark 3. Note that working with $\mathrm{P}^\star_k(G)$ instead of the convex hull of incidence vectors of all star colorings has one main advantage: $\mathrm{P}^\star_k(G)$ is full-dimensional as we will see next, while the dimension of the latter would depend on the star coloring number, which is NP-hard to compute.

Lemma 4. *Let G be an undirected graph and let $k \in \mathbb{N}$. Then* $\dim P_k^\star(G) = nk$*, i.e.,* $P_k^\star(G)$ *is full-dimensional.*

Proof. Clearly, $E^{vr} \in \mathrm{P}_k^\star(G)$ for all $v \in V$, $r \in [k]$, and $0 \in \mathrm{P}_k^\star(G)$. Thus, we have $nk+1$ affinely independent vectors, and hence $\dim \mathrm{P}_k^\star(G) = nk$. □

As mentioned above, we obtain the stable set problem for $k = 1$. Hence, $\mathrm{P}_1^\star(G)$ is isomorphic to the *stable set polytope*, which we denote by $P(G)$.

In the following, we are interested in the facial structure of $\mathrm{P}_k^\star(G)$ and especially in its facets. Before we analyze whether the inequalities given by (2)–(4) are facet inducing for the maximum k-star colorable subgraph polytope, we present two basic results on the facial structure of $\mathrm{P}_k^\star(G)$.

Observation 5. The nonnegativity constraint $x_{vr} \geq 0, v \in V$, $r \in [k]$, defines a facet of $\mathrm{P}_k^\star(G)$ for each $k \in \mathbb{N}$.

Proof. It is easy to see that $\mathrm{P}_k^\star(G)$ is down-monotone for each $k \in \mathbb{N}$, i.e., if $x \in \mathrm{P}_k^\star(G)$ then y with $0 \leq y \leq x$ (component-wise) is also included in $\mathrm{P}_k^\star(G)$. Hammer et al. [16, Prop. 2] have shown that for any full-dimensional and down-monotone polytope S, the nonnegativity inequalities define facets for S. Thus, the assertion follows due to Lemma 4. □

Furthermore, we know that the polytopes $P(G)$ and $\mathrm{P}_k^\star(G)$ are closely related, since they are isomorphic if $k = 1$. Indeed, we find that this connection can be extended for $k > 1$. The following result is based on a result of [20, Prop. 4], and their proof can be applied also in our setting.

Proposition 6. *Assume that $\langle a, y\rangle \leq \beta$ defines a non-trivial facet of $P(G)$. Then the inequality*

$$\sum_{v\in V} a_v\, x_{vr} \leq \beta,$$

defines a facet of $P_k^\star(G)$ for all $r \in [k]$.

After these two general results we investigate the inequalities of the IP-formulation ($\mathrm{IP}_k^\star(G)$). We start with packing inequalities.

Proposition 7. *Let $G = (V, E)$ be an undirected graph and $k \geq 2$ be an integer. Then the packing inequality*

$$\sum_{r\in[k]} x_{vr} \leq 1$$

defines a facet of $P_k^\star(G)$ for every $v \in V$.

Proof. We will show that there are $\dim \mathrm{P}_k^\star(G) = nk$ many affinely independent solutions that fulfill (2) with equality. As in [18], one can prove that the matrices

$$V^{ur} = \begin{cases} E^{vr}, & \text{if } u = v \\ E^{ur} + E^{v,(r \bmod k)+1}, & \text{if } u \neq v \end{cases}$$

with $(u, r) \in V \times [k]$ are affinely independent, fulfill (2) with equality, and do not violate (3) and (5). Finally, since V^{ur} has at most two 1-entries, the star inequalities (4) cannot be violated. This shows the claim. □

$$\underset{B_1}{\begin{pmatrix}0&0\\1&0\\0&1\\1&0\end{pmatrix}} \quad \underset{B_2}{\begin{pmatrix}1&0\\0&0\\1&0\\0&1\end{pmatrix}} \quad \underset{B_3}{\begin{pmatrix}0&1\\1&0\\0&0\\1&0\end{pmatrix}} \quad \underset{B_4}{\begin{pmatrix}1&0\\0&1\\1&0\\0&0\end{pmatrix}} \qquad \underset{B_9}{\begin{pmatrix}1&0\\0&1\\0&0\\1&0\end{pmatrix}} \quad \underset{B_{10}}{\begin{pmatrix}1&0\\0&0\\0&1\\1&0\end{pmatrix}}$$

$$\underset{B_5}{\begin{pmatrix}0&0\\0&1\\1&0\\0&1\end{pmatrix}} \quad \underset{B_6}{\begin{pmatrix}0&1\\0&0\\0&1\\1&0\end{pmatrix}} \quad \underset{B_7}{\begin{pmatrix}1&0\\0&1\\0&0\\0&1\end{pmatrix}} \quad \underset{B_8}{\begin{pmatrix}0&1\\1&0\\0&1\\0&0\end{pmatrix}} \qquad \underset{B_{11}}{\begin{pmatrix}0&1\\1&0\\0&0\\0&1\end{pmatrix}} \quad \underset{B_{12}}{\begin{pmatrix}0&1\\0&0\\1&0\\0&1\end{pmatrix}}$$

FIGURE 3. Feasible bicolorings of a 4-path (matrices B_1–B_{12}) and 4-cycle (matrices B_1–B_8), respectively; the labels correspond to the notation used in the proof of Lemma 9.

In contrast to packing inequalities, edge inequalities may not define facets, but they do if the corresponding edge is not contained in a larger clique, i.e., the edge is a maximal clique (a *clique* is a complete graph). For a clique subgraph C of G, we identify C with its nodes.

Proposition 8. *Let $G = (V, E)$ be an undirected graph, $k \in \mathbb{N}$, and let C be a clique in G. The* clique inequality

$$\sum_{v \in C} x_{vr} \leq 1 \tag{6}$$

for $r \in [k]$ is valid for $\mathrm{P}_k^\star(G)$ and defines a facet if and only if C is a maximal clique.

Proof. It is well-known that maximal clique inequalities define facets for the stable set polytope, see, e.g., Padberg [30]. Due to Proposition 6 this result is also true for $\mathrm{P}_k^\star(G)$ and a fixed color $r \in [k]$.

On the other hand, if C is a clique that is not maximal then there is another clique D such that $V(C) \subsetneq V(D)$ and as for the stable set polytope, the clique inequality for D dominates the inequality for C. Thus, the clique inequality for C cannot define a facet. □

Finally, we investigate star inequalities. Similar to edge inequalities, a star inequality does not always define a facet of $\mathrm{P}_k^\star(G)$. The facet defining property follows from properties of the corresponding path's neighborhood. The proofs are quite technical and require an enumeration of cases.

Lemma 9. *Let $P = (V, E)$ be a graph with $|V| = 4$ that contains a 4-path. Then each star inequality on P with distinct colors $r, s \in [k]$ defines a facet of $\mathrm{P}_k^\star(P)$ if P does not contain a 3-clique.*

Proof. Assume w.l.o.g. that the nodes of P are labeled by $1, \dots, 4$. Furthermore, we assume the rows of the matrices to be labeled by the nodes of P in order $1, \dots, 4$ and the columns to be labeled by color r and s, respectively.

If $k = 2$ and P is an induced 4-path with nodes $1, \dots, 4$ appearing in this order, one can check that the only 0/1-matrices that fulfill the star inequality with equality and satisfy inequalities (2) and (3) are the matrices B_1–B_{12} in

Figure 3. It can be verified that B_1–B_8 are affinely independent and for this reason, the star inequality on P defines a facet of $\mathrm{P}^\star_2(P)$.

Adding the additional edge $\{1,4\}$ between both endpoints of P leads to an induced 4-cycle and exactly matrices B_9–B_{12} become infeasible. Since B_1–B_8 are still feasible, the star inequality on P is also facet defining if P is a cycle. All other graphs on four nodes contain a 3-clique.

For general $k > 2$, we extend the matrices B_1–B_8 to matrices $\tilde{B}_1$–$\tilde{B}_8$ by an additional 0-column for each additional color. Furthermore, define the following $n \cdot (k-2)$ matrices

$$\hat{B}^{vt} = \tilde{B}_v + E^{vt}$$

for $v \in \{1,2,3,4\}$ and any color $t \in [k] \setminus \{r,s\}$. Obviously, $\hat{B}^{vt}$ and $\tilde{B}_1$–$\tilde{B}_8$ are affinely independent. The matrices $\hat{B}^{vt}$ are feasible for $\mathrm{P}^\star_k(P)$: Observe that row v of matrix $\tilde{B}_v$ is a 0-row. Hence, no packing inequality can be violated by $\hat{B}^{vt}$. Edge inequalities can only be violated if they are defined for color t. Similarly, star inequalities can only be violated if they use color t. But since there is exactly one 1-entry in column t this cannot happen. Hence, we have found a set of $n \cdot k = 2 \cdot k + n \cdot (k-2)$ affinely independent matrices that fulfill the star inequality with equality. For this reason, the star inequality defines a facet of $\mathrm{P}^\star_k(P)$. $\square$

Lemma 10. *Let $k \geq 2$, let G be an undirected graph, and $P \in \mathcal{P}_4(G)$. If $G[P]$ is an induced 4-path, then a star inequality on P defines a facet of $P^\star_k(G)$ if there are only nodes in the neighborhood of P that are adjacent to $G[P]$ as in Figure 4(a)–4(e). If $G[P]$ is an induced 4-cycle, the star inequality induces a facet, if there are only nodes in the neighborhood of $G[P]$ which are adjacent to $G[P]$ as in Figure 4(f) or 4(g).*

Proof. Let r and s be the colors used in the star inequality and let w.l.o.g. the nodes of P be labeled consecutively with $1,\ldots,4$. Furthermore, extend the matrices $\tilde{B}_1$–$\tilde{B}_8$ from the proof of Lemma 9 by an additional 0-row for each node $v \notin P$. For each node v in the neighborhood of $G[P]$ we define for the case of

Figure 4(a) $\quad D^{vt} = \tilde{B}_1 + E^{vt},\ t \in [k]$,

Figure 4(b) $\quad D^{vt} = \tilde{B}_2 + E^{vt},\ t \in [k]$,

Figure 4(c) $\quad D^{vt} = \tilde{B}_2 + E^{vt},\ t \in [k] \setminus \{r\}$ and $D^{vr} = \tilde{B}_6 + E^{vr}$,

Figure 4(d) $\quad D^{vt} = \tilde{B}_7 + E^{vt},\ t \in [k] \setminus \{r\}$ and $D^{vr} = \tilde{B}_3 + E^{vr}$,

Figure 4(e) $\quad D^{vt} = \tilde{B}_2 + E^{vt},\ t \in [k] \setminus \{r\}$ and $D^{vr} = \tilde{B}_6 + E^{vr}$.

