Time Consistent Risk Measure Under Stopping Time Framework and its Applications*

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February 13, 2015

Abstract

For financial investment with a pre-specified investment target, the risk can be regarded as the uncertainty of the earliest target reaching time. As a refinement of this new perspective of risk measure, we introduce a new risk measure which is the extra time cost between the current time and the earliest time that the investment achieves the target or beats the benchmark at some confidence level. The new risk measure is based on a probabilistic constraint and can be seen as a generalization of the payback period method in corporate financial management. We show that the new risk measure has good properties such as normalization, monotonicity, positive homogeneity and weak time consistency. We also discuss the relationship between the new risk measure and variance and VaR. Then we consider two applications of the new measure, the theoretical application is the introduction of arbitrage with probability, which extends the usual definition of arbitrage opportunity and can help the investor differentiate markets which are arbitrage free; the practical application is the multi-period portfolio selection. For the chance constrained multi-period investment decision problem constructed under the new measure, we use the Chebyshev inequality

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*This research was supported by the National Natural Science Foundation of China (Grant Numbers 70971109 and 71371152).
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to approximate the probabilistic constraint and transform the original problem into a problem involving means and variances at individual periods. To overcome the non-separability of the variance operator, we construct an auxiliary problem and finally obtain the analytical solution of the multi-period portfolio selection problem by generalizing relevant solution techniques in the literature. Finally, we numerically demonstrate the practicality and efficiency of the new measure and the multi-period portfolio section method.

**Keyword:** risk measure, stopping time, chance constraints, time consistency, arbitrage opportunity, portfolio selection

## 1 Introduction

The prerequisite for finding a robust and efficient investment policy is the proper description of the investment risk. The measuring and controlling of financial risk has been a very important issue in modern financial management, hundreds of risk measures have been proposed in the literature. Traditional risk measures focus on the uncertainty of investment results, which are often expressed in the form of deviation from the mean or some investment target. Typical examples include variance [19], mean-absolute moments [16] and deviation measures [28]. Due to a series of research by Föllmer, Schied, Artzner and et al. [1, 2, 10], coherent risk measures and convex risk measures have attracted much attention since 1999. These kinds of measures consider the distance of investment results from the so-called acceptance set. Typical coherent (convex) risk measures are CVaR [27] and the two-sided risk measure [7]. Since investors mainly care about the investment loss in practice, different forms of safety or down-side risk measures have been proposed, from the Roy’s safety-first criterion to semi-variance [20], lower partial moment [3], VaR [23] and so on. In reality, most investment problems are medium-term or long-term decision making problems, the above risk measures are naturally extended to the multi-period or dynamic situation. One common feature of all these risk measures is that they measure the financial investment risk during a pre-specified investment horizon.

Nevertheless, many researchers argue that the investment horizon may be uncertain. In as early as 1971, Merton [22] addressed a dynamic optimal portfolio selection
problem for an investor retiring at an uncertain time. The issue of uncertain investment exiting time has been recently considered in different financial problems such as the pricing problem [4], the optimal consumption and investment problem [8, 24] and the robust portfolio selection problem [13]. Actually, the idea of uncertain exiting time is also useful for the construction of new risk measure. In practical investment problems, investors and/or managers often have a definite investment target, they mainly care about when their investment target can be achieved. Unfortunately, due to the stochastic volatility of security markets and constantly change of economic and financial information, the earliest target reaching time is random, it then becomes the investment risk investors and/or managers want to control. This earliest target reaching problem is similar to the payback period rule in corporate financial management, the payback period method is often used by large and sophisticated companies as a screen for making the myriad of minor investment decisions they continually face [29].

In a stochastic market, the earliest target reaching time problem is essentially a stopping time problem. This new perspective of risk measure has recently attracted more and more attention because it can describe the investor’s greedy psychology, hoping to achieve his/her investment target as early as possible. For example, Karantzas and Wang [14] considered the utility maximization problem with discretionary stopping; based on the discussion of stopping time, Martellini and Urosevic [21] proposed the concept of exit time risk and discussed its relationship with the mean-variance problem; Browne [5] established four models to find the optimal investment strategies in terms of maximizing the probability beating the benchmark, minimizing the expectation of random stopping time, maximizing the expected discounted reward and minimizing the discounted penalty paid, respectively. Moreover, Kardaras and Platen [15] addressed the problem of minimizing the expected time to upcross a certain wealth level, which can be viewed as a neutral risk. For all the risk measures constructed under the earliest target reaching framework, we have not seen any discussion about the time consistency, which is now regarded as a prerequisite that any reasonable dynamic risk measure should satisfy.

We can deduce from the recent literature that using the concept of stopping time to construct new dynamic risk measures is an open and valuable topic worthwhile further investigation. Unfortunately, current risk measures based on the earliest target
reaching time or stopping time, see Browne [5] for instance, simply pay attention to the time issue, they did not consider the resulting wealth loss because of the delay in target reaching. Hence, in this paper, we propose a new risk measure based on the stopping time, and then apply it to the multi-period portfolio selection problem under the mean-risk framework. After a series of model transformations, we derive the analytical investment policy.

Our new risk measure is defined in two steps. Firstly, different from studies fully focusing on the stopping time [5], we use a probabilistic constraint to describe the investment target reaching under a specific confidence level and determine the stopping time; we then define the new risk measure as the discounted time cost from the stopping time to the planned time, which measures the wealth loss caused by the delay in achieving the investment target. Therefore, the risk measure extends both the traditional Roy’s safety-first criterion [30] and the current measures based on stopping time. Due to this, the dynamic portfolio selection model under the new risk measure considers more information, and can thus find more robust and efficient investment policy than other dynamic investment models. What’s more important, we show that the new risk measure has good properties and especially, it is time consistent.

Last but not least, based on the idea of the new risk measure, we introduce a new concept of arbitrage with probability, which extends the usual definition of arbitrage opportunity. One important advantage of the arbitrage with probability is that it can help the investor differentiate markets which are arbitrage free, and thus help the investor choose a suitable security market to invest.

The rest of this paper is organized as follows. In Section 2, we define the probabilistic constrained stopping time and introduce the new risk measure. In Section 3, we discuss properties of the new risk measure, especially its time consistency. In Section 4, we examine the relationship between the new risk measure and two classic risk measures, variance and VaR. In Section 5, we propose a new concept, the arbitrage with probability, compare it with usual arbitrage opportunity and show its function in distinguishing markets. In Section 6, we apply the new risk measure to the multi-period portfolio selection problem and demonstrate how to derive the explicit investment policy through a series of model transformations. Section 7 numerically illustrates the practicality and efficiency of the new risk measure and multi-period portfolio section.
method from different aspects. Section 8 completes the article by the conclusions.

2 Stopping time and new risk measure

In this paper, we consider an investment problem with a long enough investment period $[0, T_0]$, here $T_0$ is a sufficiently large number, $T_0$ could be the furthest time in the future the investor pays attention to. The stochastic wealth at time $t \in [0, T_0]$ is denoted as $x_t$. There exists a risk-less asset whose payback in the time period $[s, t]$ is $R_{s,t}$, which means that if one invests one unit of cash in the risk-less asset at time $s$, he/she would receive $R_{s,t}$ units of cash at time $t$. We assume $R_{s,t} \geq 1$ if $s < t$; $R_{s,t} = 1$ if $s = t$; $R_{s,t} \in (0, 1]$ if $s > t$; and $R_{s,t}R_{t,\tau} = R_{s,\tau}$.

The investor stands at time 0 with an initial wealth $x_0$, and tries to gain wealth at least $C_t$, the target wealth, by time $t$. Due to the randomness of the investment result, the investor’s target might fail to achieve. We suppose the investor measures his/her goal satisfaction under the safety-first framework. Concretely, he/she wants the probability $P(x_t \geq C_t)$ to be no less than $\alpha$ at time $t$, i.e.,

$$P(x_t \geq C_t) \geq \alpha. \quad (2.1)$$

However, as we explained in the introduction, due to the stochastic volatility of the security market, the investor can not ensure (2.1) at time $t$. To describe this phenomenon, we introduce the following definition, which can be seen as a variant of usual stopping time.

**Definition 2.1** (Stopping time). Given a confidence level $\alpha$, and an investment benchmark $C_t$, the stopping time $T$ of the stochastic wealth process $\{x_t\}$ is the earliest time when the probability constraint (2.1) holds, that is,

$$T = \min\{s|P(x_s \geq C_t) \geq \alpha, \ 0 \leq s \leq T_0\}. \quad (2.2)$$

**Remark 2.1.** We assume the stopping time $T$ in (2.2) always exists. If this stopping time can not be found in the long enough time period $[0, T_0]$, it means there is no satisfactory investment opportunity in the market for this kind of investor.
Of course, the delay in the satisfaction of the investment goal is caused by the market uncertainty. It is this delay, \( T \), that is defined as the investment risk in [5] and [15]. Here, we extend this idea by considering not only the delay in the investment target striking time, but also the resulting wealth loss. Specifically, such a delay brings some "time cost" because, otherwise, the investor can invest the money he/she supposed to earn at time \( t \) in the risk-less asset during the delayed time period and accumulate more wealth. The time cost here reflects the influence of market volatility on the earnings of investment, which can naturally be viewed as the investment risk. Combining the above two factors, we introduce the following new risk measure.

**Definition 2.2** (New risk measure). *Given a confidence level \( \alpha \) and an investment benchmark \( C_t \), the risk measure at time period \([0, t]\), with respect to the information available by time 0, is defined as the difference between \( C_t \) and the discounted time cost from the stopping time \( T \) to \( t \):

\[
\rho_t(x, C_t) = C_t - \frac{C_t}{R_{t,T}},
\]

where \( T \) is defined in (2.2).*

Figure 1 illustrates the meaning of the stopping time, the corresponding time cost, as well as the new risk measure.

It is not true until the stopping time \( T \) that the probability for the wealth being at least \( C_t \) is greater than or equal to the confidence level \( \alpha \). \( T - t \) is called the extra time for the investment achieving the pre-specified investment target \( C_t \). \( \rho_t(x, C_t) \) measures time cost for the cash flow \( C_t \) which should be realized at time \( t \), instead of the later time \( T \).

