A Flexible Coordinate Descent Method for Big Data Applications

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Abstract

In this paper we present a novel randomized block coordinate descent method for the minimization of a convex composite objective function. The method uses (approximate) partial second-order (curvature) information, so that the algorithm performance is more robust when applied to highly nonseparable or ill conditioned problems. We call the method Flexible Coordinate Descent (FCD). At each iteration of FCD, a block of coordinates is sampled randomly, a quadratic model is formed about that block and the model is minimized approximately/inexactly to determine the search direction. An inexpensive line search is then employed to ensure a monotonic decrease in the objective function and acceptance of large step sizes. We present several high probability iteration complexity results to show that convergence of FCD is guaranteed theoretically. Finally, we present numerical results on large-scale problems to demonstrate the practical performance of the method.

Keywords: large scale optimization; second-order methods; curvature information; block coordinate descent; nonsmooth problems; iteration complexity

AMS Classification: 49M15; 49M37; 65K05; 90C06; 90C25;90C53.

1 Introduction

In this work we are interested in solving the following convex composite optimization problem

$$\min_{x \in \mathbb{R}^N} F(x) := f(x) + \Psi(x),$$

(1)

where $f(x)$ is a smooth convex function and $\Psi(x)$ is a (possibly) nonsmooth, (coordinate) separable, real valued convex function (this will be defined precisely in Section 2.5). Problems of the form of (1) arise in many important scientific fields, and applications include machine learning [42], regression [37] and compressed sensing [5, 4, 10]. Often the term $f(x)$ is a data fidelity term, and the term $\Psi(x)$ represents some kind of regularization.

Frequently, problems of the form of (1) are large-scale, i.e., the size of $N$ is of the order of a million or a billion. Large-scale problems impose restrictions on the types of methods that can be employed for the solution of (1). In particular, the methods should have low per iteration computational cost, otherwise completing even a single iteration of the method might require unreasonable time. The methods must also rely only on simple operations such as matrix vector products, and ideally, they should offer fast progress towards optimality.

First order methods, and in particular randomized coordinate descent methods, have found great success in this area because they can take advantage of the underlying problem structure (separability and block structure), and satisfy the requirements of low computational cost and low storage requirements. For example, in [30] the authors show that their randomized coordinate descent method was able to solve sparse problems with millions of variables in a reasonable amount of time.

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Unfortunately, randomized coordinate descent methods have two significant drawbacks. First, due to its coordinate nature, it is efficient only on problems with a high degree of partial separability, and performance suffers when there is a high dependency between variables. Second, as a first-order method, coordinate descent methods do not usually capture essential curvature information of the problem and have been shown to struggle on complicated sparse problems [15, 14].

The purpose of this work is to overcome these drawbacks by equipping a randomized block coordinate descent method with approximate partial second-order information. In particular, at every iteration of FCD a search direction is obtained by solving a local quadratic model approximately. The quadratic model is defined by a matrix representing approximate second order information, so that essential curvature information is incorporated. The model need only be minimized approximately to give an inexact search direction, so that the process is efficient in practice. (The termination condition for the quadratic subproblem is inspired by [3], and we discuss this in more detail later.) A line search is then employed along this inexact direction, in order to guarantee a monotonic decrease in the objective function.

FCD is computationally efficient. At each iteration of FCD, a block of coordinates is randomly selected, which is inexpensive. The method allows the use of inexact search directions, i.e., the quadratic model is minimized approximately, which intuitively, should require less running time than if it needed to be minimized exactly. Also, the line search step depends on a subset of the coordinates only (corresponding to the randomly selected block of coordinates), so it is much cheaper compared with working in the full dimensional space. We note that the per iteration computational cost of the FCD method may be slightly higher than other randomized coordinate descent methods. This is due to the incorporation of the matrix representing partial second order information — the matrix gives rise to a quadratic model that is (in general) nonseparable, so the associated subproblem may not have a closed form solution. However, in the numerical results presented later in this work, we show that the method is more robust (and efficient) in practice, and often FCD will require fewer iterations to reach optimality, compared with other state-of-the-art methods. i.e., we show that FCD is able to solve difficult problems, on which other coordinate descent method may struggle.

The FCD method is supported by theoretical convergence guarantees in the form of high probability iteration complexity results. These iteration complexity results provide an explicit expression for the number of iterations $k$, which guarantees that, for any given error tolerance $\epsilon > 0$, and confidence level $\rho \in (0, 1)$, the probability that FCD achieves an $\epsilon$-optimal solution, exceeds $1 - \rho$, i.e., $P(F(x_k) - F^* \leq \epsilon) \geq 1 - \rho$.