As in the proof of Lemma 9, these matrices together with the (extended) matrices $\tilde{B}_1$–$\tilde{B}_8$ are affinely independent, since those matrices extend the matrices $\tilde{B}_1$–$\tilde{B}_8$ in pairwise different entries, they fulfill the star inequality with equality, and are feasible for $\mathrm{P}^\star_k(G)$.

If $G[P]$ is an induced cycle, we use the same argumentation but define for $v \in \Gamma(P)$

Figure 4(f) $\quad D^{vt} = \tilde{B}_2 + E^{vt},\ t \in [k]$,

Figure 4(g) $\quad D^{vt} = \tilde{B}_6 + E^{vt},\ t \in [k] \setminus \{r\}$ and $D^{5r} = \tilde{B}_2 + E^{vr}$.

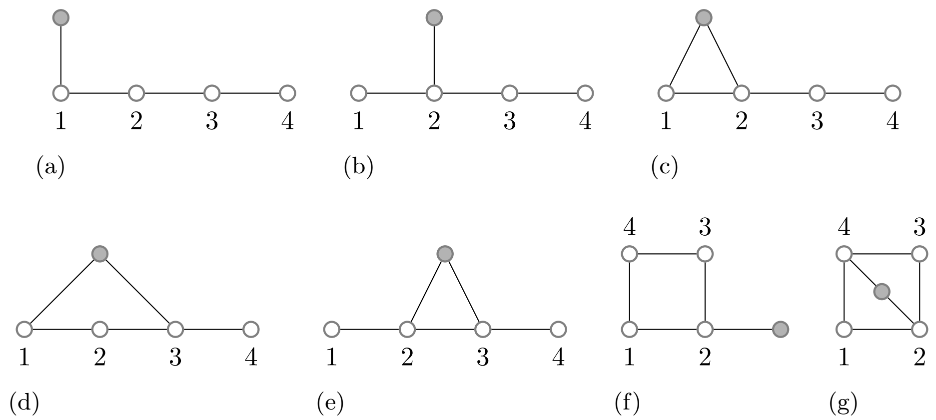


FIGURE 4. Extensions of a 4-path and a 4-cycle such that a star inequality induces a facet. The extensions are labeled grey.

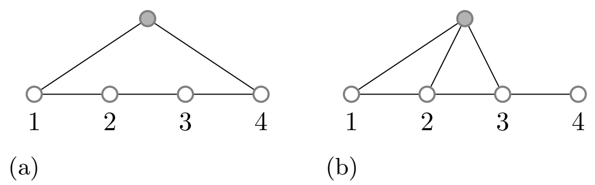


FIGURE 5. Extensions of a 4-path such that a star inequality does not induce a facet. The extensions are labeled grey.

As above, these matrices together with the (extended) matrices from Lemma 9 form a set of affinely independent matrices, they fulfill the star inequality with equality, and are feasible for $\mathrm{P}_k^\star(G)$.

Finally, we define for each $v \notin \Gamma(P)$ and each color $t \in [k]$ the matrix $V^{vt} := \tilde{B}_1 + E^{vt}$. Again, these matrices and the previously defined matrices form a set of affinely independent matrices, they fulfill the star inequality with equality, and are feasible for $\mathrm{P}_k^\star(G)$. Thus, the star inequality defines a facet of $\mathrm{P}_k^\star(G)$. $\square$

Combining both results leads to a complete characterization of the facet inducing property of a star inequality.

Proposition 11. *Let $G = (V, E)$ be an undirected graph with $\mathcal{P}_4(G) \neq \emptyset$, $k \geq 2$, and $P \in \mathcal{P}_4(G)$. Then the star inequality*

$$\sum_{v \in P} (x_{vr} + x_{vs}) \leq 3$$

on P with colors $r < s \in [k]$ defines a facet of $P_k^\star(G)$ if and only if P does not contain a 3-clique,

- *$G[P]$ is not a 4-cycle and for every node $v \in \Gamma(P)$ the induced subgraph $G[P \cup \{v\}]$ is one of the graphs in Figure 4(a) to Figure 4(e), or*
- *$G[P]$ is a 4-cycle and for every node $v \in \Gamma(P)$ the induced subgraph $G[P \cup \{v\}]$ is one of the graphs in Figure 4(f) or Figure 4(g).*

Proof. If G has the properties as given in the proposition, we can combine Lemma 9 and Lemma 10 to show that the star inequality defines a facet.

To show that it is not facet inducing if one of these properties is violated, we assume first that $G[P]$ contains a chord, i.e., it contains a 3-clique. W.l.o.g. let the nodes of $G[P]$ be labeled consecutively with $1, \ldots, 4$ and let $\{1, 3\}$ be a chord of $G[P]$. Then $\tilde{V} = \{1, 2, 3\}$ forms a 3-clique, and the star inequality is dominated by the sum of the clique inequalities for color r and s, respectively, as well as the packing inequality for node 4, i.e.,

$$\sum_{v \in P} (x_{vr} + x_{vs}) \leq \sum_{v \in \tilde{V}} x_{vr} + \sum_{v \in \tilde{V}} x_{vs} + \sum_{t \in [k]} x_{4t} \leq 3.$$

Thus, the star inequality cannot define a facet.

On the other hand, if G contains a node that is not adjacent to $G[P]$, which is isomorphic to the graphs in Figure 4, there has to be a (not necessarily induced) subgraph of G that is isomorphic to the graphs given in Figure 5(a) or Figure 5(b). We show that in this case the star inequality is dominated by the following inequalities. Experiments with the software tool polymake, see Gawrilow and Joswig [11], show that for the graph in Figure 5(a)

$$\sum_{v \in V} (x_{vr} + x_{vs}) \leq 3$$

dominates the star inequality. For this reason, lifting this inequality leads to an inequality that dominates the star inequality for G. Finally, if there is a node as in Figure 5(b), then the star inequality is dominated by the sum of the 4-clique inequalities for color r and s, respectively, and the packing inequality for node 4. □

5. Complete linear descriptions for particular graphs

In this section, we identify graph classes for which a simple complete linear description can be obtained. We will see that the derived description is only valid for (a subset of) trivially perfect graphs and we will argue that there is no simple description for path graphs.

5.1. Graphs with a complete linear description

As mentioned in Section 4, $\mathrm{P}_1^{\star}(G)$ is (isomorphic to) the stable set polytope. Hence, well known results on the stable set polytope provide cases of full linear descriptions of $\mathrm{P}_1^{\star}(G)$, e.g., $\mathrm{P}_1^{\star}(G)$ is completely described by maximal clique, packing, and nonnegativity inequalities if and only if G is perfect, see Chvátal [5].

This description can also be transferred to cases with a larger number of colors if we can ensure that each k-star subcoloring of G is an ordinary k-subcoloring of G. We will see that these three families of inequalities provide a complete linear description for $\mathrm{P}_2^{\star}(G)$ if and only if G is a *trivially perfect graph*, i.e., it neither contains an induced 4-path nor an induced 4-cycle. For $k \geq 3$ this set of inequalities will also be sufficient to describe $\mathrm{P}_k^{\star}(G)$ if and only if G is a trivially perfect graph that does not contain an induced diamond. A *diamond* is a 4-clique with one edge removed. A graph which does not contain an induced diamond is called *diamond-free*. We denote with $D = (V(D), E(D))$ the *diamond graph* with node set $V(D) = \{1, 2, 3, 4\}$ and edge set $E(D) = \{\{i, j\} : i, j \in [4], i \neq j\} \setminus \{\{1, 2\}\}$.

Theorem 12. *The polytope $P^\star_k(G)$ for $k \geq 2$ is completely described by maximal clique, packing, and nonnegativity constraints if and only if*

(1) *$k = 2$ and G is a trivially perfect graph or*
(2) *$k \geq 3$ and G is a trivially perfect and diamond-free graph.*

Proof. First, we show that maximal clique, packing and nonnegativity constraints suffice to give a complete linear description for $\mathrm{P}^\star_k(G)$ if G is a trivially perfect graph and $k = 2$ or if G is trivially perfect and diamond-free and $k \geq 3$, respectively. If we can show that each k-subcoloring is already a k-star subcoloring, we can apply a result on the facial structure of the maximum k-colorable subgraph polytope, see [20, Theorem 5], because both polytopes coincide, compare Section 3.