The new risk measure can be defined not only at the initial time 0, but also dynamically at any time \( t_0 \in [0, t] \) with a time-varying benchmark. This gives us the following dynamic version of the new risk measure in the sense of [2, 31].

**Definition 2.3** (New dynamic risk measure). *Given a confidence level \( \alpha \) and an investment benchmark \( C_{t_0,t} \) which is adapted to \( \mathcal{F}_{t_0} \), the conditional risk mapping at time period \([t_0, t]\), with respect to the information available until time \( t_0 \), is defined as*
Figure 1: Stopping time, time cost and risk measure

the difference between $C_{t_0,t}$ and the discounted time cost from the stopping time $T$ to $t$:

$$
\rho_{t_0,t}(x, C_{t_0,t}) = C_{t_0,t} - \frac{C_{t_0,t}}{R_{t,T}},
$$

where

$$
T = \min\{s | P(x_s \geq C_{t_0,t} | \mathcal{F}_{t_0}) \geq \alpha, \ 0 \leq s \leq T_0\}.
$$

The sequence of the risk mappings forms the corresponding dynamic risk measure.

Here, at each time point $t_0$, $\mathcal{F}_{t_0} \subset \mathcal{F}$ is the set of events corresponding to the information available until $t_0$.

3 Properties of the new risk measure

We discuss in this section basic properties of the new risk measure, such as its normalization, monotonicity, positive homogeneity, convexity and time consistency.

When studying static properties of the new risk measure, we use Definition 2.2, from which we know that the value of the new risk measure relies on the choice of the target wealth $C_t$ to a great extent. Therefore, when we discuss the specific property of the new risk measure in the general case, we also describe the corresponding property
for the following three commonly adopted target wealths: $C_t = R_{0,t}x_0$, which means the investor just wants to beat the risk-less return; $C_t = E[x_t]$, which means the investor wants to gain at least the average revenue; and $C_t = (1+c)x_0$, which means the investor wants to achieve a constant return rate.

Normalization describes the property of a risk function when all the wealth is invested in a risk-less asset. Generally, under the normalization condition, risk measures which measure the uncertainty of returns, such as variance, the $p$th order central moment, should be equal to 0; while risk measures defined as the distance to the acceptance set, such as CVaR and other coherent risk measures, should be equal to a constant determined by the shape of the acceptance set. The following property illustrates that, when $C_t = (1+c)x_0$, the normalization of the new risk measure is similar to that of coherent risk measures; when $C_t = R_{0,t}x_0$ or $C_t = E[x_t]$, the normalization of the new risk measure is the same as that for risk measures with respect to the investment uncertainty.

Property 3.1 (Normalization). If all the wealth is invested in a risk-less asset at time period $[t_0, t]$, we have

a) when $C_t = R_{0,t}x_0$, $\rho_t(x, C_t) = 0$,

b) when $C_t = E[x_t]$, $\rho_t(x, C_t) = 0$,

c) when $C_t = (1+c)x_0$, $\rho_t(x, C_t) < 0$ if $1 + c < R_{0,t}$, $\rho_t(x, C_t) > 0$ if $1 + c > R_{0,t}$, and $\rho_t(x, C_t) = 0$ if $1 + c = R_{0,t}$.

Monotonicity describes the order relationship between the loss/wealth and the risk value. It guarantees that a large loss (less wealth) corresponds to a high risk. The dynamic monotonicity is a key property of conditional risk functions [31]. Here, we describe monotonicity with respect to the cumulated wealth, which is stronger than that defined on the wealth flow (like that in [31]). The difference between these two versions can be found in [11]. The following proposition shows that the new risk measure is monotonic for any kind of wealth target.

Proposition 3.1. Let $x = \{x_t\}$ and $y = \{y_t\}$ be two cumulated wealth processes, $C_x^r$ ($C_y^r$) is the target wealth set for $x_t$ ($y_t$) at time $t$, then $\rho_t(x, C_x^r) \leq \rho_t(y, C_y^r)$ holds if $x_s \geq y_s$ for all $s \in [0, T_0]$, and $C_x^r \leq C_y^r$. 

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Proof. From the assumption \( x_s \geq y_s \) for all \( s \in [0, T_0] \), we know that at any time \( s \), \( 0 \leq s \leq T_0 \), \( P(y_s \geq C^y_t) \leq P(x_s \geq C^x_t) \). From the assumption \( C^x_t \leq C^y_t \), we know \( P(x_s \geq C^y_t) \leq P(x_s \geq C^x_t) \). Hence, \( P(y_s \geq C^y_t) \leq P(x_s \geq C^x_t) \) for all \( s \in [0, T_0] \), and

\[
T_y = \min\{s | P(y_s \geq C^y_t) \geq \alpha\} \geq T_x = \min\{s | P(x_s \geq C^x_t) \geq \alpha\}.
\]

This means the stopping time of the process \( x \) is always at least that of \( y \). Because \( C^x_t \leq C^y_t \), we have \( C^x_t (1 - 1/R_{t,T_x}) \leq C^y_t (1 - 1/R_{t,T_y}) \), that is, \( \rho_t(x, C^x_t) \leq \rho_t(y, C^y_t) \).

For specific forms of target wealths, we can describe the concrete monotonicity as follows.

**Property 3.2** (Monotonicity). Consider two cumulated wealth processes \( x = \{x_t\} \) and \( y = \{y_t\} \). For \( C^x_t = R_{0,t} x_0 \), \( C^y_t = R_{0,t} y_0 \), or \( C^x_t = (1 + c)x_0 \), \( C^y_t = (1 + c)y_0 \), we have \( \rho_t(x, C^x_t) \leq \rho_t(y, C^y_t) \) if \( x_0 = y_0 \) and \( x_s \geq y_s \) for all \( s \in (0, T_0] \).

However, when \( C^x_t = E[x_t] \) and \( C^y_t = E[y_t] \), the new risk measure is not monotonic, because the time-varying benchmark increases with respect to the wealth process.

In practice, investors often care about the change of risk when the wealth increases in proportion. Under strong liquidity assumption, the risk function is usually required to change in proportion to the wealth level, this is the so-called positive homogeneity. We see from (2.2) that the stopping time \( T \), thus the \( R_{t,T} \) in (2.3) is invariant with respect to the proportional change of the wealth level and the target return, it is then easy to deduce from (2.3) that the new risk measure satisfies positive homogeneity, concretely.

**Proposition 3.2.** For any wealth process \( x \) and constant \( \lambda > 0 \), \( \rho_t(\lambda x, \lambda C^x_t) = \lambda \rho_t(x, C^x_t) \).

Applying Proposition 3.2 to typical forms of wealth targets, we can derive the following property about the satisfaction of positive homogeneity.

**Property 3.3** (Positive homogeneity). Considering a wealth process \( x \),

a) if \( \lambda C^x_t = C^x_t \), for example \( C^x_t = E[x_t] \), then \( \rho_t(\lambda x, C^x_t) = \lambda \rho_t(x, C^x_t) \) for any \( \lambda > 0 \),
b) if the benchmark $C_t$ is a pre-set constant, for example $C_t = R_{0,t}x_0$ or $C_t = (1 + c)x_0$, then

$$\rho_t(\lambda x, C_t) \geq \lambda \rho_t(x, C_t)$$

if $0 < \lambda \leq 1$, $\rho_t(\lambda x, C_t) \leq \lambda \rho_t(x, C_t)$ if $\lambda \geq 1$, and

$$\rho_t(\lambda x, C_t) = \lambda \rho_t(x, C_t)$$

if and only if $\lambda = 1$.

The value of the new risk measure implicitly depends on the stopping time $T$, which is determined through a probabilistic constraint. Due to the non-convexity of general probabilistic constraints, it is impossible for us to establish the convexity of the new risk measure. Nevertheless, if the probabilistic density function of $x_t$ satisfies some conditions, we can show some generalized convexity of $\rho_t(x, C_t)$. For example, by utilizing Theorem 2.10 in [25], we can show that $\rho_t(x, C_t)$ is quasi-convex with respect to $\{x_t\}$.

**Property 3.4** (Quasi-convexity). Let $f(x_t)$ be the probability density function of $x_t$ at time $t$. If $f^{-1}(x_t)$ is convex, then $\rho_t(x, C_t)$ is quasi-convex with respect to $x = \{x_t\}$.

Examples of the random variable whose probability density function satisfies the above convexity include that random variables following the Cauchy distribution or Pareto distribution.

We have discussed some static properties of the new risk measure, now we turn to its time consistency, which is thought to be an important property for describing the dynamic relationship among risks at different periods.

The earliest notion about time consistency is now called strong time consistency, whose main idea can be described as follows: for two investment positions $X$ and $Y$, if $X$ is riskier than $Y$ under a specific risk measure from the perspective of some future time, then $X$ is riskier than $Y$ under the same measure from today’s perspective. It is demonstrated in [31] that under some assumptions, any strong time consistent multi-period risk measure can be represented in a recursive form. Hence, the requirement of strong time consistency may lead to strict limitations on possible forms of multi-period risk measure. For this reason, a weak notion of time consistency is proposed in [6] and [33]: if some position is accepted (rejected) for any scenario tomorrow, it should be already accepted (rejected) today, this is the so-called weak time consistency. The following example shows that the new risk measure is not strong time consistent, but it satisfies the weak time consistency. Here, we adopt Definition 2.3, and assume the
confidence level and the investment benchmark $C_{t_0,t}$ do not change as $t_0$ varies. We denote them as $\alpha$ and $C_t$, respectively, for simplicity.

Example 3.1.

Consider two investment wealth processes $x = \{x_t\}$ and $y = \{y_t\}$, which are adapted to a binomial scenario tree with six periods. We use $I_i^t$ to denote the $i$th node at time period $t$. At the root node $I_0$, the probability for switching to $I_1^0$ is 80%, and the probability for turning to $I_2^0$ is 20%.