1.1 Literature review

Coordinate descent methods are some of the oldest and simplest iterative methods, and they are often better known in the literature under various names such as Jacobi methods, or Gauss-Seidel methods, among others. It has been observed that these methods may suffer from poor practical performance, particularly on ill-conditioned problems, and until recently, often higher order methods have been favoured by the optimization community. However, as we enter the era of big data, coordinate descent methods are coming back into favour, because of their ability to provide approximate solutions to real world instances of very large/huge scale problems in a reasonable amount of time.

Currently, state-of-the-art randomized coordinate descent methods include that of Richtárik and Takáč [30], where the method can be applied to unconstrained convex composite optimization problems of the form (1). The algorithm is supported by theoretical convergence guarantees in the form of high probability iteration complexity results, and [30] also reports very impressive practical performance on highly separable large scale problems. Their work has also been extended to the parallel case [29], to include acceleration techniques [13], and to include the use of inexact updates [35].

Other important works on coordinate descent methods include that of Tseng and Yun in [38, 39, 40], Nesterov [24] for huge-scale problems, work in [21] and [36] that improves the complexity analysis of [30] and [29] respectively, coordinate descent methods for group lasso problems [27, 34] or for problems with general regularizers [33, 41], a coordinate descent methods that uses sequential active set strategy [31], and coordinate descent for constrained optimization problems [22].

Unfortunately, on ill-conditioned problems, or problems that are highly nonseparable, first order methods can display very poor practical performance, and this has prompted the study of methods that...
employ second order information. To this end, recently there has been a flurry of research on Newton-type methods for problems of the general form (1), or in the special case where $\Psi(x) = \|x\|_1$. For example, Karimi and Vavasis [17] have developed a proximal quasi-Newton method for $l_1$-regularized least squares problems, Lee, Sun and Saunders [19, 20] have proposed a family of Newton-type methods for solving problems of the form (1) and Scheinberg and Tang [32] present iteration complexity results for a proximal Newton-type method. Moreover, the authors in [3] extended standard inexact Newton-type methods to the case of minimization of a composite objective involving a smooth convex term plus an $l_1$-regularizer term.

Facchinei and coauthors in [7, 11, 12] have also made a significant contribution to the literature on randomized coordinate descent methods. Two new algorithms are introduced in those works, namely FLEXA [11, 12] and HyFLEXA [7]. Both methods can be applied to problems of the form (1) where $f$ is allowed to be nonconvex, and for HyFLEXA, $\Psi$ is allowed to be nonseparable. We stress that nonconvexity of $f$, and nonseparability of $\Psi$ are outside the scope of this work. Nevertheless, these methods are pertinent to the current work, so we discuss them now.

For FLEXA and HyFLEXA, as is standard, the block structure of the problem is fixed throughout the algorithm: the data vector $x \in \mathbb{R}^N$ is partitioned into $n$ blocks, $x = (x_1, \ldots, x_n) \in \mathbb{R}^N$. At each iteration, one block, all blocks, or some number in between, are selected in such a way that some specified error bound is satisfied. Next, a local convex model is formed and that model is approximately minimized to find a search direction. Finally, a convex combination of the current point, and the search direction is taken, to give a new point, and the process is repeated. We notice that the model they propose is block separable, so the subproblems for each block are independent. (See (3) in [12] for the local/block quadratic model, or F1–F3 in [7] for a description of the strongly convex local/block model). This has the advantage that the subproblem for each block can be solved/minimized in parallel, but has the disadvantage that no interaction between different blocks is captured by either algorithm. We also note that, while these algorithms are equipped with global convergence guarantees, iteration complexity results have not been established for either method, which is a significant drawback.

Another important method that has recently been proposed is a randomized coordinate decent method by Qu et al. [28]. One of the most notable features of their algorithm is that the block structure is not fixed throughout the algorithm. That is, at each iteration of their algorithm, a random subset of coordinates $S_k$ is selected from the set of coordinates $\{1, \ldots, N\}$. Then, a quadratic model is minimized exactly to obtain an update, and a new point is generated by applying the update to the current point. Unfortunately, this algorithm is only applicable to strongly convex smooth functions, or strongly convex empirical risk minimization problems, but is not suitable for general convex problems of the form (1). Moreover, the matrices used to define their quadratic model must obey several (strong) restrictions, and the quadratic model must be minimized exactly. These restrictions may make the algorithm inconvenient from a practical perspective.