Since G does not contain any induced 4-path or 4-cycle if G is trivially perfect, each $P \in \mathcal{P}_4$ has to contain at least one chord, i.e., each P contains a 3-clique. If we choose two distinct colors $r, s \in [k]$, each subcoloring of G can color at most two nodes of the 3-clique with r, s. Thus, at most three nodes of P can be colored by any 2-subcoloring of G with r and s, respectively. For this reason, each k-subcoloring of G is also a k-star subcoloring and because of [20, Theorem 5], maximal clique, packing, and nonnegativity constraints suffice to describe $\mathrm{P}^\star_k(G)$ completely.

Finally, we show that these three families of inequalities do not provide a complete linear description for $\mathrm{P}^\star_k(G)$ if G contains an induced 4-path or 4-cycle, respectively, or if $k \geq 3$ and G contains an induced diamond.

Should G contain an induced 4-path or an induced 4-cycle, then we know that a (lifted) star inequality on the path or cycle induces a facet of $\mathrm{P}^\star_k(G)$ for two distinct colors $r, s \in [k]$, see Proposition 11. If $k \geq 3$, experiments with polymake show that for the diamond graph

$$x_{1s} + x_{1t} + x_{2r} + x_{2s} + x_{3r} + x_{3s} + x_{3t} + x_{4s} \leq 3, \quad r, s, t \in [k] \text{ pairwise different}$$

defines a facet of $\mathrm{P}^\star_k(G)$. Thus, we can lift this diamond inequality to the space of $\mathrm{P}^\star_k(G)$ if $k \geq 3$ and G contains an induced diamond to obtain a facet of $\mathrm{P}^\star_k(G)$. Observe that these facets cannot coincide with a facet induced by maximal clique, packing, and nonnegativity constraints, since these inequalities have a righthand side of 1 or 0, and lifting does not change the righthand side of the star or diamond inequality, which is 3. Thus, $\mathrm{P}^\star_k(G)$ cannot be completely described by maximal clique, packing and nonnegativity constraints if $k \geq 3$ and G contains an induced diamond. □

Unfortunately, the class of trivially perfect and diamond-free graphs is not very rich: the connected components of such graphs are either cliques or at least two cliques that share exactly one common node. In fact, Bandelt and Mulder [3, Prop. 1] have shown that a graph G is diamond-free and does not contain an induced 4-cycle if and only if it is a block graph. A *block graph* is a graph of which each connected component is a clique or a set of cliques that pairwise share exactly one common node, see Harary [17]. Since we restrict attention to graphs that also do not contain an induced 4-path, every connected component of G is a combination of a set of cliques that

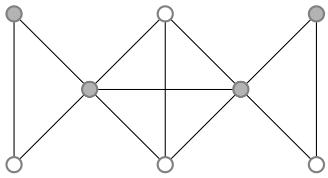


FIGURE 6. A block graph that has an induced 4-path (gray nodes).

share exactly one common node. Otherwise, G would contain an induced path of length four as can be seen in Figure 6.

5.2. LINEAR DESCRIPTIONS FOR PATH GRAPHS

In Section 5.1 we have seen complete linear descriptions of $\mathrm{P}_k^\star(G)$ for $k \geq 2$; but the corresponding classes of graphs G are quite restrictive. In order to extend these results, path graphs are natural candidates, since they form the basis of star inequalities. Hence, the question is whether a complete linear description for these graphs can be obtained by adding star inequalities to the inequalities mentioned in Theorem 12.

Surprisingly, it turns out that this extended set of inequalities never suffices, even for a 4-path G: If we concentrate on $k = 2$ and label the nodes of the 4-path consecutively by $1, \ldots, 4$, experiments with polymake show that

$$x_{21} + x_{41} + x_{12} + 2x_{22} + x_{32} + x_{42} \leq 3 \tag{7}$$

as well as seven similar inequalities define facets for $\mathrm{P}_2^\star(G)$. This proves that maximal clique, packing, nonnegativity, and star inequalities do not give a complete description of $\mathrm{P}_2^\star(G)$ for a 4-path G. Furthermore, by lifting (7) (or the seven similar inequalities) one can see the following:

Observation 13. There is no path graph G of length at least four such that $\mathrm{P}_k^\star(G), k \geq 2$, is completely described by maximal clique, packing, nonnegativity, and star inequalities.

This is surprising, since paths form one of the most simple classes of graphs. Moreover, we computed all facets of $\mathrm{P}_2^\star(G)$ for path graphs with length up to ten with polymake. It turns out that for increasing path sizes, new facets similar to (7) arise and that the corresponding inequalities define facets for paths of larger size and for any $k \geq 2$. Therefore, we conjecture that each increase of the path length leads to new types of facet defining inequalities. Unfortunately, it seems to be hard to generalize inequalities like (7) to path graphs of arbitrary length.

A similar phenomenon can be observed for 4-cycles instead of 4-paths. For this reason, we can conclude that whenever $P \in \mathcal{P}_4(G)$ such that $G[P]$ does not contain a 3-clique, i.e., the star inequalities on P define facets of $\mathrm{P}_k^\star(G)$, then also unclassified inequalities like (7) induce facets of $\mathrm{P}_k^\star(G)$. These inequalities cannot be obtained by lifting trivial inequalities as well as packing and maximal clique inequalities, since lifting does not change the righthand side. Therefore, we obtain the following negative result.

Proposition 14. *For any $k \geq 2$ there is no graph G such that maximal clique, packing, nonnegativity, and lifted star inequalities give a complete and nonredundant description of $P_k^\star(G)$.*

In Section 7, we will see how this phenomenon can be handled to obtain a complete linear description of the maximum 2-star colorable subgraph problem on path graphs. To do so, we have to reformulate the polyhedral model for $k = 2$, which allows to eliminate the unclassified facets which arise for $\mathrm{P}_2^\star(G)$.

6. Projected model for two colors

In this section, we consider a model for 2-star subcolorings that ignores the colors, but only decides which nodes to color. The following Lemma states that it is possible to efficiently recover a coloring.

Lemma 15. *Let $G = (V, E)$ be a graph that is 2-star colorable. Then there is a polynomial time algorithm that determines a 2-star coloring of G.*

Proof. Since G is 2-star colorable, the connected components of G have to be stars. Thus, we can iterate over all nodes of G and assign them color 1 if and only if the current node has degree 1 and it is not adjacent to an already colored node. Then we iterate again over all nodes and color the uncolored nodes of G with color 2. After this procedure the spokes of all connected components are colored with 1 and the hub of each connected component is colored with 2. Observe that this is also true if the connected component is an edge, since both nodes of the edge can be interpreted as hub or spoke, respectively. Furthermore, isolated nodes are colored with color 2. □

Remark 16. The situation of Lemma 15 is similar to finding a 2-coloring of a maximum bipartite graph: Finding a maximum bipartite subgraph is $\mathcal{NP}$-hard, see, e.g., Garey and Johnson [10, Problem GT21], but determining a 2-coloring for a given bipartite subgraph is easy.

Given a matrix $x \in \{0,1\}^{V\times[2]}$ in $\mathrm{P}_2^\star(G)$, we can detect which nodes are colored in x by applying the map

$$T : \mathbb{R}^{V\times[2]} \to \mathbb{R}^V, \quad x \mapsto (x_{v1} + x_{v2}), v \in V.$$

(In the following, we will identify T with the corresponding matrix, i.e., $T \in \mathbb{R}^{V\times(V\times[2])}$.) The following is easy to see.

Observation 17. Let x be any vertex of $\mathrm{P}_2^\star(G)$. Then $T(x)$ is the incidence vector of the corresponding coloring pattern, i.e., $T(x)_v$ is 1 if and only if v is colored.

This observation can be used to formulate an integer program ($\mathrm{pIP}_2^\star(G)$) that only represents which nodes are colored. For a graph $G = (V, E)$ and $k = 2$, the reformulation introduces for each node $v \in V$ a binary variable y_v that is 1 if and only if v is colored. Let $\mathcal{K}_3 = \mathcal{K}_3(G)$ denote the set of all cliques of cardinality at least 3 in G.

$$(\mathrm{pIP}_2^\star(G)) \quad \max \sum_{v\in V} w_v\, y_v \tag{8}$$

$$\sum_{v\in C} y_v \leq 2 \qquad \forall\, C \in \mathcal{K}_3, \tag{9}$$

$$\sum_{v\in P} y_v \leq 3 \qquad \forall\, P \in \mathcal{P}_4, \tag{10}$$

$$y_v \in \{0,1\} \qquad \forall\, v \in V. \tag{11}$$

The objective (8) gives the weight of the colored subgraph, Equation (10) guarantees that at most three consecutive nodes are colored; this corresponds directly to the star constraints (4) that ensure that no 4-path is bicolored. In contrast to $(\mathrm{IP}_2^\star(G))$, we cannot introduce edge inequalities (3), since we do not distinguish colors. Instead, we have to ensure that in each clique of size at least three at most two nodes can be colored. This is achieved by inequalities of type (9), which we call *projection clique inequalities* or for short *p-clique inequalities*. Observe that we do not need to define packing inequalities, because we do not concentrate on specific colors. We denote the underlying polytope of $(\mathrm{pIP}_2^\star(G))$ by

$$\mathrm{R}(G) \coloneqq \mathrm{conv}\{y \in \{0,1\}^V \,:\, x \text{ fulfills (9) and (10)}\}.$$

Observe that this model works only for $k = 2$, since in this case Lemma 15 shows how to recover a star subcoloring just by the knowledge which nodes are colored. For general k, we have to distinguish colors to avoid a bicoloring of a 4-path.