Consider two-period risk measures $\rho_{0.2}(x, C_2)$ and $\rho_{0.2}(y, C_2)$. We assume that the target wealth is $C_2 = 100$, the confidence level is $\alpha = 90\%$, and the risk-less payback is $R_{t_1,t_2} = 1.05^{(t_2-t_1)}$. Under the conditions of $I_1^1$ and $I_2^1$, the probabilities for $x_t^i \geq C_2$ and $y_t^i \geq C_2$ at the six periods are given in the first two sections of Table 1. From the law of total probability,

$$P(x_t \geq C_2 | I_0) = P(I_1^1)P(x_t \geq C_2 | I_1^1) + P(I_2^1)P(x_t \geq C_2 | I_2^1),$$

$$P(y_t \geq C_2 | I_0) = P(I_1^1)P(y_t \geq C_2 | I_1^1) + P(I_2^1)P(y_t \geq C_2 | I_2^1),$$

we can determine probabilities for $x_t^i \geq C_2$ and $y_t^i \geq C_2$ under the conditions of $I_0$, which are shown in the last sections of Table 1.

<table>
<thead>
<tr>
<th>Table 1: Target reaching probabilities at different periods and stopping time</th>
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<tbody>
<tr>
<td>$t$</td>
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<tr>
<td>-------</td>
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<tr>
<td>$P(x_t \geq C_2</td>
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<td>$P(y_t \geq C_2</td>
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<td>$P(x_t \geq C_2</td>
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<td>$P(y_t \geq C_2</td>
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</tbody>
</table>

From the probabilities in Table 1, it is not hard to determine through (2.2) the stopping times under the conditions $I_1^1$, $I_2^1$, and $I_0$, respectively, which are shown in the last column of Table 1. Furthermore, we have the values of the dynamic risk measure at period 1 and period 0 as follows, respectively.

$$\rho_{1,2}^1(x, C_2) = C_2(1 - \frac{1}{R_{2,3}}) = 4.762, \quad \rho_{1,2}^0(y, C_2) = C_2(1 - \frac{1}{R_{2,3}}) = 4.762,$$
\[
\rho_{1,2}(x, C_2) = C_2(1 - \frac{1}{R_{2,5}}) = 13.616, \quad \rho_{1,2}(y, C_2) = C_2(1 - \frac{1}{R_{2,4}}) = 9.297,
\]
\[
\rho_{0,2}(x, C_2) = C_2(1 - \frac{1}{R_{2,3}}) = 4.762, \quad \rho_{0,2}(y, C_2) = C_2(1 - \frac{1}{R_{2,4}}) = 9.297.
\]

These results show that \(\rho_{i,2}(x, C_2) \geq \rho_{i,2}(y, C_2), \ i = 1, 2\), while \(\rho_{0,2}(x, C_2) \leq \rho_{0,2}(y, C_2)\). Example 3.1 tells us that \(\rho_{0,t}(x, C_t)\) is not strong time consistent \([2, 32]\).

However, if we compare \(x\) or \(y\) with the deterministic process, 0, we can see that these two-period risk measures and their conditional risk measures at period 1 are all positive. Furthermore, we consider 1-, 2-, 3-, 4-, 5- and 6-period risk measures from the perspective of both period 0 and period 1. We still assume the confidence level and the investment benchmark is not changed as \(t_0\) varies and let \(C_t = 100, t = 1, ..., 5\).

Table 2 shows the signs of the risk measures of the random wealth process \(x\) or \(y\), with different period numbers.

<table>
<thead>
<tr>
<th>(t)</th>
<th>(t=1)</th>
<th>(t=2)</th>
<th>(t=3)</th>
<th>(t=4)</th>
<th>(t=5)</th>
<th>(t=6)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\rho_{1,t}(x, C_t))</td>
<td>&gt; 0</td>
<td>&gt; 0</td>
<td>= 0</td>
<td>&lt; 0</td>
<td>&lt; 0</td>
<td>&lt; 0</td>
</tr>
<tr>
<td>(\rho_{2,t}(x, C_t))</td>
<td>&gt; 0</td>
<td>&gt; 0</td>
<td>&gt; 0</td>
<td>&gt; 0</td>
<td>= 0</td>
<td>&lt; 0</td>
</tr>
<tr>
<td>(\rho_{0,t}(x, C_t))</td>
<td>&gt; 0</td>
<td>&gt; 0</td>
<td>= 0</td>
<td>&lt; 0</td>
<td>&lt; 0</td>
<td>&lt; 0</td>
</tr>
<tr>
<td>(\rho_{1,t}(y, C_t))</td>
<td>&gt; 0</td>
<td>&gt; 0</td>
<td>= 0</td>
<td>&lt; 0</td>
<td>&lt; 0</td>
<td>&lt; 0</td>
</tr>
<tr>
<td>(\rho_{2,t}(y, C_t))</td>
<td>&gt; 0</td>
<td>&gt; 0</td>
<td>&gt; 0</td>
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<td>&lt; 0</td>
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<tr>
<td>(\rho_{0,t}(y, C_t))</td>
<td>&gt; 0</td>
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<td>= 0</td>
<td>&lt; 0</td>
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</tbody>
</table>

From Table 2, we can observe that, if the values of the conditional risk at period 1 are less than or equal to 0 in any possible scenario, the value of risk at the current period is also less than or equal to 0. The above phenomenon means that the new risk measure satisfies the weak time consistency \([6]\). Actually, we have:

**Property 3.5** (Weak time consistency). Assume the confidence level and the investment benchmark \(C_t\) do not change with \(t_0\). If \(\rho_{s+1,t}(x, C_t) \leq 0\), then \(\rho_{s,t}(x, C_t) \leq 0\), here \(0 \leq s < t \leq T_0\).

*Proof.* \(\rho_{s+1,t}(x, C_t) \leq 0\) means that \(P(x_t \geq C_t|I_{s+1}) \geq \alpha\) holds for any information set \(I_{s+1}\) available at period \(s + 1\), \(I_{s+1} \supseteq I_s\). From the law of total probability, we have

\[
P(x_t \geq C_t|I_s) = E[P(x_t \geq C_t|I_{s+1})|I_s] \geq E[\alpha|I_s] = \alpha,
\]

which means in return that \(\rho_{s,t}(x, C_t) \leq 0\). \(\square\)
The above discussion tells us that the satisfaction of some property relies on the choice of target wealth. Hence, it is important that the investor chooses the target wealth properly in order to construct a good risk measure. Moreover, the proper choice of target wealth also affects the existence of arbitrage opportunity with probability and the boundedness of the corresponding portfolio selection problem, which will be demonstrated in Section 5 and Section 6, respectively. Before that, we first discuss the relationship between the new risk measure and two classic risk measures, variance and VaR.

4 Relationship with variance and VaR

Although VaR and the proposed risk measure are both connected with the Roy’s safety-first principle, they are different. Given a confidence level, VaR statically measures the smallest quantile satisfying the probabilistic constraint at the period

\[
V aR_{\alpha}(x_t) = \min \{ l : P(-x_t > l) \leq 1 - \alpha \};
\]

(4.1)

on the other hand, given a confidence level and a target wealth, the risk function dynamically monitors the earliest time when the probabilistic constraint is satisfied and we then take the loss of wealth caused by the time delay in achieving the target wealth as the risk value. As for their further relationship, we have the following conclusion.

Theorem 4.1. For any wealth process \( x = \{ x_t \} \) and the confidence level \( \alpha \), we have

a) the risk measure (2.3) with benchmark \( C_t = -V aR_{\alpha}(x_t) \) is less than zero, i.e., \( \rho_t(x, -V aR_{\alpha}(x_t)) \leq 0 \);

b) \( V aR_{\alpha}(x_t) \) is the minimum benchmark making the risk measure (2.3) equal to zero, i.e., \( V aR_{\alpha}(x_t) = -\max\{ c | \rho_t(x, c) = 0 \} \);

c) \( \min_{s \in [0,t]} V aR_{\alpha}(x_s) = -\max\{ c | \rho_t(x, c) \leq 0 \} \).

Proof. a), we know from (4.1) that

\[
V aR_{\alpha}(x_t) = \min \{ l : P(-x_t > l) \leq 1 - \alpha \} = -\max\{ c : P(x_t \geq c) \geq \alpha \},
\]

(4.2)
and hence

\[ P(x_t \geq -\text{VaR}_\alpha(x_t)) \geq \alpha. \]

This means \( t \geq \min\{s | P(x_s \geq -\text{VaR}_\alpha(x_s)) \geq \alpha, 0 \leq s \leq T_0\} \), and correspondingly \( \rho_t(x, -\text{VaR}_\alpha(x_t)) \leq 0 \).

b), \( \rho_t(x, c) = 0 \) implies \( P(x_t \geq c) \geq \alpha \). From (4.2), we know \( \text{VaR}_\alpha(x_t) = -\max\{c | \rho_t(x, c) = 0\} \).

c), we denote \( c' = \max\{c : \rho_t(x, c) \leq 0\} \). \( \rho_t(x, c') \leq 0 \) implies that \( P(x_s \geq c') \geq \alpha \) holds for some \( s \in [0, t] \). For the time \( s \), \( -\text{VaR}_\alpha(x_s) = \max\{c | P(x_s \geq c) \geq \alpha\} \), which implies \( -\text{VaR}_\alpha(x_s) \geq c' \). Due to the existence of \( s \), we have \( \max_{s \in [0, t]}(-\text{VaR}_\alpha(x_s)) \geq c' \), which is equivalent to \( \min_{s \in [0, t]} \text{VaR}_\alpha(x_s) \leq -\max\{c | \rho_t(x, c) \leq 0\} \).

On the other hand, we know from (4.2) that for any time \( s \in [0, t] \), \( P(x_s \geq -\text{VaR}_\alpha(x_s)) \geq \alpha \). This means

\[ t \geq s \geq \min\{s' | P(x_{s'} \geq -\text{VaR}_\alpha(x_{s'})) \geq \alpha, 0 \leq s' \leq T_0\}, \]

and correspondingly \( \rho_t(x, -\text{VaR}_\alpha(x_s)) \leq 0 \). Hence, we have \( -\text{VaR}_\alpha(x_s) \leq \max\{c | \rho_t(x, c) \leq 0\} \), for any \( s \in [0, t] \), which means that \( \max_{s \in [0, t]}(-\text{VaR}_\alpha(x_s)) \leq \max\{c | \rho_t(x, c) \leq 0\} \).

