Multi-period robust risk measures and portfolio selection models with regime-switching

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In this paper, we first construct a multi-period worst-case risk measure, which measures the dynamic risk period-wisely from a distributionally robust perspective. To better describe the time-varying property of the dynamic risk, we further propose two multi-period robust risk measures under the regime switching framework. All the three proposed multi-period risk measures are time consistent. Under the usually adopted uncertainty set, we derive the explicit optimal investment strategy for the multi-period robust portfolio selection problem under the multi-period worst-case risk measure, and we show that the multi-period robust portfolio selection problems under two multi-period robust risk measures with regime switching can be transformed into second order cone programs, which can thus be efficiently solved in polynomial time. Numerical results demonstrate that the portfolio selection model under the multi-period worst-case risk measure is a good complement to existing multi-period robust portfolio selection models using the adjustable robust approach, and the corresponding models under robust risk measures with regime switching can flexibly help the investor make superior and robust investment strategies according to the switching of the market environment.

Key words: distributionally robust optimization, multi-period risk measure, regime switching, dynamic portfolio selection, conditional value-at-risk.

Area of review: Optimization.
1. Introduction

The standard assumption in traditional risk measure and portfolio selection is that the probability distribution of the random return is known beforehand and only the realizations are unknown at the time of decision making. However, a lot of experiences show that no distribution can perfectly describe the random investment return or loss, and the inaccuracy of the adopted distribution brings extra estimation risk in practice. In order to measure and reduce this kind of risk, the distributionally robust optimization technique was proposed and has been applied in many areas of management problems under uncertainty.

Lobo and Boyd (1999) first use the worst-case method to estimate the uncertain variance, and demonstrate that the problem of minimizing the worst-case variance can be transformed into a semi-definite programming problem. After that, the worst-case technique is frequently used to measure the investment risk when the ambiguity of the return (loss) distribution is considered. El Ghaoui et al. (2003) define the worst-case value-at-risk (VaR), they show the worst-case VaR constraint with respect to the uncertainty set with given first and second order moments is equivalent to a second order cone constraint, and apply it to a portfolio selection problem. Chen et al. (2011) consider the worst-case lower partial moment and the worst-case conditional VaR (CVaR) with given first and second order moments, and derive a tight bound for these two kinds of problem. Zhu and Fukushima (2009) construct some robust portfolio selection models by taking the worst-case CVaR as the risk measure, they prove that the robust models can be transformed into linear programs or second order cone programs, depending on the forms of uncertainty sets.

To make the robust risk measure reasonable and the resulting robust portfolio selection problem tractable, the proper choice of the uncertainty set is very important. As we mentioned above, the moment information is used frequently to construct the uncertainty set. However, in a medium-term or long-term investment problem, the investor can hardly estimate the moments which are suitable for any market environment. In order to guarantee the accuracy of the uncertainty set, Delage and Ye (2010) investigate the distributionally robust problem with uncertainty in terms of
the distribution and of its moments, they show the resulting solutions perform best over a set of distributions with high probability.

However, all the above robust risk measures and robust portfolio selection models are static. In practice, most investment problems are medium-term or long-term decision making problems, the investment risk should be measured and the investment decision should be adjusted dynamically according to the time-varying information. This requirement inspires us to discuss the distributionally robust risk measure under the multi-period setting (Huang et al. 2007; Gulpinar and Rustem 2007; Chen et al. 2011).

The multi-period robust problem concerns the ambiguity of the multi-variate joint distribution stretching over multiple periods, which is difficult to quantify. Hence, tractability becomes one of the most important factors in constructing multi-period robust models. Ben-Tal et al. (2009) discuss the robust Markov decision process, where the system’s dynamic is described by the state transition, and the uncertain transition probability is assumed to be within a given uncertainty set. Such a process is further extended to a more generalized case, called the adjustable robust approach in Ben-Tal et al. (2009). The adjustable robust approach allows one to solve the multi-period robust problem in a computationally efficient way by adapting the dynamic programming technique (Shapiro, 2011). However, when the adjustable robust approach is applied to the multi-period risk measure and portfolio selection problems, the investor has to make a worst estimation at the current period on the basis of the worst estimation at the next period, this leads to an excessively conservative investment strategy.

Besides the tractability of the multi-period model, time consistency is also an important issue when we construct the multi-period risk measure or portfolio selection model. Time consistency guarantees the investor does not make a decision which he/she will regret in the future. Several forms of time consistency have been proposed in Wang (1999), Weber (2006) and Shapiro (2009). They can be divided into two groups: the time consistency of the dynamic investment strategy which requires the consistency between the global optimum strategy and the local optimum decision,
and the time consistency of the dynamic risk measure which requires some specific relationship among the risk values at adjacent periods. Chen et al. (2013) point out that for a risk-averse portfolio selection problem in a specific mean-risk framework, the time consistency of the dynamic investment strategy can be ensured by the time consistency of the chosen dynamic risk measure. When the robustness is introduced in the multi-period risk measure or portfolio selection problem, does time consistency still hold? This is still an unsettled problem, which has not been investigated systematically.

The tractability and time consistency of dynamic risk measures and/or multi-period portfolio selection problems significantly rely on the description framework of the dynamic information process or the time-varying market environment. In this respect, the regime switching technique has been used in many financial studies. Regime switching describes the nonlinear dynamic relationship of market environments among different time periods. It can reflect the trend of macro economy and the dynamic correlation of random returns or losses in different economic cycles. Through elaborately distinguishing economic states, regime switching can even describe the dynamic variation of the skewness, kurtosis and heavy-tail properties of the return (loss) distribution. One can refer to Hamilton (1989) about the introduction of regime switching and papers like Ma et al. (2011) and Elliott and Siu (2009) for its financial application. Recently, Liu and Chen (2014) adopt the regime switching method to describe the time-varying property of the uncertainty set with the first and second order moment information, from which they propose two forms of static robust risk measures with regime switching. Until now, we have not seen any research about the application of the regime switching technique to the construction of multi-period robust risk measures and/or multi-period robust portfolio selection models.

Considering the above issues, we aim to introduce in this paper some proper multi-period risk measures with distributional robustness, such that the multi-period risk measure is time consistent and the corresponding multi-period portfolio selection problem is tractable. In detail, we first extend the static worst-case risk measure to the multi-period situation by adopting a separable expected
conditional form, which is proved to be time consistent. And the multi-period portfolio selection problem with the multi-period worst-case risk measure can be solved analytically. Furthermore, in order to better describe the dynamic property of the uncertainty set with respect to different market environments, we assume the uncertainty set is time-varying under the regime switching framework. We define two kinds of regime-dependent robust risk measures, which are also proved to be time consistent. When applying the new risk measures to the multi-period portfolio selection problem incorporating transaction costs, we show that they can be equivalently transformed into second order cone programs.

Compared with the static robust risk measure models in Liu and Chen (2014), the multi-period robust risk measure and associated multi-period portfolio selection model can better reflect the dynamic property of the investment risk and resulting investment policy. Especially, the time consistency of the multi-period robust risk measure guarantees the robustness of the optimal investment policy. Compared with the adjustable robust approach and its applications in portfolio selection (Ben-Tal et al. 2009; Shapiro 2011; Chen et al. 2011), our new robust model considers the time-varying robustness period-wisely, therefore, the resulting robust portfolio selection model is not overly conservative. Furthermore, the introduction of regime switching makes the multi-period portfolio selection model more robust and flexible in terms of reflecting the constantly varying market environment. At last, our multi-period portfolio selection models can be solved analytically or numerically in polynomial time as long as the uncertainty set is properly chosen. Therefore, the proposed new models can help the investor adjust his/her investment policy rapidly according to the latest information.

The remainder of the paper is organized as follows. After briefly reviewing the worst-case risk measure in the static case, Section 2 introduces three kinds of multi-period robust risk measures without and with regime switching, respectively. By taking CVaR as an example, we apply in Section 3 the proposed multi-period robust risk measures to multi-period portfolio selection problems and derive their analytical optimal solution or design efficient solution method. Section 4 presents some empirical results to show the practicality and efficiency of our new models. Section 5 concludes the paper.
2. Multi-period robust risk measures

As a preparation for introducing multi-period robust risk measures in the latter two subsections, we briefly review the single-period risk measure and the worst-case risk measure under the static case.

2.1. Single-period risk measures

Consider a loss function \( x = f(u, r) \) associated with the investment decision \( u \in U \subseteq \mathbb{R}^n \), here \( U \) is the set of feasible investment decisions, and the random return rate \( r \) defined on a probability space \( (\Omega, \mathcal{F}, P) \). We assume \( E[|x|] < +\infty \) for any \( u \in U \).

The traditional single-period risk measure can be regarded as an aggregation function \( \rho(x) : L^p(\mathcal{F}) \rightarrow \mathbb{R} \) with respect to the probability \( P \), here \( 1 \leq p < \infty \). A typical single-period risk measure is value-at-risk (VaR), which is defined as

\[
VaR(x) = \min\{\gamma \in \mathbb{R} : P\{\gamma \leq x\} \leq \epsilon\},
\]

here \( \epsilon \in (0, 1] \) is a given loss tolerant probability (say, 1% or 5%). The corresponding conditional value-at-risk (CVaR) is the expected value of losses that exceed \( VaR(x) \):

\[
CVaR(x) = E[x \mid x \geq VaR(x)] = \frac{1}{\epsilon} \int_{x \geq VaR(x)} xdP.
\]

To demonstrate the properties that a reasonable single-period risk measure should possess, Artzner et al. (1999) introduce the concept of coherent risk measure as the risk measure that satisfies the following four axioms:

(A1) subadditivity: for all \( x, y \in L^p(\mathcal{F}) \), \( \rho(x + y) \leq \rho(x) + \rho(y) \),

(A2) monotonicity: \( x \geq y \) implies \( \rho(x) \geq \rho(y) \),

(A3) positive homogeneity: for a positive constant \( \lambda \in \mathbb{R} \), \( \rho(\lambda x) = \lambda \rho(x) \),

(A4) translation invariance: for a constant \( a \in \mathbb{R} \), \( \rho(x + a) = \rho(x) + a \).

It is easy to verify that CVaR is a coherent risk measure, while VaR is not due to the lack of subadditivity.
In general, the computation of the investment risk value requires the information about the underlying probability \( P \) of the random return rate. However, due to the prediction error and the measurement error, we can hardly obtain the full distribution information. An efficient way to copy with the uncertainty of the distribution information is to employ a robust strategy. That is, we assume \( P \) belongs to an uncertainty set \( \mathcal{P} \) and estimate \( \rho \) by choosing the \( P \) under which the worst estimation is made. This gives us the following worst-case risk measure:

**Definition 1.** For a single-period risk measure \( \rho \), the worst-case risk measure with respect to \( \mathcal{P} \) is defined as

\[
\text{w}\rho(x) \equiv \sup_{P \in \mathcal{P}} \rho(x).
\]

A classic example of \( \mathcal{P} \) is \( \mathcal{P} = \{ P | E_P[r] = \mu, \text{Cov}_P[r] = \Gamma \} \), which is the set of all probability distributions with given mean vector \( \mu \) and covariance matrix \( \Gamma \).

