Stochastic versus Robust Optimization for a Transportation Problem

Francesca Maggioni
Department of Management, Economics and Quantitative Methods, University of Bergamo, Bergamo, Italy.
E-mail: francesca.maggioni@unibg.it
Florian A. Potra
Department of Mathematics & Statistic, University of Maryland, Baltimore County, U.S.A.
E-mail: potra@math.umbc.edu
Marida Bertocchi
Department of Management, Economics and Quantitative Methods, University of Bergamo, Bergamo, Italy.
E-mail: marida.bertocchi@unibg.it

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Abstract. In this paper we consider a transportation problem under uncertainty related to *gypsum* replenishment for a cement producer. The problem is to determine the number of vehicles to book at the beginning of each week to replenish gypsum at all the cement factories of the producer in order to minimize the total cost, given by the sum of the transportation costs and buying cost from external sources in extreme situations. Two sources of uncertainty are considered: the demand of gypsum at cement factories of the producer and the buying costs from external sources. We solve the problem both via a two-stage stochastic programming and different robust optimization models. The proposed robust formulations have the advantage to be solvable in polynomial time and to have theoretical guarantees for the quality of their solutions, which is not the case for the stochastic formulation. Numerical experiments show that the robust approach results in larger objective function values than the stochastic approach due to the certitude of constraints satisfaction and more conservative decision strategies on the number of booked vehicles. Conversely, the computational complexity is higher for the stochastic approach.
Keywords. stochastic programming, robust optimization, adjustable robust optimization, supply, transportation.

1 Introduction

The problem of transporting goods or resources from a set of supply points (production plants) to a set of demand points (destination factories or customers) is an important component of the planning activity of a manufacturing firm. Critical parameters such as customer demands, row material prices, and resource capacity are quite uncertain in real problems. An important issue is then represented by the decision on quantities to acquire and store at each destination factory before actual demands reveal themselves. This is involved in the tactical planning of the firm supply chain operations. The significance of uncertainty has prompted a number of works addressing random parameters in tactical level supply chain planning involving distribution of raw material and products (see for example [9], [23], [20], [10], [11] and [24]).

In this paper we analyze the effect of two modelling approaches, stochastic programming (SP) and robust optimization (RO), to a real case of a transportation problem under uncertainty. To the best of our knowledge, a direct comparison between SP and RO on such a class of problems has not been addressed yet in literature. SP and RO are the main alternative techniques to deal with uncertain data both in a single period and in a multi-period decision making process. The main difficulty associated with the former is the need to provide the probability distribution functions of the underlying stochastic parameters. This requirement creates a heavy burden on the user because in many real world situations, such information is unavailable or hard to obtain (see for example [8] and [21]).

On the other hand RO addresses the uncertain nature of the problem without making specific assumptions on probability distributions: the uncertain parameters are assumed to belong to a deterministic uncertainty set. RO adopts a min-max approach that addresses uncertainty by guaranteeing the feasibility and optimality of the solution against all instances of the parameters within the uncertainty set. A vast literature about the hypotheses that have to be imposed on the structure of the uncertainty set in order to have computationally tractable problems are available, see [22] and [6] for polyhedral uncertainty sets and [3] for ellipsoidal uncertainty sets. For some interesting discussions of the robust optimization modeling framework see [19] and [12]. The original RO model deals with static problems where all the decision variables have to be determined before any of the uncertain parameters are realized. This is not the typical situation in most transportation problems that are multiperiod in nature, and where a decision for any period can and should account for data realizations in previous periods. An extension of RO to a dynamic framework was introduced in [3] via the concept of Affinely Adjustable Robust Counterpart (AARC), where part of the decision variables, the so-called adjustable variables, have to be determined after a portion of the uncertain data is realized. The dependence of the adjustable variables on the realized data is represented by an affine function. Other contributions along this line may be found in [2] and in [7]. In [13] the authors discuss the gap between primal and dual formulations of stochastic linear program when recourse decisions are modeled as linear decision rules.

The transportation problem considered, is inspired by a real case of *gypsum* replenishment in Italy, provided by the primary Italian cement producer. The problem consists in determining the number of vehicles to book, at the end of each week, from each plant of the set of suppliers, to replenish gypsum at cement factories in order to minimize the total cost, given by the sum of the transportation costs from origin to destinations (including the discount for vehicles booked but not used) and the cost of buying units of product from external sources in case of inventory shortage. The uncertainty comes from gypsum demand and buying costs from external sources in case of inventory shortage.

The problem described can be classified as a *transportation problem under uncertainty* where a set of retailers is served by a set of suppliers. A particular case is given by the so-called single-sink transportation problem, in which a single retailer is served by a set of suppliers. This problem has been extensively studied, in particular when the total cost is given by the sum of a variable transportation cost and a fixed charge cost to use the supplier ([15], [14], [1] and [17]).

We solve the problem both via a two-stage stochastic programming and robust optimization models with different uncertainty sets. For the former the goal is to compute the minimum expected cost based on the specific probability distribution of the uncertain demand of gypsum at the cement factories and buying cost from external sources based on a set of possible scenarios. Scenarios of demand, for all destinations are built on historical data directly. On the other hand, scenarios of buying costs have been generated sampling from a uniform distribution.

Since the gypsum demand is highly affected by the economic conditions of the public and private medium and large-scale construction sector, a reliable forecast and reasonable estimates of probability distributions are difficult to obtain. This is the main reason that lead us to consider also robust optimization approaches. First we consider static approaches where the uncertain parameters belong to box or box-ellipsoidal uncertainty sets, and then dynamic approaches, via the concept of AARC. A robust solution at the tactical level allows to find a feasible solution for the operational planning problem for each possible realization of demand in the uncertainty set considered.

The choice of the box uncertainty set is preferable only if the feasibility of all the constraints is highly required. In order to get a less conservative outcome, a box-ellipsoid uncertainty set which considers a box for the demand and an ellipsoid for the buying cost is also implemented.

Both SP and RO allow us to determine the nonadjustable variables, i.e., the number of vehicles to book at the end of each week, using the information available at that time. During the following week, as new information on gypsum demand and buying costs from external sources become available, the adjustable (or recourse) decision variables have to be determined using the new information. We describe six strategies for updating the adjustable variables given the values of the already determined values of the nonadjustable variables and the newly available data and report the respective costs.

The paper is organized as follows: Section 2 describes the supply transportation problem and its deterministic formulation. Section 3 discusses the two-stage stochastic programming formulation, while Section 4 deals with the robust formulation with linear uncertainty set, ellipsoidal and mixed uncertainty set. An adjustable robust optimization approach has been described in Section 5 with several methods to determine the adjustable variables. Finally, Section 6 discusses the numerical results. Conclusions follow.

2 Problem description: a supply transportation problem

This problem is inspired by a real case of *gypsum* replenishment in Italy, provided by the primary Italian cement producer. The logistic system is organized as follows: a set $\mathscr{K}$ of suppliers, each of them composed of a set of several plants $\mathscr{O}_k$, $k \in \mathscr{K}$ (origins) located all around Italy have to satisfy the demand of gypsum of a set $\mathscr{D}$ of cement factories (destinations) belonging to the same cement company producer. The demand d_j of gypsum at cement factory $j \in \mathscr{D}$ is considered stochastic. We assume a uniform fleet of vehicles with capacity q and allow only full-load shipments. Shipments are performed by capacitated vehicles which have to be booked in advance, before the demand is revealed. When the demand becomes known, there is an option to discount vehicles booked but not actually used $x_{ijk} - z_{ijk}$ from supplier $i \in \mathscr{O}_k,\ k \in \mathscr{K}$ to plant $j \in \mathscr{D}$. The cancellation fee is given as a proportion α, $0 \leq \alpha \leq 1$, of the transportation costs t_{ijk}, so the transportation cost of each vehicle from the supplier $i \in \mathscr{O}_k$ to destination $j \in \mathscr{D}$ is qt_{ijk} if the vehicle is booked and then used, or αqt_{ijk} if the vehicle is booked, but later cancelled. If the quantity shipped from the suppliers using the booked vehicles is not enough to satisfy the demand of factory j, residual product qy_j is purchased from an external company at a higher price b_j, $j \in \mathscr{D}$, which is considered stochastic. The problem is to determine the number of vehicles x_{ijk} to book from each plant $i \in \mathscr{O}_k$, of the set of suppliers $k \in \mathscr{K}$, to replenish gypsum at cement factory j in order to minimize the total cost, given by the sum of the transportation costs t_{ijk} from origin i to destination j (including the discount α for vehicles booked but not used) and the cost of buying $q \cdot y_j$ units of product from external sources in extreme situations.

Figure 1: Location of destinations $j \in \mathscr{D}$ (cement factories) all around Italy.

2.1 The expected value model

We first consider a deterministic model, which does not take into account of the uncertainty on demand and buying cost. The notation adopted is the following:
Sets:

$$\begin{aligned} \mathscr{K} &= \{k : k = 1, \ldots, K\} \quad , \quad \text{set of suppliers;} \\ \mathscr{O}_k &= \{i : i = 1, \ldots, O_k\} \quad , \quad \text{set of plant locations of supplier } k \in \mathscr{K}; \\ \mathscr{D} &= \{j : j = 1, \ldots, D\} \quad , \quad \text{set of destination plants (cement factories);} \end{aligned}$$

Parameters:

$$\begin{aligned} t_{ijk} \quad &, \quad \text{unit transportation cost from supplier } i \in \mathscr{O}_k, k \in \mathscr{K} \text{ to plant } j \in \mathscr{J}; \\ \bar{b}_j \quad &, \quad \text{average buying cost from an external source for plant } j \in \mathscr{J}; \\ q \quad &, \quad \text{vehicle capacity;} \\ g_j \quad &, \quad \text{unloading capacity at the customer } j \in \mathscr{D}; \end{aligned}$$

v_k , maximum requirement capacity of supplier $k \in \mathscr{K}$;
r_k , minimum requirement capacity of supplier $k \in \mathscr{K}$;
l_j^0 , initial inventory of product at customer $j \in \mathscr{D}$;
α , discount;
$\bar{d}_j$, average demand of customer j.

Variables:

$x_{ijk} \in \mathbb{N}$, number of vehicles booked from supplier $i \in \mathscr{O}_k,\ k \in \mathscr{K}$ to plant $j \in \mathscr{D}$;
$z_{ijk} \in \mathbb{N}$, number of vehicles actually used from supplier $i \in \mathscr{O}_k,\ k \in \mathscr{K}$ to plant $j \in \mathscr{D}$;
$y_j \in \mathbb{R}^+$, volume of product(normalized by q), to purchase from an external source for plant $j \in \mathscr{D}$;

Above we have denoted by $\mathbb{N}$ the set of all nonnegative integers and by $\mathbb{R}^+$ the set of all nonnegative real numbers.

When the demand and the buying cost from external sources are fully known, we get the following linear model:

$$\min_{(x_{ijk}),(y_j),(z_{ijk})} q\sum_{k=1}^{K}\sum_{i=1}^{O_k}\sum_{j=1}^{D} t_{ijk}x_{ijk}+\left[q\sum_{j=1}^{D}\bar{b}_j\, y_j-\alpha q\sum_{k=1}^{K}\sum_{i=1}^{O_k}\sum_{j=1}^{D} t_{ijk}(x_{ijk}-z_{ijk})\right] \tag{1}$$

$$s.t. \qquad q\sum_{k=1}^{K}\sum_{i=1}^{O_k} x_{ijk} \leq g_j\ , \quad j \in \mathscr{D}, \tag{2}$$

$$l_j^0 + q\left(\sum_{k=1}^{K}\sum_{i=1}^{O_k} z_{ijk} + y_j\right) - \bar{d}_j \geq 0\ , \quad j \in \mathscr{D}, \tag{3}$$

$$z_{ijk} \leq x_{ijk}\ , i \in \mathscr{O}_k,\, k \in \mathscr{K},\, j \in \mathscr{D}, \tag{4}$$

$$r_k \leq q\sum_{i\in O_k}\sum_{j=1}^{D} z_{ijk} \leq v_k\ , \quad k \in \mathscr{K}, \tag{5}$$

$$x_{ijk} \in \mathbb{N}\ , i \in \mathscr{O}_k,\, k \in \mathscr{K},\, j \in \mathscr{D}, \tag{6}$$

$$y_j \in \mathbb{R}^+\ , j \in \mathscr{D}, \tag{7}$$

$$z_{ijk} \in \mathbb{N}\ , i \in \mathscr{O}_k,\, k \in \mathscr{K},\, j \in \mathscr{D}, \tag{8}$$

The first sum in the objective function (1) denotes the expected booking costs of the vehicles, while the second sum represents the expected recourse actions, consisting of buying gypsum from external sources (qy_j); Constraint (2) guarantees that the total quantity delivered from the suppliers to customer j is not greater than the j-customer's unloading capacity g_j, inducing thus an upper bound on the total number of vehicles. Constraint (3) ensures that the j-customer's demand is satisfied. Constraint (4) guarantees that the number of vehicles actually used at most equal to the number booked in advance. Constraint (5) ensures that the number of vehicles serving supplier k

does not exceed the production capacity v_k of supplier $k \in \mathscr{K}$ and satisfies the lowest requirement capacity r_k established in the contract. Finally, (6)–(8) define the decision variables of the problem.

Notice that since in a deterministic setting the future demand $\bar{d}_j$ and the buying cost $\bar{b}_j$ from external sources are fully known, the number of actually used vehicles corresponds to the number of ordered vehicles $x_{ijk} = z_{ijk}$, so the third term in the objective function is zero. The model (1)-(8) reduces to:

$$\min_{(x_{ijk}),(y_j)} q\sum_{k=1}^{K}\sum_{i=1}^{O_k}\sum_{j=1}^{D} t_{ijk}x_{ijk} + q\sum_{j=1}^{D}\bar{b}_j\, y_j \tag{9}$$

$$\begin{aligned} s.t. \qquad q\sum_{k=1}^{K}\sum_{i=1}^{O_k} x_{ijk} &\leq g_j\ , \quad j \in \mathscr{D}, &(10)\\ l_j^0 + q\left(\sum_{k=1}^{K}\sum_{i=1}^{O_k} x_{ijk} + y_j\right) - \bar{d}_j &\geq 0\ , \quad j \in \mathscr{D}, &(11)\\ r_k \leq q\sum_{i\in O_k}\sum_{j=1}^{D} x_{ijk} &\leq v_k\ , \quad k \in \mathscr{K}, &(12)\\ x_{ijk} &\in \mathbb{N}\ , i \in \mathscr{O}_k, k \in \mathscr{K}, j \in \mathscr{D}, &(13)\\ y_j &\in \mathbb{R}^+\ , j \in \mathscr{D}. &(14) \end{aligned}$$

We refer to problem (1)-(8) as the *expected value problem* (EV).

3 A two stage stochastic optimization model

In this section we introduce a two-stage stochastic optimization model for the problem described above. Besides the sets, parameters and variables introduced before, we consider the following notation:

Sets:

$$\mathscr{S} = \{s : s = 1, \ldots, S\} \quad , \quad \text{set of scenarios;}$$

Stochastic Parameters:

- p^s , probability of scenario $s \in \mathscr{S}$;
- d_j^s , demand of customer j at scenario $s \in \mathscr{S}$;
- b_j^s , buying cost from external sources for customer j at scenario $s \in \mathscr{S}$.