The purpose of this work is to build upon the positive features of some existing coordinate descent methods and create a flexible framework that can be used to solve any problem of the form (1). The goal is to make the framework general enough to recover (most of) the algorithms that have been mentioned so far. We will adopt some of the ideas in [3, 12, 28] (variable block structure; the incorporation of partial approximate second order information; the practicality of approximate solves and inexact updates; cheap line search) and create a new flexible randomized coordinate descent framework that is not only computationally practical, but is also supported by strong theoretical guarantees.

### 1.2 Major Contributions

In this section we list several of the major contributions that we make in this paper, which correspond to the central ideas of our FCD algorithm.

- **Blocks can vary throughout iterations.** Existing randomized coordinate descent methods initially partition the data $x \in \mathbb{R}^N$ into $n$ blocks $(x = (x_1, \ldots, x_n) \in \mathbb{R}^N)$ and at each iteration, one/all/several of the blocks are selected according to some probability distribution, and those blocks are then updated. For those methods, the block partition is fixed throughout the entire algorithm. So, for example, coordinates in block $x_1$ will always be updated all together as a group, independently of any other block $x_i$, $i \neq 1$. One of the main contributions of this work is that we allow the blocks to vary throughout the algorithm. i.e., the method is not restricted to a fixed block
structure. No partition of $x$ need be initialized. Rather, when FCD is initialized, a parameter $1 \leq \tau \leq N$ is chosen and then, at each iteration of FCD, an index subset $S \subseteq \{1, \ldots, N\}$ of size $|S| = \tau$ is randomly selected. To the best of our knowledge, the only other paper that allows for this random mixing of coordinates/varying blocks strategy, is the work by Qu et al. [28].

This variable block structure is crucial with regards to our next major contribution.

**Remark 1.** We note that the algorithms of Tseng and Yun [38, 39, 40] also allow a certain amount of coordinate mixing. However, their algorithms enforce deterministic rules for coordinate subset selection (either a ‘generalized Gauss-Seidel’, or Gauss-Southwell rule), which is different from our randomized selection rule. This difference is important because their deterministic strategy can be expensive/inconvenient to implement in practice, whereas our random scheme is simple and cheap to implement.

2. **Incorporation of partial second order information.** FCD uses a quadratic model to determine the search direction. The model is defined by any positive definite matrix $H^S(x_k)$, which depends on both the subset of coordinates $S$, and also on the iteration counter $k$, i.e., $H^S(x_k)$ is not fixed and can change throughout the algorithm, regardless of the iteration $k$ or subset $S$. We stress that this is a key advantage over most existing methods, which enforce the positive definite matrix defining their quadratic model to be fixed with respect to the fixed block structure, and/or iteration counter. In fact, the only works that allow the matrix $H^S(x_k)$ to vary with respect to the subset $S$ is this one (FCD), and the work by Qu et al. [28]. A crucial observation is that, if $H^S(x_k)$ approximates the Hessian, our approach allows every element of the Hessian to be accessed. On the other hand, all other existing methods can only access blocks along the diagonal of (some perturbation of) the Hessian, (including [7, 11, 12, 29, 30, 36].)

Furthermore, the only restriction we make on the matrix $H^S(x_k)$ is that it be positive definite. This is a much weaker assumption than made by Qu et al. [28], who insist that the matrix defining their quadratic model be a principle submatrix of some fixed/predefined $N \times N$ matrix $M$ say, and the large matrix $M$ must be sufficiently positive definite. (See Section 2.1 in [28].)

3. **Inexact search directions.** Another key ingredient of FCD is the search direction, which is obtained by minimizing the quadratic model. To ensure that this method is computationally practical, it is imperative that the iterates are inexpensive, and FCD achieves this through the use of *inexact* updates. That is, the quadratic model may be only approximately minimized, giving *inexact search directions*. This makes intuitive sense because, if the current point is far from the true problem solution, it may be wasteful/unnecessary to exactly minimize the model. Any algorithm can be used to approximately minimize the quadratic model, and if $H^S(x_k)$ approximates the Hessian, then the search direction obtained by minimizing the model is an approximate Newton-type direction. Importantly, the stopping conditions for the inner solve are *easy to verify*; they depend upon quantities that are easy/inexpensive to obtain, or may be available as a byproduct of the inner search direction solver.

**Remark 2.** The precise form of our inexactness termination criterion is important because, not only is it computationally practical (i.e., easy to verify), it also allowed us to derive iteration complexity results for FCD (see point 5). We considered several other termination criterion formulations, but we were unable to obtain corresponding iteration complexity results (although global convergence results were possible). Currently, the majority of randomized CD methods require exact solves for their inner subproblems and we believe that a major reason for this is because iteration complexity results are significantly easier to obtain in the exact case. Coming up with practical inexact termination criteria for randomized CD methods that also allow the derivation of iteration complexity results seems to be a major hurdle in this area, although some progress is being made, e.g., [6, 9, 8, 35].