By choosing different single-period risk measures, we can obtain different versions of the worst-case risk measure, such as worst-case variance (Gulpinar and Rustem, 2007), worst-case VaR (El Ghaoui et al., 2003) and worst-case CVaR (Zhu and Fukushima, 2009). Zhu and Fukushima (2009) prove that if the \( \rho \) associated with the crisp probability measure \( P \) is a coherent risk measure, then the corresponding \( \text{w}\rho \) associated with the ambiguous probability measure \( \mathcal{P} \) remains a coherent risk measure.

### 2.2. Multi-period worst-case risk measure

In practice, most investment problems are medium-term or long-term decision making problems, which can be formulated as dynamic portfolio selection problems. Correspondingly, the financial risk over the investment horizon should be measured dynamically, where all the risks from the uncertain losses at intermediary periods should be involved. In this paper, we consider the investment risk of a random loss process over an investment horizon \( [0, T] \), which has \( T + 1 \) time points: \( 0, 1, 2, \ldots, T \), and thus \( T \) consecutive investment periods. The random loss process is adapted to a filtration \( (\mathcal{F}_1, \cdots, \mathcal{F}_T) \) with \( \mathcal{F}_T = \mathcal{F} \), which is an increasing sequence of \( \sigma \)-algebras, i.e. \( F_t \subseteq F_{t+1} \).

Moreover, we define the probability space of the random loss \( x_t \) at period \( t \) as

\[
\mathcal{L}_t = \mathcal{L}_p(\Omega, \mathcal{F}_t, P_t),
\]
\[ p \in [1, +\infty], \] with \( P_t \) a probability measure assigning to any event \( B \) in \( \mathcal{F}_t \) its probability \( P_t(B) \). For the random loss process \( (x_1, \cdots, x_T) \), we denote \( \mathcal{L}_{t,T} = \mathcal{L}_1 \times \cdots \times \mathcal{L}_T \), and \( x_{t,T} = (x_t, \cdots, x_T) \in \mathcal{L}_{t,T} \) for notational simplicity.

Under the multi-period setting, the investment risk of the random loss process between period \( t+1 \) and period \( T \) observed at time point \( t \) can be measured by a conditional risk mapping \( \rho_{t,T}(\cdot) : \mathcal{L}_{t+1,T} \to \mathcal{L}_t \) (Ruszczyński, 2010). As a special case, \( \rho_t(\cdot) : \mathcal{L}_t \to \mathcal{L}_{t-1} \) is the single-period risk measure at period \( t-1 \). A sequence of conditional risk mappings \( \rho_{t,T} \) from time point 0 to time point \( T-1 \) is called a multi-period risk measure.

A typical multi-period risk measure is the separable expected conditional (SEC for short) mapping introduced in Pflug and Römisch (2007):

\[
\rho_{t,T}(x_{t+1,T}) = \sum_{i=t+1}^{T} E \left[ \rho_i(x_i) \mid \mathcal{F}_i \right], \quad t = 0, 1, \cdots, T-1.
\]

Kovacevic and Pflug (2009) prove that the SEC mapping is time consistent under the definition in Wang (1999). This kind of time consistency requires an order relationship between risks at later periods and risks at earlier periods. It says, for any \( 0 \leq \tau < \theta \leq T-1 \) and \( x_{\tau+1,T}, y_{\tau+1,T} \in \mathcal{L}_{\tau+1,T} \), if \( x_{\tau+1,\theta} \leq y_{\tau+1,\theta} \) and \( \rho_{\theta,T}(x_{\theta+1,T}) \leq \rho_{\theta,T}(y_{\theta+1,T}) \) imply \( \rho_{\tau,T}(x_{\tau+1,T}) \leq \rho_{\tau,T}(y_{\tau+1,T}) \), then \( \{\rho_{t,T}\}_{t=0}^{T-1} \) is called time consistent.

Similarly to the single-period case, the computation of the multi-period risk measure depends on the probability distributions at \( T \) periods. Meanwhile, the precise prediction of the distribution process is even harder than the prediction of the distribution in the static case, due to the dynamic relationship among periods. Hence, it is necessary to make a robust estimation for the multi-period risk measure to account for the estimation risk. Different from the adjustable robust approach, we make the worst-case estimation period-wisely in a separable expected conditional mapping form. Specifically, we assume the probability distribution \( P_t \) of the random loss \( x_t \) belongs to an uncertainty set \( \mathcal{P}_t \), which is observed at time point \( t-1 \). By choosing the distribution which generates the worst estimation for the single-period conditional risk mapping \( \rho_t(x_t) \), we obtain a robust estimation of the risk at period \( t \), then all the estimations of risks at different periods are
combined together in the separable expected conditional mapping form. This gives us the following multi-period worst-case risk measure.

**Definition 2 (Multi-period worst-case risk measure).** For \( t = 0, 1, \ldots, T - 1 \) and \( x_{t+1,T} \in \mathcal{L}_{t+1,T} \),

\[
\rho_{t,T}(x_{t+1,T}) = \sum_{i=t+1}^{T} E \left[ \sup_{P_i \in \mathcal{P}_i} \rho_i(x_i) \mid \mathcal{F}_t \right]
\]

is called the conditional worst-case risk mapping. The sequence of the risk mappings \( \{\rho_{t,T}\}_{t=0}^{T-1} \) is called the multi-period worst-case risk measure.

The multi-period worst-case risk measure can be rewritten in a dynamic equation form:

\[
\rho_{t-1,T}(x_{t,T}) = \left( \sup_{P_t \in \mathcal{P}_t} \rho_t(x_t) \right) + E[\rho_{t,T}(x_{t+1,T}), \mathcal{F}_{t-1}], \quad t = 1, 2, \ldots, T.
\]

From the above dynamic equation, we can see that the multi-period worst-case risk measure at each period is constituted by two parts: the first part measures the risk brought by the random loss at the current period, the second part measures the cumulated risk from the next period to the final period. Unlike the adjustable robust approach which makes a worst-case estimation for the above two parts of risk together, our multi-period robust model only makes the worst-case estimation for the first part. And the worst-case estimation will not be cumulated to the earlier period. Consequently, our multi-period robust model seeks a compromise between the static robust model and the adjustable robust approach.

We can easily verify from the dynamic equation form that the multi-period worst-case risk measure is time consistent.

**Proposition 1 (Time consistency).** If \( \rho_t \) associated with the crisp probability distribution \( P_t \in \mathcal{P}_t \) is monotone, \( t = 1, 2, \ldots, T \), then the corresponding multi-period worst-case risk measure \( \{\rho_{t,T}\}_{t=0}^{T-1} \) is time consistent.

**Proof.** For any \( 0 \leq \tau < \theta \leq T - 1 \), we consider two sequences \( x_{\tau+1,T}, y_{\tau+1,T} \) such that \( x_{\tau+1,\theta} \leq y_{\tau+1,\theta} \) and \( \rho_{\theta,T}(x_{\theta+1,T}) \leq \rho_{\theta,T}(y_{\theta+1,T}) \). The monotonicity of \( \rho_t, t = \tau + 1, \ldots, \theta \), implies \( \rho_t(x_t) \leq \rho_t(y_t) \).
\[ \rho_t(y_t) \] with respect to the crisp probability distribution \( P_t \in \mathcal{P}_t, \ t = \tau + 1, \cdots, \theta, \) which gives \( \sup_{P_t \in \mathcal{P}_t} \rho_t(x_t) \leq \sup_{P_t \in \mathcal{P}_t} \rho_t(y_t), \ t = \tau + 1, \cdots, \theta. \) Combining the above inequalities and applying the projection property of the conditional expectation, we obtain

\[
\sum_{t=\tau+1}^{\theta} E \left[ \sup_{P_t \in \mathcal{P}_t} \rho_t(x_t) \right] + E[w_{\rho_0,T}(x_{\theta+1},T) | \mathcal{F}_\tau] 
\leq \sum_{t=\tau+1}^{\theta} E \left[ \sup_{P_t \in \mathcal{P}_t} \rho_t(y_t) \right] + E[w_{\rho_0,T}(y_{\theta+1},T) | \mathcal{F}_\tau],
\]

that is, \( w_{\rho_{\tau,T}}(x_{\tau+1},T) \leq w_{\rho_{\tau,T}}(y_{\tau+1},T). \) □

Furthermore, if \( \rho_t \) associated with the crisp probability distribution \( P_t \in \mathcal{P}_t \) is coherent, the corresponding multi-period worst-case risk measure is dynamic coherent (Artzner et al., 2007).

This means that, for any \( t = 0, 1, \cdots, T - 1, \) and \( x_{t+1},T, y_{t+1},T \in \mathcal{L}_{t+1,T}, \) the multi-period worst-case risk measure \( \{ w_{\rho_{i,T}} \}_{i=0}^{T-1} \) satisfies the following four axioms:

(A1) subadditivity: \( w_{\rho_{i,T}}(x_{t+1},T + y_{t+1},T) \leq w_{\rho_{i,T}}(x_{t+1},T) + w_{\rho_{i,T}}(y_{t+1},T), \)

(A2) monotonicity: \( x_{t+1},T \leq y_{t+1},T \) implies \( w_{\rho_{i,T}}(x_{t+1},T) \leq w_{\rho_{i,T}}(y_{t+1},T), \)

(A3) positive homogeneity: for any positive \( \lambda \in \mathcal{L}_t, \) \( w_{\rho_{i,T}}(\lambda x_{t+1},T) = \lambda w_{\rho_{i,T}}(y_{t+1},T), \)

(A4) translation invariance: for any \( z \in \mathcal{L}_t \) and \( s = t + 1, \cdots, T, \) \( w_{\rho_{i,T}}(x_{t+1}, \cdots, x_s + z, \cdots, x_T) = z + w_{\rho_{i,T}}(x_{t+1},T). \)

### 2.3. Regime dependent multi-period robust risk measures

In this subsection, we utilize the regime switching technique to make a robust measure of the multi-period risk, and propose two kinds of regime dependent multi-period robust risk measures.