Proposition 18. *Let G be a graph. Then $R(G)$ is the projection of $P_2^\star(G)$ via the linear map T, i.e., $T(P_2^\star(G)) = R(G)$.*

Proof. First, we show $T(\mathrm{P}_2^\star(G)) \subseteq R(G)$. To this end, let $y \in T(\mathrm{P}_2^\star(G))$. Then there is $x \in \mathrm{P}_2^\star(G)$ such that $Tx = y$. Since $\mathrm{P}_2^\star(G)$ is the convex hull of the incidence matrices of star subcolorings, there are vertices $x^1, \dots, x^\ell$ of $\mathrm{P}_2^\star(G)$ and $\lambda_1, \dots, \lambda_\ell \geq 0$ such that $x = \lambda_1 x^1 + \dots + \lambda_\ell x^\ell$ and $\lambda_1 + \dots + \lambda_\ell = 1$. Due to Observation 17, $Tx^i \in \mathrm{R}(G)$ for all $i \in [\ell]$, and thus, $y = Tx \in \mathrm{R}(G)$.

It remains to show that $T(\mathrm{P}_2^\star(G)) \supseteq R(G)$: Let y be any vertex of $\mathrm{R}(G)$. By the recovery routine described in Lemma 15, we can construct a vertex x of $\mathrm{P}_2^\star(G)$ such that $y = Tx$. Thus, $y \in T(\mathrm{P}_2^\star(G))$. Because of the convexity of $\mathrm{R}(G)$ this concludes the proof. □

In the following, we will show a connection between color symmetric faces of $\mathrm{P}_2^\star(G)$ and the faces of $\mathrm{R}(G)$. A face F of $\mathrm{P}_2^\star(G)$ is called *color symmetric* if there is an inequality $\langle a, x\rangle \coloneqq \sum_{v\in V}\sum_{r\in[2]} a_{vr}x_{vr} \leq \beta$ defining F such that $a_{v1} = a_{v2}$ for all $v \in V$.

Proposition 19 below will help us to describe all color symmetric faces for $\mathrm{P}_2^\star(G)$. Because of the packing inequalities (2), each color symmetric inequality $\langle a, x\rangle \leq \beta$ that is valid for $\mathrm{P}_2^\star(G)$ can be transformed into the valid inequality $\sum_{v\in V} a_{v1}y_v \leq \beta$ for $\mathrm{R}(G)$. Vice versa, one can easily verify that each valid inequality $\langle a, y\rangle \leq \beta$ for $\mathrm{R}(G)$ can be transformed into a valid color symmetric inequality

$$\langle T^\top a, x\rangle \leq \beta \tag{12}$$

for $\mathrm{P}_2^\star(G)$. Observe that we interpret T as a matrix and $T^\top$ transforms the vector a into the matrix

$$\begin{pmatrix} a_1 & a_1 \\ \vdots & \vdots \\ a_n & a_n \end{pmatrix}.$$

If an inequality in (12) has the property that it is valid for $\mathrm{P}_2^\star(G)$, we call it *projection compatible*. In our setting, the color symmetric faces are exactly the faces that can be described by projection compatible inequalities. To keep notation short, we abbreviate the face lattice of a polyhedron P by $\mathcal{L}(P)$.

Proposition 19. *There is an order preserving bijection between the color symmetric faces of $P_2^\star(G)$ and the faces of $R(G)$. Furthermore, this bijection maps color symmetric facets of $P_2^\star(G)$ to facets of $R(G)$.*

Proof. Kaibel has shown, see [22, Theorem 16] or [21, Theorem 1], that there is a bijection between the sublattice of projection compatible faces of a polyhedron and the face lattice of its linear projection. Since we deal with color symmetric faces of $\mathrm{P}_2^\star(G)$, we can apply his result to obtain the first part of the proposition.

To show that this bijection maps each color symmetric facet of $\mathrm{P}_2^\star(G)$ to a facet of $R(G)$, we use (as in the proof of Kaibel [21, Theorem 1]) an argument on the sublattice of projection compatible faces of $\mathcal{L}(\mathrm{P}_2^\star(G))$ and $\mathcal{L}(R(G))$. Obviously, the bijection maps $\mathrm{P}_2^\star(G)$ in the sublattice of $\mathcal{L}(\mathrm{P}_2^\star(G))$ onto $R(G)$ in $\mathcal{L}(R(G))$.

Assume for contradiction that there is a color symmetric facet F of $\mathrm{P}_2^\star(G)$ that is not mapped onto a facet of $R(G)$, but a lower dimensional face $F' \coloneqq T(F)$. Since there is a bijection between the color symmetric faces of $\mathrm{P}_2^\star(G)$ and the faces of $R(G)$, see above, there has to be a proper color symmetric face H of $\mathrm{P}_2^\star(G)$ such that $F' \subsetneq H' \coloneqq T(H)$. But because F and H are color symmetric, there are inequalities $\langle T^\top a^F, x\rangle \leq \beta_F$ and $\langle T^\top a^H, x\rangle \leq \beta_H$ that induce F and H, respectively. For each $\tilde{x} \in F$ we have $T\tilde{x} \in F' \subsetneq H'$ and thus,

$$\beta_H = \langle a^H, (T\tilde{x})\rangle = \langle T^\top a^H, \tilde{x}\rangle.$$

Hence, $\tilde{x} \in H$ and for this reason, $F \subseteq H \neq \mathrm{P}_2^\star(G)$. Consequently, F and H have to coincide, since F is a facet and thus cannot be contained in any nontrivial face of $\mathrm{P}_2^\star(G)$. But this is a contradiction, because we assumed $T(F) \subsetneq T(H)$. □

Proposition 19 allows to describe all color symmetric facets of $\mathrm{P}_2^\star(G)$, whenever we know a complete linear description of $\mathrm{R}(G)$. But observe that there are facets of $\mathrm{R}(G)$ that do not have a color symmetric facet as pre-image but a lower dimensional face: Consider, e.g., the nonnegativity inequality $y_v \geq 0$ for any $v \in V$. This inequality defines a facet of $\mathrm{R}(G)$, but its pre-image in the sublattice is the face described by $x_{v1} + x_{v2} \geq 0$, which does not define a facet (it is dominated by $x_{v1} \geq 0$, which is not color symmetric).

Furthermore, Proposition 19 motivates to use the same name for an inequality for $\mathrm{P}_2^\star(G)$ and its image for $\mathcal{L}(\mathrm{R}(G))$, i.e., we will call inequality (10) a star inequality and $y_v \leq 1$, $v \in V$, a packing inequality.

7. A complete linear description of the projected polytope for path graphs

As we have mentioned in the end of Section 5, the facial structure of $\mathrm{P}_2^\star(G)$ seems to be rather complicated for path graphs of arbitrary length, and we were not able to give a complete linear description although path graphs form such a simple class of graphs. Fortunately, we can use the projected model developed in the last section to find a complete linear description for path graphs for the projected polytope. We then also know all color symmetric facets of $\mathrm{P}_2^\star(G)$ for a path graph. For the rest of this section, let G be a path graph of length n and let its nodes be labeled consecutively with $1, \ldots, n$. Furthermore, let $P_i := \{i-3, i-2, i-1, i\}$, $i = 4, \ldots, n$.

To investigate the facial structure of $\mathrm{R}(G)$, we first consider its linear relaxation:

$$\begin{aligned}
\text{(P)} \quad \max \sum_{v \in V} w_v\, y_v & & \\
\sum_{v \in P} y_v &\leq 3 & \forall\, P \in \mathcal{P}_4, & \qquad (\mu_P)\\
y_v &\leq 1 & \forall\, v \in V, & \qquad (\lambda_v)\\
y_v &\geq 0 & \forall\, v \in V. &
\end{aligned}$$

Observe that no p-clique inequality appears since the maximal clique size of G is 2. For the dual problem (D) of (P), we denote the set of all $P \in \mathcal{P}_4$ that contain node v by $\mathcal{P}_4^v := \{P \in \mathcal{P}_4 : v \in P\}, v \in V$. Let $\mu_P, P \in \mathcal{P}_4$, be the dual variables for the projected star inequalities (star variables) and let $\lambda_v, v \in V$, be the dual variables for the upper bounds (packing variables). The dual of (P) is

$$\begin{aligned}
\text{(D)} \quad \min \sum_{v \in V} \lambda_v + 3 \sum_{P \in \mathcal{P}_4} \mu_P & & \\
\lambda_v + \sum_{P \in \mathcal{P}_4^v} \mu_P &\geq w_v & \forall\, v \in V, & \qquad (13)\\
\lambda_v &\geq 0 & \forall\, v \in V, &\\
\mu_P &\geq 0 & \forall\, P \in \mathcal{P}_4. &
\end{aligned}$$

In the remainder of this section, we will show that the linear relaxation (P) suffices to describe $\mathrm{R}(G)$ completely for path graphs and the key result for this is the following theorem.

Theorem 20. *Let G be a path graph. Then the system in (P) is* totally dual integral *(TDI), i.e., (D) has an integer optimal solution for any choice of $w \in \mathbb{Z}^n$.*

To show Theorem 20, we use Algorithm 1 that solves (D) for each choice of w such that the obtained solution is integral.