Variables:

- $x_{ijk} \in \mathbb{N}$, number of vehicles booked from supplier $i \in \mathscr{O}_k,\ k \in \mathscr{K}$ to plant $j \in \mathscr{D}$, (first stage decision variables);
- $z_{ijk}^s \in \mathbb{N}$, number of vehicles actually used from supplier $i \in \mathscr{O}_k,\ k \in \mathscr{K}$ to plant $j \in \mathscr{D}$, at scenario $s \in \mathscr{S}$ (second stage decision variables);
- $y_j^s \in \mathbb{R}^+$, volume of product to purchase from an external source normalized by q, for plant $j \in \mathscr{D}$, at scenario $s \in \mathscr{S}$ (second stage decision variables);

In the two-stage (one-period) case, we get the following mixed-integer stochastic programming model with recourse:

$$\min_{(x_{ijk}),(y_j^s),(z_{ijk}^s)} q\sum_{k=1}^{K}\sum_{i=1}^{O_k}\sum_{j=1}^{D} t_{ijk}x_{ijk} + \sum_{s=1}^{S} p^s \left[\sum_{j=1}^{D} qb_j^s\, y_j^s - \alpha q \sum_{k=1}^{K}\sum_{i=1}^{O_k}\sum_{j=1}^{D} t_{ijk}\left(x_{ijk} - z_{ijk}^s\right)\right] \quad (15)$$

$$s.t. \qquad q\sum_{k=1}^{K}\sum_{i=1}^{O_k} x_{ijk} \leq g_j\ , \quad j \in \mathscr{D}, \quad (16)$$

$$l_j^0 + q\left(\sum_{k=1}^{K}\sum_{i=1}^{O_k} z_{ijk}^s + y_j^s\right) - d_j^s \geq 0\ , \quad j \in \mathscr{D}, \quad s \in \mathscr{S} \quad (17)$$

$$z_{ijk}^s \leq x_{ijk}\ , i \in \mathscr{O}_k, k \in \mathscr{K}, j \in \mathscr{D}, s \in \mathscr{S}, \quad (18)$$

$$r_k \leq q\sum_{i\in O_k}\sum_{j=1}^{D} z_{ijk}^s \leq v_k\ , \quad k \in \mathscr{K}, \quad s \in \mathscr{S} \quad (19)$$

$$x_{ijk} \in \mathbb{N}\ , i \in \mathscr{O}_k, k \in \mathscr{K}, j \in \mathscr{D}, \quad (20)$$

$$y_j^s \in \mathbb{R}^+\ , j \in \mathscr{D}, s \in \mathscr{S} \quad (21)$$

$$z_{ijk}^s \in \mathbb{N}\ , i \in \mathscr{O}_k, k \in \mathscr{K}, j \in \mathscr{D}, s \in \mathscr{S} \quad (22)$$

The first sum in the objective function (15) denotes the expected booking costs of the vehicles, while the second sum represents the expected recourse actions, consisting of buying gypsum from external sources (qy_j^s) and canceling unwanted vehicles. The meaning of constraints (16)–(22) is similar to the meaning of constraints (2)–(8).

From now on we will refer to problem (15)-(22) as the stochastic *recourse problem* (RP).

4 Robust optimization models

In this section we introduce several robust formulations (RO) for the problem described in Section 2. This would be the case when it is impossible, or not practical, to give reasonable estimates of probability distributions for the random parameters given by the demand of gypsum at the cement factories and the cost of buying from external sources. Moreover, some RO formulations can be solved in polynomial time and have theoretical guarantees for the quality of the solution which is not the case with the SP formulations. In order to achieve polynomial complexity we have to properly choose the uncertainty sets and to relax the integer constraints on the variables in case of an ellipsoidal formulation.

There are three well known formulations of RO problems in literature; these are given by [22], [4]-[5] and [6]. They all share the advantage that minimal assumptions about the nature of the uncertainties have to be made and they differ in the ways the uncertainty sets are represented. More specifically, the formulations by Soyster [22] and by Bertsimas and Sim [6] use polyhedral uncertainty sets, while the formulation by Ben-Tal and Nemirovski [4]-[5] considers an ellipsoidal uncertainty set, transforming the original LP problem into a Second Order Cone Programming (SOCP) problem.

We consider different selections of the uncertainty set for the objective function and constraints involving the uncertain demands and buying costs. More precisely we assume that $b \in \mathscr{U}_b$ and

$d \in \mathscr{U}_d$ for some uncertainty sets $\mathscr{U}_b \subset \mathbb{R}^D$ and $\mathscr{U}_d \subset \mathbb{R}^D$. For any such uncertainty sets the robust optimization formulation of our problem becomes

$$\min_{(w),(x_{ijk}),(y_j),(z_{ijk})} w \tag{23}$$

$$\text{s.t.} \quad w \geq q\sum_{k=1}^{K}\sum_{i=1}^{O_k}\sum_{j=1}^{D} t_{ijk}x_{ijk} + \left[\sum_{j=1}^{D} qb_j\, y_j - \alpha q \sum_{k=1}^{K}\sum_{i=1}^{O_k}\sum_{j=1}^{D} t_{ijk}(x_{ijk} - z_{ijk})\right] \ \forall b \in \mathscr{U}_b, \tag{24}$$

$$q\sum_{k=1}^{K}\sum_{i=1}^{O_k} x_{ijk} \leq g_j\ , \quad j \in \mathscr{D}, \tag{25}$$

$$l_j^0 + q\left(\sum_{k=1}^{K}\sum_{i=1}^{O_k} z_{ijk} + y_j\right) - d_j \geq 0\ , \quad j \in \mathscr{D}, \quad \forall d \in \mathscr{U}_d, \tag{26}$$

$$z_{ijk} \leq x_{ijk}\ , i \in \mathscr{O}_k, k \in \mathscr{K}, j \in \mathscr{D}, \tag{27}$$

$$r_k \leq q\sum_{i \in O_k}\sum_{j=1}^{D} z_{ijk} \leq v_k\ , \quad k \in \mathscr{K}, \tag{28}$$

$$x_{ijk} \in \mathbb{N}\ , i \in \mathscr{O}_k, k \in \mathscr{K}, j \in \mathscr{D}, \tag{29}$$

$$y_j \in \mathbb{R}^+\ , j \in \mathscr{D}, \tag{30}$$

$$z_{ijk} \in \mathbb{N}\ , i \in \mathscr{O}_k, k \in \mathscr{K}, j \in \mathscr{D}. \tag{31}$$

where an auxiliary variable w has been introduced.

4.1 Box uncertainty

In this subsection we adopt the methodology from [5] to construct a box uncertainty model for the cost vector b. We assume that this vector belongs to the uncertainty set

$$\mathscr{U}_{b,box,L} = \left\{\bar{b} + \sum_{\ell=1}^{L} \zeta_\ell b^\ell \ : \ \forall \zeta = [\zeta_1; ...; \zeta_L] \in \mathbb{R}^L,\ \|\zeta\|_\infty \leq 1\right\}, \tag{32}$$

where $b^1, \ldots, b^L$ are vectors representing possible perturbations of the vector average cost $\bar{b}$. It is easily shown that, for a given y

$$\max y^T b = y^T\bar{b} + \max_{\|\zeta\|_\infty \leq 1} \sum_{\ell=1}^{L} \zeta_\ell (y^T b^\ell) = y^T\bar{b} + \sum_{l=1}^{L} |y^T b^\ell| \quad \forall b \in \mathscr{U}_{b,box,L}. \tag{33}$$

By introducing the auxiliary variables $u_1, \ldots, u_L$, the robust optimization constraint (24), with $\mathscr{U}_{b,box,L}$ instead of $\mathscr{U}_b$, can be replaced by the constraints

$$w \geq q\sum_{k=1}^{K}\sum_{i=1}^{O_k}\sum_{j=1}^{D} t_{ijk}x_{ijk} + \left[\sum_{j=1}^{D} q\bar{b}_j\, y_j - \alpha q \sum_{k=1}^{K}\sum_{i=1}^{O_k}\sum_{j=1}^{D} t_{ijk}(x_{ijk} - z_{ijk})\right] + q\sum_{\ell=1}^{L} u_\ell, \tag{34}$$

$$-u_\ell \leq y^T b^\ell \leq u_\ell, \quad \ell = 1, \ldots, L. \tag{35}$$

If we choose $L = D$ and the perturbation vectors

$$b^\ell = \rho_2 F_\ell e^\ell,\ \ell = 1, \ldots, D\,, \tag{36}$$

where e^ℓ is the $\ell-$th vector from the standard basis of $\mathbb{R}^D$ then $y^T b^\ell = \rho_2 F_\ell y_\ell$. Here the positive number F_ℓ represents the uncertainty scale and $\rho_2 > 0$ is the uncertainty level. By making the change of variables $u_j = \rho_2 F_j W_j,\ j = 1, \ldots, D$, (34)-(35) becomes

$$w \geq q\sum_{k=1}^{K}\sum_{i=1}^{O_k}\sum_{j=1}^{D} t_{ijk}x_{ijk} + \left[\sum_{j=1}^{D} q\bar{b}_j\, y_j - \alpha q\sum_{k=1}^{K}\sum_{i=1}^{O_k}\sum_{j=1}^{D} t_{ijk}(x_{ijk} - z_{ijk})\right] + q\rho_2\sum_{j=1}^{D} F_j W_j \tag{37}$$

$$-W_j \leq y_j \leq W_j, \quad j = 1, \ldots, D. \tag{38}$$

Assume now that $b \in \mathscr{U}_{b,box,L}$ where the perturbation vectors are chosen as in (36). Then the components of b are given by

$$b_j = \bar{b}_j + \zeta_j \rho_2 F_2, \quad |\zeta_j| \leq 1\,, \quad j = 1, \ldots, D\,.$$

This shows that with the choice (36) the uncertainty set (32) coincides with the simple box

$$\mathscr{U}_{b,box} = \left\{b \in \mathbb{R}^D : |b_j - \bar{b}_j| \leq \rho_2 F_j,\ j \in \mathscr{D}\right\}. \tag{39}$$

Of course, for other choices of the perturbation vectors we get different results.

Similarly, we assume that the demand vector d belongs to an uncertainty set of the form

$$\mathscr{U}_{d,box,L} = \left\{\bar{d} + \sum_{\ell=1}^{L}\zeta_\ell\, d^\ell \ :\ \forall\zeta = [\zeta_1; \ldots; \zeta_L] \in \mathbb{R}^L,\ \|\zeta\|_\infty \leq 1\right\}, \tag{40}$$

for given perturbation vectors $d^1, \ldots, d^L$. We have

$$\max_{d\in\mathscr{U}_{d,box,L}} d_j = \bar{d}_j + \max_{\|\zeta\|_\infty\leq 1}\sum_{\ell=1}^{L}\zeta_\ell d_j^\ell = \bar{d}_j + \sum_{l=1}^{L}|d_j^\ell|. \tag{41}$$

As above, it is easily seen that with the choice

$$L = D, \quad d^\ell = \rho_1 G_\ell e^\ell,\ \ell = 1, \ldots, D\,, \tag{42}$$

the uncertainty set $\mathscr{U}_{d,box,L}$ reduces to the simple box

$$\mathscr{U}_{d,box} = \left\{d \in \mathbb{R}^D : |d_j - \bar{d}_j| \leq \rho_1 G_j,\ j \in \mathscr{D}\right\}. \tag{43}$$

We clearly have $\max_{d\in\mathscr{U}_{d,box}} d_j = \bar{d}_j + \rho_1 G_j$. Using the uncertainty sets (39) and (43) for the uncertain vectors b and d, the robust formulation (23) of our problem can be written as the following linear mixed-integer problem:

$$\min_{(w),(x_{ijk}),(y_j),(z_{ijk})} w \tag{44}$$

$$\text{s.t.} \quad w \geq q\sum_{k=1}^{K}\sum_{i=1}^{O_k}\sum_{j=1}^{D} t_{ijk}x_{ijk} + \left[\sum_{j=1}^{D}(q\bar{b}_j\, y_j + q\rho_2 F_j\, W_j) - \alpha q\sum_{k=1}^{K}\sum_{i=1}^{O_k}\sum_{j=1}^{D} t_{ijk}(x_{ijk} - z_{ijk})\right], \tag{45}$$

$$q\sum_{k=1}^{K}\sum_{i=1}^{O_k} x_{ijk} \leq g_j \ , \quad j \in \mathscr{D}, \tag{46}$$

$$l_j^0 + q\left(\sum_{k=1}^{K}\sum_{i=1}^{O_k} z_{ijk} + y_j\right) \geq \bar{d}_j + \rho_1 G_j \ , \quad j \in \mathscr{D}, \tag{47}$$

$$z_{ijk} \leq x_{ijk} \ , i \in \mathscr{O}_k, k \in \mathscr{K}, j \in \mathscr{D}, \tag{48}$$

$$r_k \leq q\sum_{i \in O_k}\sum_{j=1}^{D} z_{ijk} \leq v_k \ , \quad k \in \mathscr{K}, \tag{49}$$

$$|y_j| \leq W_j \ , j \in \mathscr{D}, \tag{50}$$

$$x_{ijk} \in \mathbb{N} \ , i \in \mathscr{O}_k, k \in \mathscr{K}, j \in \mathscr{D}, \tag{51}$$

$$y_j \in \mathbb{R}^+ \ , j \in \mathscr{D}, \tag{52}$$

$$z_{ijk} \in \mathbb{N} \ , i \in \mathscr{O}_k, k \in \mathscr{K}, j \in \mathscr{D}, \tag{53}$$

$$W_j \in \mathbb{R}^+ \ , j \in \mathscr{D} \ . \tag{54}$$

Because of the nonnegativity constraints (52) we can dispense of the auxiliary variables W_j, $i \in \mathscr{D}$ and write the above optimization problem equivalently as

$$\min_{(w),(x_{ijk}),(y_j),(z_{ijk})} w \tag{55}$$

$$\text{s.t.} \quad w - \left\{ q\sum_{k=1}^{K}\sum_{i=1}^{O_k}\sum_{j=1}^{D} t_{ijk}x_{ijk} + \left[\sum_{j=1}^{D} q\bar{b}_j\, y_j - \alpha q\sum_{k=1}^{K}\sum_{i=1}^{O_k}\sum_{j=1}^{D} t_{ijk}(x_{ijk} - z_{ijk})\right]\right\} \geq \sum_{j=1}^{D} q\rho_2 F_j y_j \tag{56}$$

$$q\sum_{k=1}^{K}\sum_{i=1}^{O_k} x_{ijk} \leq g_j \ , \quad j \in \mathscr{D}, \tag{57}$$

$$l_j^0 + q\left(\sum_{k=1}^{K}\sum_{i=1}^{O_k} z_{ijk} + y_j\right) \geq \bar{d}_j + \rho_1 G_j \ , \quad j \in \mathscr{D}, \tag{58}$$

$$z_{ijk} \leq x_{ijk} \ , i \in \mathscr{O}_k, k \in \mathscr{K}, j \in \mathscr{D}, \tag{59}$$

$$r_k \leq q\sum_{i \in O_k}\sum_{j=1}^{D} z_{ijk} \leq v_k \ , \quad k \in \mathscr{K}, \tag{60}$$

$$x_{ijk} \in \mathbb{N} \ , i \in \mathscr{O}_k, k \in \mathscr{K}, j \in \mathscr{D}, \tag{61}$$

$$y_j \in \mathbb{R}^+ \ , j \in \mathscr{D}, \tag{62}$$

$$z_{ijk} \in \mathbb{N} \ , i \in \mathscr{O}_k, k \in \mathscr{K}, j \in \mathscr{D}, \tag{63}$$

We note that the above model is very conservative. Indeed let us assume that

$$\zeta_1, \ldots, \zeta_L \text{ are zero mean independent random variables with values in the interval } [-1, 1], \tag{64}$$

and let us consider the random vectors

$$b = b(\zeta) = \bar{b} + \sum_{\ell=1}^{D} \zeta_\ell \, b^\ell, \quad d = d(\zeta) = \bar{d} + \sum_{\ell=1}^{D} \zeta_\ell \, d^\ell \,, \tag{65}$$

where the perturbation vectors b^ℓ and d^ℓ are given by (36) and (42) respectively. Consider now a feasible solution of the RO problem (44)-(54). Then

$$\mathrm{Prob}_{\zeta\sim P}\left\{\zeta \;:\; w \geq q\sum_{k=1}^{K}\sum_{i=1}^{O_k}\sum_{j=1}^{D} t_{ijk}x_{ijk}+\left[\sum_{j=1}^{D} qb_j(\zeta)\, y_j - \alpha q\sum_{k=1}^{K}\sum_{i=1}^{O_k}\sum_{j=1}^{D} t_{ijk}(x_{ijk}-z_{ijk})\right]\right\} = 1 \tag{66}$$

and

$$\mathrm{Prob}_{\zeta\sim P}\left\{\zeta \;:\; l_j^0 + q(\sum_{k=1}^{K}\sum_{i=1}^{O_k} z_{ijk} + y_j) - d_j(\zeta) \geq 0\right\} = 1 \,, \quad j \in \mathscr{D} \tag{67}$$

for *any* probability distribution P that is compatible with (64). This certitude of constraints satisfaction will result in a high cost for the optimal solution of the RO problem (44)-(54) (or, equivalently, (55)-(63)).