We suppose the regime at the initial time is \( s_0, \) and the regimes during the following \( T \) investment periods are \( s_1, \cdots, s_T. \) Like that in Ma et al. (2011) and Elliott and Siu (2009), we assume the regime switching is Markovian, and the set of possible regimes \( S_t \) at period \( t, \ t = 1, \cdots, T, \) is constituted by \( J \) regimes: \( s^1, \cdots, s^J. \) \( P_{s^t|s_0}^s = \text{Prob}\{s_t = s^j | s_{t-1} = s^j\} \) represents the transition probability from
regime $s^i$ at period $t-1$ to regime $s^j$ at period $t$. In this paper, we assume the Markovian regime switching process is stationary. That means, for any period $t$, the transition probability matrix is

$$P^s = \begin{pmatrix}
P^s_{s^1 s^1} & P^s_{s^1 s^2} & \cdots & P^s_{s^1 s^J} \\
P^s_{s^2 s^1} & P^s_{s^2 s^2} & \cdots & P^s_{s^2 s^J} \\
\vdots & \vdots & \ddots & \vdots \\
P^s_{s^J s^1} & P^s_{s^J s^2} & \cdots & P^s_{s^J s^J} 
\end{pmatrix}. $$

We assume that the uncertainty set $\mathcal{P}_t$ at period $t$ is associated with the regime $s_t \in S_t$, then the worst-case measure of the single-period risk at period $t$, $\sup_{P_t \in \mathcal{P}_t(s_t)} \rho_t(x_t)$, is regime dependent.

Depending on how to combine these regime dependent risk measures together, we can define different multi-period robust risk measures. In what follows, we propose two kinds of combination method. The first strategy is to choose the greatest risk value among all possible regimes as the robust measure of the risk at period $t$. These robust risk measures at different periods are then added together through the SEC mapping, which leads to the following multi-period worst-regime risk measure.

**Definition 3 (multi-period worst-regime risk measure).** For $t = 0, 1, \cdots, T-1$ and $x_{t+1, T} \in \mathcal{L}_{t+1, T}$,

$$wr \rho_{t,T}(x_{t+1,T}) = \sum_{i=t+1}^{T} E \left[ \sup_{s_i \in S_i, P_i \in \mathcal{P}_i(s_i)} \rho_i(x_i) \mid \mathcal{F}_{t-1} \right] $$

is called the conditional worst-regime risk mapping. And the sequence of the conditional worst-regime risk mappings $\{wr \rho_{t,T}\}_{t=0}^{T-1}$ is called the multi-period worst-regime risk measure.

The multi-period worst-regime risk measure can also be rewritten in the following dynamic equation form:

$$wr \rho_{t-1, T}(x_{t,T}) = \left( \sup_{s_t \in S_t} \left( \sup_{P_t \in \mathcal{P}_t(s_t)} \rho_t(x_t) \right) \right) + E \left[ wr \rho_{t,T}(x_{t+1,T}) \mid \mathcal{F}_{t-1} \right], \quad t = 1, 2, \cdots, T. $$

We can see from (2) and (3) that the multi-period worst-regime risk measure only cares about the worst regime at a period and ignores other regimes. Meanwhile, it does not take into account
the probability that a regime may appear. Therefore, the multi-period worst-regime risk measure probably generates a very conservative risk evaluation. This inspires us to integrate all the risks under different regimes by considering their appearing probabilities. Here, we replace the supreme operator in the multi-period worst-regime risk measure with the weighted summation with respect to the regime transition probability, which gives the following multi-period mixed worst-case risk measure:

**Definition 4 (multi-period mixed worst-case risk measure).** For \( t = 0, 1, \ldots, T - 1 \) and \( x_{t+1,T} \in \mathcal{L}_{t+1,T} \),

\[
mw\rho_{t,T}(x_{t+1,T}) = \sum_{i=t+1}^{T} \mathbb{E} \left[ \sum_{s_i \in S_i} \left( P_{s_i-1,s_i} \sup_{P_i \in \mathcal{P}_i(s_i)} \rho_i(x_i) \right) \bigg| \mathcal{F}_t \right]
\]

is called the conditional mixed worst-case risk mapping. And the sequence of the conditional mixed worst-case risk mappings \( \{mw\rho_{t,T}\}_{t=0}^{T-1} \) is called the multi-period mixed worst-case risk measure.

Like the previous risk measures, the multi-period mixed worst-case risk measure can be rewritten in the following dynamic equation form:

\[
mw\rho_{t-1,T}(x_{t,T}) = \left( \sum_{s_t \in S_t} P_{s_t-1,s_t} \left( \sup_{P_t \in \mathcal{P}_t(s_t)} \rho_t(x_t) \right) \right) + \mathbb{E} \left[ mw\rho_{t,T}(x_{t+1,T}) \big| \mathcal{F}_{t-1} \right], \quad t = 1, 2, \ldots, T.
\]

Based on the dynamic equation forms (3) and (5), we can establish the time consistency of the two multi-period robust risk measures.

**Proposition 2 (time consistency).** If \( \rho_t \) associated with the crisp probability distribution \( P_t \in \mathcal{P}_t(s_t) \) is monotone for any \( s_t \), \( t = 1, 2, \ldots, T \), then the corresponding multi-period robust risk measures \( \{w\rho_{t,T}\}_{t=0}^{T-1} \) and \( \{mw\rho_{t,T}\}_{t=0}^{T-1} \) are time consistent.

**Proof.** At any period \( t \), suppose \( x_t \leq y_t \), \( w\rho_{t,T}(x_{t+1,T}) \leq w\rho_{t,T}(y_{t+1,T}) \) and \( mw\rho_{t,T}(x_{t+1,T}) \leq mw\rho_{t,T}(y_{t+1,T}) \). From the monotonicity of the expectation operator, we have

\[
E[w\rho_{t,T}(x_{t+1,T}) | \mathcal{F}_{t-1}] \leq E[w\rho_{t,T}(y_{t+1,T}) | \mathcal{F}_{t-1}] \quad \text{and} \quad E[mw\rho_{t,T}(x_{t+1,T}) | \mathcal{F}_{t-1}] \leq E[mw\rho_{t,T}(y_{t+1,T}) | \mathcal{F}_{t-1}].
\]

Moreover, from the monotonicity of \( \rho_t \) with respect to the crisp probability distribution \( P_t \in \mathcal{P}_t(s_t) \) and regime \( s_t \), we have \( \sup_{P_t \in \mathcal{P}_t(s_t)} \rho_t(x_t) \leq \sup_{P_t \in \mathcal{P}_t(s_t)} \rho_t(y_t) \) for any
By taking the supremum and expectation to both sides of this inequality with respect to $s_t$, respectively, we have

$$\sup_{s_t \in S_t} \left( \sup_{P_t \in \mathcal{P}(s_t)} \rho_t(x_t) \right) \leq \sup_{s_t \in S_t} \left( \sup_{P_t \in \mathcal{P}(s_t)} \rho_t(y_t) \right)$$

and

$$\sum_{s_t \in S_t} P^s_{s_{t-1}s_t} \left( \sup_{P_t \in \mathcal{P}(s_t)} \rho_t(x_t) \right) \leq \sum_{s_t \in S_t} P^s_{s_{t-1}s_t} \left( \sup_{P_t \in \mathcal{P}(s_t)} \rho_t(y_t) \right).$$

Then it is easy to deduce from the above inequalities that $w_r\rho_{t-1,T}(x_{t,T}) \leq w_r\rho_{t-1,T}(y_{t,T})$ and $m_w\rho_{t-1,T}(x_{t,T}) \leq m_w\rho_{t-1,T}(y_{t,T})$, which gives the time-consistency at period $t$. Applying these results recursively between any two periods gives the time consistency of the two multi-period robust risk measures.

Furthermore, if $\rho_t$ associated with the crisp probability distribution $P_t \in \mathcal{P}(s_t)$ is coherent for any $s_t$, $t = 1, 2, \cdots, T$, both $\{w_r\rho_{t,T}\}_{T-1}^{T-1}$ and $\{m_w\rho_{t,T}\}_{T-1}^{T-1}$ are dynamic coherent risk measures. By specifying the single-period risk mapping $\rho_t$, we can obtain concrete multi-period robust risk measures. Especially, taking CVaR as an example, we can construct the multi-period worst-case CVaR, the multi-period worst regime CVaR and the multi-period mixed worst-case CVaR, we denote them as $wCVaR$, $wrCVaR$, and $mwCVaR$, respectively. From Definitions 3 and 4, we see that when there is only one regime in the market, $wrCVaR$ and $mwCVaR$ degenerate into $wCVaR$.

### 3. Multi-period robust portfolio selection models

In this section, the proposed multi-period robust risk measures are used to establish multi-period robust portfolio selection models. Without loss of generality, we will take $wCVaR$, $wrCVaR$, and $mwCVaR$ as typical examples.

#### 3.1. Multi-period robust portfolio selection model under $wCVaR$

Firstly, we investigate the multi-period robust portfolio selection model with $wCVaR$ as the risk measure. Suppose that there are $n$ risky assets in the security market. The investment horizon is divided into $T$ consecutive periods. $r_t = [r_t^1, \cdots, r_t^n]^\top$ is the random return rate vector of the $n$ assets at period $t$, $t = 1, 2, \cdots, T$. What we know about the return rates at $T$ periods are only...
the information about their first and second order moments. We denote the mean return vector at period $t$ as $\mu_t = E(r_t)$ and the corresponding co-variance matrix at period $t$ as $\Gamma_t = \text{Cov}(r_t)$. Considering the ambiguity of the distribution of the return rate vector, we assume that the distribution of $r_t$ at period $t$ belongs to an uncertainty set with the given first two order moments:

$$\mathcal{P}_t = \left\{ P \mid P \left[ r_t \right] = \mu_t, \text{Cov}_P \left[ r_t \right] = \Gamma_t \right\}. \tag{6}$$

Suppose an investor joins the market at time 0 with an initial wealth $w_0$, he/she plans to invest his/her wealth in the security market for $T$ consecutive periods in order to maximize his/her final wealth and to minimize his/her investment risk measured by wCVaR. At the beginning of each period, the current wealth can be reallocated among the $n$ risky assets. Let $u_{t-1} = [u_{1t-1}, \cdots, u_{nt-1}]^\top$ be the vector of cash amounts invested in the risky assets at the beginning of period $t$, and let $w_t$ be the total wealth at the end of period $t$. Then $-w_t$ can be viewed as the possible loss at period $t$. We assume that the whole investment process is self-financing, hence we have

$$e^\top u_{t-1} = w_{t-1}, \quad t = 1, \cdots, T. \tag{7}$$

$$r_t^\top u_{t-1} = w_t, \quad t = 1, \cdots, T. \tag{8}$$

Here, $e = [1, \cdots, 1]^\top$.

Similar to the classic mean-variance framework, the problem of maximizing the expected final wealth and minimizing wCVaR simultaneously can be written in a mean-risk form as follows:

$$\max_u E[w_T] - \lambda \sum_{t=1}^T E \left[ \sup_{P_t \in \mathcal{P}_t} \text{CVaR}_t(-w_t) \right], \tag{9}$$

subject to

$$e^\top u_{t-1} = w_{t-1}, \quad t = 1, \cdots, T, \quad (7) - (8). \tag{10}$$

Here, $\lambda$ is the risk avverse coefficient, and the confidence level at period $t$ is $\epsilon_t$. We denote the above model as mean-wCVaR in what follows.