Algorithm 1: Solving (D) on path graphs

input : A path graph $G = (V, E)$, a weight vector $w \in \mathbb{Z}^V$
output: An optimal solution (λ, μ) for (D)

1 initialize all variables μ_P and λ_v with 0;
2 copy all weights w_v to weights w'_v;
3 **if** $n \leq 3$ **then**
4 $\quad \lambda_i = \max\{w_i, 0\},\ \forall i \in [n]$;
5 **else**
6 $\quad$ **for** $i \leftarrow 4$ **to** n **do**
7 $\qquad \alpha_i \leftarrow \min\{\max(w'_j,\ 0) \ :\ j \in \{i-3, \dots, i\}\}$;
8 $\qquad w'_j \leftarrow w'_j - \alpha_i,\ \forall j \in \{i-3, \dots, i\}$;
9 $\qquad \mu_{P_i} \leftarrow \alpha_i$;
10 $\quad$ **for** $j \leftarrow 1$ **to** n **do**
11 $\qquad \lambda_j \leftarrow \max\{w'_j, 0\}$;

Observe that the weights w'_v, $v \in V$, may decrease during Algorithm 1 (line 8). To distinguish between the original weights and the modified weights, the original weights are copied in line 2 to weights w'_v.

Lemma 21. *The solution produced by Algorithm 1 is feasible for (D).*

Proof. Since all values produced by Algorithm 1 are nonnegative, the left-hand side of each inequality in (13) is nonnegative and hence, any inequality with nonpositive weight w_v on the righthand side is satisfied. Thus, consider $v \in V$ with $w_v \geq 0$. We show that the corresponding inequality is then fulfilled with equality. If $n \leq 3$, we have $\mathcal{P}_4 = \emptyset$. Hence, (D) does not contain any star variable and the constraints of (D) have no common variables. Thus, Algorithm 1 sets λ_v to the corresponding weight w_v, and the inequality is fulfilled with equality.

Consider now the case $n \geq 4$. After the first for-loop in Algorithm 1, w_v is iteratively modified to a weight $0 \leq w'_v \leq w_v$, i.e.,

$$\sum_{P \in \mathcal{P}_4^v} \mu_P = w_v - w'_v.$$

The second for-loop in Algorithm 1 sets $\lambda_v = w'_v$, and we get

$$\lambda_v + \sum_{P \in \mathcal{P}_4^v} \mu_P = w'_v + (w_v - w'_v) = w_v.$$

For this reason, Algorithm 1 computes a feasible solution for (D). $\square$

For the remainder of this section, we need the following notation. Let $\varphi(n)$ be the objective value of the solution produced by Algorithm 1 for a path graph of length n, and let $\varphi^\star(n)$ be the objective value of an optimal integer solution of (P).

Proposition 22. *Algorithm 1 computes an optimal solution of (D) if all weights w_v are positive. The objective value of this solution equals $\varphi^\star(n)$.*

Proof. We prove this proposition by induction on the length n of G. For path graphs with length $n \leq 3$, we can star color all nodes by assigning the two colors alternatingly to nodes of G. Since G does not contain any 4-path, all packing variables λ_v are set to the corresponding node weight. Obviously, this solution has the same weight as the optimal primal solution value (packing variables have an objective coefficient of 1), i.e., $\varphi^\star(n) = \varphi(n)$. Due to the duality theorem of linear programming the solution produced by Algorithm 1 is optimal.

For the inductive step we assume that Algorithm 1 optimally solves (D) for path graphs of length at most n and that the objective value of this solution coincides with $\varphi^\star(n)$. Let G be a path graph of length $n+1$ with positive integer weights w_i, $i \in [n+1]$, and let the nodes of G be labeled consecutively by $1, \ldots, n+1$.

Claim 23. *If we use Algorithm 1 on G, there is at least one node $v \in \{n-2, \ldots, n+1\}$ with a reduced weight $w'_v = 0$.*

Proof. Assume that after assigning star variables $\mu_{P_{n-2}}, \mu_{P_{n-1}}$, and μ_{P_n} their values by Algorithm 1, the last 4-path P_{n+1} has still all (reduced) weights positive. Then $\mu_{P_{n+1}}$ is assigned the smallest value of the reduced weights on P_{n+1} (compare lines 7, 9, and 8 of Algorithm 1), i.e.,

$$\mu_{P_{n+1}} = \min\{w'_{n-2}, w'_{n-1}, w'_n, w'_{n+1}\}.$$

Hence, there is at least one node $v \in \{n-2, \ldots, n+1\}$ with a reduced weight

$$w'_v \leftarrow w'_v - \mu_{P_{n+1}} = w'_v - w'_v = 0.$$

$\square$

Moreover, since a node weight w'_v can only be reduced by a star variable of a path containing this node (compare Algorithm 1 line 8), we obtain one of the following four cases:

$$\begin{aligned}
&\mu_{P_{n+1}} - w_{n+1} = 0,\\
&\mu_{P_{n+1}} + \mu_{P_n} - w_n = 0,\\
&\mu_{P_{n+1}} + \mu_{P_n} + \mu_{P_{n-1}} - w_{n-1} = 0,\\
&\mu_{P_{n+1}} + \mu_{P_n} + \mu_{P_{n-1}} + \mu_{P_{n-2}} - w_{n-2} = 0.
\end{aligned} \tag{14}$$

Considering the structure of an optimal primal integer solution x, we can observe that a star coloring of a path graph with two colors assigns at most three consecutive nodes a color since otherwise a 4-path would be bicolored. Thus, at least one of the last four nodes of G is uncolored, say $u \in \{n-2, n-1, n, n+1\}$. Furthermore, the subgraph induced by all nodes with a smaller label than u must also be colored optimally, because otherwise we could recolor this subpath optimally in x and obtain another feasible coloring with larger objective (since u is uncolored). Obviously, all nodes with a larger label than u must be colored in an optimal integer primal solution due to the choice of u. Hence, we can use this observation to give lower bounds on $\varphi^\star(n+1)$ by checking the cases with one of the last four nodes being uncolored. Additionally, we need the following result.

Claim 24. $\varphi(n+1) = \varphi(n) + (w_{n+1} - \mu_{P_{n+1}})$.

Proof. Assume we know the value $\varphi(n)$ produced by Algorithm 1. If $\mu_{P_{n+1}} > 0$, this objective is increased by α_{n+1} (lines 7 and 8) and simultaneously it is decreased by the same value in the packing variables λ_v of nodes $n-2, n-1$, and n. Hence, an increase in the objective can only be achieved by assigning the packing variable of the last node $n+1$ a positive value. Since the weight of node $n+1$ can only be decreased by the star variable $\mu_{P_{n+1}}$, the objective value is increased exactly by $\lambda_{n+1} = w'_{n+1} = w_{n+1} - \mu_{P_{n+1}}$.

On the other hand, if $\mu_{P_{n+1}} = 0$, the values formerly assigned to the packing variables on P_{n+1} are not changed, and the packing variable λ_{n+1} is set to $w'_{n+1} = w_{n+1}$. For this reason, $\varphi(n+1) = \varphi(n) + (w_{n+1} - \mu_{P_{n+1}})$. □

We are now able to lower bound the optimal primal objective by the objective that is obtained by leaving a node $v^\star \in \{n-2, \dots, n+1\}$ uncolored, coloring all nodes with a label $v < v^\star$ by the optimal coloring pattern for this subpath, and coloring all nodes with a larger label than the uncolored node $v^\star$. If the last node $n+1$ is uncolored, we obtain the general result that $\varphi^\star(n+1) \geq \varphi^\star(n)$:

$$\begin{aligned} \varphi^\star(n+1) \quad &\geq \quad \varphi^\star(n) = \varphi(n) \overset{\text{Claim 24}}{=} \varphi(n+1) - (w_{n+1} - \mu_{P_{n+1}}) \\ &= \quad \varphi(n+1) + \mu_{P_{n+1}} - w_{n+1}. \end{aligned}$$

Assuming nodes $n, n-1$ and $n-2$ to be uncolored, in turn, we obtain a similar result as if node $n+1$ is uncolored. If n is uncolored, we have

$$\begin{aligned} \varphi^\star(n+1) \quad &\geq \quad \varphi^\star(n-1) + w_{n+1} = \varphi(n-1) + w_{n+1} \\ &\overset{\text{Claim 24}}{=} \quad \varphi(n) - (w_n - \mu_{P_n}) + w_{n+1} \\ &\overset{\text{Claim 24}}{=} \quad \varphi(n+1) + \mu_{P_{n+1}} + \mu_{P_n} - w_n. \end{aligned}$$

Analogously, we get for uncolored node $n-1$

$$\varphi^\star(n+1) \quad \geq \quad \varphi(n+1) + \mu_{P_{n+1}} + \mu_{P_n} + \mu_{P_{n-1}} - w_{n-1},$$

and for uncolored node $n-2$

$$\varphi^\star(n+1) \quad \geq \quad \varphi(n+1) + \mu_{P_{n+1}} + \mu_{P_n} + \mu_{P_{n-1}} + \mu_{P_{n-2}} - w_{n-2}.$$

Due to (14), we get that at least one of those four inequalities has a right hand side equal to $\varphi(n+1)$, i.e., $\varphi^\star(n+1) \geq \varphi(n+1)$. Since Algorithm 1 produces a feasible solution for (D), we know that the optimal dual solution value $\varphi(n+1)$ is at most $\varphi^\star(n+1)$. Thus, by the duality theorem of linear programming, $\varphi(n+1) = \varphi^\star(n+1)$. Hence, Algorithm 1 computes an optimal solution for (D). □

We are now able to prove Theorem 20.