4.2 Box-Ellipsoidal Uncertainty

In this subsection we study the case were the uncertainty set for the buying costs is given by

$$\mathscr{U}_{b,ell} = \left\{\bar{b} + \sum_{\ell=1}^{L} \zeta_\ell \, b^\ell \;:\; \forall \zeta = [\zeta_1; ...; \zeta_L] \in \mathbb{R}^L,\; \|\zeta\|_2 \leq \Omega\right\}, \tag{68}$$

and the uncertainty set for the demand vector d is the box (43).

Using the Cauchy-Schwarz inequality we obtain, for a given y

$$\max y^T b = y^T \bar{b} + \max_{\|\zeta\|_2 \leq \Omega} \sum_{\ell=1}^{L} \zeta_\ell (y^T b^\ell) = y^T \bar{b} + \Omega \sqrt{\sum_{l=1}^{L} (y^T b^\ell)^2} \quad \forall b \in \mathscr{U}_{b,ell} \,. \tag{69}$$

By choosing the perturbation vectors as in (36) and relaxing the integer constraints on variables x_{ijk} and z_{ijk} we obtain the following RO model with box and ellipsoidal uncertainty set for our problem:

$$\min_{(w),(x_{ijk}),(y_j),(z_{ijk})} w \tag{70}$$

$$w - \left\{q\sum_{k=1}^{K}\sum_{i=1}^{O_k}\sum_{j=1}^{D} t_{ijk}x_{ijk}+ \left[\sum_{j=1}^{D} q\bar{b}_j \, y_j - \alpha q\sum_{k=1}^{K}\sum_{i=1}^{O_k}\sum_{j=1}^{D} t_{ijk}(x_{ijk}-z_{ijk})\right]\right\} \geq \Omega \cdot \sqrt{\sum_{j=1}^{D} (q\rho_2 F_j y_j)^2}, \tag{71}$$

$$q\sum_{k=1}^{K}\sum_{i=1}^{O_k} x_{ijk} \le g_j \ , \quad j \in \mathscr{D}, \tag{72}$$

$$l_j^0 + q\left(\sum_{k=1}^{K}\sum_{i=1}^{O_k} z_{ijk} + y_j\right) \ge \bar{d}_j + \rho_1 G_j \ , \quad j \in \mathscr{D}, \tag{73}$$

$$z_{ijk} \le x_{ijk} \ , \ i \in \mathscr{O}_k, \ k \in \mathscr{K}, \ j \in \mathscr{D}, \tag{74}$$

$$r_k \le q \sum_{i\in O_k}\sum_{j=1}^{D} z_{ijk} \le v_k \ , \quad k \in \mathscr{K}, \tag{75}$$

$$x_{ijk} \in \mathbb{R}^+ \ , \ i \in \mathscr{O}_k, \ k \in \mathscr{K}, \ j \in \mathscr{D}, \tag{76}$$

$$y_j \in \mathbb{R}^+ \ , \ j \in \mathscr{D}, \tag{77}$$

$$z_{ijk} \in \mathbb{R}^+ \ , \ i \in \mathscr{O}_k, \ k \in \mathscr{K}, \ j \in \mathscr{D}. \tag{78}$$

The nonlinear constraint (71) is a second order cone constraint so that the above optimization problems is a SOCP that can be solved in polynomial time.
In fact, as shown in [5] any robust linear inequality constraint of the form

$$a^T x \le \beta, \ \forall [a;\beta] \in \left\{ [\bar{a};\bar{\beta}] + \sum_{\ell=1}^{L} \zeta_\ell [a^\ell;\beta_\ell] \ : \ \forall \zeta = [\zeta_1;...;\zeta_L] \in \mathbb{R}^L, \ \|\zeta\|_2 \le \Omega \right\},$$

can be written as

$$\left[(\bar{\beta} - \bar{a}^T x)/\Omega; \beta_1 - a^{1\,T} x; \ldots; \beta_L - a^{L\,T} x\right] \in \mathscr{L}_{L+1}, \tag{79}$$

where

$$\mathscr{L}_{L+1} = \left\{ p = [p_0; p_1; \ldots; p_L] \in \mathbb{R}^{L+1} : p_0 \ge \|p_1; \ldots; p_L\|_2 \right\} \ , \tag{80}$$

is the second order (or Lorentz) cone of $\mathbb{R}^{L+1}$. A feasible solution x of (79) can be obtained via interior point method in polynomial time. Moreover, any such feasible solution is also the solution of a related chance constrained problem. Let us consider a random vector $[a;\beta]$ defined by

$$[a;\beta] = [\bar{a};\bar{\beta}] + \sum_{\ell=1}^{L} \zeta_\ell [a^\ell;\beta_\ell], \tag{81}$$

$\zeta_1, \ldots, \zeta_L$ independent random variables with zero mean and taking values in $[-1,1]$. (82)

The following result is contained in Corollary 2.3.2. of [5]:

Proposition 4.1 *If $[a,\beta]$ is the random vector given by (81)- (82) and x is a solution of (79) then*

$$\text{Prob}\left\{ \ a^T x > \beta \right\} \le e^{-\frac{\Omega^2}{2}}. \tag{83}$$

We note that the above result holds for any probability distribution for the random vector $\zeta = [\zeta_1; \ldots; \zeta_L]$ that satisfies (82). Let us denote by $\mathcal{P}$ the family of all probability distributions that satisfy (82) and consider the ambiguous chance constrained problem where $\varepsilon \in (0;1)$ is a prespecified small tolerance:

$$\forall P \in \mathcal{P} \quad \text{Prob}_{\zeta\sim P}\left\{ \zeta \ : \ \bar{a}^T x + \sum_{\ell=1}^{L} \zeta_\ell [a^\ell]^T x \ > \ \bar{\beta} + \sum_{\ell=1}^{L} \zeta_\ell \beta_\ell \right\} \le \varepsilon. \tag{84}$$

The above problem is called an *ambiguous* chance constrained problem because we do not have any knowledge about the probability distribution P except the fact that it belongs to the class $\mathcal{P}$. From Proposition 4.1 it follows that any x satisfying the second order cone constraint (79) is a solution of (84).

Consider again the random vectors $b(\zeta)$ and $d(\zeta)$ from (65). By virtue of Proposition 4.1 it follows that for any feasible solution of the RO problem (70)-(78) we have

$$\mathrm{Prob}_{\zeta\sim P}\left\{\zeta \,:\, w \geq q\sum_{k=1}^{K}\sum_{i=1}^{O_k}\sum_{j=1}^{D} t_{ijk}x_{ijk}+\left[\sum_{j=1}^{D} qb_j(\zeta)\, y_j - \alpha q\sum_{k=1}^{K}\sum_{i=1}^{O_k}\sum_{j=1}^{D} t_{ijk}(x_{ijk}-z_{ijk})\right]\right\} \geq 1-e^{-\frac{\Omega^2}{2}}, \tag{85}$$

for all probability distributions P that are compatible with (64). Since we are using the same box uncertainty for the demand, (67) is also satisfied. From the Cauchy-Schwarz inequality we have

$$\sum_{j=1}^{D} q\rho_2 F_j y_j \leq \sqrt{D}\sqrt{\sum_{j=1}^{D}(q\rho_2 F_j y_j)^2}. \tag{86}$$

If follows that if $\Omega \geq \sqrt{D}$ then for any feasible solution of the RO problem (70)-(78) we have the stronger probability result (66). This certitude of constraint satisfaction will result in a high cost for the optimal solution of the RO problem (70)-(78).

5 Adjustable robust optimization

In the robust optimization models considered in the previous section, all the variables are treated in the same way, while in the two-stage stochastic programming model the variables x_{ijk} are considered first stage decision variables and the variables y_j and z_{ijk} are considered second stage (recourse) variables. This means that the variables x_{ijk} are to be determined "here and now", before the actual data "reveals itself". On the other hand, once the uncertain data are known the variables y_j, z_{ijk} should be able to adjust themselves by means of some decision rules $Y_j(\cdot)$ and $Z_{ijk}(\cdot)$. Therefore the decision variables x_{ijk} are called nonadjustable variables while the decision variables y_j and z_{ijk} are called adjustable variables. In this paper we assume that decision rules $Y_j(\cdot)$ and $Z_{ijk}(\cdot)$ are affine function of their arguments.

In developing an adjustable robust optimization model for our problem we will follow the simple model described in subsection 14.2.3.1 of [5], where it is assumed that all the coefficients of the adjustable variables are certain. This is not the case for our problem where the coefficients b_j of the adjustable variables y_j are uncertain. We will circumvent this difficulty by assuming as before that the cost vector b belongs to the ellipsoidal uncertainty set $\mathscr{U}_{b,ell}$ (68). We have seen that in this case, with the choice (36), our cost constraint can be written under the form (71). This is no longer a linear constraint, but a second order cone constraint. On the other hand we assume that the demand vector d belongs to the scenario-generated uncertainty set

$$\mathscr{U}_{d,\hat{\Delta}} = \left\{\sum_{s=1}^{S}\lambda_s \hat{d}^s \,:\, \lambda \in \mathscr{L}\right\}, \tag{87}$$

where,

$$\mathscr{L} = \left\{ \lambda = [\lambda_1; \ldots; \lambda_S] \in \mathbb{R}^S \ : \ \lambda_1 \geq 0, \ldots, \lambda_S \geq 0, \ \sum_{s=1}^{S} \lambda_s = 1 \right\}, \tag{88}$$

and $\hat{\Delta}$ is a given set of scenarios

$$\hat{\Delta} = \left\{ \hat{d}^1, \hat{d}^2, \ldots, \hat{d}^S \right\}. \tag{89}$$

In our applications $\hat{\Delta}$ is obtained from historical data.

Let us denote by u the vector composed of all "here and now" decision variables x_{ijk},

$$u = \text{vec}\left(x_{ijk}, i \in \mathscr{O}_k, \ k \in \mathscr{K}, \ j \in \mathscr{D} \right),$$

and by v the vector composed of all adjustable decision variables y_j, z_{ijk}.

$$v = [y; \text{vec}\left(z_{ijk}, i \in \mathscr{O}_k, \ k \in \mathscr{K}, \ j \in \mathscr{D} \right)].$$

We consider also the vector of decision rules

$$V(\cdot) = [Y_1(\cdot); \ldots; Y_D(\cdot); \text{vec}\left(Z_{ijk}(\cdot), i \in \mathscr{O}_k, \ k \in \mathscr{K}, \ j \in \mathscr{D} \right)].$$

Since decision rules $Y_j(\cdot)$ and $Z_{ijk}(\cdot)$ were assumed to be affine, so is $V(\cdot)$. The deterministic constraints of our problem are:

$$\mathscr{C}(u,v) \ : \ \left\{ \begin{array}{rcl} q \sum_{k=1}^{K} \sum_{i=1}^{O_k} x_{ijk} & \leq & g_j \ , \quad j \in \mathscr{D}, \\ z_{ijk} & \leq & x_{ijk} \ , \ i \in \mathscr{O}_k, \ k \in \mathscr{K}, \ j \in \mathscr{D}, \\ r_k \leq q \sum_{i \in O_k} \sum_{j=1}^{D} z_{ijk} & \leq & v_k \ , \quad k \in \mathscr{K}, \\ x_{ijk} & \in & \mathbb{R}^+ \ , \ i \in \mathscr{O}_k, \ k \in \mathscr{K}, \ j \in \mathscr{D}, \\ y_j & \in & \mathbb{R}^+ \ , \ j \in \mathscr{D}, \\ z_{ijk} & \in & \mathbb{R}^+ \ , \ i \in \mathscr{O}_k, \ k \in \mathscr{K}, \ j \in \mathscr{D}. \end{array} \right.$$

while our uncertain constraints are given by:

$$\widetilde{\mathscr{C}_b}(u,v) \ : \ w \geq q \sum_{k=1}^{K} \sum_{i=1}^{O_k} \sum_{j=1}^{D} t_{ijk} x_{ijk} + \sum_{j=1}^{D} q b_j \, y_j - \alpha q \sum_{k=1}^{K} \sum_{i=1}^{O_k} \sum_{j=1}^{D} t_{ijk} (x_{ijk} - z_{ijk}),$$

and

$$\widetilde{\mathscr{C}_d}(u,v) \ : \ l_j^0 + q \left(\sum_{k=1}^{K} \sum_{i=1}^{O_k} z_{ijk} + y_j \right) - d_j \geq 0 \ , \quad j \in \mathscr{D}.$$

With the above notation our uncertain problem can be written as:

$$\mathscr{R} = \min_{w,u,v} \left\{ w \ : \ \mathscr{C}(u,v), \widetilde{\mathscr{C}_b}(u,v), \widetilde{\mathscr{C}_d}(u,v), \forall b \in \mathscr{U}_{b,ell}, \forall d \in \mathscr{U}_{d,\hat{\Delta}} \right\}.$$

We note that with the choice (36) the infinite set of constraints $\widetilde{\mathscr{C}_b}(u,v), \forall b \in \mathscr{U}_{b,ell}$ reduces to the deterministic single second-order cone constraint

$$\widetilde{\mathscr{C}}(u,v) \ : \ w - q \sum_{k=1}^{K} \sum_{i=1}^{O_k} \sum_{j=1}^{D} t_{ijk} x_{ijk} - \sum_{j=1}^{D} q \bar{b}_j \, y_j + \alpha q \sum_{k=1}^{K} \sum_{i=1}^{O_k} \sum_{j=1}^{D} t_{ijk} (x_{ijk} - z_{ijk}) \geq \Omega \sqrt{\sum_{j=1}^{D} (q \rho_2 F_j y_j)^2}.$$

Therefore, our uncertain problem can be written equivalently as:

$$\mathscr{R} \quad : \quad \min_{w,u,v} \left\{ w \ : \ \mathscr{C}(u,v), \widetilde{\mathscr{C}}(u,v), \widetilde{\mathscr{C}}_d(u,v), \forall d \in \mathscr{U}_{d,\hat{\Delta}} \right\}. \tag{90}$$

The adjustable version of this problem is

$$\mathscr{A} \quad : \quad \min_{w,u,V(\cdot)} \left\{ w \ : \ \mathscr{C}(u,V(d)), \widetilde{\mathscr{C}}(u,V(d)), \widetilde{\mathscr{C}}_d(u,V(d)), \forall d \in \mathscr{U}_{d,\hat{\Delta}} \right\}, \tag{91}$$

where the minimum is taken for all decision rules $V(\cdot)$ that are affine functions of their arguments. We will show that this adjustable version is equivalent to the following tractable optimization problem

$$\mathscr{Q} \quad : \quad \min_{w,u,\{v^s\}_1^S} \left\{ w \ : \ \mathscr{C}(u,v^s), \widetilde{\mathscr{C}}(u,v^s), \widetilde{\mathscr{C}}_{\hat{d}^s}(u,v^s), s = 1,2,\ldots,S \right\}. \tag{92}$$