This is equivalent to \( \min_{s \in [0, t]} \text{VaR}_\alpha(x_s) \geq -\max\{c | \rho_t(x, c) \leq 0\} \).

The above two inequalities show the conclusion in c).

From Theorem 4.1, we know that if an investor takes the value of VaR at time \( t \) as the investment benchmark, he/she will undertake no risk in the sense of the new risk measure. If the investor hopes to gain a wealth larger than VaR, he/she will suffer a positive risk. Therefore, VaR corresponds to the critical point where the new risk measure turns to positive. This critical point is very useful for examining the existence of possible profit opportunity in a security market, which inspires us to introduce the concept of arbitrage with probability. We will discuss this in detail in section 5.

It is interesting to compare the new risk measure with the classic risk measure, variance, since the value of the new risk measure is closely related with the market volatility. We examine the relationship between the new risk measure and volatility, thus variance, by considering a typical diffusion process.
We assume the wealth process \( \{x_t\} \) follows a geometric Brownian motion with the drift parameter \( r \) and the diffusion parameter \( \delta \), i.e.,

\[
(4.3) \quad dx_t = r x_t dt + \delta x_t dW_t,
\]

where \( W_t \) is the standard Brownian motion. And there is a risk-less wealth process \( x^0 \), which satisfies

\[
(4.4) \quad dx^0_t = r^0 x^0_t dt.
\]

Solving the differential equation (4.3) gives

\[
(4.5) \quad x_t = x_0 \exp \left\{ \left( r - \frac{\delta^2}{2} \right) t + \delta W_t \right\}.
\]

Substituting (4.5) into the probabilistic constraint (2.1), we have

\[
(4.6) \quad P \left( x_0 \exp \left\{ \left( r - \frac{\delta^2}{2} \right) T + \delta W_T \right\} \geq C_t \right) \geq \alpha,
\]

or equivalently,

\[
(4.7) \quad P \left( Y_T \geq \ln \frac{C_t}{x_0} \right) \geq \alpha,
\]

where \( Y_T = (r - \frac{\delta^2}{2}) T + \delta W_T \), and hence \( Y_T \sim N \left( (r - \frac{\delta^2}{2}) T, \delta^2 T \right) \).

Let \( B_T = \frac{Y_T - (r - \frac{\delta^2}{2}) T}{\delta \sqrt{T}} \), which follows the standard normal distribution. The probabilistic constraint (4.7) is equivalent to

\[
(4.8) \quad P \left( B_T \geq \frac{\ln \frac{C_t}{x_0} - (r - \frac{\delta^2}{2}) T}{\delta \sqrt{T}} \right) \geq \alpha.
\]

We denote the cumulated density function of the standard normal distribution by \( \phi(x) \), and its inverse function by \( \phi^{-1}(\alpha) \). (4.8) can be reformulated as

\[
(4.9) \quad \frac{\ln \frac{C_t}{x_0} - (r - \frac{\delta^2}{2}) T}{\delta \sqrt{T}} \geq \phi^{-1}(\alpha).
\]

Then it is easy to see that, when \( \Delta = \delta^2 \phi^{-1}(\alpha)^2 + (4r - 2\delta^2) \ln \frac{C_t}{x_0} < 0 \), the stopping
time $T$ does not exist. When $\Delta \geq 0$, the stopping time $T$ and the corresponding risk value can be explicitly determined as

$$T = \begin{cases} 
\frac{\ln \frac{C_t}{x_0}}{\delta \phi^{-1}(\alpha)}, & \delta = \sqrt{2r}, \\
\frac{-\delta \phi^{-1}(\alpha) + \sqrt{\Delta}}{2r - \delta^2}^2, & \delta < \sqrt{2r}, \Delta \geq 0,
\end{cases}$$

and

$$\rho_t(x, C_t) = C_t - C_t \exp \left\{ r^0 \left[ t - \left( \frac{-\delta \phi^{-1}(\alpha) + \sqrt{\Delta}}{2r - \delta^2} \right)^2 \right] \right\},$$

respectively.

By setting $C_t = R_{0,t} \cdot x_0 = x_0 e^{r^0 t}$, $C_t = E[x_t] = x_0 e^{rt}$ and $C_t = 2x_0$, respectively, we can find that when $\delta$ is near zero, the stopping time decreases with respect to $\delta$; and when $\delta$ is quite large, the stopping time increases with respect to $\delta$. Therefore, compared with a market with no volatility at all, an investor can achieve his/her investment target with probability in a shorter time in a market with some volatility. And when the investor invests in a market with large volatility, he/she would need a longer time to achieve his/her investment target than that in a less volatile market. This phenomenon illustrates that a slight volatility helps reaching the investment target sometime, while a significant volatility might bring some unmanageable disaster.

Meanwhile, the stopping time turns to infinity and the risk value increases slowly when the volatility is large enough. The investor is insensitive to the stopping time when it is large enough can be explained from the discounting in the measure we defined in (2.3). If $T$ is large enough, then $1/R_{t,T} = e^{-r_0(T-t)} \approx 0$. That is also reasonable, investment decisions are made with respect to the investment target in the near future, the earning in the distant future is usually not much concerned by an investor in practice.

5 **Arbitrage with probability**

As we know, one character of modern financial markets is its unpredictable volatility, which makes it difficult, if not impossible, to find and utilize the arbitrage opportunity. On the other hand, many empirical studies [9, 12] tell us that the existence of
arbitrage opportunity is due to the improper selection of scenarios when making investment decisions, which can be avoided by adding new scenarios. These facts inspire us to introduce a more general concept than traditional arbitrage to help investors to select optimal portfolios in security markets. As an interesting theoretical application of the new risk measure, we propose in this section a new concept, arbitrage with probability, which is an extension to usual arbitrage opportunity. Concretely, we have

**Definition 5.1.** For a given confidence level $\alpha$ and target wealth series $C_t$, $t \in [0, T_0]$, if there are an investment period $[t_0, t] \subseteq [0, T_0]$ and a portfolio during the period such that

\begin{equation}
\rho_{t_0, t}(x, C_{t_0, t}) < 0,
\end{equation}

we say there is an arbitrage opportunity with probability $\alpha$ within the market. On the other hand, we call the market arbitrage free with probability $\alpha$, if there do not exist an investment period and a corresponding portfolio such that (5.1) holds.

It is easy to see that (5.1) is equivalent to $T(x, C_{t_0, t}) < t$. This means: if there is a stopping time which is less than the pre-specified investment horizon $t$, the investor can reach his/her investment target in a shorter time than expected. In other words, the investor can obtain his/her expected earnings, with some large probability, in this market. That’s why we say the market has an arbitrage opportunity with some probability for the investor.

Another reason we call the market with property (5.1) a market having an arbitrage with probability $\alpha$ is: the market satisfying (5.1) would lead the corresponding portfolio selection problem becomes unbounded. This implies that the investor may invest as much wealth as possible in risky assets, which is obviously unreasonable. We will explain this point in detail in section 6.

The arbitrage with probability can be seen as an extension to the traditional arbitrage opportunity, which is a special case of Definition 5.1. Concretely, when the target wealth process is chosen as the wealth process of the risk-free asset and $\alpha = 1$, the arbitrage with probability 1 reduces to the traditional arbitrage opportunity. This can be summarized as:
Theorem 5.1. The market has an arbitrage opportunity, if and only if the market has an arbitrage opportunity with probability 1 under the expected target wealth series \( C_t = R_{0,t} x_0, \ t \in [0, T_0] \). The similar equivalent relationship holds for the arbitrage free argument.

Under traditional arbitrage opportunity, we can not differentiate markets which are arbitrage free. Here comes an important and practical advantage of the arbitrage with probability, this new concept can tell us the difference among markets which are arbitrage free under the traditional meaning. We know from Definition 5.1 that the existence of arbitrage with probability is associated with two parameters \( C_t \ (0 \leq t \leq T_0) \) and \( \alpha \). Hence, the smallest \( C_t \) and \( \alpha \) which make a market arbitrage free can be used to describe the profitability of a market.

In order to illustrate the above argument and its practical application, we discuss the relationship between the risk measure (2.3) and the parameters \( C_t \ (0 \leq t \leq T_0) \) and \( \alpha \). Consider two risk measures, denoted as \( \rho_{t_0,t}^1 \) and \( \rho_{t_0,t}^2 \), with respect to investment target wealths \( C_t^1, C_t^2 \) and confidence levels \( \alpha_1, \alpha_2 \), respectively. We use \((C_t^1, \alpha_1)\) and \((C_t^2, \alpha_2)\) to denote these two measures. We say that risk measure \( \rho_{t_0,t}^1 \) is more risk averse than \( \rho_{t_0,t}^2 \) with respect to the wealth process \( x \) if

\[
(5.2) \quad \rho_{t_0,t}^1(x, C_t^1) \geq \rho_{t_0,t}^2(x, C_t^2)
\]

holds for any \( t_0 \) and \( t \) such that \([t_0, t] \subseteq [0, T_0]\). If (5.2) holds for any two target wealth processes, we then say risk measure \( \rho_{t_0,t}^1 \) is more risk averse than \( \rho_{t_0,t}^2 \) with respect to the market.