Proof of Theorem 20. To prove that the system in (P) is totally dual integral, we have to find an optimal integer solution of (D) with an objective equal to the primal optimal objective for each $w \in \mathbb{Z}^V$. Due to Lemma 21 the integer solution produced by Algorithm 1 is feasible, and it remains to show that this solution is optimal with same objective as the primal optimum.

If we use Algorithm 1 on a graph with negative weights, we can observe that the algorithm sets the variables λ_v and μ_P, $P \in \mathcal{P}_4^v$, to zero for each node $v \in V$ with $w_v \leq 0$. Thus, the algorithm splits the whole path G into subpaths with all nodes having a positive weight and solves the problem on

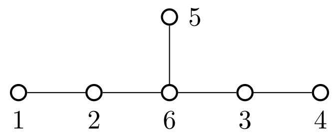


FIGURE 7. Exemplary graph for which inequality (15) defines a facet.

these subpaths. Furthermore, there is always an optimal primal solution that does not color nodes with a nonpositive weight. Thus, the optimal primal objective can be obtained by adding the optimal objectives on the subpaths obtained above. For this reason, we can assume w.l.o.g. that all nodes of G have a positive weight.

By Proposition 22 the solution computed by Algorithm 1 is optimal, and its objective coincides with the primal optimal objective. This terminates the proof. □

This result leads to the natural question whether the complete linear description (P) can be extended to trees. Unfortunately, the answer is unsatisfactory. Whenever a tree contains an induced subgraph as in Figure 7, experiments with polymake show that (the lifted version of) inequality

$$x_1 + x_2 + x_3 + x_4 + x_5 + 2x_6 \leq 5 \tag{15}$$

defines a facet of $\mathrm{R}(G)$. Since lifting does not change the righthand side of the inequality, the obtained complete linear description for path graphs does not suffice to give a complete linear description for general trees.

8. Further facets of $\mathrm{P}_k^\star(G)$

In this section, we give some examples for further facets of the maximum k-star colorable subgraph polytope. Especially, we will concentrate on (nearly) complete bipartite subgraph inequalities, since they can be interpreted as a generalization of star inequalities.

We have seen in Lemma 9 that a star inequality on $P \in \mathcal{P}_4$ defines a facet of $\mathrm{P}_k^\star(P)$ if $G[P]$ does not contain a 3-clique. Hence, $G[P]$ has to be an induced 4-path or an induced 4-hole. In both cases, $G[P]$ is a bipartite graph. We call graphs that become complete bipartite if we add exactly one edge *nearly complete bipartite*. On the other hand, $G[P]$ is not bipartite if it contains a 3-clique. In this section, we present a generalization of star inequalities to inequalities based on (nearly) complete bipartite graphs. For two positive integers p, q we define $K_{p,q}$ to be the complete bipartite graph with p nodes in one and q nodes in the other partition set; $\hat{K}_{p,q}$ represents a nearly complete bipartite graph with partition sets of cardinality p and q, respectively, i.e., $K_{p,q}$ with an arbitrary edge removed.

Lemma 25. *Let $k, p, q \geq 2$ be integers and let G be an undirected graph. Then*

$$\sum_{v \in V(K_{p,q})} (x_{vr} + x_{vs}) \leq \max\{p, q\} + 1 \tag{16}$$

is a valid inequality for $P_k^\star(G)$ for any $K_{p,q} \subseteq G$ and distinct colors $r, s \in [k]$. The same holds if we replace $K_{p,q}$ by $\hat{K}_{p,q}$.

Proof. We first concentrate on complete bipartite subgraph inequalities.

W.l.o.g. let $p \leq q$. Obviously, we can use a color in at most one set, since $K_{p,q}$ is *complete* bipartite. Thus, using both colors within the same partition set leads to one uncolored partition set. Hence, at most q nodes can be colored. However, if we use one color, say r, to color the partition set of cardinality q, we can still color one node with s in the other set. A second node cannot be colored with s because this would lead to a bicolored 4-hole.

By a similar argumentation we get the same upper bound for the nearly complete bipartite graph, since coloring at least two nodes in each partition set with the same color would lead to a bicolored induced 4-path or 4-hole, respectively. ☐

As for star inequalities, it is possible to show that (16) defines a facet of $\mathrm{P}^\star_k(K_{p,q})$ and $\mathrm{P}^\star_k(\hat{K}_{p,q})$ if $p = q$.

Proposition 26. *If $k, q \geq 2$, the (nearly) complete bipartite subgraph inequality (16) defines a facet of $P^\star_k(K_{q,q})$ and $P^\star_k(\hat{K}_{q,q})$.*

Proof. We concentrate on complete bipartite graphs first. Due to Lemma 25 we know that (16) is valid for the maximum k-star colorable subgraph polytope. Since $\mathcal{A} \coloneqq \{x \in \mathrm{P}^\star_k(K_{q,q}) : x \text{ fulfills (16) with equality}\}$ is nonempty, (16) defines a face of $\mathrm{P}^\star_k(K_{q,q})$, and we have to prove that it indeed defines a facet. Instead of constructing $2nk$ affinely independent solutions that fulfill (16) with equality, we use an indirect approach.

Let $\langle b, x\rangle \leq \beta$, $b \in \mathbb{R}^{V\times[k]}$, be a facet inducing inequality for $\mathrm{P}^\star_k(K_{q,q})$ with

$$\mathcal{A} \subseteq \{x \in \mathrm{P}^\star_k(K_{q,q}) : \langle b, x\rangle = \beta\} =: \mathcal{B}.$$

We show that $\langle b, x\rangle \leq \beta$ has to be a positive multiple of (16), and hence $\mathcal{A} = \mathcal{B}$. As an immediate consequence we get that (16) has to define a facet.

To show that $\langle b, x\rangle \leq \beta$ and (16) are multiples, it suffices to show that all entries of b are equal. In the following, we assume the rows of b being labeled with the $2q$ nodes and the columns labeled with the colors from $[k]$. Let V_1 and V_2 be the partition sets of $K_{q,q}$. W.l.o.g. let the first q rows of b correspond to nodes from V_1, and let the last q rows be labeled with nodes from V_2. First, we assume $k = 2$.

The columns of b are constant on V_1 and V_2: Define a feasible matrix that fulfills (16) with equality (in the following just called a coloring) for $u \in V_1$ by

$$\chi_u \coloneqq \left(\begin{array}{cc} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ \hline 0 & 1 \\ \vdots & \vdots \\ 0 & 1 \end{array}\right) + E^{u,1} \in \mathcal{A} \subseteq \mathcal{B},\ u \in V_1.$$

The horizontal line separates the entries of V_1 and V_2. Then $\langle b, \chi_u\rangle = \langle b, \chi_v\rangle$ for all $v \in V_1 \setminus \{u\}$. Because χ_u and χ_v differ in exactly one entry, we get $b_{u,1} = b_{v,1}$. Since u and v are arbitrary, we get that the first column of b is constant on V_1. By changing the role of both colors, we obtain the same result for the second column. Additionally, changing the role of V_1 and V_2 leads to constant columns on V_2. Thus, b has the structure

$$\begin{pmatrix} \delta^1 & \delta^2 \\ \vdots & \vdots \\ \delta^1 & \delta^2 \\ \hline \delta^3 & \delta^4 \\ \vdots & \vdots \\ \delta^3 & \delta^4 \end{pmatrix}.$$

The first column on V_1 is equal to the second column on V_2: Define a feasible coloring for $w \in V_2$ by

$$\xi_w = \begin{pmatrix} 1 & 0 \\ \vdots & \vdots \\ 1 & 0 \\ \hline 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix} + E^{w,2} \in \mathcal{A},\ w \in V_2,$$

where the only node that has color 2 is $w \in V_2$. Then for any $u \in V_1$ and $w \in V_2$ we get

$$\begin{aligned} \langle b, \xi_w \rangle = \langle b, \chi_u \rangle \quad &\Leftrightarrow \quad \sum_{v \in V_1} b_{v,1} + b_{w,2} = \sum_{v \in V_2} b_{v,2} + b_{u,1} \\ &\Leftrightarrow \quad \sum_{v \in V_1 \setminus \{u\}} b_{v,1} = \sum_{v \in V_2 \setminus \{w\}} b_{v,2}. \end{aligned}$$

Since the columns are constant on V_1 and V_2, we obtain that the first column of b on V_1 is equal to the second column on V_2 and vice versa. For this reason, $\delta^1 = \delta^4$ and $\delta^2 = \delta^3$. This suffices to conclude the proof: Take χ_u for an arbitrary node $u \in V_1$ and the coloring $\bar{\chi}_u$ that swaps the first and second column of χ_u. Then

$$\langle b, \chi_u \rangle = \langle b, \bar{\chi}_u \rangle \ \Leftrightarrow\ (q+1)\delta^1 = (q+1)\delta^2 \ \Leftrightarrow\ \delta^1 = \delta^2.$$

Thus, all entries of b are equal, and $\langle b, x \rangle \leq \beta$ is a multiple of the complete bipartite subgraph inequality.

Since all of the above colorings are feasible and fulfill the nearly complete bipartite subgraph inequality with equality, the same arguments can be applied for this inequality.