The equivalence is understood in the sense that the optimal values of $\mathscr{A}$ and $\mathscr{Q}$ are equal and that any feasible solution of $\mathscr{Q}$ can be augmented to a feasible solution of $\mathscr{A}$. More specifically let $\hat{w}, \hat{u}, \{\hat{v}^s\}_1^S$ be a feasible solution of $\mathscr{Q}$ and consider a vector $d \in \mathscr{U}_{d,\hat{\Delta}}$. Then there is a vector $\lambda(d) = [\lambda_1(d); \ldots; \lambda_S(d)] \in \mathscr{L}$ such that

$$d = \sum_{s=1}^{S} \lambda_s(d) \hat{d}^s, \tag{93}$$

and the adjustable variables are defined by the decision rule

$$v = \hat{V}(d) := \sum_{s=1}^{S} \lambda_s(d) \hat{v}^s. \tag{94}$$

The feasibility of $\hat{w}, \hat{u}, \{\hat{v}^s\}_1^S$ means that the following constraints are satisfied

$$\mathscr{C}(\hat{u},\hat{v}^s), \widetilde{\mathscr{C}}(\hat{u},\hat{v}^s), \widetilde{\mathscr{C}}_{\hat{d}^s}(\hat{u},\hat{v}^s), s = 1,2,\ldots,S.$$

Multiplying the constraints $\mathscr{C}(\hat{u},\hat{v}^s)$ and $\widetilde{\mathscr{C}}_{\hat{d}^s}(\hat{u},\hat{v}^s)$ by $\lambda_s(d)$, adding, and using the linearity of those constraints we deduce that the constraints $\mathscr{C}(\hat{u},\hat{V}(d))$, and $\widetilde{\mathscr{C}}_d(\hat{u},\hat{V}(d))$ are also satisfied. It also follows that the next inequalities are satisfied for $s = 1,2,\ldots,S$:

$$\hat{w} - q\sum_{k=1}^{K}\sum_{i=1}^{O_k}\sum_{j=1}^{D} t_{ijk}\hat{x}_{ijk} - \sum_{j=1}^{D} q\bar{b}_j\, \hat{y}_j^s + \alpha q \sum_{k=1}^{K}\sum_{i=1}^{O_k}\sum_{j=1}^{D} t_{ijk}\left(\hat{x}_{ijk} - \hat{z}_{ijk}^s\right) \geq \Omega \sqrt{\sum_{j=1}^{D} (q\rho_2 F_j \hat{y}_j^s)^2}.$$

Multiplying each inequality by $\lambda_s(d)$, adding, and using the convexity of $\sqrt{\sum_{j=1}^{D}(q\rho_2 F_j y_j)^2}$ as a function of $y_1, \ldots, y_D$ we obtain successively

$$\begin{aligned}
&\hat{w} - q\sum_{k=1}^{K}\sum_{i=1}^{O_k}\sum_{j=1}^{D} t_{ijk}\hat{x}_{ijk} - \sum_{j=1}^{D} q\bar{b}_j \sum_{s=1}^{S} \lambda_s(d)\hat{y}_j^s + \alpha q \sum_{k=1}^{K}\sum_{i=1}^{O_k}\sum_{j=1}^{D} t_{ijk}\left(\hat{x}_{ijk} - \sum_{s=1}^{S}\lambda_s(d)\hat{z}_{ijk}^s\right) \\
= \ &\sum_{s=1}^{S}\lambda_s(d)\left[\hat{w} - q\sum_{k=1}^{K}\sum_{i=1}^{O_k}\sum_{j=1}^{D} t_{ijk}\hat{x}_{ijk} - \sum_{j=1}^{D} q\bar{b}_j\, \hat{y}_j^s + \alpha q \sum_{k=1}^{K}\sum_{i=1}^{O_k}\sum_{j=1}^{D} t_{ijk}\left(\hat{x}_{ijk} - \hat{z}_{ijk}^s\right)\right] \\
\geq \ &\sum_{s=1}^{S}\lambda_s(d)\Omega\sqrt{\sum_{j=1}^{D}(q\rho_2 F_j \hat{y}_j^s)^2} \ \geq \ \Omega\sqrt{\sum_{j=1}^{D}\left(q\rho_2 F_j \sum_{s=1}^{S}\lambda_s(d)\hat{y}_j^s\right)^2},
\end{aligned}$$

which shows that the constraint $\widetilde{\mathscr{C}}(\hat{u}, \hat{V}(d))$ is also satisfied. Therefore $\hat{w}, \hat{u}, \hat{V}(\cdot)$ is feasible for $\mathscr{A}$. In particular, this proves that the optimal solution of $\mathscr{A}$ is less than or equal to the optimal solution of $\mathscr{Q}$. To prove the reverse inequality, we remark that if $(w, u, V(\cdot))$ is feasible for $\mathscr{A}$, then $w, u, \left\{V(\hat{d}^s)\right\}_1^S$ is clearly feasible for $\mathscr{Q}$.

In conclusion, in order to solve our adjustable robust optimization model we first find the optimal solution

$$x^*_{ijk},\ y^{*\,s}_j,\ z^{*\,s}_{ijk}, \quad i \in \mathscr{O}_k,\, k \in \mathscr{K},\, j \in \mathscr{D},\, s \in \mathscr{S} \tag{95}$$

of the second order cone optimization problem

$$\min_{(w),(x_{ijk}),(y^s_j),(z^s_{ijk})} \quad w \tag{96}$$

$$s.t.$$

$$w - q\sum_{k=1}^{K}\sum_{i=1}^{O_k}\sum_{j=1}^{D} t_{ijk}x_{ijk} - \sum_{j=1}^{D} q\bar{b}_j\, y^s_j + \alpha q\sum_{k=1}^{K}\sum_{i=1}^{O_k}\sum_{j=1}^{D} t_{ijk}\left(x_{ijk} - z^s_{ijk}\right) \geq \Omega\sqrt{\sum_{j=1}^{D}(q\rho_2 F_j y^s_j)^2},\ s \in \mathscr{S} \tag{97}$$

$$q\sum_{k=1}^{K}\sum_{i=1}^{O_k} x_{ijk} \ \leq\ g_j\ , \quad j \in \mathscr{D}, \tag{98}$$

$$l^0_j + q\left(\sum_{k=1}^{K}\sum_{i=1}^{O_k} z^s_{ijk} + y^s_j\right) - \hat{d}^s_j \ \geq\ 0\ , \quad j \in \mathscr{D}, \quad s \in \mathscr{S}, \tag{99}$$

$$z^s_{ijk} \ \leq\ x_{ijk}\ ,\ i \in \mathscr{O}_k,\, k \in \mathscr{K},\, j \in \mathscr{D},\, s \in \mathscr{S}, \tag{100}$$

$$r_k \leq q\sum_{i=1}^{O_k}\sum_{j=1}^{D} z^s_{ijk} \ \leq\ v_k\ , \quad k \in \mathscr{K}, \quad s \in \mathscr{S}, \tag{101}$$

$$x_{ijk} \ \in\ \mathbb{R}^+\ ,\ i \in \mathscr{O}_k,\, k \in \mathscr{K},\, j \in \mathscr{D}, \tag{102}$$

$$y^s_j \ \in\ \mathbb{R}^+\ ,\ j \in \mathscr{D},\, s \in \mathscr{S}, \tag{103}$$

$$z^s_{ijk} \ \in\ \mathbb{R}^+\ ,\ i \in \mathscr{O}_k,\, k \in \mathscr{K},\, j \in \mathscr{D},\, s \in \mathscr{S}. \tag{104}$$

When the uncertain demand d reveals itself we try to find a vector $\lambda(d) = [\lambda_1(d); \ldots; \lambda_S(d)] \in \mathscr{L}$ satisfying (93) by solving the following optimization problem in λ

$$\min_{\lambda \in \mathscr{L}} f(\lambda) := \sum_{j=1}^{D}\left(d_j - \sum_{s=1}^{S}\lambda_s \hat{d}^s_j\right)^2 . \tag{105}$$

The optimal value of the objective function is equal to zero, i.e., $f(\lambda(d)) = 0$, if and only if $d \in \mathscr{U}_{d,\hat{\Delta}}$. In this case the adjustable variables are given by

$$y^*_j = \sum_{s=1}^{S}\lambda_s(d)y^{*\,s}_j,\ z^*_{ijk} = \sum_{s=1}^{S}\lambda_s(d)z^{*\,s}_{ijk} \quad i \in \mathscr{O}_k,\, k \in \mathscr{K},\, j \in \mathscr{D}. \tag{106}$$

From the above considerations it follows that

$$x^*_{ijk},\ y^*_j,\ z^*_{ijk}, \quad i \in \mathscr{O}_k,\, k \in \mathscr{K},\, j \in \mathscr{D}, \tag{107}$$

is an optimal solution of the robust adjustable optimization problem (91). This is no longer the case when $f(\lambda(d)) > 0$. As noted in [5], the scenario-generated uncertainty set $\mathscr{U}_{d,\hat{\Delta}}$ from (87) usually "too small" to be of much interest. Nevertheless, as shown in the next section the values

$$x^*_{ijk}, \quad i \in \mathscr{O}_k,\, k \in \mathscr{K},\, j \in \mathscr{D}, \tag{108}$$

could be used to find a solution of our transportation problem even when $d \notin \mathscr{U}_{d,\hat{\Delta}}$.

5.1 Determining the adjustable variables

Suppose that the values (108) of the nonadjustable (or first stage) variables have been obtained by one of the optimization problems considered in the previous sections of this paper. When the previously unknown cost and demand vectors, b and d become available we can always try to compute the values of the adjustable (or second stage) variables as the optimal solutions of the following linear optimization problem:

$$\min_{(y_j),(z_{ijk})} \quad q\sum_{k=1}^{K}\sum_{i=1}^{O_k}\sum_{j=1}^{D} t_{ijk}x^*_{ijk} + q\sum_{j=1}^{D} b_j\, y_j - \alpha q \sum_{k=1}^{K}\sum_{i=1}^{O_k}\sum_{j=1}^{D} t_{ijk}\big(x^*_{ijk} - z_{ijk}\big) \tag{109}$$

$s.t.$

$$q\sum_{k=1}^{K}\sum_{i=1}^{O_k} x^*_{ijk} \ \le\ g_j\,, \quad j \in \mathscr{D}, \tag{110}$$

$$l^0_j + q\left(\sum_{k=1}^{K}\sum_{i=1}^{O_k} z_{ijk} + y_j\right) - d_j \ \ge\ 0\,, \quad j \in \mathscr{D}, \tag{111}$$

$$z_{ijk} \ \le\ x^*_{ijk}\,,\ i \in \mathscr{O}_k,\, k \in \mathscr{K},\, j \in \mathscr{D}, \tag{112}$$

$$r_k \le q\sum_{i \in O_k}\sum_{j=1}^{D} z_{ijk} \ \le\ v_k\,, \quad k \in \mathscr{K}, \tag{113}$$

$$y_j \in \mathbb{R}^+\,,\ j \in \mathscr{D}, \tag{114}$$

$$z_{ijk} \in \mathbb{R}^+\,,\ i \in \mathscr{O}_k,\, k \in \mathscr{K},\, j \in \mathscr{D}. \tag{115}$$

Depending on the values of b, d and the way the "optimal" values (108) of the nonadjustable variables were obtained, the above optimization problem may be feasible or not. If it is feasible then it has an optimal solution

$$y^*_j,\ z^*_{ijk}, \quad i \in \mathscr{O}_k,\, k \in \mathscr{K},\, j \in \mathscr{D}. \tag{116}$$

The values (108) and (116) are then feasible for our transportation problem

$$\min_{(x_{ijk}),(y_j),(z_{ijk})} \quad q\sum_{k=1}^{K}\sum_{i=1}^{O_k}\sum_{j=1}^{D} t_{ijk}x_{ijk} + q\sum_{j=1}^{D} b_j\, y_j - \alpha q \sum_{k=1}^{K}\sum_{i=1}^{O_k}\sum_{j=1}^{D} t_{ijk}(x_{ijk} - z_{ijk}) \tag{117}$$

$s.t.$

$$q\sum_{k=1}^{K}\sum_{i=1}^{O_k} x_{ijk} \ \le\ g_j\,, \quad j \in \mathscr{D}, \tag{118}$$

$$l_j^0 + q\left(\sum_{k=1}^{K}\sum_{i=1}^{O_k} z_{ijk} + y_j\right) - d_j \;\geq\; 0\;, \quad j \in \mathscr{D}, \tag{119}$$

$$z_{ijk} \;\leq\; x_{ijk}\;,\; i \in \mathscr{O}_k,\; k \in \mathscr{K},\; j \in \mathscr{D}, \tag{120}$$

$$r_k \leq q \sum_{i \in O_k} \sum_{j=1}^{D} z_{ijk} \;\leq\; v_k\;, \quad k \in \mathscr{K}, \tag{121}$$

$$x_{ijk} \;\in\; \mathbb{R}^+\;,\; i \in \mathscr{O}_k,\; k \in \mathscr{K},\; j \in \mathscr{D}, \tag{122}$$

$$y_j \;\in\; \mathbb{R}^+\;,\; j \in \mathscr{D}, \tag{123}$$

$$z_{ijk} \;\in\; \mathbb{R}^+\;,\; i \in \mathscr{O}_k,\; k \in \mathscr{K},\; j \in \mathscr{D}. \tag{124}$$

Therefore the optimal cost given by the optimization problem (109)-(115) is greater or equal than the optimal cost given by the optimization problem (117)-(124) and in some cases it may be significantly larger. However we have to remember that (117)-(124) can be solved only when the vectors b and d are known, while the nonadjustable (or first stage, or here-and-now) decision variables x_{ijk} have to be determined *before* b and d are known. As we mentioned before the nonadjustable variables can be computed by any of the methods described in the previous sections of this paper. In order to have a fair comparison we will relax the integer constraints in all the methods. Namely, whenever a variable is constrained to belong to $\mathbb{N}$ we will require that it to belong to $\mathbb{R}^+$. Let us define precisely the methods under consideration.

There are many ways in which we can compare the performance of the various methods. In what follows we will compare them in a scenario based framework. As in (89), let

$$\hat{\Delta} = \left\{\hat{d}^1, \hat{d}^2, \ldots, \hat{d}^S\right\} \tag{125}$$

a given set of demand scenarios of dimension D from historical data. We consider a set of indices

$$\bar{\mathscr{S}} = \left\{1, \ldots, \bar{S}\right\} \subset \mathscr{S}\;, \tag{126}$$

with cardinality $\bar{S} < S$. For each such $\tau = \bar{S}, \ldots, S-1$ we compute the quantities

$$\bar{d} = \frac{1}{\bar{S}} \sum_{s=1}^{\bar{S}} \hat{d}^s, \quad \rho_1 G_j = \max_{s \in \bar{\mathscr{S}}} \hat{d}_j^s - \bar{d}_j,\; j \in \mathscr{D}. \tag{127}$$

We do not have historical data for the cost vectors, but we know a D-dimensional vector $\bar{b}$ of average costs. Then we generate a set of vectors

$$\hat{B} = \left\{\hat{b}^1, \hat{b}^2, \ldots, \hat{b}^S\right\}, \tag{128}$$

with components $\hat{b}_j^s$ obtained by sampling from a uniform distribution in the interval $\left[\bar{b}_j - \sigma \cdot \bar{b}_j, \bar{b}_j + \sigma \cdot \bar{b}_j\right]$ with a given deviation level of σ. For each $\tau = \bar{S}, \ldots, S-1$ we compute the following quantities

$$\rho_2 F_j = \max_{s \in \bar{\mathscr{S}}} \hat{b}_j^s - \bar{b}_j,\; j \in \mathscr{D}. \tag{129}$$

With the above notation we are ready to describe the methods that we want to compare:

$\mathbf{M_1}$ is the EV optimization problem (9)-(14) without integer constraints;

$\mathbf{M_2}$ is the RP stochastic optimization problem (15)-(22) without integer constraints;

$\mathbf{M_3}$ is the robust optimization problem with box constraints (44) -(54) without integer constraints;

$\mathbf{M_4}$ is the robust optimization problem with box-ellipsoidal constraints (70) -(78);

$\mathbf{M_5}$ is the adjustable robust optimization problem (96) -(104),

For each $\tau = \bar{S}, \ldots, S-1$ and for each method $\mathbf{M_m},\ \mathbf{m} = \mathbf{1}, \ldots, \mathbf{5}$ we find the optimal solution of the corresponding optimization problem using only the information contained in the vectors

$$\hat{d}^1, \hat{d}^2, \ldots, \hat{d}^\tau, \quad \hat{b}^1, \hat{b}^2, \ldots, \hat{b}^\tau. \tag{130}$$

From this optimal solution we obtain the nonadjustable decision variables

$$x^*_{ijk}, \quad i \in \mathscr{O}_k, \, k \in \mathscr{K}, \, j \in \mathscr{D}.$$

They are determined by using only the information contained in (130). Assume now that the vectors $\hat{d}^{\tau+1}, \hat{b}^{\tau+1}$ become available. Then we can solve the optimization problem (109)-(115) with $d = \hat{d}^{\tau+1}, b = \hat{b}^{\tau+1}$ to obtain the adjustable variables (116). The optimal value of the objective function (109) is denoted by $\mathbf{cost_{m,\tau}}$. It represents the optimal cost of our transportation problem with the adjustable strategy considered in this section when using method $\mathbf{M_m}$ for determining the nonadjustable variables. If the optimization problem (109)-(115) is infeasible we set $\mathbf{cost_{m,\tau}} = \infty$. We also denote by $\mathbf{CPU_{m,\tau}}$ the total CPU time, in seconds, spent in solving the optimization problems $\mathbf{M_m}$ and (109)-(115), including the possible infeasibility detection of the latter.