Suppose there are two investors, investor 1 and investor 2, who adopt the above two risk measures \( \rho_{t_0,t}^1 \) and \( \rho_{t_0,t}^2 \), respectively, to decide their investment strategies. If the risk measure \( \rho_{t_0,t}^1 \) investor 1 chooses is more risk averse, he/she would be more conservative and invests less in risky assets, as the probabilistic constraint under \( \rho_{t_0,t}^1 \) in the portfolio selection model is stricter than that of the constraint under \( \rho_{t_0,t}^2 \), corresponding to investor 2. For an absolutely risk averse investor, he/she could not find any investment opportunity at all in a highly volatile market, because the probabilistic constraint can not be satisfied during the acceptable investment horizon. Therefore, finding an index to reflect the possibility to earn profit in the market under different
degrees of risk aversion is very useful for investors to discriminate and choose suitable security markets. With the help of the arbitrage with probability in Definition 5.1, we define the efficient risk averse level as the pair \((C_t, \alpha)\) such that the market is arbitrage free in the sense of Definition 5.1 with respect to \((C_t, \alpha)\), and there is no \((C'_t, \alpha') > (C_t, \alpha)\) such that the market is arbitrage free in the same sense. Here \((C'_t, \alpha') > (C_t, \alpha)\) means \(C'_t \geq C_t, \alpha' \geq \alpha\), but \(C'_t = C_t\) and \(\alpha' = \alpha\) do not hold at the same time.

Due to the monotonicity of \(\rho_{t_0, t}(\cdot, \cdot)\) with respect to \(C_t\) and \(\alpha\), in practice, we can determine the efficient risk averse level by minimizing \(\rho_{t_0, t}(\cdot, \cdot)\) with respect to \(C_t\) for fixed \(\alpha\), and minimizing \(\rho_{t_0, t}(\cdot, \cdot)\) with respect to \(\alpha\) for fixed \(C_t\). Similar to the mean-variance efficient frontier, we call the set of all efficient risk averse levels the efficient risk averse frontier, which can be used to discriminate the profitability of different markets.

Analogous to the analysis framework at the end of section 4, we illustrate the application of the above theoretical result by assuming the wealth process follows the Brownian motion defined in (4.3). Specifically, we try to determine the efficient risk averse frontiers of American stock market and Chinese stock market, respectively. To simplify calculation, all 500 constituent stocks in the S&P 500 index constitute the candidate stock set of American stock market, and all 300 constituent stocks in the H&S 300 index constitute the candidate stock set of Chinese stock market. The returns and volatilities of these stocks are estimated by using their daily return rates from January 01, 2013 to November 01, 2013, which are downloaded from Yahoo finance\(^1\).

The efficient risk averse frontiers of American stock market and Chinese stock market are shown in Figure 2, from which we can see that the efficient risk averse frontier of American stock market is always above that of Chinese stock market. This is an expected and reasonable result, because of the bull market in US equities in 2013 and the bear market in China equities in 2013. And this means, under the same confidence level, the investor can achieve a higher wealth target in American stock market than that in Chinese stock market during the same time period. Hence, for an investor who aims at gaining stable returns, he/she can achieve his/her investment target earlier and better in American stock market than that in Chinese stock market.

\(^{1}\text{http://finance.yahoo.com}\)
Such a comparison can be made among security markets under different forms of target, which represent different kinds of risk averse investors. After showing how to use our model and/or new concept to select a suitable market to invest, we will apply the new risk measure to the dynamic portfolio selection problem in order to guide investors making their optimal investment strategies in a specific stock market.

6 Multi-period portfolio selection

In practice, most investment problems are long-term decision making problems, which can be formulated as dynamic portfolio selection problems. Nevertheless, due to typical market frictions, a practical way is to describe dynamic investment decision making problems as multi-period, rather than continuous time, portfolio selection problems. Hence, in this section, we consider the application of the new risk measure to the multi-period portfolio selection problem with a long enough investment horizon, say $T_0$ consecutive investment periods. We consider a security market with $n+1$ assets, which include $n$ risky securities, with random rates of return, and one risk-free bond, with a deterministic rate of return. At each period $s$, $r^0_s$ denotes the deterministic return rate of the risk-less bond, and $r^i_s$ denotes the random return rate of risky security $i$. Then, for any two times $s_1$, $s_2$, $s_1 \leq s_2$, the payback during the time period $[s_1, s_2]$
is
\[ R_{s_1, s_2} = (1 + r_{s_1}^0)(1 + r_{s_1+1}^0) \ldots (1 + r_{s_2-1}^0). \]

For notational convenience, we denote \( e_s = 1 + r_s^0, s = 0, 1, \ldots, T_0 - 1, \) and
\[ P_s = [p_1^s, p_2^s, \ldots, p_n^s]' = [(r_1^s - r_s^0), (r_2^s - r_s^0), \ldots, (r_n^s - r_s^0)]', s = 0, 1, \ldots, T_0 - 1, \]
which represents the vector of excess return rates of risky securities.

Suppose an investor joins the market at time 0 with an initial wealth \( x_0 \) and plans to invest his/her wealth in the stock market for \( t \) consecutive periods. At the beginning of each period, the current wealth can be reallocated among the \( n + 1 \) assets. Let \( x_s \) be the total wealth at the beginning of period \( s \), and let \( u_i^s, i = 1, 2, \ldots, n, \) be the cash amount invested in the \( i \)th risky security at period \( s \). Correspondingly, the cash amount invested in the risk-free bond \( u_s^0 \) at the beginning of period \( s \) is equal to \( x_s - \sum_{i=1}^{n} u_i^s \).

We assume that the whole investment process is self-financing, hence we have
\[
x_{s+1} = \sum_{i=1}^{n} (1 + r_i^s)u_i^s + e_s(x_s - \sum_{i=1}^{n} u_i^s) = e_s x_s + P_s' u_s, \quad s = 0, 1, \ldots, T_0 - 1.
\]

Here \( u_s = [u_1^s, u_2^s, \ldots, u_n^s]' \) is the vector of cash amounts invested in risky securities at period \( s \).

The investor wants to seek the best investment strategy during the investment periods \( 0, 1, \ldots, t - 1 \) in order to maximize the weighted difference between the expected value of \( x_t \) and the investment risk \( \rho_t(x, C_t) \). Such an investment strategy selection problem can be described as the following mix-integer chance constrained programming problem, denoted as \( P(1) \) in what follows:
\[
\max \quad E[x_t] - \lambda_t (C_t - \frac{C_t}{R_{t,T}}) \\
\text{s.t.} \quad P(x_T \geq C_t) \geq \alpha, \\
x_{s+1} = e_s x_s + P_s' u_s, \quad s = 0, 1, \ldots, T_0 - 1, \\
T \in \mathbb{Z}, \quad 0 \leq T \leq T_0.
\]

Here \( \lambda_t \geq 0 \) is the risk-aversion coefficient. Considering the investment process in
reality, we assume the possibly stopping time $T$ is a nonnegative integer, which can be regarded as some nonnegative integer multiple of the minimum trading time unit in practice. This assumption is also consistent with the theoretical result in section 5 about the arbitrage with probability.

It is known from the above illustration and discussion in section 5 that, in a market which is arbitrage free with probability $\alpha$, we only need to consider possible stopping times at discrete time points $t, t+1, \ldots, T_0$. Therefore, to solve problem $P(1)$, we can first solve the following programming problem $P(2)$ with given $T$, here $T = t, t+1, \ldots, T_0$,

$$\max \ E[x_t]$$

$$P(2) \quad \text{s.t.} \quad P(x_T \geq C_t) \geq \alpha,$$

$$x_{s+1} = e_s x_s + P'_s u_s, \quad s = 0, 1, \ldots, T - 1,$$

and obtain the optimal investment strategy $u^*(T)$ and the optimal value $G(T)$. Let $\pi(T)$ denote the optimal multi-period investment strategy, and $\Pi(T)$ the set of all the optimal strategies. Then we can find the optimal stopping time $T^*$ of problem $P(1)$ by solving

$$P(3) \quad \max_T \ G(T) - \lambda_t(C_t - \frac{C_t}{R_{t,T^*}})$$

$$\text{s.t.} \quad T \in \mathbb{Z}, \ t \leq T \leq T_0.$$

Finally, $u^*(T^*)$ is the globally optimal dynamic investment strategy, $G(T^*) - \lambda_t(C_t - C_t/R_{t,T^*})$ is the globally optimum value of problem $P(1)$.

In the next four subsections, we will show in detail how to derive the analytical solution to problem $P(2)$.

## 6.1 Sub-problem with fixed stopping time

For the general wealth process $\{x_t\}$, the probabilistic constraint in $P(2)$ makes problem $P(2)$ highly nonlinear and nonconvex. It is impossible to directly find its analytical optimal solution, neither to numerically solve it quickly. For this reason, one practical way to solve chance constrained programs is to approximate the probabilistic constraint by a deterministic constraint, with the help of some probability inequality.
This solution technique has been used in many papers like [17], [30], [34] and [36]. Similarly to these studies, we use the following Chebyshev inequality

\[ P(x_T \leq C_t) \leq \frac{1}{1 + (E[x_T] - C_t)^2/Var(x_T)} \]

to approximate the probabilistic constraint in problem \( P(2) \). Then problem \( P(2) \) can be transformed into:

\[
\begin{align*}
\max & \quad E[x_t] \\
\text{s.t.} & \quad Var(x_T) \leq \kappa(\alpha)(E[x_T] - C_t)^2, \\
& \quad x_{s+1} = x_s + P'_s u_s, \quad s = 0, 1, ..., T - 1,
\end{align*}
\]

where \( \kappa(\alpha) = (1 - \alpha)/\alpha. \)

Let \( G_0(T) \) denote the optimum value of problem \( P_0(2) \) and \( \Pi_0(T) \) the set of optimal solutions. We show that the optimal solution of problem \( P_0(2) \) must lay on the boundary of constraint in \( P_0(2) \).