From the time consistency of the multi-period worst-case risk measure wCVaR, we know that the order relationship holds between adjacent periods. Hence, the maximum objective value at the current period can be found on the basis of the maximum objective value at the next period. This
observation reminds us to use the dynamic programming technique to find the optimal investment strategy of problem (9)-(10). For this purpose, we introduce the following notation:

\[
a_t = e^\top \Gamma_t^{-1} e, \quad b_t = e^\top \Gamma_t^{-1} \mu_t, \quad c_t = \mu_t^\top \Gamma_t^{-1} \mu_t, \quad \kappa_t = \sqrt{1 - \epsilon_t}, \quad t = 1, \ldots, T, \quad z_T = 1,
\]

\[
z_{t-1} = (\lambda + z_t)s_t - \lambda \kappa_t \sqrt{\frac{1}{a_t c_t - b_t^2} (c_t - 2b_t s_t + a_t s_t^2)}, \quad t = 2, \ldots, T,
\]

\[
h_t = \left(\frac{\lambda}{\lambda + z_t}\right)^2 \frac{1}{a_t c_t - b_t^2}, \quad \Delta_t = 4(h_t a_t - 1)(a_t c_t - b_t^2), \quad s_t = \frac{2b_t (a_t h_t - 1) + \Delta_t}{2a_t (a_t h_t - 1)}, \quad t = 1, \ldots, T.
\]

Then we have

**Theorem 1.** Suppose that the wealth \( w_t \) at each period \( t \) is non-negative, and the investor is risk averse such that \( \lambda + z_t \) is always non-negative. Then, if \( a_t h_t - 1 \geq 0 \) for all \( t = 1, \ldots, T \), the optimal investment policy for problem (9)-(10) is

\[
u_{t-1} = (\Gamma_t^{-1} e)^\top \Gamma_t^{-1} \mu_t \frac{1}{a_t c_t - b_t^2} \begin{pmatrix} c_t & -b_t \\ -b_t & a_t \end{pmatrix} \begin{pmatrix} 1 \\ s_t \end{pmatrix} w_{t-1}, \quad t = 1, \ldots, T.
\]

If \( a_t h_t - 1 < 0 \) for some \( t, 1 \leq t \leq T \), the optimal portfolio at period \( t - 1 \) trends to infinity, and the problem (9)-(10) is unbounded.

**Proof.** We start from the period \( T \). For given \( w_{T-1} \), the optimal objective value can be found by the following dynamic equation:

\[
f_T(w_{T-1}) = \max_{w_T} \left\{ E_{T-1}[w_T] - \lambda \cdot \sup_{P_T \in P_T} CVaR_T(-w_T) \right\}, \quad (11)
\]

\[
s.t. \quad u_T^\top e = w_{T-1}, \quad (12)
\]

\[
u_T^\top r_T = w_T. \quad (13)
\]

By substituting (13) into (11), we have

\[
f_T(w_{T-1}) = \max_{w_T} \left\{ E_{T-1}[u_T^\top r_T] - \lambda \cdot \sup_{P_T \in P_T} CVaR_T(-u_T^\top r_T) \right\}, \quad (14)
\]

\[
s.t. \quad u_T^\top e = w_{T-1}. \quad (15)
\]
We know from Appendix 1 in Chen et al. (2011) that problem (14)-(15) is equivalent to

\[
\begin{align*}
\bar{f}_T(w_{T-1}) &= \max_{u_{T-1}} \left\{ (1 + \lambda)u_{T-1}^\top \mu_T - \lambda \cdot \kappa_T \sqrt{u_{T-1}^\top \Gamma_T u_{T-1}} \right\}, \\
\text{s.t.} \quad u_{T-1}^\top e &= w_{T-1},
\end{align*}
\]

(16)

here, \(\kappa_T = \sqrt{\frac{1 - \epsilon_T}{\epsilon_T}}\).

By setting \(s = u_{T-1}^\top \mu_T\), problem (16)-(17) is equivalent to the following unconstrained problem

\[
\begin{align*}
\bar{f}_T(w_{T-1}) &= \max_s \left\{ (1 + \lambda)s - \lambda \cdot \kappa_T \sqrt{g(s)} \right\}
\end{align*}
\]

(18)

where

\[
\begin{align*}
g(s) &= \min_{u_{T-1}} u_{T-1}^\top \Gamma_T u_{T-1}, \\
\text{s.t.} \quad u_{T-1}^\top e &= w_{T-1}, \\
u_{T-1}^\top \mu_T &= s.
\end{align*}
\]

(19)

(20)

(21)

The Lagrange function for problem (19)-(21) is

\[
L(u_{T-1}, \eta_1, \eta_2) = u_{T-1}^\top \Gamma_T u_{T-1} - \eta_1 (u_{T-1}^\top e - w_{T-1}) - \eta_2 (u_{T-1}^\top \mu_T - s).
\]

By solving the first order necessary optimality condition:

\[
\begin{align*}
\begin{cases}
2\Gamma_T u_{T-1} - \eta_1 e - \eta_2 \mu_T = 0, \\
u_{T-1}^\top e - w_{T-1} = 0, \\
u_{T-1}^\top \mu_T - s = 0,
\end{cases}
\end{align*}
\]

we can obtain the optimal solution for problem (19)-(21) as

\[
\begin{align*}
u_{T-1}^* = (\Gamma_T^{-1} e \Gamma_T^{-1} \mu_T) \frac{1}{a_T c_T - b_T^2} \begin{pmatrix} c_T & -b_T \\ -b_T & a_T \end{pmatrix} \begin{pmatrix} w_{T-1} \end{pmatrix},
\end{align*}
\]

(22)
and the corresponding optimum value is
\[
g(s) = \frac{1}{a_T c_T - b_T^2} (a_T s^2 - 2b_T w_{T-1} s + c_T w_{T-1}^2),
\]  
here, \(a_T = e^T \Gamma_T^{-1} e, \ b_T = e^T \Gamma_T^{-1} \mu_T, \ c_T = \mu^T \Gamma_T^{-1} \mu_T\).

Substituting (23) back into (18) gives
\[
f_T(w_{T-1}) = \max_s \left\{ (1 + \lambda) s - \lambda \kappa_T \sqrt{\frac{a_T s^2 - 2b_T w_{T-1} s + c_T w_{T-1}^2}{a_T c_T - b_T^2}} \right\}.
\]  
As \(w_{T-1} > 0\), we have from the first order optimality condition for problem (24) that
\[
\frac{\lambda \kappa_T}{1 + \lambda} \sqrt{\frac{1}{a_T c_T - b_T^2}} \frac{a_T s - b_T w_{T-1}}{\sqrt{a_T s^2 - 2b_T w_{T-1} s + c_T w_{T-1}^2}} - 1 = 0.
\]  
When \(h_T a_T - 1 \geq 0\), \(h_T = \left(\frac{\lambda \kappa_T}{1 + \lambda}\right)^2 \frac{1}{a_T c_T - b_T^2}\), the optimal solution to problem (24) is \(s^* = s_T w_{T-1}\), where
\[
s_T = \frac{2b(a_T h_T - 1) + \sqrt{\Delta_T}}{2a_T(a_T h_T - 1)},
\]  
and \(\Delta_T = 4(h_T a_T - 1)(a_T c_T - b_T^2)\); when \(h_T a_T - 1 < 0\), the optimal solution and optimum value for problem (24) tend to infinity.

For the first case, taking the optimal \(s^*\) back into (22) gives the optimal portfolio at period \(T - 1\), and taking it back into (24) gives
\[
f_T(w_{T-1}) = z_{T-1} w_{T-1},
\]  
where
\[
z_{T-1} = (1 + \lambda) s_T - \lambda \kappa_T \sqrt{\frac{1}{a_T c_T - b_T^2} (c_T - 2b_T s_T + a_T s_T^2)}.
\]  
At intermediary periods, we suppose
\[
f_{t+1}(w_t) = z_tw_t,
\]  
holds for some period \(t\). Specially, \(f_T(w_{T-1}) = z_{T-1} w_{T-1}\) as we have found above. Then the dynamic equation at period \(t - 1\) becomes
\[
f_t(w_{t-1}) = \max_{u_{t-1}} \left\{ -\lambda \cdot \sup_{R_t \in \mathcal{R}_t} CVaR_t(-w_t) + E_{t-1}[f_{t+1}(w_t)] \right\},
\]  
\[
\text{s.t. } u_{t-1}^T e = w_{t-1},
\]  
\[
u_{t-1}^T r_t = w_t
\]
which can be rewritten as

\[
 f_t(w_{t-1}) = \max_{u_{t-1}} \left\{ z_t E_t[w_{t-1} r_t] - \lambda \cdot \sup_{P_t \in \mathcal{P}_t} CVaR_t(-u_{t-1} r_t) \right\},
\]  

(26)

s.t. \( u_{t-1} e = w_{t-1} \).

(27)

By using the similar transformation technique as above, problem (26)-(27) is equivalent to the following programming problem:

\[
 f_t(w_{t-1}) = \max_{u_{t-1}} \left\{ (z_t + \lambda) u_{t-1} \mu_t - \lambda \cdot \kappa_t \sqrt{u_{t-1} \Gamma_t u_{t-1}} \right\},
\]  

(28)

s.t. \( u_{t-1} e = w_{t-1} \).

(29)

here, \( \kappa_t = \sqrt{\frac{1 - \epsilon_t}{\epsilon_t}} \).

For problem (28)-(29), we can find that when \( h_t a_t - 1 \geq 0 \), the optimal portfolio at period \( t \) is

\[
 u_{t-1}^* = \left( \Gamma_t^{-1} e \right)^{-1} \frac{1}{a_t c_t - b^2_t} \begin{pmatrix} c_t & -b_t \\ -b_t & a_t \end{pmatrix} \begin{pmatrix} 1 \\ s_t \end{pmatrix} w_{t-1},
\]  

(30)

here, \( a_t = e^\top \Gamma_t^{-1} e, b_t = e^\top \Gamma_t^{-1} \mu_t, c_t = \mu_t^\top \Gamma_t^{-1} \mu_t, s_t = \frac{2k(a_t h_t - 1) + \sqrt{4k}}{2a_t(a_t h_t - 1)}, h_t = \left( \frac{\lambda \kappa_t}{\lambda + \kappa} \right)^2 \frac{1}{a_t c_t - b^2_t}, \Delta_t = 4(h_t a_t - 1)(a_t c_t - b^2_t) \). And the corresponding optimum value is

\[
 f_t(w_{t-1}) = z_{t-1} w_{t-1},
\]

where

\[
 z_{t-1} = (z_t + \lambda) s_t - \lambda \kappa_t \sqrt{\frac{1}{a_t c_t - b^2_t} (c_t - 2b_t s_t + a_t s_t^2)},
\]

which proves the correctness of (25) at period \( t - 1 \).

And when \( h_t a_t - 1 < 0, \Delta_t < 0 \), problem (28)-(29) does not have a finite solution, and hence the optimal portfolio and the optimum value at period \( t \) tend to infinity.