We will also consider the following affinely adjustable robust optimization problem:

$\mathbf{M_6}$:

(a) Solve the optimization problem (96) -(104) using only the information (130);
(b) Solve the optimization problem (105) with $S = \tau$ and $d = \hat{d}^{\tau+1}$;
(c) Define the adjustable variables as in (106).

This method has the advantage that the adjustable variables are obtained by solving only the very simple optimization problem (105). However this method only works if the optimal objective function of the latter problem $f\left(\lambda\left(\hat{d}^{\tau+1}\right)\right)$ is equal to zero. In this case the optimal cost given by this method, $\mathbf{cost_{6,\tau}}$, is equal to the optimal value of the objective function (96). If $f\left(\lambda\left(\hat{d}^{\tau+1}\right)\right) > 0$ we set $\mathbf{cost_{6,\tau}} = \infty$. The CPU time required for running $\mathbf{M_6}$ is denoted by $\mathbf{CPU_{6,\tau}}$.

Of course, when $\hat{d}^{\tau+1}, \hat{b}^{\tau+1}$ become available we can also solve the optimization problem (117)-(124) with $d = \hat{d}^{\tau+1}, b = \hat{b}^{\tau+1}$. However, this optimization problem determines the optimal values of both the adjustable and nonadjustable variables, while in our setting the nonadjustable variables have to be determined before the $\hat{d}^{\tau+1}, \hat{b}^{\tau+1}$ become available. We note that the optimization problem (117)-(124) is always feasible as long as the deterministic constraints (118), (120), (121) are not contradictory, because the uncertain constraint (119) is always satisfied for y_j large enough. We denote by $\mathbf{cost_\tau}$ the optimal value of the objective function (117) and by $\mathbf{CPU_\tau}$ the total CPU time, in seconds, spent in solving the optimization (117)-(124). The results of our numerical experiments are reported in Table 17 and 18 at the end of the next section.

6 Numerical results

In this section we discuss numerical results for the deterministic, stochastic and robust modelling approaches applied to the supply transportation problem. Deterministic and stochastic parameters values for the problem are reported below: Table 1 lists the set of suppliers $\mathscr{K}$ and the set of their plants $\mathscr{O}_k,\ k \in \mathscr{K}$. The list of destinations (cement factories) are shown in Table 2 with relative emergency costs and unloading capacities (see the points in Figure 1). Table 3 refers to the minimum and maximum requirement capacity of supplier $k \in \mathscr{K}$ in the expected value problem. We suppose to have an initial inventory level $l_j^0 = 0$ at all the destinations $j \in \mathscr{D}$ and to use vehicles with fixed capacity $q = 31$ Ton. The cancellation fee α is fixed to the value of 0.7.

Table 1: Set of suppliers $\mathscr{K}$ and set of their plants $O_k,\ k \in \mathscr{K}$.

	Supplier $k \in \mathscr{K}$	**Plant** $i \in \mathscr{O}_k$
1)	SAINT-GOBAIN PPC ITALIA SPA	Guglionesi Montenero Bisaccia Montiglio Murisengo Novafeltria di Sassofeltrio Riolo Terme
2)	F.LLI CORTESE SRL	Canolo
3)	DAMOS SRL	Pieve di Cadore
4)	ESTRAZIONE GESSO SNC	Murisengo
5)	F.LLI FUSCA' AUTOTRASPORTI SNC	Mineo
6)	LAGES SPA	Pisogne Riolo Terme
7)	AUTOTRASPORTI PIGLIACELLI SPA	Guglionesi
8)	AUTOTRASP. CARMINE SALERNO SRL	Guglionesi Montenero Bisaccia
9)	SIEM INDUSTRIA DEL GESSO SRL	Marcellinara
10)	FOGLIA GIUSEPPE	Giffoni Valle Piana
11)	SAMA SRL	Cava Ripari
12)	FASSA SPA	Cava di Calliano
13)	LOGISTICA BOCCATO GHIAIA SRL	Secchiano
14)	SOMEL PICCOLA SOC.COOP. A RL	Giffoni Valle Piana
15)	NUOVA DARSENA SRL	Pieve di Cadore
16)	BAIGUINI ALBERTO & C SNC	Rogno
17)	GESSI ROCCASTRADA SRL	Roccastrada
18)	BONACIA SRL	Mineo
19)	VOLANO SRL	Novafeltria di Sassofeltrio Secchiano
20)	MATERIA SERVIZI DI PROTTI DENIS	Novafeltria di Sassofeltrio
21)	VITO ALTERIO GESSI SNC	Anzano di Puglia
22)	PADUA ANGELO	Licodia Eubea
23)	ITALSAB SRL	Canolo
24)	FOGLIA GROUP SRL	Giffoni Valle Piana

Table 2: List of destinations (cement factories) with relative expected emergency costs $\bar{b}_j$ and unloading capacities g_j, $j \in \mathscr{D}$.

	Destination $j \in \mathscr{D}$	Expected emergency cost $\bar{b}_j$	Unloading capacity g_j
1)	BORGO SAN DALMAZZO	72.61	422.95
2)	CALUSCO D'ADDA	70.58	2054.55
3)	REZZATO	68.01	1330.67
4)	MONSELICE	64.94	453.64
5)	TRIESTE	73.52	613.41
6)	SALERNO	58.57	695.24
7)	SARCHE DI CALAVINO	69.83	443.14
8)	CASTROVILLARI	66.32	815.36
9)	MATERA	62.63	933.33
10)	NOVI LIGURE	68.22	319.79
11)	SCAFA	48.92	443.11
12)	COLLEFERRO	50.04	760.11
13)	VIBOVALENTIA	73.07	381.20
14)	RAVENNA	59.93	498.33
15)	GUARDIAREGIA	55.63	232411.75

Scenarios of demand at the first week of March 2014, are built on historical data directly, using all the weekly values in March, April, May and June of the years 2011, 2012 and 2013 (see Figure

Table 3: Minimum r_k and average maximum v_k requirement capacity of supplier $k \in \mathscr{K}$.

	Supplier $k \in \mathscr{K}$	r_k	v_k
1)	SAINT-GOBAIN PPC ITALIA SPA	1057.69	1500
2)	F.LLI CORTESE SRL	0	200
3)	DAMOS SRL	0	200
4)	ESTRAZIONE GESSO SNC	0	66
5)	F.LLI FUSCA' AUTOTRASPORTI SNC	0	200
6)	LAGES SPA	0	376.92
7)	AUTOTRASPORTI PIGLIACELLI SPA	0	100
8)	AUTOTRASP. CARMINE SALERNO SRL	0	100
9)	SIEM INDUSTRIA DEL GESSO SRL	0	150
10)	FOGLIA GIUSEPPE	0	94
11)	SAMA SRL	0	280
12)	FASSA SPA	0	100
13)	LOGISTICA BOCCATO GHIAIA SRL	0	100
14)	SOMEL PICCOLA SOC.COOP. A RL	0	100
15)	NUOVA DARSENA SRL	0	100
16)	BAIGUINI ALBERTO & C SNC	0	100
17)	GESSI ROCCASTRADA SRL	0	100
18)	BONACIA SRL	0	100
19)	VOLANO SRL	0	100
20)	MATERIA SERVIZI DI PROTTI DENIS	0	100
21)	VITO ALTERIO GESSI SNC	0	100
22)	PADUA ANGELO	0	100
23)	ITALSAB SRL	0	50
24)	FOGLIA GROUP SRL	0	100

2). Values of $\rho_1 G_j$ as in (127) are reported in Table 14. On the other hand, scenarios of buying costs from external sources have been generated sampling from a uniform distribution in the interval $\left[\bar{b}_j - \sigma \cdot \bar{b}_j, \bar{b}_j + \sigma \cdot \bar{b}_j\right]$ with a given deviation level of $\sigma = 20\%$. Values of $\rho_2 F_j$ as in (129) are reported in Table 14.

In this way a scenario tree composed of 48 leaves has been built. Scenarios are supposed to be equiprobable. The considered models aim to find, for each plant of the 24 suppliers, the number of vehicles to be booked for replenishing gypsum in order to minimize the total cost, during the first week of March 2014.

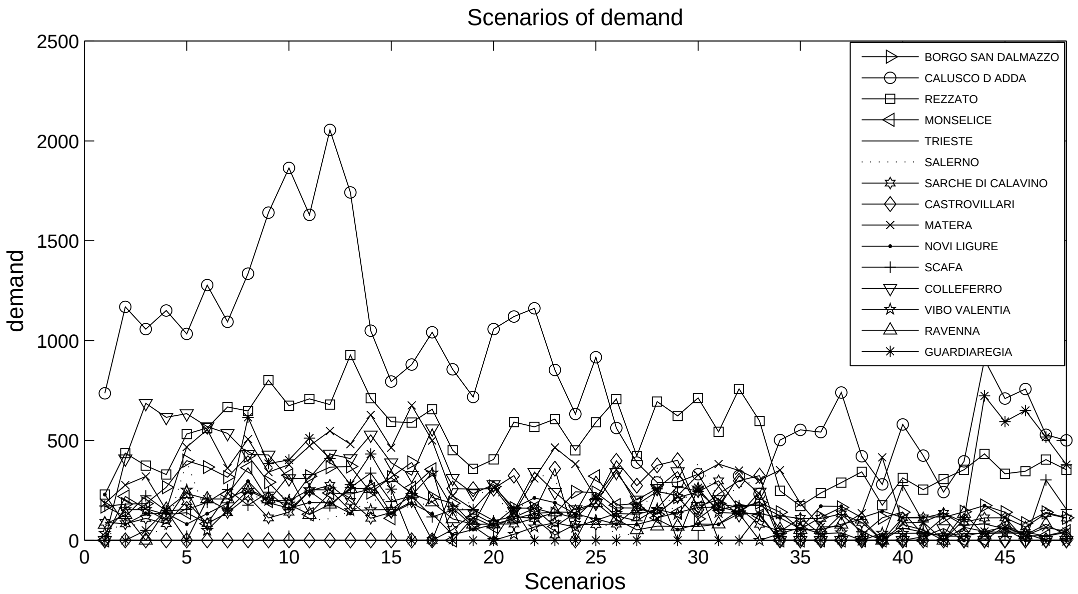

Figure 2: Demand scenarios d_j^s of cement factories $j \in \mathscr{D}$.

Both the Expected Value EV problem and stochastic optimization problem SP are modeled in AMPL and solved using the CPLEX 12.5.1.0. solver. The expected value problem, (1)-(8) (with a unique scenario of demand, $\bar{d}_j$) converges to the solution in 51 MIP iterations. It is composed of 975 variables of which 960 are integer. AMPL presolve eliminates 23 constraints, the remaining ones are 535, all linear. The computing time is negligible (see the first line of Table 7).

On the other side, the two-stage stochastic programming problem (15)-(22), converges to the solution in 16 609 MIP iterations. The computing time is much higher than the EV (see the second line of Table 7). The RP problem is composed of 24 240 variables of which 23 520 integer. AMPL presolve eliminates 1261 constraints, the remaining ones are 24 818, all linear inequalities.

Solution of the stochastic model, (15)-(22) RP, is compared with the solution of the *expected value problem*, (1)-(8) EV (see Table 7). Solutions to the deterministic mixed-integer model EV are reported in Table 5: the model will always book the exact number of vehicles needed for the next period (so $\bar{x}_{ijk} = \bar{z}^s_{ijk}$, $i \in \mathscr{O}_k$, $k \in \mathscr{K}$, $j \in \mathscr{J}$, $s \in \mathscr{S}$); it sorts the suppliers and their plants according to the transportation costs and books a full production capacity from the cheapest one, following by the next-cheapest. As long as we have enough transportation capacity, the model will never purchase extra gypsum from external sources (i.e. $y_j = 0$ for some destinations, see Table 4).

Table 4: Optimal solutions y_j for the deterministic "supply transportation problem" with mixed-integer variables.

Destinations	y_j
Calusco D'Adda	0.34
Guardiaregia	0.12
Rezzato	0.09
Trieste	0.28

First stage solutions for the stochastic mixed-integer model are reported in Table 6.

Table 5: Optimal solution of the expected value model EV with mixed-integer variables. The table shows the optimal number of booked vehicles x_{ijk} from plant $i \in \mathscr{O}_k$ of supplier k to destination $j \in \mathscr{D}$.

Destination	Supplier	Plant	x_{ijk}
BORGO SAN DALMAZZO	SAINT-GOBAIN PPC ITALIA SPA	Montiglio	7
CALUSCO D'ADDA	ESTRAZIONE GESSO SNC	Murisengo	2
CALUSCO D'ADDA	FASSA SPA	Cava di Calliano	3
CALUSCO D'ADDA	SAINT-GOBAIN PPC ITALIA SPA	Montiglio	18
CALUSCO D'ADDA	BAIGUINI ALBERTO & C SNC	Rogno	3
CASTROVILLARI	PADUA ANGELO	Licodia Eubea	3
COLLEFERRO	SAINT-GOBAIN PPC ITALIA SPA	Novafeltria di Sassofeltrio	7
COLLEFERRO	VOLANO SRL	Novafeltria di Sassofeltrio	1
GUARDIAREGIA	FOGLIA GIUSEPPE	Giffoni Valle Piana	2
GUARDIAREGIA	VITO ALTERIO GESSI SNC	Anzano di Puglia	3
MATERA	F.LLI CORTESE SRL	Canolo	6
MATERA	FOGLIA GROUP SRL	Giffoni Valle Piana	3
MONSELICE	SAINT-GOBAIN PPC ITALIA SPA	Novafeltria di Sassofeltrio	4
NOVI LIGURE	SAINT-GOBAIN PPC ITALIA SPA	Montiglio	4
RAVENNA	SAINTGOBAIN	Novafeltria di Sassofeltrio	4
REZZATO	LAGES SPA	Pisogne	12
REZZATO	MATERIA SERVIZI DI PROTTI DENIS	Novafeltria di Sassofeltrio	3
REZZATO	VOLANO SRL	Novafeltria di Sassofeltrio	1
SALERNO	FOGLIA GIUSEPPE	Giffoni Valle Piana	1
SALERNO	SOMEL PICCOLA SOC.COOP.A RL	Giffoni Valle Piana	3
SARCHE DI CALAVINO	SAINT-GOBAIN PPC ITALIA SPA	Novafeltria di Sassofeltrio	4
SCAFA	SAMA SRL	Cava Ripari	5
TRIESTE	MATERIA SERVIZI DI PROTTI DENIS	Novafeltria di Sassofeltrio	0
VIBO VALENTIA	SIEM INDUSTRIA DEL GESSO SRL	Marcellinara	4

Table 6: Optimal solution of the stochastic model RP with mixed-integer variables. The table shows the optimal number of booked vehicles x_{ijk} from plant $i \in \mathscr{O}_k$ of supplier k to destination $j \in \mathscr{D}$.