**Theorem 6.1.** A necessary condition for \( \pi_0 \in \Pi_0(T) \) is

\[
(6.2) \quad Var(x_T) - \kappa(\alpha)(E[x_T] - C_t)^2|_{\pi_0} = 0.
\]

**Proof.** By substituting the self-financing constraint \( x_{s+1} = e_s x_s + P'_s u_s, \ s = 0, 1, ..., T - 1, \) into the objective function \( E[x_t] \), we have

\[
E[x_t] = \prod_{k=0}^{t-1} e_k x_0 + \sum_{s=0}^{t-1} \left( \prod_{k=s+1}^{t-1} e_k \right) E[P'_s u_s],
\]

which is a linear function of \( E[u_s], \ s = 0, ..., T - 1. \) If the constraint in \( P_0(2) \) is not active at \( \pi_0, \) it will lead to an infinite optimum value of problem \( P_0(2) \) and thus an infinite optimal solution. Hence, if \( \pi_0 \) is a finite optimal solution, the constraint in \( P_0(2) \) must be active at \( \pi_0, \) this means \( Var(x_T) - \kappa(\alpha)(E[x_T] - C_t)^2|_{\pi_0} = 0. \)

Because the approximate constraint in \( P_0(2) \) is stronger than the probabilistic constraint in \( P(2) \) in terms of restricting \( x_T, \) the optimal solution of problem \( P_0(2) \) must be a feasible solution of the original problem \( P(2), \) and we have the following conclusion:
Theorem 6.2. For any given $T$, the optimum value of problem $P_0(2)$ is a lower bound to the optimum value of problem $P(2)$, i.e., $G_0(T) \leq G(T)$.

Furthermore, we can move the constraint in $P_0(2)$ into the objective function of problem $P_0(2)$ by using the Lagrange multiplier $w \geq 0$, which deduces the following auxiliary problem:

$$P_1(2) \quad \max \quad E[x_t] - w(Var(x_T) - \kappa(\alpha)(E[x_T] - C_t)^2),$$

s.t. $x_{s+1} = e_s x_s + P'_s u_s, \ s = 0, 1, ..., T - 1.$

By simple calculation, the objective function of problem $P_1(2)$ can be rewritten as

$$E[x_t] - wE[x_T^2] + w(1 + \kappa(\alpha))E^2[x_T] - 2w\kappa(\alpha)C_t E[x_T] + C_t^2 wk(\alpha).$$

For given $w$ and $T$, let $\Pi_1(w, T)$ denote the set of optimal solutions of the auxiliary problem $P_1(2)$, and $G_1(w, T)$ the corresponding optimum value. The following lemma shows the relationship between problems $P_1(2)$ and $P_0(2)$, which can be easily deduced from the dual theory of nonlinear programming.

Lemma 6.1. There must exist a $w \geq 0$ such that $G_0(T) = G_1(w, T)$.

Due to the non-separability of $E^2[x_T]$, problem $P_1(2)$ can not be solved by the dynamic programming technique. Hence, we adopt a linear-quadratic framework similar to that suggested in [17] and [35] to embed problem $P_1(2)$ into the following auxiliary problem,

$$P_2(2) \quad \max \quad E[x_t] - wE[x_T^2] + \lambda E[x_T],$$

s.t. $x_{s+1} = e_s x_s + P'_s u_s, \ s = 0, 1, ..., T - 1.$

For given $\lambda$, $w$ and $T$, let $\Pi_2(\lambda, w, T)$ denote the set of optimal solutions of problem $P_2(2)$ and $G_2(\lambda, w, T)$ the corresponding optimum value. Then we can establish the following relationship between problems $P_1(2)$ and $P_0(2)$.

Theorem 6.3. For any $w$, $T$ and $\pi \in \Pi_2(\lambda, w, T)$, a necessary condition for $\pi \in \Pi_2(\lambda, w, T)$ is
\( \Pi_1(w, T) \) is

\[
\lambda = -2w\kappa(\alpha)C_t + 2w(1 + \kappa(\alpha))E[x_T]|_\pi.
\]

**Proof.** For given \( w \) and \( T \), the solution \( \pi \in \Pi_2(\lambda, w, T) \) can be parameterized by \( \lambda \). While the objective function of problem \( P_2(2) \) can be expressed by three terms, i.e., \( E[x_t(\lambda, w)] \), \( E[x_T(\lambda, w)] \) and \( E[x_T^2(\lambda, w)] \). If \( \pi \) also solves problem \( P_1(2) \), \( \lambda \) should be the optimal solution of the following maximization problem with respect to \( \lambda \):

\[
\max_\lambda \ E[x_t(\lambda, w)] - wE[x_T^2(\lambda, w)] + w(1 + \kappa(\alpha))E[x_T(\lambda, w)]
\]

\[
-2w\kappa(\alpha)C_t E[x_T(\lambda, w)].
\]

The first-order necessary optimality condition for this problem is

\[
\frac{\partial E[x_t(\lambda, w)]}{\partial \lambda} - w \frac{\partial E[x_T^2(\lambda, w)]}{\partial \lambda} - (2w\kappa(\alpha)C_t + 2w(1 + \kappa(\alpha))E[x_T]|_\pi) \frac{\partial E[x_T(\lambda, w)]}{\partial \lambda} = 0.
\]

On the other hand, \( \pi \) solves problem \( P_2(2) \), hence it satisfies the optimality condition for problem \( P_2(2) \), then we have from [26] that

\[
\frac{\partial E[x_t(\lambda, w)]}{\partial \lambda} - w \frac{\partial E[x_T^2(\lambda, w)]}{\partial \lambda} + \lambda \frac{\partial E[x_T(\lambda, w)]}{\partial \lambda} = 0.
\]

To ensure (6.5) and (6.6) hold simultaneously, we must have

\[
\lambda = -2w\kappa(\alpha)C_t + 2w(1 + \kappa(\alpha))E[x_T]|_\pi.
\]

Lemma 6.1 and Theorem 6.3 guarantee that the optimal solution of problem \( P_0(2) \) is contained in the set of optimal solutions of problem \( P_2(2) \) as long as \( \lambda \), \( w \) and \( T \) satisfy (6.3). From Theorem 6.1, we know the optimal solution of problem \( P_0(2) \) satisfies (6.2). Consequently, the solution \( \pi \in \Pi_2(\lambda, w, T) \) satisfying both (6.2) and (6.3) will solve problem \( P_0(2) \), of course, when such solutions exist.
6.2 Analytical solution of the auxiliary problem

In this subsection, we use the dynamic programming technique to solve the auxiliary problem \( P_2(2) \). For given \( x_s \), let \( J_s(u_s|x_s) \) denote the induced objective function at period \( s, s = 0, 1, ..., T - 1 \). The dynamic programming method starts from period \( T - 1 \). For given \( x_{T-1} \), the remaining sub-problem is

\[
(6.7) \quad \max J_{T-1}(u_{T-1}|x_{T-1}) = E[x_t] + \max E\{-wx_T^2 + \lambda x_T\} = E[x_t] + \max E\{-w(e_{T-1}x_{T-1} + P_{T-1}^t u_{T-1})^2 + \lambda(e_{T-1}x_{T-1} + P_{T-1}^t u_{T-1})\} = E[x_t] + \max \{-we_{T-1}^2x_{T-1}^2 + \lambda e_{T-1}x_{T-1} + (\lambda E(P_{T-1}^t) - 2w \cdot x_{T-1}E(e_{T-1}P_{T-1}^t)) \cdot u_{T-1} - wu_{T-1}' E(P_{T-1}^t P_{T-1}^t) u_{T-1}\}.
\]

By solving the first-order optimality condition

\[
\nabla J_{T-1}(u_{T-1}|x_{T-1}) = 0,
\]

we can derive the optimal solution \( u_{T-1}^* \) to problem (6.7) as

\[
u_{T-1}^* = E^{-1}(P_{T-1}P_{T-1}^t)[E(P_{T-1}) \frac{\lambda}{2w} - E(e_{T-1}P_{T-1})x_{T-1}].\]

Substituting \( u_{T-1}^* \) back into \( J_{T-1}(u_{T-1}|x_{T-1}) \) gives the induced objective function at period \( T - 1 \),

\[
J_{T-1}^*(u_{T-1}|x_{T-1}) = E[-w_{T-1}x_{T-1}^2 + \lambda_{T-1}x_{T-1} + c_{T-1} + x_t],
\]

where

\[
w_{T-1} = \kappa_2 \kappa_{T-1} - E(e_{T-1}P_{T-1}^t)E^{-1}(P_{T-1}P_{T-1}^t)E(e_{T-1}P_{T-1}),
\]

\[
\lambda_{T-1} = \kappa_2 \kappa_{T-1} - E(P_{T-1}^t)E^{-1}(P_{T-1}P_{T-1}^t)E(e_{T-1}P_{T-1}),
\]

\[
c_{T-1} = \frac{\lambda_2^2}{4w} E(P_{T-1}^t)E^{-1}(P_{T-1}P_{T-1}^t)E(e_{T-1}P_{T-1}).
\]
Inspired by this, we now derive the optimal decision for period \( s, s = 0, 1, ..., T - 1 \), and the resulting induced objective function. For simplicity, we first introduce the following notation:

\[
A_s^1 = e_s - E(P'_s E^{-1}(P_s P'_s) E(e_s P_s)), \quad s = 0, 1, ..., T - 1, \\
A_s^2 = e_s^2 - E(e_s P'_s E^{-1}(P_s P'_s) E(e_s P_s)), \quad s = 0, 1, ..., T - 1, \\
L_s = E^{-1}(P_s P'_s) E(P_s), \quad K_s = E^{-1}(P_s P'_s) E(e_s P_s), \quad s = 0, 1, ..., T - 1,
\]

and

\[
w_s = w_{s+1} A_s^2, \quad s = 0, 1, ..., T - 1, \\
\lambda_s = \begin{cases} \\
\lambda_{s+1} A_s^1, & \text{when } s \neq t, \\
\lambda_{s+1} A_s^1 + 1, & \text{when } s = t.
\end{cases}
\]

We prove by induction. Suppose the induced objective function at period \( s + 1, s > t \), is

\[
J^*_s(u_{s+1} | x_{s+1}) = E[-w_{s+1} x_{s+1}^2 + \lambda_{s+1} x_{s+1} + c_{s+1} + x_t].
\]