Starting from period \( T \), we can do the above argument recursively until period 1, which then establishes the conclusion of Theorem 1. \( \square \)
3.2. Multi-period robust portfolio selection models under wrCVaR and mwCVaR

In this subsection, the new multi-period robust risk measures, wrCVaR and mwCVaR, are adopted to establish multi-period robust portfolio selection models. In order to obtain greater realism in our portfolio selection models, we will take transaction costs into account.

Similarly, we suppose that there are $n$ risky assets in the security market, and the investment horizon is divided into $T$ periods. The investor joins the market at time 0 with an initial cash $w_0$, he/she can trade these assets at the beginning of each period. $r_t = [r^1_t, \cdots, r^n_t]^\top, t = 1, 2, \cdots, T$, is the random return rate vector of the $n$ assets at period $t$, which is adapted to the information process $\{F_t\}$. $u_t = [u^1_t, \cdots, u^n_t]^\top$ is the portfolio vector of the wealth invested in the $n$ assets, which is predictable to the information process $\{F_t\}$. $w_t, t = 1, 2, \cdots, T$, is the resulting wealth process. The investor has to pay transaction costs to purchase or sell the risky assets, with the transaction cost ratio vector being $\alpha$ and $\beta$, respectively. We assume the investment process is self-financing, hence we have

$$w_0 = u_0^\top e + \alpha^\top (u_0)^+ + \beta^\top (u_0)^-, \quad (31)$$

$$w_t = u_t^\top e + \alpha^\top (u_t - u_{t-1})^+ + \beta^\top (u_t - u_{t-1})^-, \quad t = 1, \cdots, T - 1, \quad (32)$$

$$w_{t+1} = u_t^\top r_{t+1}, \quad t = 0, \cdots, T - 1, \quad (33)$$

where, for $\xi \in \mathbb{R}^n$, $(\xi)^+ = [\max\{0, \xi_1\}, \cdots, \max\{0, \xi_n\}]^\top$ and $(\xi)^- = [-\min\{0, \xi_1\}, \cdots, -\min\{0, \xi_n\}]^\top$ are the positive and negative parts of $\xi$, respectively. Moreover, in order to reflect the real market environment, we restrict $u_t$ to a bounded interval as follows:

$$u^\alpha \leq u_t \leq \bar{u}, \quad t = 0, \cdots, T - 1, \quad (34)$$

here, $u^\alpha \in \mathbb{R}^n$ and $\bar{u} \in \mathbb{R}^n$ are the lower bound vector and the upper bound vector, respectively.

Under the mean-risk framework, the problem of maximizing the expected terminal wealth and minimizing wrCVaR or mwCVaR simultaneously can be formulated as the following optimization problems:

$$\max_u \left\{ E[w_T] - \lambda \cdot wrCVaR_{0,T}(-w_{1,T}) \right\}, \quad (35)$$

s.t. $\ (31) - (34). \quad (36)$
and

\[
\max_u \left\{ E[w_T] - \lambda \cdot mwCVaR_{0,T}(-w_{1,T}) \right\},
\]

\[
\text{s.t. } (31) - (34).
\]

(37)

(38)

here, \( \lambda \) is the risk averse coefficient. The confidence level at period \( t \) is \( \epsilon_t(s_t) \), which changes with the regime \( s_t \). We denote the above two robust mean-risk models as the mean-wrCVaR model and the mean-mwCVaR model, respectively, in what follows.

Due to the nonlinear relationship of regime switching between periods, it is hard to find the closed-form optimal solutions for the mean-wrCVaR and mean-mwCVaR models. To ensure its practicality and flexibility, we adopt the scenario tree technology to derive a tractable transformation for the above two optimization problems. Before that, we need to introduce some notation.

Firstly, we assume the regime switching during the \( T \) periods is a stationary Markovian process and adopt a scenario tree to represent the structure of the regime process. The initial (root) node in the scenario tree represents the current regime \( s_0 \), each of its child nodes represents a possible regime \( s_1 \) appearing at period 1, here the branching probability is equal to the transition probability from \( s_0 \) to \( s_1 \), and so on. In the scenario tree, \( K^+ \) denotes the set of all nodes at periods \( 1, 2, \cdots, T \), \( N(K^+) \) is the number of nodes in \( K^+ \); \( K^- \) denotes the set of all nodes at periods \( 0, 1, \cdots, T - 1 \), \( N(K^-) \) is the number of nodes in \( K^- \). For a node \( k \) in the scenario tree, we use \( t(k) \) to denote the number of the period it belongs to, \( s(k) \) to denote the corresponding regime, and \( P_s(k) \) to denote its appearing probability in the tree. For a node \( k \in K^+ \), it has exactly one direct predecessor, denoted as \( k^- \); for a node \( k \in K^- \), it has more than one direct successor, whose set is denoted as \( k^+ \).

For a node \( k \in K^+ \), let \( \mu(s(k)) \) denote the estimated expectation value of the return rate vector \( r_t \) of the \( n \) risky assets at period \( t(k) \), under the available information at its predecessor node \( k^- \), and let \( \Gamma(s(k)) \) denote the estimation value of the conditional covariance matrix, here we assume \( \Gamma(s(k)) \) is always positive definite. As \( s(k) \) is uniquely associated with node \( k \), we simply denote
\( \mu(s(k)) \) and \( \Gamma(s(k)) \) as \( \mu(k) \) and \( \Gamma(k) \), respectively. The similar simplification of notations will be adopted in what follows. Then for a node \( k \) at period \( t \), the uncertainty set with respect to its regime \( s(k) \) can be defined as

\[
\mathcal{P}(k) = \{ P \mid E_P[ r_{t-1} | \mathcal{F}_{t-1}] = \mu(k), \Gamma_P[ r_{t-1} | \mathcal{F}_{t-1}] = \Gamma(k) \}.
\]  

(39)

Here, the uncertainty set is built by fixing the conditional expected returns and the conditional covariance matrix, which can be observed at period \( t-1 \). Therefore, this definition meets the basic requirement of the dynamic uncertainty set.

Secondly, under the regime switching framework, we need to modify the following dynamic equation for decision variables between two adjacent periods,

\[
u_{t-1}^T r_t = u_t^T e + \alpha^T (u_t - u_{t-1})^+ + \beta^T (u_t - u_{t-1})^-,
\]

which is a combination of (32) and (33). For this purpose, we use \( \mu(k) \) to represent the forecasted value of the return rate vector at node \( k \) of stage \( t \), then the above dynamic equation can be extended to

\[
u(k^-)^T \mu(k) = u(k)^T e + \alpha^T (u(k) - u(k^-))^+ + \beta^T (u(k) - u(k^-))^-,
\]

\( k \in K^- \setminus \{0\} \).

(40)

With the above notations, we can reformulate the mean-mwCVaR model (37) and (38) as a second order cone programming problem. Concretely,

**Theorem 2.** With the uncertainty set (39) and the dynamic equation (40), the mean-mwCVaR model (37)-(38) is equivalent to the following cone programming problem:

\[
\begin{align*}
\max_{u, y, x, g, u^+, u^-} & \left\{ (1 + \lambda)w_0 + \sum_{k \in K^+} (1 + (T - t(k^-) - 1)\lambda) P^*(k)(\mu(k) - e)^T u(k^-) \\
& - \lambda \sum_{k \in K^+} P^*(k)y(k) - (1 + T\lambda)(\alpha^T u^+(0) + \beta^T u^-(0)) \\
& - \sum_{k \in K^- \setminus \{0\}} (1 + (T - t(k))\lambda)[\alpha^T u^+(k) + \beta^T u^-(k)] \right\} \\
\text{s.t.} & \quad \Gamma^{1/2}(k)u(k) = z(k), \quad k \in K^+,
\end{align*}
\]

(41)
\[ (\mu(k) - e)^\top u(k^-) + y(k) = \kappa(k)g(k), \ k \in K^+, \]
\[ \|z(k)\|_2 \leq g(k), \ k \in K^+, \]
\[ u(0) = u^+(0) - u^-(0), \]
\[ w_0 = u(0)^\top e + \alpha^\top u^+(0) + \beta^\top u^-(0), \]
\[ u(k) - u(k^-) = u^+(k) - u^-(k), \ k \in K^\\backslash\{0\}, \]
\[ u(k^-)^\top \mu(k) = u(k)^\top e + \alpha^\top u^+(k) + \beta^\top u^-(k), \ k \in K^\\backslash\{0\}, \]
\[ u^+(k), u^-(k) \geq 0, \ k \in K^-, \]
\[ u \leq u(k) \leq \bar{u}, \ k \in K^-, \]

with \((n + 2)N(K^+) + 3nN(K^-)\) variables, \((n + 1)N(K^+) + (n + 1)N(K^-)\) linear constraints and \(N(K^+)\) standard second order cone constraints.

**Proof.** We can deduce from (31)-(33) that

\[ w_t = w_{t-1} + u_{t-1}^\top (r_t - e) - \alpha^\top (u_{t-1} - u_{t-2})^+ - \beta^\top (u_{t-1} - u_{t-2})^-. \]  

(51)

Applying (51) recursively gives

\[ w_t = w_0 + \sum_{s=0}^{t-1} u_s^\top (r_{s+1} - e) - \sum_{s=1}^{t-1} \left( \alpha^\top (u_s - u_{s-1})^+ + \beta^\top (u_s - u_{s-1})^- \right) - \alpha^\top (u_0)^+ - \beta^\top (u_0)^-. \]

(52)

By substituting (51) into (4), we have

\[ mw_{0,T}(-w_{1,T}) = \sum_{t=1}^T E \left[ \sum_{s_t \in S_t} \left( P_{s_{t-1} s_t}^{s_t} \sup_{P_t \in \mathcal{P}(s_t)} \text{CVaR}_t \left( -w_{t-1} - u_{t-1}^\top (r_t - e) \right) \right) \right] + \sum_{t=1}^T E \left[ \sum_{s_t \in S_t} \left( P_{s_{t-1} s_t}^{s_t} \sup_{P_t \in \mathcal{P}(s_t)} \text{CVaR}_t \left( -u_{t-1}^\top (r_t - e) \right) \right) \right] - \sum_{t=0}^{T-1} E[w_t] \]

\[ + \left( \alpha^\top (u_0)^+ + \beta^\top (u_0)^- \right) + \sum_{t=1}^{T-1} E \left[ \alpha^\top (u_t - u_{t-1})^+ + \beta^\top (u_t - u_{t-1})^- \right], \]

here, we have utilized the translation invariance property of CVaR. And then, we take it back into (37), and find that the objective function in (37) can be divided into three parts: the risk part, the return part, and the transaction cost part. Specifically, we have

\[ E[w_T] - \lambda \cdot mw_{0,T}(-w_{1,T}) = \text{Return} - \lambda \text{Risk} - \text{Transaction}, \]

(53)
where

\[
\text{Return} = (1 + T\lambda)w_0 + \sum_{t=0}^{T-1} (1 + (T - t - 1)\lambda)E[u_t^\top (r_{t+1} - e)],
\]

\[
\text{Risk} = \sum_{t=1}^{T} E \left[ \sum_{s_t \in S_t} P_{s_t \rightarrow s_{t-1}} \sup_{\mathcal{P}_t(s_t)} \mathrm{CVaR}_t \left( -u_{t-1}^\top (r_t - e) \right) \right],
\]

\[
\text{Transaction} = (1 + T\lambda)(\alpha^\top (u_0)^+ + \beta^\top (u_0)^-) + \sum_{t=1}^{T-1} (1 + (T - t)\lambda)
\]

\[
E[\alpha^\top (u_t - u_{t-1})^+ + \beta^\top (u_t - u_{t-1})^-].
\]

Firstly, we can reformulate the return part as follows

\[
\text{Return} = (1 + T\lambda)w_0 + \sum_{k \in K^+} \left( 1 + (T - t(k) - 1)\lambda \right) P^*(k)(\mu(k) - e)^\top u(k^-),
\]

which is a linear function of decision variables.