Destination	Supplier	Plant	x_{ijk}
BORGO SAN DALMAZZO	SAINT-GOBAIN PPC ITALIA SPA	Montiglio	9
CALUSCO D'ADDA	FASSA SPA	Cava di Calliano	3
CALUSCO D'ADDA	SAINTGOBAIN PPC ITALIA SPA	Montiglio	22
CALUSCO D'ADDA	DAMOS SRL	Pieve di Cadore	1
CALUSCO D'ADDA	ESTRAZIONE GESSO SNC	Murisengo	2
CALUSCO D'ADDA	BAIGUINIALBERTOCSNC	Rogno	3
CALUSCO D'ADDA	LAGES SPA	Pisogne	4
CASTROVILLARI	PADUA ANGELO	LicodiaEubea	3
CASTROVILLARI	BONACIA SRL	Mineo	1
CASTROVILLARI	F.LLI FUSCA' AUTOTRASPORTI SNC	Mineo	1
CASTROVILLARI	SIEM INDUSTRIA DEL GESSO SRL	Marcellinara	3
COLLEFERRO	MATERIA SERVIZI	Novafeltria	1
COLLEFERRO	SAINT-GOBAIN PPC ITALIA SPA	Novafeltria di Sassofeltrio	4
COLLEFERRO	SAMA SRL	Cava Ripari	2
COLLEFERRO	GESSI ROCCASTRADA SRL	Roccastrada	3
GUARDIAREGIA	FOGLIA GROUP SRL	Giffoni Valle Piana	1
GUARDIAREGIA	FOGLIA GIUSEPPE	Giffoni Valle Piana	1
GUARDIAREGIA	AUTOTRASP. CARMINE SALERNO SRL	Montenero Bisaccia	3
GUARDIAREGIA	VITO ALTERIO GESSI SNC	Anzano di Puglia	3
MATERA	FOGLIA GIUSEPPE	Giffoni Valle Piana	1
MATERA	FOGLIA GROUP SRL	Giffoni Valle Piana	1
MATERA	F.LLI CORTESE SRL	Canolo	6
MATERA	F.LLI FUSCA' AUTOTRASPORTI SNC	Mineo	3
MATERA	ITALSAB SRL	Canolo	1
MATERA	SIEM INDUSTRIA DEL GESSO SRL	Marcellinara	1
MONSELICE	MATERIA SERVIZI	Novafeltria di Sassofeltrio	1
MONSELICE	SAINT GOBAIN PPC ITALIA SPA	Novafeltria di Sassofeltrio	1
MONSELICE	DAMOS SRL	Pieve di Cadore	1
MONSELICE	LOGISTICA BOCCATO GHIAIA SRL	Secchiano	3
NOVI LIGURE	SAINT-GOBAIN PPC ITALIA SPA	Montiglio	6
RAVENNA	MATERIA SERVIZI	Novafeltria di Sassofeltrio	1
RAVENNA	SAINT-GOBAIN PPC ITALIA SPA	Novafeltria di Sassofeltrio	3
RAVENNA	VOLANO SRL	Novafeltria di Sassofeltrio	3
REZZATO	SAINT-GOBAIN PPC ITALIA SPA	Novafeltria di Sassofeltrio	8
REZZATO	NUOVA DARSENA SRL	Pieve di Cadore	3
REZZATO	LAGES SPA	Pisogne	8
REZZATO	DAMOS SRL	Pieve di Cadore	2
SALERNO	FOGLIA GIUSEPPE	Giffoni Valle Piana	2
SALERNO	FOGLIA GROUP SRL	Giffoni Valle Piana	2
SALERNO	SOMEL PICCOLA SOC.COOP A RL	Giffoni Valle Piana	3
SARCHE DI CALAVINO	MATERIA SERVIZI	Novafeltria di Sassofeltrio	1
SARCHE DI CALAVINO	SAINT-GOBAIN PPC ITALIA SPA	Novafeltria di Sassofeltrio	4
SARCHE DI CALAVINO	DAMOSSRL	Pieve di Cadore	1
SCAFA	SAMA SRL	Cava Ripari	7
TRIESTE	MATERIA SERVIZI	Novafeltria di Sassofeltrio	1
VIBOVALENTIA	SIEM INDUSTRIA DEL GESSO	Marcellinara	3
VIBOVALENTIA	BONACIA SRL	Mineo	2
VIBOVALENTIA	F.LLI FUSCA' AUTOTRASPORTI SNC	Mineo	3

Table 7: Optimal solutions for the "supply transportation problem" with mixed-integer variables where total optimal costs and CPU costs are reported.

	Objective value (€)	CPU seconds
EV	75 425.54	0.109
RP (with $\vert\mathscr{S}\vert = 48$)	107 244.67	16.19
EEV	infeasible	
WS	84 472.21	4.47
RO box constraints	391928.16	0.109

A direct comparison of the total number of booked vehicles at each destination plant $\sum_{k\in\mathscr{K}}\sum_{i\in\mathscr{O}_k} x_{ijk}\ \forall\, j \in \mathscr{D}$, in the expected value solution and in the stochastic one is shown in Table 8. The deterministic model books much fewer vehicles, resulting in a solution costing 70.33% of the stochastic counterpart (see Table 7). However, when using the EV solution, the *expectation expected value* problem EEV is

Table 8: Comparison of the total number of booked vehicles $\sum_{k\in\mathscr{K}}\sum_{i\in\mathscr{O}_k} x_{ijk}$ $\forall j \in \mathscr{D}$ in the expected value solution EV and in the stochastic solution RP with mixed-integer variables.

Destination	deterministic solution $\bar{x}_{ijk}$	stochastic solution x_{ijk}
BORGO SAN DALMAZZO	7	9
CALUSCO D'ADDA	26	35
CASTROVILLARI	3	8
COLLEFERRO	8	10
GUARDIAREGIA	6	8
MATERA	9	13
MONSELICE	4	6
NOVI LIGURE	4	6
RAVENNA	4	7
REZZATO	16	21
SALERNO	4	7
SARCHE DI CALAVINO	4	6
SCAFA	5	7
TRIESTE	0	1
VIBO VALENTIA	4	8

infeasible (see Table 7) resulting in an infinite *Value of Stochastic Solution*

$$VSS = EEV - RP = \infty \ . \tag{131}$$

The infinite VSS value shows that the expected value solution is not appropriate in a stochastic setting since constraint (19) is no longer satisfied because the number of vehicles booked from the supplier SAINT-GOBAIN PPC ITALIA SPA along scenarios 31, 41 and 42 is too low. This results in a violation of the minimum requirement capacity $r_k = 1057.69$ (see [16] and [18]).

Figure 3 shows the objective function values of the deterministic problems solved separately over each scenario $s \in \mathscr{S}$ from the historical data. The total cost of the worst scenario is 205 705.09. The corresponding *Wait-and-See* value WS is reported in Table 7 showing the advantage of having the information about future demand at the first stage. The *Expected Value of Perfect Information* EVPI reduces to:

$$EVPI = RP - WS = 107\,244.67 - 84\,472.21 = 22\,772.46 \ . \tag{132}$$

Notice that the RP problem considered above uses only 48 scenarios. Another important issue in SP is to check how the value of the total cost varies with the number of scenarios. For this reason figure 4 shows the convergence of the optimal function value (total cost) of the stochastic model where the integrality constraints have been relaxed, for an increasing number of scenarios. From the results we note that the optimal function value stabilizes for scenario trees with more than 200 scenarios.

Since the gypsum demand is highly affected by the economic conditions of the public and private medium and large-scale construction sector, a reliable forecast and reasonable estimates of probability distributions are difficult to obtain. This is the main reason that lead us to consider also robust optimization approaches. In the following results we first consider static approaches with uncertainty parameters respectively belonging to box, ellipsoidal uncertainty sets or mixture of them, and secondly dynamic approaches, via the concept of adjustable robust counterpart.

We first considered the robust optimization model with box uncertainty, (55)-(63). As done for the previous approaches, the problem is modeled in AMPL and solved using the CPLEX 12.5.1.0. solver. It converges to the solution in 83 MIP iterations. The problem is composed of 976 variables of which 960 are integer. AMPL presolve eliminates 23 constraints, the remaining ones are 521 inequality constraints, all linear. Robust solution of the mixed-integer model with box constraints are reported in Table 7 at the last line. Results show that the approach is very conservative having

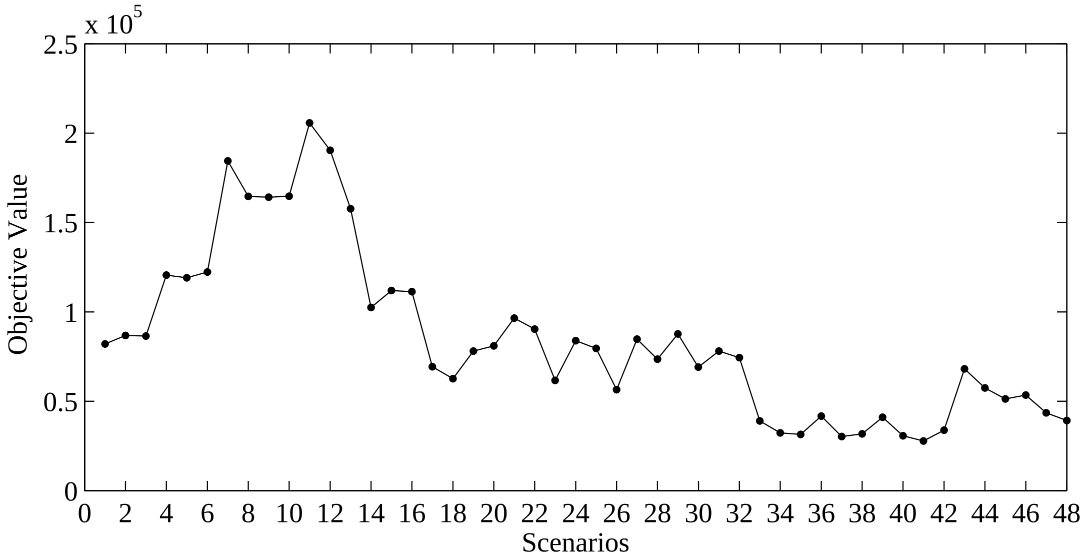

Figure 3: Objective function values of the deterministic problems over 48 scenarios with mixed-integer variables.

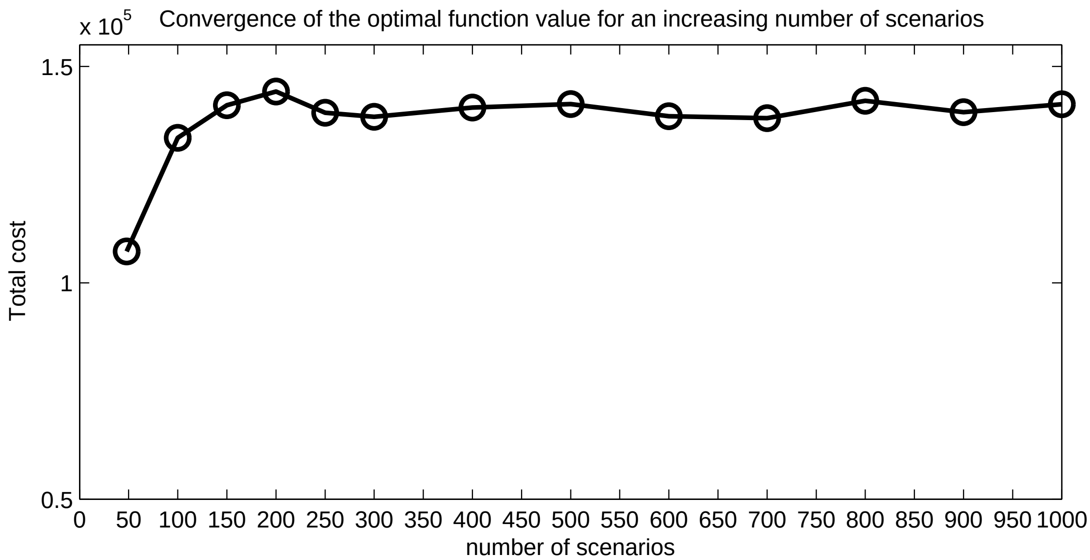

Figure 4: Convergence of the optimal function value of the stochastic model with all continuous variables for an increasing number of scenarios.

a total cost of about 365% larger than the expected cost obtained by solving RP. Tables 9 and 10 report the optimal solutions of the mixed-integer robust problem with box constraints given by the normalized volume y_j for destination plant $j \in \mathscr{D}$ and number of booked vehicles x_{ijk} from plant $i \in \mathscr{O}_k$ of supplier k to destination $j \in \mathscr{D}$. Due to the suppliers' maximum requirement constraint and largest demand, the model forces the company to buy from external sources. This happens for almost all the destinations, with exception of Trieste and Vibovalentia where the demand is fully satisfied by the orders x_{ijk}. On the other hand, the demand of Colleferro, Guardiaregia and Rezzato, are satisfied only by external orders $y_j > 0$ with a consequent larger cost. The choice of the box uncertainty set is preferable only if the feasibility of all the constraints is highly required. One can try to use a different uncertainty set in order to get a less conservative outcome. We show the results obtained by using a box-ellipsoid uncertainty set which, as mentioned in Section 4.2, requires the solution of a second-order cone program. To make a fair comparison with the previous models, we need to relax the integer variables hypothesis on x_{ijk} and z_{ijk}.

Table 9: Optimal solution of the robust mixed-integer box optimization model. The table shows the optimal normalized volume y_j for destination plant $j \in \mathscr{D}$.

Destination	y_j
BORGO SAN DALMAZZO	0.643847
CALUSCO D'ADDA	17.2759
CASTROVILLARI	0.931387
COLLEFERRO	22.1082
GUARDIAREGIA	23.3377
MATERA	15.7849
MONSELICE	8.7287
NOVI LIGURE	0.542244
RAVENNA	6.09235
REZZATO	29.9154
SALERNO	0.705345
SARCHE DI CALAVINO	0.595318
SCAFA	1.83631
TRIESTE	0
VIBO VALENTIA	0

Table 10: Optimal solution of the robust mixed-integer box optimization model. The table shows the optimal number of booked vehicles x_{ijk} from plant $i \in \mathscr{O}_k$ of supplier k to destination $j \in \mathscr{D}$.