For the general case $k \geq 2$, we know that there are at least $4q$ affinely independent colorings that fulfill the inequality with equality. For each further color, we can add a zero-column to these matrices without influencing the value of the inequality and affine independence. Since for every node $v \in K_{q,q}$, there is one of the colorings defined above (with corresponding matrix representation I^v) that does not color v, we can define

$$V^{vt} = I^v + E^{vt}$$

for each $v \in K_{q,q}$ and $t \in [k] \setminus \{r, s\}$ to obtain feasible matrices for $\mathrm{P}_k^\star(G)$. Obviously, the matrices V^{vt} fulfill (16) with equality, since the additional entry does not use color r or s. With the same argumentation as in the proof of Lemma 9, there are $2mk$ affinely independent matrices that fulfill (16) with equality. This concludes the proof of the facet property of (16). □

Observe that a (nearly) complete bipartite subgraph inequality may not define a facet for $\mathrm{P}_k^\star(G)$, where G is a general graph that contains a (nearly) complete bipartite subgraph (compare Proposition 11). In general, lifting of the inequality has to be applied to preserve the facet property.

Furthermore, one can show that if $p \neq q$, the (nearly) complete bipartite subgraph inequality does not define a facet of $\mathrm{P}_2^\star(K_{p,q})$ or $\mathrm{P}_2^\star(\hat{K}_{p,q})$, since the larger partition set has to be colored completely to fulfill the upper bound. For this reason, it can be proved that there do not exist $2|V|$ affinely independent matrices that fulfill the inequality with equality.

(Nearly) complete bipartite subgraph inequalities can be generalized to a larger class of inequalities if we use *complete multipartite graphs* instead of complete bipartite graphs. A multipartite graph of order ℓ is an undirected graph $G = (V, E)$ for which V is partitioned into sets $V_1, \dots, V_\ell$ such that there is no edge connecting two nodes from the same partition set. We call it complete multipartite if $E = \{\{u, v\} : u \in V_{m_1}, v \in V_{m_2}, m_1 \neq m_2 \in [\ell]\}$, i.e., all possibles edges between all different partition sets exist. Analogously to the above argumentation, we obtain the following result.

Proposition 27. *Let $G = (V, E)$ be an undirected graph, $k \geq \ell$, and let K be a complete multipartite subgraph of order ℓ for G. Then the following* complete multipartite subgraph inequality *is valid for $P_k^\star(G)$:*

$$\sum_{r \in [k]} \sum_{v \in V} x_{vr} \leq \max\{|V_i| : i \in [\ell]\} + \ell - 1.$$

Furthermore, it defines a facet of $P_k^\star(K)$ if all partition sets of K have the same cardinality greater or equal 2.

The inequalities described in this section are so called rank inequalities for the independence system (IS) of k-star subcolorings of G on the ground set M of all tuples $(v, r) \in V \times [k]$. The down-monotone polytope $\mathrm{P}_k^\star(G)$ induces an IS $\mathcal{I}$ by taking the set of all subsets of M whose corresponding incidence matrix is contained in $\mathrm{P}_k^\star(G)$. The *rank* $r(N)$ of a subset $N \subseteq M$ is the maximal number of elements in N that are contained in any $I \in \mathcal{I}$. For two distinct colors s and t, it is easy to prove that the rank of a complete bipartite subgraph K w.r.t. s and t is the maximal number of nodes in K that can be colored with s and t. Hence, the rank inequality

$$\sum_{(v,c) \in V(K) \times \{s,t\}} x_{vc} \leq r(K)$$

coincides with the complete bipartite subgraph inequality. A similar argumentation can be used for complete multipartite subgraph inequalities. For further reading on independence systems and rank inequalities, we refer the reader to, e.g., Laurent [24].

9. Computational results

In order to evaluate the performance of a branch-and-cut-and-propagate approach for the maximum k-star colorable subgraph problem, we extended the implementation of such an algorithm for the maximum k-colorable subgraph problem from [19]. The new algorithm does not use graph symmetries, but it eliminates color symmetries by using orbitopes, cf. [23]. The algorithm is written in `C++` and is based on `SCIP 3.1.0`, see Achterberg [1] and [31]. We use `CPLEX 12.6` as LP-Solver. The experiments were run on a linux cluster with Intel i3 3.2GHz dual core processors and 8GB memory; the code was run using one thread and running a single process at a time.

The basic algorithm and its implementation was already described in [19]. To be able to use it as a solver for the maximum k-star colorable subgraph problem, we implemented a constraint handler for the star inequalities. At the beginning of the algorithm we compute $\mathcal{Q}_4$, and we add a star inequality for $P \in \mathcal{Q}_4$ and two distinct colors $r, s \in [k]$ if the inequality is violated by the current LP-solution. To avoid adding a large number of violated star inequalities, we bound the number of inequalities that can be added in each separation round.

Furthermore, we implemented a separation algorithm for the (nearly) complete bipartite subgraph inequalities to obtain further cutting planes. This algorithm generates an initial list of (nearly) complete balanced bipartite graphs and tests whether an inequality for a graph in this list is violated. If it finds a violated inequality, the inequality is added. If it does not find any violated cut for graphs in the list, the algorithm tries to find a new bipartite subgraph for which a corresponding inequality is violated.

The initial list of complete balanced bipartite subgraphs is generated in the following way: First, we initialize two empty sets V_1 and V_2 that will contain the partition sets of the bipartite subgraph. For each node $v \in V$ we add v to list V_1. Then we iterate over the neighbors of v and we add an adjacent node w to V_2 if and only if we did not add a node to V_2 that is adjacent to w before. In a second loop we extend the list V_1 by determining a node $u \in V_2$ with highest degree and adding iteratively all neighbors of u to V_1 which are not adjacent to any node in V_1. So far, $G[V_1 \cup V_2]$ is a bipartite graph that is not necessarily complete or balanced. To achieve (nearly) completeness and balancedness, we delete nodes from V_1 or V_2 until the subgraph becomes balanced and (nearly) complete bipartite.

As mentioned above, we try to extend the list of bipartite subgraphs whenever we cannot find a violated cut based on the currently known subgraphs. Since we are interested in finding a subgraph such that the corresponding inequality is violated (its lefthand side is as large as possible), we use a procedure similar to the one described above. The only difference is that we do not add (or delete) nodes in an arbitrary order. Before adding the neighbors of a node, we sort its neighbors decreasingly w.r.t. their weights of the current LP relaxation and two distinct colors and try to add the neighbors in this order. If we have to delete a node, we sort the nodes increasingly w.r.t. the LP relaxation and delete nodes with a low weight first.

TABLE 1. Statistics of the graphs of the test instances.

Name	Node	Edges	Average Degree	Max. clique size	$\|\mathcal{Q}_4\|$
1-FullIns_4	93	593	12	3	82824
1-Insertions_4	67	232	6	2	8652
4-FullIns_3	114	541	9	6	49476
4-Insertions_3	79	156	3	2	1365
DSJC125.9	125	6961	111	34	481928
miles250	128	387	6	8	3523
queen8_8	64	728	22	8	105160
2-FullIns_3	52	201	7	4	8835
anna	138	493	7	11	27777
fpsol2.i.1	496	11654	46	65	4208702
miles1500	128	5198	81	73	774496
queen6_6	36	290	16	6	15588
1-FullIns_3	30	100	6	3	2496
mulsol.i.1	197	3925	39	49	657770
queen5_5	25	160	12	5	4088
myciel4	23	71	6	2	1810
jean	80	254	6	10	5088
zeroin.i.1	211	4100	38	49	603893
david	87	406	9	11	9962

Additionally, we implemented a variant of this separation algorithm that uses only the list of subgraphs that is computed in the beginning; it does not compute further subgraphs if no cut could be separated.

9.1. Experiments for arbitrary graphs

For our first experiments we use a constant objective function, i.e., we are maximizing the size of an induced subgraph that can be star colored with at most k colors.

In the experiments, we use a time limit of 3600 seconds, and we initialized each instance with an optimal solution that was computed beforehand to minimize the influence of heuristics and to test the impact of the (nearly) complete bipartite subgraph inequalities. The algorithm was tested on instances from the Color02 symposium [8] with different choices of colors. The instances were chosen in such a way that our algorithm can solve most of the instances in less than an hour if we initialize each instance with an optimal solution. Table 1 gives a summary of the number of nodes, edges, the average node degree, the maximal size of a clique that could be found, and the number of paths in $\mathcal{Q}_4$ of the graphs in our test set.

We tested three different settings on these test instances: In the default settings, we do not use (nearly) complete bipartite subgraph cuts whereas these cuts were turned on in settings A and B. These settings have in common that we initialize in the beginning of the algorithm a list of (nearly) complete bipartite subgraphs. Setting A uses only the bipartite subgraphs we computed in the beginning of the algorithm. The separation procedure for the bipartite cuts is called in each depth level of the branch-and-bound tree that is an integer multiple of 10. In setting B, we generate the cuts dynamically, i.e., we are looking for new bipartite subgraphs using the current solution and we call the separator in each second depth level of the branch-and-bound tree. To bound the number of cuts added in each separation

TABLE 2. Results for instances of the Color02 symposium. Instances which cannot be solved within 3600 seconds are marked with tl (time limit). To avoid a geometric mean of 0, we replaced 0-entries by 1.