Secondly, the risk part can be rewritten as

\[
\text{Risk} = \min_y \sum_{k \in K^+} P^*(k)y(k) \quad (55)
\]

\[
s.t. \quad \sup_{\mathcal{P}_t(k)} \mathrm{CVaR}_t(k)(-r(k) - e)^\top u(k^-) \leq y(k), \quad k \in K^+,
\]

here, we introduce auxiliary variables \(y(k), \quad k \in K^+\), which are adapted to \(\mathcal{F}_t\).

Under \(\mathcal{P}(k)\), the constraint (56) at node \(k\) is actually a single-period conditional worst-case CVaR constraint with the information about first two order moments. Then, we know from Theorem 2.9 in Chen et al. (2011) that it is equivalent to

\[
\kappa(k)||\Gamma^{1/2}(k)u(k^-)||_2 - (\mu(k) - e)^\top u(k^-) \leq y(k), \quad k \in K^+,
\]

where \(\kappa(k) = \sqrt{\frac{1 - \epsilon(k)}{\epsilon(k)}}\).

Through introducing auxiliary vectors \(z(k) = [z_1(k), \cdots, z_u(k)]^\top, \quad k \in K^+\), and auxiliary variables \(g(k), \quad k \in K^+\), the constraint (57) can further be equivalently described as the following two groups of linear constraints and a group of standard second order cone constraints:

\[
\Gamma^{1/2}(k)u(k^-) = z(k), \quad k \in K^+,
\]

\[
(\mu(k) - e)^\top u(k^-) + y(k) = \kappa(k)g(k), \quad k \in K^+,
\]

\[
||z(k)||_2 \leq g(k), \quad k \in K^+.
\]
Thirdly, the transaction cost part can be rewritten as:

\[
\text{Transaction} = \min_{u^+, u^-} \left\{ (1 + T\lambda)(\alpha^\top u^+(0) + \beta^\top u^-(0)) + \sum_{k \in K^- \setminus \{0\}} (1 + (T - t(k))\lambda)[\alpha^\top u^+(k) + \beta^\top u^-(k)] \right\}
\]

\[
\text{s.t. } u(0) = u^+(0) - u^-(0),
\]

\[
u(k) - u(k^-) = u^+(k) - u^-(k), \quad k \in K^\setminus \{0\},
\]

\[
u^+(k), u^-(k) \geq 0, \quad k \in K^-.
\]

Finally, we can obtain the second order cone programming problem (41)-(50) by taking all (31), (40), (53), (54), (55), (58)-(60) and (61)-(65) into (37)-(38), which completes the proof. □

By using the similar demonstration as the proof of Theorem 2, we can show that the mean-wrCVaR model can also be transformed into a second order cone programming problem. It is summarized in the following theorem.

**Theorem 3.** With the uncertainty set (39) and the dynamic equation (40), the mean-wrCVaR model (35)-(36) is equivalent to the following cone programming problem:

\[
\min_{u, y, z, g, u^+, u^-} \left\{ (1 + \lambda)w_0 + \sum_{k \in K^+} (1 + (T - t(k) - 1)\lambda)P^s(k)(\mu(k) - e)^\top u(k^-) - \lambda \sum_{k \in K^-} P^s(k)g(k) - (1 + T\lambda)(\alpha^\top u^+(0) + \beta^\top u^-(0)) + \sum_{k \in K^- \setminus \{0\}} (1 + (T - t(k))\lambda)[\alpha^\top u^+(k) + \beta^\top u^-(k)] \right\}
\]

\[
\text{s.t. } (\mu(k) - e)^\top u(k^-) + g(k) = \kappa(k)g(k), \quad k \in K^+,
\]

\[
(42), (44) - (50).
\]

with \((n + 1)N(K^+) + (3n + 1)N(K^-)\) variables, \((n + 1)N(K^+) + (n + 1)N(K^-)\) linear constraints and \(N(K^+)\) standard second order cone constraints.

Theorems 2 and 3 mean that both of our robust portfolio selection models (37)-(38) and (35)-(36) can be efficiently solved in polynomial time by using interior point methods (Alizadeh and Goldfarb, 2003).
Remark: The multi-period worst-case risk measure is a special case of the multi-period worst regime risk measure or the multi-period mixed worst-case risk measure with the number of regimes being one. Therefore, if transaction costs are taken into account, the corresponding multi-period portfolio selection model under the multi-period worst-case risk measure can also be efficiently solved by utilizing the second order cone programming problem (41)-(50) or (66)-(68).

4. Numerical results

In this section, we first carry out some simulations to show the advantages and good performance of the proposed mean-wCVaR model when it is compared with two typical multi-period portfolio selection models in the literature. Then, we will empirically show how to specify market regimes to find optimal portfolios under the introduced mean-wCVaR, mean-wrCVaR and mean-mwCVaR models by using the real trading data in American stock markets, and to analyze the differences among the optimal investment policies. These empirical results and the out-of-sample performance evaluation will demonstrate the practicality, efficiency and robustness of our new multi-period robust portfolio selection models.

4.1. Simulation results

To intuitively compare the performances of the optimal portfolio obtained with the proposed mean-wCVaR model and that got with the dynamic MV model in Li et al. (2000) and that got under the multistage portfolio selection model with robust second order lower partial moment (LPM2) as the risk measure in Chen et al. (2011), we consider the case study in Example 2 of Li et al. (2000) by assuming a stationary four-period return process with $T = 4$. The market is constituted by three risky securities, A, B, and C. The expected returns for these risky securities are $E(r_t^A) = 1.162$, $E(r_t^B) = 1.246$ and $E(r_t^C) = 1.228$, $t = 1, \cdots, 4$, respectively, and the corresponding covariance matrix of the return vector is

$$\text{Cov}(r_t) = \begin{bmatrix} 0.0146 & 0.0187 & 0.0145 \\ 0.0187 & 0.0854 & 0.0104 \\ 0.0145 & 0.0104 & 0.0289 \end{bmatrix}, t = 1, \cdots, 4.$$
By directly applying the result in Example 2 of Li et al. (2000), the conclusion in Theorem 3.1 in Chen et al. (2011) and the explicit investment policy in Theorem 1 to the above example, we can obtain the optimal portfolios under the three models. The optimal portfolios at the root node are

\[
u_0(MV) = \begin{bmatrix} 2.731 \\ 7.642 \\ -9.373 \end{bmatrix}, \quad u_0(LPM2) = \begin{bmatrix} 1.4857 \\ -0.1723 \\ -0.3134 \end{bmatrix}, \quad u_0(wCVaR) = \begin{bmatrix} 0.9455 \\ -0.0278 \\ 0.0823 \end{bmatrix},
\]

respectively. Here, wCVaR denotes the mean-wCVaR model, MV denotes the dynamic MV model in Li et al. (2000) and LPM2 denotes the multistage portfolio selection model in Section 3 of Chen et al. (2011).

Compared with \(u_0(LPM2)\) and \(u_0(wCVaR)\), the optimal portfolio \(u_0(MV)\) is rather extreme in the sense that it assigns large proportions in both buying and short-selling risky assets. Although the dynamic MV model can theoretically control the variance of the terminal wealth by hedging among risky assets, it might suffer large losses under extreme scenarios. The following simulation will illustrate this point. We generate 100 four-period scenario trees. For each tree, there are 10 branches at every non-leaf node. The sample return rates of the three assets at each node are generated by the normal distribution with the above given mean and variance. By applying the three groups of optimal portfolios to the four-period portfolio selection problem under each scenario tree, we can determine the distribution of the terminal wealth under the three models. We repeat the process for 100 times, and obtain 100 groups of the mean, variance and Sharpe ratio of the terminal wealth under each of the three models. For the 100 values of mean, variance and the Sharpe ratio got under each of the three portfolio selection models, we calculate the smallest value, the largest value and the average value of each character, which are presented in Table 1.

We can observe from Table 1 that the optimal portfolio under the MV model gains a rather high expected wealth in the best case, while it suffers an extreme large wealth loss under the worst case; meanwhile, the corresponding variance of the terminal wealth got under the MV model is always very large, this means that it is quite dubious about the MV model to control the risk in
Table 1  Characteristics of the final wealths got under the three models

<table>
<thead>
<tr>
<th></th>
<th>mean</th>
<th>variance</th>
<th>Sharpe ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>wCVaR</td>
<td>MV</td>
<td>LPM2</td>
</tr>
<tr>
<td>minimum</td>
<td>1.8387</td>
<td>-1.9208</td>
<td>1.1080</td>
</tr>
<tr>
<td>maximum</td>
<td>2.1989</td>
<td>6.0885</td>
<td>1.2659</td>
</tr>
<tr>
<td>average value</td>
<td>2.0184</td>
<td>1.8296</td>
<td>1.1875</td>
</tr>
</tbody>
</table>

practice. On the contrary, both the optimal portfolios obtained under wCVaR and LPM2 models perform much better in controlling the investment risk under all three cases. It illustrates that taking the robust technology into the multi-period portfolio selection model is quite worthwhile, it can efficiently reduce the expected wealth loss and investment risk under extreme cases.

By comparing the performances between two robust portfolio selection models, we can find that the optimal terminal wealth’s variance under the LPM2 model is the smaller one in both the best and average cases. We know from Chen et al. (2011) that the LPM2 model is constructed in the spirit of the adjustable robust approach, which makes a worst estimation at the current period on the basis of the worst estimation at the next period. Such a strategy leads to an overly conservative investment policy. Furthermore, we can see from the last block of Table 1 that the Sharpe ratio of the optimal portfolio obtained with the wCVaR model is always the largest one among those of three optimal portfolios. This illustrates that our robust portfolio selection model with the multi-period worst-case risk measure is not that extremely conservative as the adjustable robust approach, and it makes a good balance between providing a high terminal wealth and controlling the extreme risk. All these simulation results illustrate that our new robust portfolio selection model is a good complement to the existing multi-period portfolio selection models.