Destination $j \in \mathscr{D}$	**Supplier** $k \in \mathscr{K}$	**Plant** $i \in \mathscr{O}_k$	x_{ijk}
BORGO SAN DALMAZZO	SAINT-GOBAIN PPC ITALIA SPA	Montiglio	13
CALUSCO D'ADDA	ESTRAZIONE GESSO SNC	Murisengo	2
CALUSCO D'ADDA	FASSA SPA	Cava di Calliano	3
CALUSCO D'ADDA	SAINT-GOBAIN PPC ITALIA SPA	Montiglio	29
CALUSCO D'ADDA	BAIGUINI ALBERTO CSNC	Rogno	3
CALUSCO D'ADDA	LAGES SPA	Pisogne	12
CASTROVILLARI	PADUA ANGELO	Licodia Eubea	3
CASTROVILLARI	F.LLI FUSCA' AUTOTRASPORTI SNC	Mineo	5
CASTROVILLARI	SIEM INDUSTRIA DEL GESSO SRL	Marcellinara	4
MATERA	AUTOTRASP. CARMINE SALERNO SRL	Montenero Bisaccia	3
MATERA	VITO ALTERIO GESSI SNC	Anzano di Puglia	1
MONSELICE	LOGISTICA BOCCATO GHIAIA SR	Secchiano	3
NOVI LIGURE	SAINT-GOBAIN PPC ITALIA SPA	Montiglio	6
NOVI LIGURE	GESSI ROCCA STRADA SRL	Roccastrada	3
RAVENNA	MATERIA SERVIZI DI PROTTI DENIS	Novafeltria	1
RAVENNA	VOLANO SRL	Novafeltria di Sassofeltrio	3
SALERNO	FOGLIA GIUSEPPE	Giffoni Valle Piana	3
SALERNO	SOMEL PICCOLA SOC. COOP A RL	Giffoni Valle Piana	3
SALERNO	AUTOTRASPORTI PIGLIACELLI SPA	Gugionesi	3
SALERNO	FOGLIA GROUP SRL	Giffoni Valle Piana	3
SARCHE DI CALAVINO	DAMOS SRL	Pieve di Cadore	4
SARCHE DI CALAVINO	MATERIA SERVIZI DI PROTTI DENIS	Novafeltria	2
SARCHE DI CALAVINO	NUOVA DARSENA SRL	Pieve di Cadore	3
SCAFA	SAMA SRL	Cava Ripari	9
TRIESTE	DAMOS SRL	Pieve di Cadore	2
VIBO VALENTIA	BONACIA SRL	Mineo	3
VIBO VALENTIA	F.LLI CORTESE SRL	Canolo	6
VIBO VALENTIA	F.LLI FUSCA' AUTOTRASPORTI SNC	Mineo	1
VIBO VALENTIA	ITALSAB SRL	Canolo	1

Problem (70)-(78) with box-ellipsoidal uncertainty was modeled in AMPL and solved using the MOSEK solver. The problem is composed of 976 variables. AMPL presolve eliminates 23 constraints, the remaining ones are 518, all linear except one. Since in our application the number of destination $D = 15$, condition (66) is satisfied for $\Omega \geq \sqrt{15} = 3.873$. Table 11 reports total costs of the robust

Table 11: Optimal solutions (costs, probability of constraint satisfaction (Prob), and computing time in CPU seconds) for the "supply transportation problem" with all continuous variables.

Optimization method	Objective value (€)	Prob	CPU seconds
EV	72 751.90		0.015
RP (with $\|\mathscr{S}\| = 48$)	104 971.09=RP		12.2461
RP (with $\|\mathscr{S}\| = 100$)	133 499.31		32.94
RP (with $\|\mathscr{S}\| = 150$)	141 015.19		98.32
RP (with $\|\mathscr{S}\| = 200$)	144 189.72		114.13
RP (with $\|\mathscr{S}\| = 250$)	139 277.54		196.60
RP (with $\|\mathscr{S}\| = 300$)	138 347.25		214.84
RP (with $\|\mathscr{S}\| = 400$)	140 493.67		383.98
RP (with $\|\mathscr{S}\| = 500$)	141 311		593.84
RP (with $\|\mathscr{S}\| = 600$)	138 466.34		711.87
RP (with $\|\mathscr{S}\| = 700$)	138 051.99		826.97
RP (with $\|\mathscr{S}\| = 800$)	142 047.69		769.13
RP (with $\|\mathscr{S}\| = 900$)	139 410.84		1762.14
RP (with $\|\mathscr{S}\| = 1000$)	141 258.22		1851.25
RO box constraints	381 634.90	1	0.015
RO with $\bar{b}_j$ and box constraints for d_j	343 849.19	0	0.015
RO box-ellipsoidal constraints ($\Omega = 0$)	343 849.19	0	0.2028
RO box-ellipsoidal constraints ($\Omega = 0.2$)	346 986.47	0.0198	0.2028
RO box-ellipsoidal constraints ($\Omega = 0.4$)	349 992.72	0.0769	0.2028
RO box-ellipsoidal constraints ($\Omega = 0.6$)	352 963.73	0.1647	0.2028
RO box-ellipsoidal constraints ($\Omega = 0.8$)	355 840.43	0.2739	0.2028
RO box-ellipsoidal constraints ($\Omega = 1$)	358 616.46	0.3935	0.2028
RO box-ellipsoidal constraints ($\Omega = 1.2$)	361 347.71	0.5132	0.2028
RO box-ellipsoidal constraints ($\Omega = 1.4$)	364 056.06	0.6247	0.2028
RO box-ellipsoidal constraints ($\Omega = 1.6$)	366 735.70	0.7220	0.2028
RO box-ellipsoidal constraints ($\Omega = 1.8$)	369 385.09	0.8021	0.2028
RO box-ellipsoidal constraints ($\Omega = 2$)	372 002.76	0.8647	0.2028
RO box-ellipsoidal constraints ($\Omega = 2.2$)	374 591.53	0.9111	0.2028
RO box-ellipsoidal constraints ($\Omega = 2.4$)	377 141.44	0.9439	0.2028
RO box-ellipsoidal constraints ($\Omega = 2.6$)	379 655.26	0.9660	0.2028
RO box-ellipsoidal constraints ($\Omega = 2.75$)	381 520.16	0.9772	0.2028
RO box-ellipsoidal constraints ($\Omega = 3.873$)	395 095.068	1	0.2028

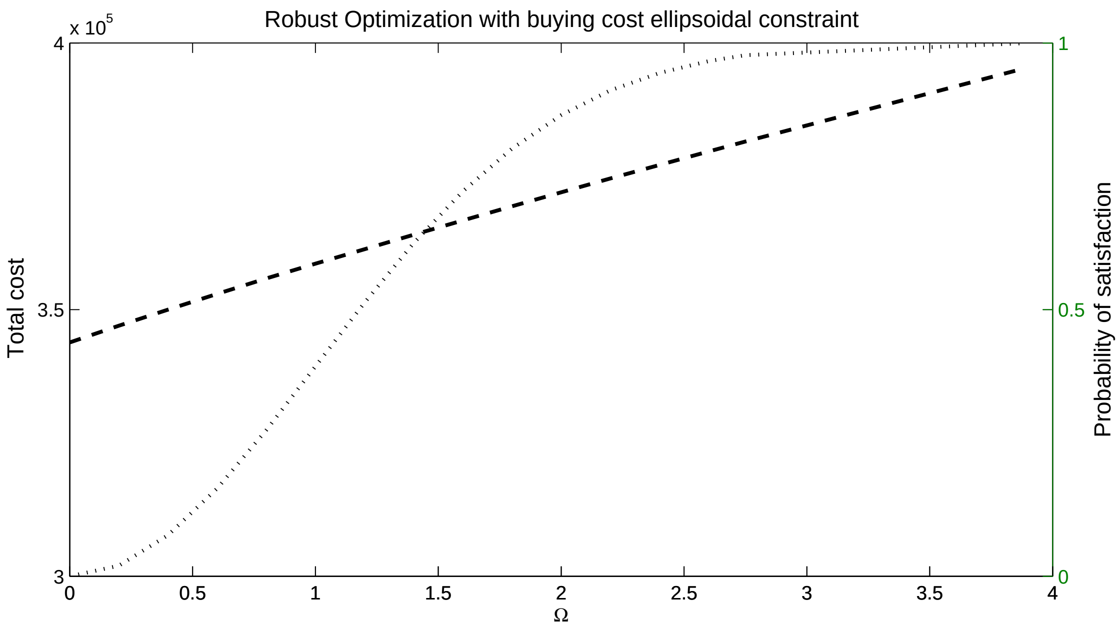

Figure 5: Total cost of robust optimization model with all continuous variables and ellipsoidal constraint. The figure shows for increasing values of Ω the total cost and probability of satisfaction of constraint (71).

Table 12: Optimal solution of the robust box optimization model with all continuous variables. The table shows the optimal normalized volume y_j for destination plant $j \in \mathscr{D}$.

Destination	y_j
BORGO SAN DALMAZZO	0
CALUSCO D'ADDA	17.1097
CASTROVILLARI	0
COLLEFERRO	22.1082
GUARDIAREGIA	23.3333
MATERA	13.2951
MONSELICE	8.5029
NOVI LIGURE	0
RAVENNA	5.513
REZZATO	29.9154
SALERNO	0
SARCHE DI CALAVINO	0
SCAFA	1.80405
TRIESTE	0
VIBO VALENTIA	0

Table 13: Optimal solution of the robust box optimization model with all continuous variables. The table shows the optimal number of booked vehicles x_{ijk} from plant $i \in \mathscr{O}_k$ of supplier k to destination $j \in \mathscr{D}$.

Destination $j \in \mathscr{D}$	**Supplier** $k \in \mathscr{K}$	**Plant** $i \in \mathscr{O}_k$	x_{ijk}
BORGO SAN DALMAZZO	SAINT-GOBAIN PPC ITALIA SPA	Montiglio	13.6438
CALUSCO D'ADDA	ESTRAZIONE GESSO SNC	Murisengo	2.12903
CALUSCO D'ADDA	FASSA SPA	Cava di Calliano	3.22581
CALUSCO D'ADDA	SAINT-GOBAIN PPC ITALIA SPA	Montiglio	28.4268
CALUSCO D'ADDA	BAIGUINI ALBERTO & C SNC	Rogno	3.22581
CALUSCO D'ADDA	LAGES SPA	Pisogne	12.1587
CASTROVILLARI	PADUA ANGELO	Licodia Eubea	3.22581
CASTROVILLARI	F.LLI FUSCA' AUTOTRASPORTI SNC	Mineo	4.86687
CASTROVILLARI	SIEM INDUSTRIA DEL GESSO SRL	Marcellinara	4.83871
GUARDIAREGIA	AUTOTRASPORTI PIGLIACELLI SPA	Guglionesi	0.00433257
MATERA	AUTOTRASP. CARMINE SALERNO SRL	Montenero Bisaccia	3.22581
MATERA	F.LLI CORTESE SRL	Canolo	0.425199
MATERA	ITALSAB SRL	Canolo	1.6129
MATERA	VITO ALTERIO GESSI SNC	Anzano di Puglia	3.22581
MONSELICE	LOGISTICA BOCCATO GHIAI ASR	Secchiano	3.22581
NOVI LIGURE	SAINT-GOBAIN PPC ITALIA SPA	Montiglio	6.31644
NOVI LIGURE	GESSI ROCCA STRADA SRL	Roccastrada	3.22581
RAVENNA	MATERIA SERVIZI DI PROTTI DENIS	Novafeltria di Sassofeltrio	3.22581
RAVENNA	VOLANO SRL	Novafeltria	1.35355
SALERNO	FOGLIA GIUSEPPE	Giffoni Valle Piana	3.03226
SALERNO	SOMEL PICCOLA SOC. COOP A RL	Giffoni Valle Piana	3.22581
SALERNO	AUTOTRASPORTI PIGLIACELLI SPA	Gugionesi	3.22147
SALERNO	FOGLIA GROUPSRL	Giffoni Valle Piana	3.22581
SARCHE DI CALAVINO	DAMOS SRL	Pieve di Cadore	4.49725
SARCHE DI CALAVINO	VOLANO SRL	Novafeltria di Sassofeltrio	1.87226
SARCHE DI CALAVINO	NUOVA DARSENA SRL	Pieve di Cadore	3.22581
SCAFA	SAMA SRL	Cava Ripari	9.03226
TRIESTE	DAMOS SRL	Pieve di Cadore	1.95436
VIBO VALENTIA	BONACIA SRL	Mineo	3.22581
VIBO VALENTIA	F.LLI CORTESE SRL	Canolo	6.02641
VIBO VALENTIA	F.LLI FUSCA' AUTOTRASPORTI SNC	Mineo	1.58474

box-ellipsoidal optimization with the probability of satisfaction of the second order cone constraint (71) and CPU seconds, for increasing values of the parameter $0 \leq \Omega \leq 3.873$ (see also Figure 5).

For a fair comparison, total costs and CPU time of EV and RP models refer to the case in which the integrality on the decision variables is relaxed.

The results show that for $\Omega = 0$, the total cost of the RO box-ellipsoidal approach is the same than the RO model with average buying cost $\bar{b}_j$ and box constraint requirement for the demand d_j. As Ω increases to 2.75 the total cost reaches approximately the same value, 381 520.16, of the box model case with a probability of constraint satisfaction close to one. In this case the optimal cost of the box-ellipsoidal model is only 114.74 lower than the box model, where the probability of constraint satisfaction is exactly one.

For a comparative analysis, tables 15 and 16 report the solution variables x_{ijk} and y_j in the

Table 14: Values used in defining the box constraints (43) for the demand and (39) for buying cost.

Destination	$\rho_1 G_j$	$\rho_2 F_j$
BORGO SAN DALMAZZO	226.288561	9.702845442
CALUSCO D'ADDA	1237.994565	11.17820427
REZZATO	428.5384258	10.75140176
MONSELICE	241.6982616	10.19631396
TRIESTE	92.40921985	11.28079203
SALERNO	264.6183199	10.96670549
SARCHE DI CALAVINO	174.0835254	11.6578062
CASTROVILLARI	312.291651	10.32202013
MATERA	397.6409753	10.65633647
NOVI LIGURE	175.7510858	9.943717501
SCAFA	181.3741261	12.39780253
COLLEFERRO	326.825566	10.03871316
VIBO VALENTIA	228.3892089	9.799237344
RAVENNA	191.7861022	10.97611094
GUARDIAREGIA	564.6712725	10.94525171

case of the box-ellipsoidal robust approach with $\Omega = 2.75$ whereas tables 12 and 13 refer to the optimal solutions of the continuous box case. While the two approaches have approximately the same total costs, their solution strategies have some differences: the box-ellipsoidal solution do not make any order only for Colleferro, deciding to satisfy their maximum demand by external sources $y_{12} = 22.10$. On the other side the box solution does not make any order both for Rezzato and Colleferro making orders from external sources. The box solution tries to satisfy the demand of Borgo San Dalmazzo, Castrovillari, Novi Ligure, Salerno, Sarche di Calavino, Trieste and Vibo Valentia only by booking vehicles x_{ijk} from the set of suppliers while the box-ellipsoidal solution requires for all the destinations, with exception of Vibo Valentia, to buy from external sources.

The solution of the box-ellipsoidal model with $\Omega = \sqrt{D} = 3.873$ is guaranteed to satisfy the second order cone constraint with probability one. This fact has been remarked at the end of Section 4. In this case, the ellipsoidal uncertainty set $\mathscr{U}_{b,ell}$, given by (68) and (36), includes the box uncertainty set $\mathscr{U}_{b,box}$ defined in (39). Therefore the RO with box-ellipsoidal constraints results in a cost that is 13 575 larger than the RO with box constraints.

In terms of CPU complexity, column 4 of Table 11 shows that the stochastic approach is significantly more expensive than the conservative robust one. The stochastic approach with 200 scenarios, where the stability of the solution is reached (see Figure 4), requires 114 CPU seconds versus 0.2 of the box-ellipsoidal formulation.

In order to make a fair comparison with the stochastic programming methodology, as proposed in Section 5, we compute total costs associated to a dynamic approach via the concept of adjustable robust counterpart.

Table 17 and Figure 6 show total costs $\mathbf{cost_{1}}_{,\tau}, \dots, \mathbf{cost_{6}}_{,\tau}$ obtained by solving model (109)-(115) using the nonadjustable decision variables x_{ijk} respectively given by methods $\mathbf{M_1}, \dots, \mathbf{M_6}$, for $\tau = 24, \dots, 47$. Results are average values over 1000 simulations. In general the number of booked vehicles x_{ijk} (nonadjustable variables) from the stochastic approach $\mathbf{M_2}$ is higher than the one from the expected value approach $\mathbf{M_1}$. This implies an interesting behaviour of the two solutions in terms of $\mathbf{cost_{1}}_{,\tau}$ and $\mathbf{cost_{2}}_{,\tau}$: for $\tau = 24, \dots, 31$, $\mathbf{cost_{1}}_{,\tau} > \mathbf{cost_{2}}_{,\tau}$, since the low orders x_{ijk} from $\mathbf{M_1}$ force the model to buy from external sources with an higher cost. On the contrary, for $\tau = 32, \dots, 47$, since the observed demand d_{t+1} is lower than the ones observed in the scenario set $\bar{\mathscr{S}}$, a cancellation fee has to be paid for all vehicles booked from model $\mathbf{M_2}$ but not actually used and consequently $\mathbf{cost_{2}}_{,\tau} > \mathbf{cost_{1}}_{,\tau}$.