		Default			A				B			
Name	Colors	Nodes	Time	Sep. P4s	Nodes	Time	Sep. P4s	Sep. Bip.	Nodes	Time	Sep. P4s	Sep. Bip
1-FullIns_4	2	168230	444.7	4023694	165665	437.3	3570830	1	129561	321.6	2739749	115
1-Insertions_4	2	160490	165.2	1436431	156402	161.3	1432078	67	177073	208.8	1508564	113
4-FullIns_3	2	322315	1062.1	6306024	235225	751.9	4915282	23	205850	675.4	4749253	14360
4-Insertions_3	2	355320	348.2	1466892	355320	348.9	1466892	0	358089	337.1	1327695	297
DSJC125.9	2	5057	304.4	39135	4865	272.9	36716	1	5588	335.7	39129	98
miles250	2	4125354	tl	8483912	4104371	tl	8371066	29	3189102	2473.7	2017653	1067220
queen8_8	2	669133	1311.9	4187686	669133	1312.0	4187686	1	671774	1389.5	4265540	2844
2-FullIns_3	3	1972968	tl	25144990	2020649	tl	27061948	1	1936511	3520.1	18572255	39987
anna	3	580571	1159.8	5256247	580571	1164.2	5256247	0	304327	580.4	2061210	113528
fpsol2.i.1	3	81350	tl	346207	81159	tl	346029	0	8703	340.5	17203	5693
miles1500	3	106800	1650.6	424971	106800	1656.6	424971	0	35328	595.9	171425	1962
queen6_6	3	115836	130.3	1093899	114275	125.3	1084438	0	121745	144.8	1043688	3993
1-FullIns_3	4	112447	118.3	2681034	112335	119.0	2684685	9	116448	129.3	2660861	4398
1-FullIns_3	5	126461	153.4	3896841	126009	152.5	3901241	3	140311	176.9	3653782	8558
mulsol.i.1	5	3749	80.0	26822	3749	80.3	26822	0	2819	62.9	39343	121
queen5_5	5	239255	263.8	2352726	238980	269.4	2363690	2	234814	279.9	2144856	13657
1-FullIns_3	6	300659	386.1	6056174	300368	383.6	6081999	4	557561	833.3	12643757	17880
mulsol.i.1	6	16582	443.6	120905	16582	442.4	120905	0	11128	357.5	184211	376
myciel4	6	146245	135.5	2515109	144647	134.6	2505160	5	152336	143.6	2624014	100
queen5_5	6	1180721	1529.0	11560063	1165538	1518.9	11477224	5	1145008	1641.7	10371802	71756
jean	7	67126	185.5	1572446	67126	185.0	1572446	0	56395	181.2	1390091	81679
myciel4	7	208783	213.8	3078418	207809	213.6	3069882	4	209998	217.1	3111950	16
zeroin.i.1	7	1270	52.6	36824	1270	52.7	36824	0	1464	75.6	58358	271
1-FullIns_3	8	22798	34.1	284340	39773	64.5	460976	1	21624	36.0	414416	270
myciel4	8	223568	267.3	2881541	223081	266.8	2846332	4	208567	256.3	2758141	4
zeroin.i.1	8	1167	59.2	71127	1167	59.1	71127	0	602	31.9	43542	20
david	9	180637	1201.3	7635618	155801	1082.0	7107964	466	141889	1149.5	5530799	271468
zeroin.i.1	9	563	56.3	122964	563	56.3	122964	0	1892	145.4	167999	99
miles250	10	375914	tl	9758453	62866	425.5	1209597	1120	124811	1643.4	2575537	801119
david	12	22847	180.6	832891	10005	80.1	342953	4	14171	123.0	545723	3433
jean	12	204271	713.5	1705397	204271	713.7	1705397	0	176140	774.6	1374363	50444
geom. mean		92935.9	374.6	1294694.3	85362.9	340.7	1176678.8	3.3	76046.4	324.2	1006241.5	2341.8
unsolved instances			4			3				0		

round, setting A allows at most 250 cuts to be added and setting B allows at most 500.

Our experiments show that (nearly) complete balanced bipartite subgraph inequalities can speed up the branch-and-cut procedure: In the default settings our algorithms needs 374.6 seconds on average to solve an instance, and there are four instances that cannot be solved within one hour. With settings A and B we can achieve an average speed-up of 9.1% and 13.5%, respectively. The number of unsolvable instances for setting A is three and with setting B we are able to prove optimality for all instances within one hour. Table 2 sums up the experiments.

Focusing on the number of separated (nearly) complete balanced bipartite subgraph inequalities, we can observe that in setting A in more than one third of all instances no cuts are added and that in these cases the solving time remains almost unchanged in comparison to the default setting. If we compare the results just on the instances that could be solved with setting A as well as with the default setting, we obtain an average speed-up of about 2.9% (in geometric mean).

Using setting B leads to an increase in the number of separated cuts. Since we do not use only the subgraphs which we computed in the beginning of the algorithm, we can detect further violated inequalities based on the current LP solution. These further inequalities help to reduce the running time of the algorithm for 16 instances in comparison to the default settings. Furthermore, we can solve all of the instances, whereas with the default settings there are four instances that cannot be solved within 3600 seconds. If we concentrate again just on the instances that can be solved by the default settings and setting B, we can achieve an average speed-up of 3.4%, but we can see that the cuts can help us to solve instances that were unsolvable without these cuts.

9.2. Experiments on path graphs

In Section 7 we studied a projective model for $\mathrm{P}^{\star}_{k}(G)$ with $k = 2$ and we gave a complete linear description of the underlying polytope for path graphs. Thus, we can solve the maximum 2-star colorable subgraph problem on path graphs by solving an LP. In this section, we want to analyze the behavior of our algorithm on path graphs for the original formulation in comparison with the performance of a pure LP (IP) solver for the reduced model.

For this reason, we built a test set of path graphs with two different weight functions. In the first set, node weights were chosen randomly between 0 and 19. To allow a better comparison between the instances with random node weights, we computed random weights for the path graph of length 500 and extended them periodically on further nodes. In the second set, the weight function assigns a uniform weight of 1 to the node. As in our first experiments, we are looking with the second set for the maximal number of nodes that can be 2-star colored in a path graph.

The results for the original formulation were obtained by using the algorithm described in the last section with its default setting. Since the reduced model for path graphs can be solved by solving an LP, see Section 7, we used SCIP with CPLEX as LP solver. Again, all instances were initialized with

Table 3. Comparison of running time of the maximum 2-star colorable subgraph problems for paths graphs by using $\mathrm{P}_2^\star(G)$ with the time for the reduced model $\mathrm{R}(G)$

	random weights			uniform weights		
Length	Time $\mathrm{P}_2^\star(G)$	Sep. P4s	Time $\mathrm{R}(G)$	Time $\mathrm{P}_2^\star(G)$	Sep. P4s	Time $\mathrm{R}(G)$
10000	26.5	4357	0.9	40.8	9939	20.2
10500	37.2	4565	0.9	1975.2	10107	22.4
11000	9.0	4325	0.9	1842.3	11695	25.0
11500	9.8	4624	1.0	2855.3	13687	26.8
12000	46.5	6223	1.0	1263.2	12569	29.3
12500	50.4	5935	1.1	5809.9	20304	32.1
13000	62.4	5650	1.0	3941.8	19818	34.6
13500	52.9	5867	1.1	160.9	14946	37.3
14000	18.1	5707	1.1	162.1	15405	40.3
14500	78.8	6821	1.1	93.9	15399	43.5
15000	86.4	6552	1.2	161.3	16496	46.9

an optimal solution which was computed beforehand. The time limit for the path graph instances was set to 7200, seconds and all the instances could be solved within this limit in the root node of the branch-and-bound tree. Table 3 collects the time we needed to solve the maximum 2-star colorable subgraph problem on path graphs with lengths between 10000 and 15000 in the original formulation $\mathrm{P}_2^\star(G)$ and the reduced model $\mathrm{R}(G)$. Furthermore, it gives the number of separated star inequalities for the original formulation.

The results show that the reduced model for the path graph instances can be solved faster than the original formulation $\mathrm{P}_2^\star(G)$: The reduced randomly weighted instances can all be solved in about one second, whereas the solving time for the original formulation varies between 9.0 and 86.4 seconds. For the uniformly weighted instances also for the reduced model the solving times fluctuate in an interval between 20.2 and 46.9 seconds. We can identify the trend that for increasing path graph lengths the solution time increases. This monotone behavior cannot be observed for the solving time of our algorithm. The time of the solution process seems to be independent from the path length, since it increases and decreases arbitrarily and the fluctuation is much larger than for the reduced model (fluctuations between 40.8 and 5809.9 seconds). Finally, the experiments show that the number of star inequalities separated in the solving process of our algorithm to prove optimality grows with increasing path size.

10. Conclusions

This paper presents a first approach to investigate the polyhedral properties of star colorings. We characterize the cases when the defining inequalites of the corresponding polytope (2)–(4) define facets of $\mathrm{P}_k^\star(G)$. However, we have seen in Section 5 that these inequalities suffice just for some very restrictive graph classes to give a complete linear description of $\mathrm{P}_k^\star(G)$. In particular, we have seen that there is no graph class for which $\mathrm{P}_k^\star(G)$ can be completely described by packing, maximal clique, star and trivial inequalities. Thus, whenever there is a subcoloring of G that is not a star coloring, the facial structure becomes much more complicated.

However, we have seen in Section 6 how to deal with this problem if $k = 2$ by using a projected formulation. This eliminates some unclassified facets and allows to obtain a complete linear description for the maximum 2-star colorable subgraph problem on a path graph.

It would be interesting to identify further techniques that reduce the facial complexity to allow finding further complete linear descriptions of the maximum k-star colorable subgraph problem also for $k > 2$.

Our computational results have shown that bipartite subgraph cutting planes may have a positive influence on the performance of a branch-and-cut procedure. For this reason, an identification of further cutting planes and the analysis of their impact onto branch-and-cut may be an interesting future topic. Furthermore, we could observe that the projective model from Section 6 allows us to solve the maximum 2-star colorable subgraph problem on path graphs much faster than in the original formulation.

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