4.2. Empirical results

To empirically test the practicality and superiority of the proposed new robust portfolio selection models, we randomly choose 10 stocks from different industries in both Dow Jones Industrial Average and S&P 500 Indexes, they are DIS, DOW, ED, GE, IBM, MRK, MRO, MSI, PEP and
JNJ. We use the adjusted daily close-prices of these stocks on every Monday to compute their weekly logarithmic return rates from February 14, 1977 to January 30, 2012. The original data are downloaded from Yahoo finance$^1$.

As usually do in the literature, we use three regimes to represent the states of market: the bull regime, denoted as $s^1$, indicates the market is going up; the consolidation regime, denoted as $s^2$, means the market is in the transitional period between recovery and recession; and the bear regime, denoted as $s^3$, means the market is going down.

We use the data of MKT-RF (Fama and French, 1993), a market index which is the value-weighted excess returns on all NYSE, AMEX and NASDAQ stocks minus the 30 day US T-Bill yield, to determine the market regime during each week. The corresponding weekly data of MKT-RF are downloaded from the Kenneth R. French Data Library$^2$. In detail, to determine the regime of a specific week, we prescribe an effective time window with 28 weeks, centered on the examining week, and add all the data of MKT-RF in the effective time window together. If the sum is larger than 1.0, we say the week is under the bull regime; if the sum is smaller than -1.0, we say the week is under the bear regime; if the sum is between -1.0 and 1.0, we say the week is under the consolidation regime. The time window approach can eliminate the influence of big short-term fluctuations and can improve the estimation accuracy in practice. By applying the above method to each week in the sample period, we can get a historical regime switching series. Then the regime transition probabilities can be estimated by counting the relevant historical transition times, which gives us the following transition matrix:

$$
P^* = \begin{bmatrix}
0.9475 & 0.0336 & 0.0189 \\
0.3333 & 0.3148 & 0.3519 \\
0.0471 & 0.0634 & 0.8895 
\end{bmatrix}.
$$

From the diagonal elements in $P^*$, we see that it is stable to stay in the bull or bear regime, but there is a relatively high possibility to switch from the consolidation regime into the bull or bear regime. This phenomenon is consistent with the real market situation.
We consider a three-period investment problem, \( T = 3 \), with the corresponding regime switching process (tree) shown in Figure 1.

Under each regime, we estimate the moment information of stock return rates by using the historical return data in weeks belonging to that regime. The expected return rates of the 10 assets under the three regimes are shown in Table 2. The expected return rates estimated with the whole sample data are also listed in the last line of Table 2 as a comparison. Due to the space limitation, we only show in Table 3 the estimated variances of the return rates of the 10 assets under the three regimes and those variances estimated with the whole sample data.

<table>
<thead>
<tr>
<th>DIS</th>
<th>DOW</th>
<th>ED</th>
<th>GE</th>
<th>IBM</th>
<th>MRK</th>
<th>MRO</th>
<th>MSI</th>
<th>PEP</th>
<th>JNJ</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mu(s^1) )</td>
<td>0.2486</td>
<td>0.1845</td>
<td>0.1165</td>
<td>0.2260</td>
<td>0.1290</td>
<td>0.1884</td>
<td>0.1639</td>
<td>0.2291</td>
<td>0.1825</td>
</tr>
<tr>
<td>( \mu(s^2) )</td>
<td>0.0206</td>
<td>-0.0116</td>
<td>0.1413</td>
<td>0.0110</td>
<td>-0.1879</td>
<td>0.1027</td>
<td>0.2251</td>
<td>0.0817</td>
<td>0.1653</td>
</tr>
<tr>
<td>( \mu(s^3) )</td>
<td>-0.1921</td>
<td>-0.1583</td>
<td>0.0897</td>
<td>-0.1545</td>
<td>0.0035</td>
<td>-0.0691</td>
<td>-0.0274</td>
<td>-0.2706</td>
<td>-0.0199</td>
</tr>
<tr>
<td>( \mu )</td>
<td>0.1004</td>
<td>0.0681</td>
<td>0.1098</td>
<td>0.0970</td>
<td>0.0718</td>
<td>0.1046</td>
<td>0.1009</td>
<td>0.0676</td>
<td>0.1196</td>
</tr>
</tbody>
</table>

It is easy to see from Tables 2 and 3 that both the expected return rates and the variances change significantly among different regimes. Under the bull regime, the expected return rates are the highest and always positive; under the bear regime, they are the lowest and mostly negative;
and under the consolidation regime, they are in the middle. Correspondingly, the variances are always the largest under the bear regime, the smallest under the consolidation regime, and in the middle under the bull regime. These phenomena match the real market well: the investment under the bull (bear) regime is usually active, which leads to a rather high (low) return rate with large volatilities; while the investment under the consolidation regime is not that active, which leads to an intermediate return rate with a small volatility.

Having obtained the information of first and second order moments of the return rates under the three regimes, we can then determine the optimal investment policies under the mean-wrCVaR model and mean-mwCVaR model by solving problem (41)-(50) and problem (66)-(68), respectively. Correspondingly, we can also find the optimal investment policy under the mean-wCVaR model by solving problem (41)-(50) or (66)-(68), here we need to set $J = 1$, $\mu(s^1) = \mu$ and $\sigma^2(s^1) = \sigma^2$.

We assume that the initial wealth is 1.0; at each period, the confidence level is chosen as $\epsilon_t(s_t) = 0.05$ under all the three regimes; the risk-averse coefficient is set to $\lambda = 20$; the lower bounds for portfolio weights are 0.0, the short-selling is thus forbidden; the upper bounds for portfolio weights are 0.3. We use the Mosek package in Matlab 7.6.0 (2008a) to solve problem (41)-(50) and problem (66)-(68), all the numerical experiments are carried out on a Lenovo PC with 2.98G RAM, 2.93GHz Dual Core CPU.

By setting the current regime (at node 0 in Fig. 1) to the bull, bear and consolidation regime in succession, we can find three groups of optimal portfolios under the mean-wCVaR, mean-wrCVaR and mean-mwCVaR models, respectively.

For the multi-period portfolio selection problem, the optimal portfolio at the current period is most important, because it will be applied immediately. Hence, we first show in Table 4 the optimal

<table>
<thead>
<tr>
<th></th>
<th>DIS</th>
<th>DOW</th>
<th>ED</th>
<th>GE</th>
<th>IBM</th>
<th>MRK</th>
<th>MRO</th>
<th>MSI</th>
<th>PEP</th>
<th>JNJ</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\sigma^2(s^1)$</td>
<td>0.2677</td>
<td>0.3009</td>
<td>0.1012</td>
<td>0.2186</td>
<td>0.2182</td>
<td>0.2529</td>
<td>0.3545</td>
<td>0.4989</td>
<td>0.1954</td>
<td>0.1841</td>
</tr>
<tr>
<td>$\sigma^2(s^2)$</td>
<td>0.2531</td>
<td>0.1491</td>
<td>0.1065</td>
<td>0.1478</td>
<td>0.1667</td>
<td>0.2003</td>
<td>0.2842</td>
<td>0.4683</td>
<td>0.1982</td>
<td>0.1478</td>
</tr>
<tr>
<td>$\sigma^2(s^3)$</td>
<td>0.4489</td>
<td>0.4363</td>
<td>0.1599</td>
<td>0.3224</td>
<td>0.3099</td>
<td>0.2974</td>
<td>0.5075</td>
<td>0.8822</td>
<td>0.2577</td>
<td>0.2225</td>
</tr>
<tr>
<td>$\sigma^2$</td>
<td>0.3259</td>
<td>0.3355</td>
<td>0.1194</td>
<td>0.2489</td>
<td>0.2437</td>
<td>0.2645</td>
<td>0.3975</td>
<td>0.6187</td>
<td>0.2152</td>
<td>0.1938</td>
</tr>
</tbody>
</table>
portfolios at the root node determined under the mean-wCVaR, wrCVaR and mwCVaR models, here the optimal portfolios under the mean-wCVaR or mean-wrCVaR model are the same for the three current regimes. Furthermore, we show in Table 5 the optimal portfolios at recourse periods for the mean-mwCVaR model when the current regime is set to the bull regime, as an illustration to show the variation of optimal portfolios among periods.

**Table 4** Root optimal portfolios under mean-wCVaR, wrCVaR and mwCVaR models

<table>
<thead>
<tr>
<th></th>
<th>DIS</th>
<th>DOW</th>
<th>ED</th>
<th>GE</th>
<th>IBM</th>
<th>MRK</th>
<th>MRO</th>
<th>MSI</th>
<th>PEP</th>
<th>JNJ</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u_{wCVaR}^*(s_0)$</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.3000</td>
<td>0.0000</td>
<td>0.3000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.2995</td>
<td>0.1005</td>
</tr>
<tr>
<td>$u_{wCVaR}^*(s_0)$</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.3000</td>
<td>0.0000</td>
<td>0.3000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.1367</td>
<td>0.2633</td>
</tr>
<tr>
<td>$u_{mwCVaR}^*(s_1)$</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.3000</td>
<td>0.0000</td>
<td>0.3000</td>
<td>0.0000</td>
<td>0.1385</td>
<td>0.0000</td>
<td>0.2615</td>
<td>0.0000</td>
</tr>
<tr>
<td>$u_{mwCVaR}^*(s_2)$</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.3000</td>
<td>0.0000</td>
<td>0.3000</td>
<td>0.0000</td>
<td>0.0550</td>
<td>0.0000</td>
<td>0.3000</td>
<td>0.0450</td>
</tr>
<tr>
<td>$u_{mwCVaR}^*(s_3)$</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.3000</td>
<td>0.0000</td>
<td>0.3000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.1492</td>
<td>0.2508</td>
</tr>
</tbody>
</table>