Then for given \( x_s \), the optimal decision should maximize \( J_s(u_s | x_s) \). That is,

\[
\max J_s(u_s | x_s) = \max J^*_s(u_{s+1} | x_{s+1}) \\
= \max E\{-w_{s+1} x_{s+1}^2 + \lambda_{s+1} x_{s+1} + c_{s+1} + x_t\} \\
= \max E\{-w_{s+1} (e_s x_s + P'_s u_s)^2 + \lambda_{s+1} (e_s x_s + P'_s u_s)\} + c_{s+1} + x_t \\
= \max \{-w_{s+1} (e_s x_s^2 + \lambda_{s+1} e_s x_s) + \{\lambda_{s+1} E(P'_s) - 2 w_{s+1} \cdot x_s E(e_s P'_s)\} \cdot u_s \}
\]

By solving the first-order optimality condition,

\[
\frac{\nabla J_s(u_s | x_s)}{\nabla u_s} = 0,
\]

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we can obtain the optimal solution \( u_s^* \) to problem (6.9):

\[
    u_s^* = E^{-1}(P_sP'_s)|E(P_s)\frac{\lambda_{s+1}}{2w_{s+1}} - E(e_sP_s)x_s| = -K_s x_s + L_s \frac{\lambda_{s+1}}{2w_{s+1}}.
\]

Substituting \( u_s^* \) into \( J_s(\lambda, w) \) yields

\[
    J_s^*(u_s|x_s) = E[-w_s x_s^2 + \lambda_s x_s + c_s + x_s], \text{ when } s > t,
\]

where

\[
    c_s = \frac{\lambda^2_{s+1}}{4w_{s+1}}E(P'_s)E^{-1}(P_sP'_s)E(e_sP_s).
\]

This demonstrates the reasonability of (6.8). When \( s \leq t \), it is not difficult to derive that

\[
    J_s^*(u_s|x_s) = E[-w_s x_s^2 + \lambda_s x_s + c_s].
\]

And for any time period \([s, s+1]\), \( s = 0, 1, ..., T - 1 \), the corresponding optimal investment decision is

\[
    u_s^* = -K_s x_s + L_s \frac{\lambda_{s+1}}{2w_{s+1}}
    \begin{cases}
        -K_s x_s + L_s \frac{\prod_{k=s+1}^{T-1} A_1^k \cdot \lambda}{2 \prod_{k=s+1}^{T-1} A^2_k \cdot w}, & s > t - 1, \\
        -K_{t-1} x_{t-1} + L_{t-1} \frac{\prod_{k=t}^{T-1} A_1^k \cdot \lambda + 1}{2 \prod_{k=t}^{T-1} A^2_k \cdot w}, & s = t - 1, \\
        -K_s x_s + L_s \frac{(\prod_{k=s+1}^{T-1} A_1^k \cdot \lambda + 1) \prod_{k=s+1}^{T-1} A^1_k}{2 \prod_{k=s+1}^{T-1} A^2_k \cdot w}, & s < t - 1,
    \end{cases}
\]

where in (6.10) and all the relevant equations hereinafter, we define \( \prod_{k=t}^{T-1} A^1_k = 1 \), \( \prod_{k=t}^{T-1} A^2_k = 1 \), \( \prod_{k=T}^{T-1} A^1_k = 1 \), and \( \prod_{k=T}^{T-1} A^2_k = 1 \).

Substituting (6.10) back into the self-financing constraint (6.1), we obtain the optimal wealth process:

\[
    x_{s+1}(\lambda, w) = (e_s - P'_s K_s)x_s(\lambda, w) + P'_s L_s \frac{\lambda_{s+1}}{2w_{s+1}}.
\]
Taking the expectation operation to both sides of (6.11), we have

\begin{equation}
E[x_{s+1}(\lambda, w)] = A_s^1 E[x_s(\lambda, w)] + E(P_s')L_s \frac{\lambda_{s+1}}{2w_{s+1}}. \tag{6.12}
\end{equation}

By applying the equation (6.12) recursively, we can deduce that

\begin{equation}
E[x_t(\lambda, w)] = (T-1) \prod_{s=0}^{T-1} A_{s+1}^1 x_0 + \sum_{s=0}^{T-1} \left( \prod_{k=s+1}^{T-1} A_{k}^1 \right) E(P_s')L_s \frac{\lambda_{s+1}}{2w_{s+1}},
\end{equation}

\begin{equation}
E[x_T(\lambda, w)] = \left( \prod_{s=0}^{T-1} A_{s+1}^1 \right) x_0 + \sum_{s=0}^{T-1} \left( \prod_{k=s+1}^{T-1} A_{k}^1 \right) E(P_s')L_s \frac{\lambda_{s+1}}{2w_{s+1}}.
\end{equation}

where

\begin{align*}
\hat{\mu}_0 &= \prod_{s=0}^{T-1} A_{s+1}^1, \\
\hat{\mu}_1 &= \sum_{s=0}^{T-1} E(P_s')L_s \left( \prod_{k=s+1}^{T-1} A_k^1 \right) \frac{\prod_{k=s+1}^{T-1} A_k^1}{2(\prod_{k=s+1}^{T-1} A_k^2)}, \\
\hat{\mu}_2 &= \sum_{s=0}^{T-1} E(P_s')L_s \frac{\prod_{k=s+1}^{T-1} A_k^2}{2(\prod_{k=s+1}^{T-1} A_k^2)} + \frac{1}{2} E(P_{t-1}')L_{t-1}, \\
\mu_0 &= \prod_{s=0}^{T-1} A_{s+1}^1, \\
\mu_1 &= \sum_{s=0}^{T-1} E(P_s')L_s \left( \prod_{k=s+1}^{T-1} A_k^1 \right)^2 \frac{\prod_{k=s+1}^{T-1} A_k^1}{2(\prod_{k=s+1}^{T-1} A_k^2)}, \\
\mu_2 &= \sum_{s=0}^{T-1} E(P_s')L_s \frac{\prod_{k=s+1}^{T-1} A_k^2}{2(\prod_{k=s+1}^{T-1} A_k^2)} - \frac{1}{2} E(P_{t-1}')L_{t-1},
\end{align*}

\[
\triangle s = \begin{cases} 
0, & s > t - 1, \\
1, & s = t - 1, \\
\prod_{k=s+1}^{t-1} A_k^1, & s < t - 1.
\end{cases}
\]

Correspondingly, taking the square on both sides of (6.11) and then taking the
6.3 Optimal multipliers

To find the optimal multiplier, we substitute (6.13), (6.14) and (6.16) back into (6.2), then (6.2) can be equivalently described as follows:

\[
\text{Var}(x_T) - \kappa(\alpha)(E[x_T] - C_I)^2|_{\pi^*} = 0
\]

\[
\iff E[x_T^2] - E^2[x_T] - \kappa(\alpha)(E[x_T] - C_I)^2|_{\pi^*} = 0
\]

\[
\iff (v_0 x_0^2 - \kappa(\alpha)C_t^2 - (1 + \kappa(\alpha))\mu_0^2x_0^2 + 2\alpha C_I \mu_0 x_0) + (v_1 - (1 + \kappa(\alpha))\mu_1^2)\frac{\lambda^2}{w^2} + (v_2 - 2(1 + \kappa(\alpha))\mu_1 \mu_2)\frac{\lambda}{w} + (v_3 - (1 + \kappa(\alpha))\mu_2^2)\frac{1}{w^2} + (2\kappa(\alpha)C_I \mu_1 - (1 + \kappa(\alpha))2\mu_0 \mu_2 x_0)\frac{\lambda}{w} + (2\kappa(\alpha)C_I \mu_2 - (1 + \kappa(\alpha))2\mu_0 \mu_3 x_0)\frac{1}{w} = 0
\]

\[
(6.17) \iff d_0 + d_1 \frac{\lambda^2}{w^2} + d_2 \frac{\lambda}{w^2} + d_3 \frac{1}{w^2} + d_4 \frac{\lambda}{w} + d_5 \frac{1}{w} = 0
\]
where

\[ d_0 = v_0 x_0^2 - \kappa(\alpha) C_t^2 - (1 + \kappa(\alpha)) \mu_0^2 x_0 + 2 \kappa(\alpha) C_t \mu_0 x_0, \]

\[ d_1 = v_1 - (1 + \kappa(\alpha)) \mu_1^2, \]

\[ d_2 = v_2 - 2(1 + \kappa(\alpha)) \mu_1 \mu_2, \]

\[ d_3 = v_3 - (1 + \kappa(\alpha)) \mu_2^2, \]

\[ d_4 = 2 \kappa(\alpha) C_t \mu_1 - (1 + \kappa(\alpha)) 2 \mu_0 \mu_1 x_0, \]

\[ d_5 = 2 \kappa(\alpha) C_t \mu_2 - (1 + \kappa(\alpha)) 2 \mu_0 \mu_2 x_0. \]

Hence, problem \( P_0(2) \) can be transformed into the following nonlinear programming with respect to \( \lambda \) and \( w \):

\[
\begin{align*}
\min & \quad \hat{\mu}_0 x_0 + \mu_1 \frac{\lambda}{w} + \mu_2 \frac{1}{w}, \\
\text{s.t.} & \quad d_0 + d_1 \frac{\lambda^2}{w^2} + d_2 \frac{\lambda}{w^2} + d_3 \frac{1}{w^2} + d_4 \frac{\lambda}{w} + d_5 \frac{1}{w} = 0.
\end{align*}
\]

It is not difficult to find from the first order optimality condition that:

\begin{align*}
(6.18) & \quad \lambda = aw + b, \\
\end{align*}

where

\[ a = \frac{d_3 \mu_1 - d_4 \mu_2}{2 d_1 \mu_2 - d_2 \mu_1}, \quad b = \frac{2 d_3 \mu_1 - d_2 \mu_2}{2 d_1 \mu_2 - d_2 \mu_1}, \]

when \( 2 d_1 \mu_2 - d_2 \mu_1 \neq 0 \).