Larger costs are observed for the robust methods with box constraints $\mathbf{M_3}$ and with box-ellipsoidal constraints $\mathbf{M_4}$ ($\Omega = 3.873$). An intermediate cost is obtained for the adjustable robust optimization $\mathbf{M_5}$. On the other hand, an affinely adjustable approach is no longer obtained since $f\left(\lambda\left(\bar{d}^{\tau+1}\right)\right) > 0$

Table 15: Optimal solution of the robust box-ellipsoidal optimization model with $\Omega = 2.75$. The table shows the optimal number of booked vehicles x_{ijk} from plant $i \in \mathscr{O}_k$ of supplier k to destination $j \in \mathscr{D}$.

Destination $j \in \mathscr{D}$	**Supplier** $k \in \mathscr{K}$	**Plant** $i \in \mathscr{O}_k$	x_{ijk}
BORGO SAN DALMAZZO	SAINTGOBAIN	Montiglio	9.27
CALUSCO D'ADDA	FASSA SPA	Cava di Calliano	3.23
CALUSCO D'ADDA	ESTRAZIONE GESSO SNC	Murisengo	2.13
CALUSCO D'ADDA	SAINT-GOBAIN PPC ITALIA SPA	Montiglio	38.05
CALUSCO D'ADDA	LAGES SPA	Pisogne	10.52
CALUSCO D'ADDA	BAIGUINI ALBERTO & C SNC	Rogno	3.23
CASTROVILLARI	PADUA ANGELO	Licodia Eubea	3.23
CASTROVILLARI	F.LLI FUSCA' AUTOTRASPORTI SNC	Mineo	1.96
CASTROVILLARI	BONACIA SRL	Mineo	1.26
CASTROVILLARI	SIEM INDUSTRIA DEL GESSO SRL	Marcellinara	4.84
GUARDIAREGIA	AUTOTRASP CARMINE SALERNO SRL	Montenero Bisaccia	3.21
GUARDIAREGIA	SAMA SRL	Cava Ripari	5.19
GUARDIAREGIA	AUTOTRASPORTI PIGLIACELLI SPA	Guglionesi	3.23
MATERA	F.LLI CORTESE SRL	Canolo	2.89
MATERA	FOGLIA GIUSEPPE	Giffoni Valle Piana	1.21
MATERA	FOGLIA GROUP SRL	Giffoni Valle Piana	1.26
MATERA	ITALSAB SRL	Canolo	0.86
MATERA	SOMEL PICCOLA SOC COOP A RL	Giffoni Valle Piana	1.26
MATERA	VITO ALTERIO GESSI SNC	Anzano di Puglia	3.23
MONSELICE	LOGISTICA BOCCATO GHIAIA SR	Secchiano	3.23
NOVI LIGURE	SAINT-GOBAIN PPC ITALIA SPA	Montiglio	1.06
RAVENNA	MATERIA SERVIZI DI PROTTI DENIS	Novafeltria di Sassofeltrio	0.11
RAVENNA	VOLANO SRL	Novafeltria di Sassofeltrio	0.11
REZZATO	SAMA SRL	Cava Ripari	2.16
REZZATO	DAMOS SRL	Pieve di Cadore	5.88
REZZATO	LAGES SPA	Pisogne	1.64
REZZATO	MATERIA SERVIZI DI PROTTI DENIS	Novafeltria di Sassofeltrio	0.84
REZZATO	NUOVA DARSENA SRL	Pieve di Cadore	2.72
REZZATO	VOLANO SRL	Novafeltria di Sassofeltrio	0.84
REZZATO	GESSI ROCCA STRADA SRL	Roccastrada	3.23
SALERNO	FOGLIA GIUSEPPE	Giffoni Valle Piana	1.83
SALERNO	FOGLIA GROUP SRL	Giffoni Valle Piana	1.96
SALERNO	SOMEL PICCOLA SOC COOP A RL	Giffoni Valle Piana	1.97
SARCHE DI CALAVINO	MATERIA SERVIZI DI PROTTI DENIS	Novafeltria di Sassofeltrio	2.27
SARCHE DI CALAVINO	VOLANO SRL	Novafeltria di Sassofeltrio	2.27
SCAFA	SAMA SRL	Cava Ripari	1.68
TRIESTE	DAMOS SRL	Pieve di Cadore	0.57
TRIESTE	NUOVA DARSENA SRL	Pieve di Cadore	0.5
VIBO VALENTIA	F.LLI CORTESE SRL	Canolo	3.56
VIBO VALENTIA	ITALSAB SRL	Canolo	0.82
VIBO VALENTIA	BONACIA SRL	Mineo	1.97
VIBO VALENTIA	F.LLI FUSCA' AUTOTRASPORTI SNC	Mineo	4.49

Table 16: Optimal solution of the robust box-ellipsoidal optimization model with $\Omega = 2.75$. The table shows the optimal normalized volume y_j for destination plant $j \in \mathscr{D}$.

Destination	y_j
BORGO SAN DALMAZZO	4.36
CALUSCO D'ADDA	9.12
CASTROVILLARI	1.64
COLLEFERRO	22.10
GUARDIAREGIA	11.70
MATERA	11.13
MONSELICE	8.50
NOVI LIGURE	8.48
RAVENNA	9.86
REZZATO	12.60
SALERNO	6.94
SARCHE DI CALAVINO	5.05
SCAFA	9.15
TRIESTE	2.18
VIBO VALENTIA	0.00

and consequently $\mathbf{cost}_{\mathbf{6},\tau} = \infty$ for $\tau = 24, \ldots, 47$. As expected the lowest cost ($\mathbf{cost}_\tau$) is given by the problem (117)-(124), since a full information on the realization of vectors b and d is available.

Total CPU time, in seconds, spent in solving the optimization problems $\mathbf{M_m}$, $m = 1, \ldots, 6$ and (109)-(115) are reported in Table 18. Results show the higher computational complexity of the stochastic approach $\mathbf{M_2}$ and adjustable $\mathbf{M_5}$ and $\mathbf{M_6}$ with respect to the robust box $\mathbf{M_3}$ or the box-ellispoidal $\mathbf{M_4}$.

Table 17: Optimal value of the objective function (117) with the adjustable strategy considered in the previous section where method $\mathbf{M_m}$ $m = 1, \ldots, 6$ is adopted to determine the nonadjustable variables x_{ijk}. Last column ($\mathbf{cost}_\tau$) refers to the cost of problem (117)-(124) where a full information on the realization of vectors b and d is available. Results are average values over 1000 simulations.

τ	$\mathbf{cost_{1,\tau}}$	$\mathbf{cost_{2,\tau}}$	$\mathbf{cost_{3,\tau}}$	$\mathbf{cost_{4,\tau}}$	$\mathbf{cost_{5,\tau}}$	$\mathbf{cost_{6,\tau}}$	$\mathbf{cost_\tau}$
24	112887	106268	149596	131733	139920	∞	76464
25	86156	78537	108328	99046	105893	∞	53386
26	122055	113678	157730	139411	146097	∞	80399
27	111713	101405	138061	130867	140536	∞	70792
28	125465	112587	156452	145235	147368	∞	83548
29	98605	91334	144716	125087	128931	∞	65283
30	104883	96586	151069	139699	137865	∞	73541
31	97507	92735	147384	133558	133085	∞	70808
32	56512	66738	84055	76295	72009	∞	37361
33	51712	61685	69863	68364	66863	∞	30987
34	50320	58994	71826	70761	64654	∞	29690
35	56012	65425	78257	79132	71442	∞	39029
36	49406	59721	75545	73450	66142	∞	28702
37	54452	62597	77775	75307	67223	∞	30513
38	58937	69873	83871	84397	78665	∞	39240
39	50336	60691	72534	73205	68384	∞	28646
40	49092	58604	78762	74779	69311	∞	26157
41	50588	60879	86342	75022	69542	∞	31475
42	103184	107178	127786	114733	110211	∞	65190
43	81849	93369	117811	99104	94021	∞	54748
44	80367	88607	113692	87892	92033	∞	49120
45	79641	84985	113013	97798	98013	∞	51223
46	66136	72613	105263	78852	83042	∞	40894
47	54122	66353	98546	84675	81543	∞	36623

Table 18: Total CPU time, in seconds, spent in solving the optimization problems $\mathbf{M_m}$, $m = 1, \ldots, 6$ and (109)-(115). Last column ($\mathbf{cost}_\tau$) refers to the CPU time of problem (117)-(124) where a full information on the realization of vectors b and d is available. Results are average values over 1000 simulations.

τ	$\mathbf{CPU_{1,\tau}}$	$\mathbf{CPU_{2,\tau}}$	$\mathbf{CPU_{3,\tau}}$	$\mathbf{CPU_{4,\tau}}$	$\mathbf{CPU_{5,\tau}}$	$\mathbf{CPU_{6,\tau}}$	$\mathbf{CPU_\tau}$
24	0.0312	8.8356	0.0312	0.2156	32.0156	32.0156	0.0156
25	0.0312	9.1856	0.0312	0.1856	29.8856	29.8856	0.0156
26	0.0312	8.1656	0.0312	0.2156	30.0156	30.0156	0.0156
27	0.0312	9.8856	0.0312	0.2256	32.9856	32.9856	0.0156
28	0.0312	10.0156	0.0312	0.2256	77.2756	77.2756	0.0156
29	0.0312	10.9756	0.0312	0.1956	83.5056	83.5056	0.0156
30	0.0312	11.8956	0.0312	0.2156	31.1456	31.1456	0.0156
31	0.0312	11.1456	0.0312	0.1956	59.3956	59.3956	0.0156
32	0.0312	10.7456	0.0312	0.2156	73.3956	73.3956	0.0156
33	0.0312	12.1756	0.0312	0.1956	71.2056	71.2056	0.0156
34	0.0312	11.4256	0.0312	0.2156	76.5256	76.5256	0.0156
35	0.0312	10.8256	0.0312	0.2156	65.4056	65.4056	0.0156
36	0.0312	11.5856	0.0312	0.1856	58.2256	58.2256	0.0156
37	0.0312	12.3156	0.0312	0.1956	91.2756	91.2756	0.0156
38	0.0312	11.3956	0.0312	0.2256	53.5856	53.5856	0.0156
39	0.0312	10.7256	0.0312	0.2156	83.5456	83.5456	0.0156
40	0.0312	11.1056	0.0312	0.2156	39.6156	39.6156	0.0156
41	0.0312	10.9856	0.0312	0.2026	65.8256	65.8256	0.0156
42	0.0312	10.9656	0.0312	0.2156	62.3156	62.3156	0.0156
43	0.0312	11.2756	0.0312	0.2156	89.1056	89.1056	0.0156
44	0.0312	11.2856	0.0312	0.2026	75.4256	75.4256	0.0156
45	0.0312	11.6056	0.0312	0.2156	89.4656	89.4656	0.0156
46	0.0312	11.3356	0.0312	0.2026	73.7656	73.765	0.0156
47	0.0312	14.0056	0.0312	0.2156	104.4256	104.4256	0.0156

7 Conclusions

In this paper we have discussed the effect of two modelling approaches, stochastic programming (SP) and robust optimization (RO) to a real case of a transportation problem under uncertainty. The problem consists in determining the number of vehicles to book at the beginning of each week to replenish gypsum at all the cement factories of the producer so that the total cost is minimized. The uncertainty comes from the demand of gypsum and buying costs from external sources in case of inventory shortage. The problem has been solved both via a two-stage stochastic programming and

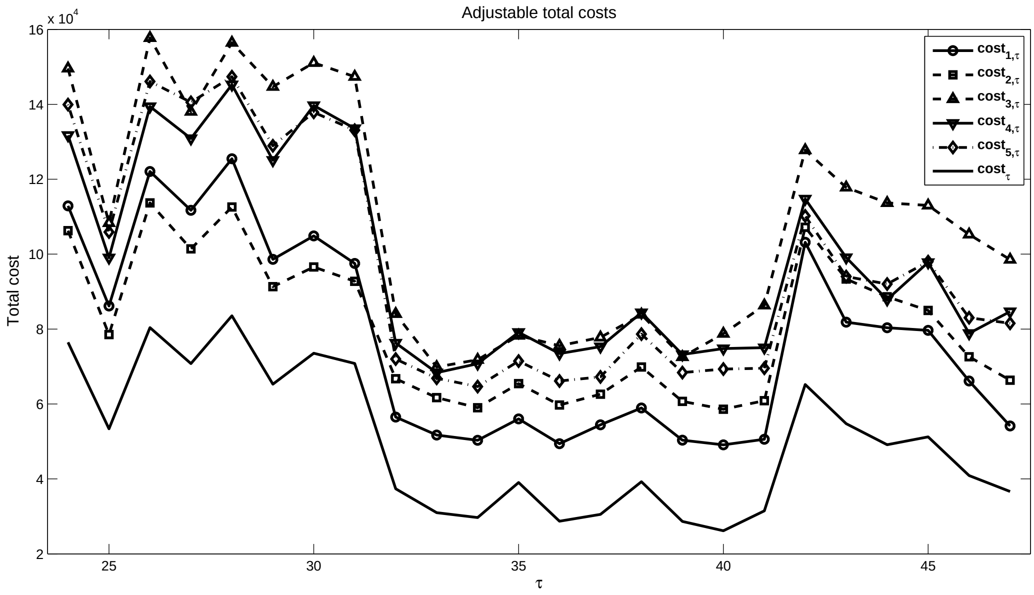

Figure 6: Adjustable total costs $\mathbf{cost_{1}}_{,\tau}, \ldots, \mathbf{cost_{5}}_{,\tau}$ obtained by solving model (117)-(124) using the nonadjustable decision variables x_{ijk} respectively given by methods $\mathbf{M_1}, \ldots, \mathbf{M_5}$, for an increasing value of $\tau = 24, \ldots, 47$. Results are compared with the cost of problem (117)-(124) where a full information on the realization of vectors b and d is available ($\mathbf{cost}_\tau$). Results are average values over 1000 simulations.

robust optimization models with different uncertainty sets.

The goal of SP is to compute the minimum expected cost based on the specific probability distribution of the uncertain parameters based on a set of scenarios. The optimal solution is firstly compared with the *Expected Value* (EV) problem under the unique average scenario. The *Value of Stochastic Solution* VSS, and the *Expected Value of Perfect Information* EVPI are then computed.

For RO we have firstly considered static approaches with random parameters belonging to box or ellipsoidal uncertainty sets, and secondly dynamic approaches, via the concept of affinely adjustable robust counterpart. The choice of the box uncertainty set is preferable only if the feasibility of all the constraints is highly required, but this certainty of constraint satisfaction results in a higher cost of the transportation problem. A less conservative outcome has been obtained with a box-ellipsoidal uncertainty set that requires the solution of a *second-order cone program* SOCP. The main advantage of the RO formulations considered, is that they can be solved in polynomial time and theoretical guarantees for the quality of the solution are provided, which is not the case with the aforementioned SP formulations. In order to make a fair comparison between the robust and the stochastic programming methodology, a dynamic approach via the concept of adjustable robust counterpart has been also considered. The variables have been partitioned in nonadjustable and adjustable variables and several methods to find the nonadjustable variables have been proposed and compared in terms of total cost and CPU time. Numerical experiments show that the robust approach results in larger objective function values at the optimal solutions due to the certitude of constraint satisfaction. Conversely, the computational complexity is higher for the stochastic approach which has no guarantees for the quality of the provided solution.

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