**Table 5** Optimal portfolios at recourse periods under the mean-mwCVaR model under $s_0 = s^1$

<table>
<thead>
<tr>
<th>node</th>
<th>regime</th>
<th>DIS</th>
<th>DOW</th>
<th>ED</th>
<th>GE</th>
<th>IBM</th>
<th>MRK</th>
<th>MRO</th>
<th>MSI</th>
<th>PEP</th>
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<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.3000</td>
<td>0.0000</td>
<td>0.3000</td>
<td>0.0000</td>
<td>0.1461</td>
<td>0.0000</td>
<td>0.2553</td>
<td>0.0000</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.3000</td>
<td>0.0000</td>
<td>0.3000</td>
<td>0.0000</td>
<td>0.1243</td>
<td>0.0000</td>
<td>0.2771</td>
<td>0.0000</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.3000</td>
<td>0.0000</td>
<td>0.3000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.2615</td>
<td>0.1396</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.3000</td>
<td>0.0000</td>
<td>0.3000</td>
<td>0.0000</td>
<td>0.1553</td>
<td>0.0000</td>
<td>0.2476</td>
<td>0.0000</td>
</tr>
<tr>
<td>5</td>
<td>2</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.3000</td>
<td>0.0000</td>
<td>0.3000</td>
<td>0.0000</td>
<td>0.1461</td>
<td>0.0000</td>
<td>0.2568</td>
<td>0.0000</td>
</tr>
<tr>
<td>6</td>
<td>3</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.3000</td>
<td>0.0000</td>
<td>0.3000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.2553</td>
<td>0.1473</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.3000</td>
<td>0.0000</td>
<td>0.3000</td>
<td>0.0000</td>
<td>0.1258</td>
<td>0.0000</td>
<td>0.2771</td>
<td>0.0000</td>
</tr>
<tr>
<td>8</td>
<td>2</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.3000</td>
<td>0.0000</td>
<td>0.3000</td>
<td>0.0000</td>
<td>0.1243</td>
<td>0.0000</td>
<td>0.2785</td>
<td>0.0000</td>
</tr>
<tr>
<td>9</td>
<td>3</td>
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<td>0.0000</td>
<td>0.3000</td>
<td>0.0000</td>
<td>0.3000</td>
<td>0.0000</td>
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<td>0.0000</td>
<td>0.2771</td>
<td>0.0014</td>
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<td>0.0000</td>
<td>0.3000</td>
<td>0.0000</td>
<td>0.3000</td>
<td>0.0000</td>
<td>0.0014</td>
<td>0.0000</td>
<td>0.2615</td>
<td>0.1396</td>
</tr>
<tr>
<td>11</td>
<td>2</td>
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<td>0.0000</td>
<td>0.3000</td>
<td>0.0000</td>
<td>0.3000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.2629</td>
<td>0.1396</td>
</tr>
<tr>
<td>12</td>
<td>3</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.3000</td>
<td>0.0000</td>
<td>0.3000</td>
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<td>0.0000</td>
<td>0.0000</td>
<td>0.2615</td>
<td>0.1410</td>
</tr>
</tbody>
</table>

We can derive at least the following three observations from the optimal portfolios in Table 4 and Table 5.
Firstly, all the optimal portfolios obtained under the three models allocate the available wealth among ED, IBM, MRO, PEP and JNJ. We know from Tables 2 and 3 that these five assets provide high return rates with small variances. As risk-averse investment models, it is rather natural that the optimal portfolios always prefer assets which generate stable returns with little uncertainty.

Secondly, besides ED and IBM in which both the mean-wCVaR and mean-wrCVaR models invest with the maximum quota, the mean-wCVaR model invests most in PEP and the mean-wrCVaR model invests most in JNJ at the current period. From Table 2 and Table 3, we know that PEP performs better (bad) in bull and consolidation regimes (in bear regime) than JNJ. This illustrates that the mean-wrCVaR model pays much attention to the risk under the worst regime, while the mean-wCVaR model controls the risks under all the regimes simultaneously. Hence, the mean-wrCVaR model is more robust than the mean-wCVaR model in controlling extreme losses.

Thirdly, the mean-wCVaR (mean-wrCVaR) model generates the same portfolios at the current period under all the three regimes. On the contrary, the mean-mwCVaR model provides us three optimal portfolios at the current period with respect to different regimes. When the current regime is the bear regime, the optimal portfolio obtained under the mean-mwCVaR model is similar to that obtained under the mean-wrCVaR model; when the current regime is the consolidation regime, the optimal portfolio obtained under the mean-mwCVaR model is similar to that portfolio got under the mean-wCVaR model; and when the current regime is the bull regime, the optimal portfolio obtained under the mean-mwCVaR model invests more in MRO and PEP. The change of the optimal portfolio obtained under the mean-mwCVaR model with respect to the market regime is because the estimation of mwCVaR relies on the regime appearing probability, while wrCVaR only focuses on the risk under the worst regime and wCVaR views the market as a whole one. Besides the optimal portfolio at the current period, the optimal portfolios at recourse periods obtained under the mean-mwCVaR model also vary with respect to different regimes in a similar way, which can be observed from Table 5.

The optimal portfolios at recourse periods obtained under the mean-wCVaR or wrCVaR models are very similar to that at the current period, they do not change with respect to different regimes. Hence, we do not show them in detail, which can be provided upon requirement.
Finally, we investigate the robustness of the three-period mean-wCVaR, mean-wrCVaR and mean-mwCVaR models by examining the out-of-sample performance of the optimal portfolio at the current period. To this end, the historical weekly data are separated into two parts: the in-sample period is from February 22, 1977 to March 1, 2010, and the out-of-sample period is from March 1, 2010 to January 30, 2012. We carry out the out-of-sample test in a rolling forward way. That is, we use the data in the in-sample period to find the three optimal portfolios corresponding to the three models, respectively, just like what we did in the above analysis. We invest with the optimal portfolio at the current period for the next one week starting from March 1, 2010, and compute the return rates of the three optimal portfolios in the first week of the out-of-sample period with the actual return data in that week. Then, we move the in-sample period one week forward by adding the new week and deleting the first week, re-solve the resulting three portfolio selection problems, find the optimal portfolios, and compute their return rates in the second week of the out-of-sample period, and so on. We carry out the out-of-sample test by rolling forward weekly until January 30, 2012, this provides us three return rate series with 100 out-of-sample weekly return rates, corresponding to the mean-wCVaR, mean-wrCVaR and mean-mwCVaR models, respectively.

Table 6 shows some statistic characteristics, including the maximum, the minimum, the mean, the variance and the skewness, of these three return series.

<table>
<thead>
<tr>
<th>model</th>
<th>mean-wCVaR</th>
<th>mean-wrCVaR</th>
<th>mean-mwCVaR</th>
</tr>
</thead>
<tbody>
<tr>
<td>maximum (%)</td>
<td>1.1020</td>
<td>1.0683</td>
<td>1.2713</td>
</tr>
<tr>
<td>minimum (%)</td>
<td>-1.4588</td>
<td>-1.4586</td>
<td>-1.2030</td>
</tr>
<tr>
<td>mean (%)</td>
<td>0.1229</td>
<td>0.1234</td>
<td>0.1627</td>
</tr>
<tr>
<td>variance (×1.0e-4)</td>
<td>0.2639</td>
<td>0.2688</td>
<td>0.2957</td>
</tr>
<tr>
<td>skewness</td>
<td>-0.4449</td>
<td>-0.4343</td>
<td>-0.1873</td>
</tr>
</tbody>
</table>

We can find from Table 6 that the optimal portfolios got under the mean-wCVaR and mean-wrCVaR models have similar performance, while the optimal portfolio got under the mean-mwCVaR model provides much higher return rate than the other two in terms of the maximum and
mean. Although the variance of the return series obtained with the mean-mwCVaR model is a bit larger than those obtained with the other two models, the distribution of the out-of-sample return rates got with the mean-mwCVaR model is less left-skewed and it has much less extreme loss, as the value of skewness shows. In modern risk management, controlling extreme losses is as important as earning high returns. Hence, the above out-of-sample result illustrates that the mean-mwCVaR model is more suitable for the investor to earn robust and superior profits, while the mean-wCVaR and mean-wrCVaR models are more suitable for the investor to reduce the investment volatility.

To illustrate the above results more intuitively, we compute the out-of-sample accumulated wealth processes under the obtained three return rate series and draw them in Figure 2, which again show the significantly superior out-of-sample performance of the mean-mwCVaR model.

In order to examine the out-of-sample performances of the three models under different regimes, we divide the out-of-sample period into three parts with respect to the three regimes. And we estimate the mean and variance of the return rates in the corresponding part with respect to the three regimes, respectively. The results are listed in Table 7. We can see from this table that, under
the consolidation regime, the performances of the optimal portfolios got under the three models are similar. Under the bear regime, the optimal portfolio obtained under the mean-wrCVaR model or mean-mwCVaR model performs better than that under the mean-wCVaR model. That is because the two models take the regime switching information into consideration and control the risk under the worst regime. Under the bull regime, the optimal portfolio obtained under the mean-mwCVaR model performs much better than those got under the other two models. That is because the mean-mwCVaR model takes the information under all the three regimes into consideration, while the mean-wrCVaR model only controls the risk under the worst regime. From the above results, we can conclude that, the mean-wrCVaR model gives us more robust optimal portfolio than the mean-wCVaR model, while the mean-mwCVaR model can find regime dependent portfolios which are suitable for varied market environments. Both of the two models are good supplements to the mean-wCVaR model.

Table 7  Statistics of out-of-sample return rates under different regimes and different models

<table>
<thead>
<tr>
<th>model</th>
<th>regime</th>
<th>bull</th>
<th>consolidation</th>
<th>bear</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>weight (weeks)</td>
<td>69</td>
<td>6</td>
<td>25</td>
</tr>
<tr>
<td>mean-wCVaR</td>
<td>mean (%)</td>
<td>0.1421</td>
<td>0.2729</td>
<td>0.0339</td>
</tr>
<tr>
<td></td>
<td>variance (×1.0e-4)</td>
<td>0.2455</td>
<td>0.3133</td>
<td>0.3129</td>
</tr>
<tr>
<td>mean-wrCVaR</td>
<td>mean (%)</td>
<td>0.1370</td>
<td>0.2401</td>
<td>0.0579</td>
</tr>
<tr>
<td></td>
<td>variance (×1.0e-4)</td>
<td>0.2542</td>
<td>0.3230</td>
<td>0.3129</td>
</tr>
<tr>
<td>mean-mwCVaR</td>
<td>mean (%)</td>
<td>0.1938</td>
<td>0.2588</td>
<td>0.0535</td>
</tr>
<tr>
<td></td>
<td>variance (×1.0e-4)</td>
<td>0.2902</td>
<td>0.3421</td>
<td>0.3087</td>
</tr>
</tbody>
</table>

5. Conclusions

Different from the current adjustable robust approach, we propose the multi-period worst-case risk measure. It provides a robust estimation of the multi-period risk and is an efficient complement to the adjustable robust approach. Furthermore, we take the regime switching into consideration and introduce two forms of multi-period robust risk measures with regime switching. The corresponding
multi-period portfolio selection models under proposed new risk measures are established, and their efficient solution is investigated. The numerical experiments demonstrate the practicality, efficiency and robustness of our new robust multi-period investment models with regime switching.

In this paper, we have specifically examined the wCVaR, wrCVaR and mwCVaR risk measures and their application to multi-period portfolio selection problems, mainly due to the good property of CVaR and the tractability of the corresponding portfolio selection problem. An interesting topic for future research is to apply the introduced multi-period robust risk measures with other forms of single-period risk mappings, such as LPM, two-sided coherent risk measure, and consider their application to multi-period portfolio selection problems. Of course, it is also worthwhile constructing other complex and flexible uncertainty sets according to the available information, and further discussing the tractability of the resulting multi-period robust risk measures and corresponding portfolio selection problems.

Endnotes


Acknowledgments

This research was supported by the National Natural Science Foundation of China under Grant Numbers 70971109 and 71371152.

References