Solving the equations (6.17) and (6.18), we can find that: when \( D_2^2 - 4 D_0 D_1 < 0 \) or \( D_0 \leq 0 \), the optimal \( w^* \) and \( \lambda^* \) do not exist and hence the corresponding programming problem \( P_2(2) \) is infeasible or unbounded. When \( D_2^2 - 4 D_0 D_1 \geq 0 \) and \( D_0 > 0 \), the optimal \( w^* \) and \( \lambda^* \) are

\begin{align*}
(6.19) & \quad w^* = \frac{-D_2 + \sqrt{D_2^2 - 4 D_0 D_1}}{2 D_0}, \\
(6.20) & \quad \lambda^* = aw^* + b,
\end{align*}

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where $D_0 = d_0 + d_1a^2 + d_3a$, $D_1 = d_1b^2 + d_2b + d_3$ and $D_2 = 2d_1ab + d_2a + d_4b + d_5$.

Substituting $w^*$ in (6.19) and $\lambda^*$ in (6.20) back into (6.13), we can finally find the optimum value of problem $P_2(2)$, $G(T)$, and obtain the optimal strategy $u^*(T)$ by substituting (6.19) and (6.20) back into (6.10). For the space reason, we will not present here the expanded expressions for $G(T)$ and $u^*(T)$.

### 6.4 The optimal stopping time

Having derived the optimal investment strategies under different stopping times $T$, and the corresponding optimum value function $G(T)$, we can substitute the explicit expression of $G(T)$ back into problem $P(3)$ and find the optimal stopping time $T^*$, which makes $G(T)$ attain its largest value among all those values corresponding to $T = t, \ldots, T_0$. $T^*$ and the corresponding optimal strategy under $T^*$ then constitute an optimal solution to problem $P(1)$.

Finally, we examine the specific value of the optimal stopping time $T^*$. In our portfolio selection problem, $T^*$ is required to be greater than or equal to $t$ and less than or equal to $T_0$. Here, $t$ represents the earliest time reaching the wealth target if the market constantly grows. Therefore, if $T^* < t$, it would imply that the investor has chosen an overly low benchmark which is easy to reach in the market. From the perspective of portfolio selection, it means the investor would put all his/her wealth into the market to gain excess returns in the sense of arbitrage with probability. On the contrary, if $T^* > T_0$, which corresponds to the infeasibility of the portfolio selection problem, it means that the investor has preset an overly high wealth target which can not be reached even after the acceptably longest investment time. This result means that he/she can not find his/her investment opportunity in the market, therefore, the investor should lower his/her target wealth in order to adapt to the market, or he/she should find another market which is more suitable for his/her investment propose.

### 7 Numerical illustrations

We numerically demonstrate the practicality and efficiency of our theoretical results from two perspectives. Firstly, as a continuation of the empirical test in section 5, we consider the application of our explicit solutions to the American stock market.
Secondly, we compare our theoretical results with those obtained under the dynamic MV model with deterministic investment horizon [18].

### 7.1 Application to American stock market

Consider a multi-period investment process with $T_0 = 10$. We randomly choose three stocks in American stock market, FTI, HCP and JCP. The investor can allocate his/her wealth among these three risky securities and a risk-less bond with a deterministic annualized return rate of $e_s = 1 + r^0_s = 1.0025$ at every period. We assume that the return processes of risky securities are stationary. Then, with their daily trading data from January 1, 2013 to November 1, 2013, we can compute the expected annualized returns of three risky securities as $E(r^FTI_s) = -0.1530$, $E(r^HCP_s) = -0.2102$ and $E(r^JCP_s) = 0.1106, s = 0, 1, 2, ..., 10$, the covariance matrix of the excess return vector $P_s = [r^FTI_s - r^0_s, r^HCP_s - r^0_s, r^JCP_s - r^0_s]$ is

$$\text{Cov}(P_s) = \begin{bmatrix}
0.0514 & 0.0313 & 0.0162 \\
0.0313 & 0.0806 & 0.0238 \\
0.0162 & 0.0238 & 0.0562
\end{bmatrix}, s = 0, 1, 2, ..., 10.$$

We adopt the mean-risk model $P(1)$ with $t = 2$, $\lambda_t = 1$, $C_t = 1.2$ and $\alpha = 0.01$. It is not difficult to know from (6.19) and (6.20) that problem $P(1)$ is feasible only when $T \geq 3$. Applying the analytical optimal solution in the objective function of $P(1)$ and (6.13), we can find the optimum objective values at different stopping times, which are shown in Table 3.

<table>
<thead>
<tr>
<th>$T$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E(x_t) - \lambda_t \rho_{0,t}(x)$</td>
<td>NaN</td>
<td>NaN</td>
<td>1.1593</td>
<td>1.2220</td>
<td>1.2616</td>
<td>1.8592</td>
<td>1.5337</td>
<td>1.4522</td>
<td>1.4177</td>
<td>1.4001</td>
</tr>
</tbody>
</table>

NaN denotes problem $P(1)$ is infeasible.

From Table 3, we can see that the feasible stopping time is larger than or equal to 3, because the chosen three assets construct a no-arbitrage market with probability $\alpha$. When $T \leq 6$, the optimum objective value is increasing with $T$. The reason should be: as the investment horizon becomes longer, it is more flexible to adjust portfolios among periods in order to reach a higher wealth at $t$. On the other hand, when $T \geq 6$, the
discounting factor in the new risk measure dominates, and the optimum objective value decreases with $T$. Comparing all the optimum objective values at different stopping times within $[0, T_0]$, we can find the optimal stopping time is $T^* = 6$, and the global optimum objective value is 1.8592. After determining the optimal stopping time, we can obtain from (6.10) the global optimal investment strategy for $T^* = 6$:

$$u_s^* = -x_s K_s + v_s,$$

where

$$K_s = \begin{bmatrix} -1.0569 \\ -1.1638 \\ 1.6011 \end{bmatrix}, \quad s = 0, 1, ..., 5, \quad v_0 = \begin{bmatrix} -1.7583 \\ -1.9362 \\ 2.6636 \end{bmatrix}, \quad v_1 = \begin{bmatrix} -1.7627 \\ -1.9411 \\ 2.6703 \end{bmatrix}.$$

$$v_2 = \begin{bmatrix} -1.7671 \\ -1.9459 \\ 2.6769 \end{bmatrix}, \quad v_3 = \begin{bmatrix} -1.7715 \\ -1.9508 \\ 2.6836 \end{bmatrix}, \quad v_4 = \begin{bmatrix} -1.7760 \\ -1.9556 \\ 2.6903 \end{bmatrix}, \quad \text{and} \quad v_5 = \begin{bmatrix} -1.7804 \\ -1.9605 \\ 2.6971 \end{bmatrix}.$$

### 7.2 Comparison with dynamic MV model

To intuitively illustrate the relationship between our results and those got with dynamic MV models, we consider the Example 2 in [18]. For this four period MV problem, Li and Ng [18] found the optimal investment policy whose expected terminal wealth and variance are $E(x_4) = 10.1043$ and $Var(x_4) = 2.2336$, respectively. The policy lays on the efficient frontier:

$$Var(x_4) = 0.02798[E(x_4) - 1.1699]^2.$$

If the wealth at period 4 follows the normal distribution, the above frontier is equivalent to that determined by the chance constraint:

$$P(x_4 \geq 1.1699) \geq 1 - 0.02798,$$

which means the terminal wealth is greater than or equal to 1.1699 with probability $1 - 0.02798 = 0.97202$. Hence, if we solve the mean-risk problem $P(2)$ with $t = 4$, $\alpha = 0.02798$ and $C_t = 1.1699$, $T = 4$ is then a feasible stopping time with $E(x_4) = 10.1043$. 

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and $Var(x_4) = 2.2336$; meanwhile, it is not difficult to verify that, when $T = 1, 2, 3$, problem $P(2)$ is not feasible. Therefore, $T = 4$ is the earliest target reaching time. Furthermore, by fixing $\alpha = 0.02798$, we can find out the relationship between the period number $t$ and the maximum $C_t$ satisfying $P(x_t \geq C_t) \geq 1 - 0.02798$, that is,

$$C_t \triangleq \max\{C_t|P(x_t \geq C_t) \geq 1 - 0.02798\}.$$

Table 4 shows the results for $t \leq 10$.

<table>
<thead>
<tr>
<th>$t$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_t$</td>
<td>0</td>
<td>-1.0153</td>
<td>-0.9710</td>
<td>1.1699</td>
<td>9.4436</td>
<td>34.4324</td>
<td>103.1586</td>
<td>283.6515</td>
<td>745.7417</td>
<td>1911.2162</td>
</tr>
</tbody>
</table>

Naturally, we can see from Table 4 that, the maximum target wealth one can obtain is monotonically increasing with respect to the period number. According to our discussion in section 5, for fixed $\alpha = 0.02798$, each pair $(\bar{C}_t, \alpha)$ would lie on the corresponding efficient risk averse frontier. At period 4, for chosen $\alpha = 0.02798$, the $(E(x_4), Var(x_4))$ of the $x_4$ determined under $\bar{C}_4$ is exactly on the efficient frontier of the four period MV problem in [18]. As the approximate problem derived through the Chebyshev inequality is stricter than the original problem, the multi-period MV efficient frontier is below the efficient risk averse frontier defined in Section 5.

8 Conclusions

To describe the degree of satisfaction of the investor about his/her investment target, we define the stopping time as the earliest target reaching time, derived from which is the new risk measure. The new risk measure has good properties and can efficiently reflect the change of information by time. Later on, we introduce the concept of arbitrage with probability, and show its close relationship with the new risk measure and its advantages over the traditional arbitrage. Especially, we can discriminate different security markets by using the arbitrage with probability and help the investor to find a better market to invest. Moreover, we establish multi-period portfolio selection problems under the new risk measure. By constructing a series of auxiliary optimization problems, we show how to find the optimal stopping time easily, and thus explicitly
solve the corresponding portfolio selection problem. Some examples and comparisons are interspersed in the paper to illustrate the value of the new risk measure and the reasonability and efficiency of our solution method for the resulting multi-period portfolio selection problem.

The results in this paper can be further extended from several perspectives, these include the consideration of other stopping time criteria, the influence of different distribution assumptions on the investment strategy, the extension of our results to the continuous time framework, the application of our theoretical results to dynamic pricing problem, as well as the application of our solution method to the optimal investment and consumption problem or the pension plan problem.

References