4. **Line search.** The algorithm includes a line search step to ensure a monotonic decrease of the objective function as iterates progress. (It is needed because we only make weak assumptions

\footnote{An earlier version of this work, which, to the best of our knowledge, was the first to propose varying random block selection, is cited by [28].}
on the matrix $H^S(x_k)$.) The line search is inexpensive to perform because, at each iteration, it depends on a subset of coordinates $S$ only. One of the major advantages of incorporating second-order information combined with line search is to allow in practice the selection of large step sizes (close to one). This is because unit step sizes can substantially improve the practical efficiency of a method. In fact, for all experiments that we performed, unit step sizes were accepted by line search for the majority of the iterations.

5. Convergence theory. We provide a complete convergence theory for FCD. In particular, we provide high probability iteration complexity guarantees for the algorithm in both the convex and strongly convex cases, and in the cases of both inexact or exact search directions. Our results show that FCD converges at a sublinear rate for convex functions, and at a linear rate for strongly convex functions.

1.3 Paper Outline

The paper is organised as follows. In Section 2 we introduce the notation and definitions that are used throughout this paper, including the definition of the quadratic model, and stationarity conditions. A thorough description of the FCD algorithm is presented in Section 3, including how the blocks are selected/sampled at each iteration (Section 3.1), a description of the search direction, several suggestions for selecting the matrices $H^S(x_k)$, and analysis of the subproblem/model termination conditions (Section 3.2), a definition of the line search (Section 3.3) and we also present several concrete problem examples (Section 3.4). In Section 4 we show that the line search in FCD will always be satisfied for some positive step $\alpha$, and that the objective function value is guaranteed to decrease at every iteration. Section 5 introduces several technical results, and in Section 6 we present our main iteration complexity results. Finally, several numerical experiments are presented in Section 7, which show that the algorithm performs very well in practice, even on ill-conditioned problems.

2 Preliminaries

In this section we introduce the notation and definitions that are used in this paper, and we also present some important technical results. Throughout the paper $\| \cdot \| \equiv \sqrt{\langle \cdot, \cdot \rangle}$ and $\| \cdot \|_A \equiv \sqrt{\langle \cdot, A \cdot \rangle}$, where $A$ is a positive definite matrix. Moreover, $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ denote the smallest and largest eigenvalue of $\cdot$, respectively.

2.1 Subgradient and subdifferential

For a function $\Phi : \mathbb{R}^N \to \mathbb{R}$ the elements $s \in \mathbb{R}^N$ that satisfy

$$\Phi(y) \geq \Phi(x) + \langle s, y - x \rangle \quad \forall y \in \mathbb{R}^N$$

are called the subgradients of $\Phi$ at point $x$. In words, all elements defining a linear function that supports the function $\Phi$ at point $x$ are subgradients. The set of all $s$ at a point $x$ is called the subdifferential of $\Phi$ and it is denoted by $\partial \Phi(x)$.

2.2 Convexity

A function $\Phi : \mathbb{R}^N \to \mathbb{R}$ is strongly convex with convexity parameter $\mu_\Phi > 0$ if

$$\Phi(y) \geq \Phi(x) + \langle s, y - x \rangle + \frac{\mu_\Phi}{2} \|y - x\|^2 \quad \forall x, y \in \mathbb{R}^N,$$

where $s \in \partial \Phi(x)$. An equivalent definition can be written without using subgradients. Let $t \in [0,1]$, function $\Phi : \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}$ is strongly convex if

$$\Phi(tx + (1-t)y) \leq t\Phi(x) + (1-t)\Phi(y) - \frac{\mu_\Phi t(1-t)}{2} \|x - y\|^2 \quad \forall x, y \in \mathbb{R}^N.$$  

If $\mu_\Phi = 0$, then function $\Phi$ is said to be convex. In this paper we will make use of both definitions.
2.3 Convex conjugate and proximal mapping

The proximal mapping of a convex function \( \Psi \) at the point \( x \in \mathbb{R}^N \) is defined as follows:

\[
\prox_{\Psi}(x) := \arg \min_{y \in \mathbb{R}^N} \Psi(y) + \frac{1}{2}\|y - x\|^2.
\] (4)

Furthermore, for a convex function \( \Phi : \mathbb{R}^N \to \mathbb{R} \), its convex conjugate \( \Phi^* \) is defined as \( \Phi^*(y) := \sup_{u \in \mathbb{R}^N} \langle u, y \rangle - \Phi(u) \). (Using this definition is straightforward to define the convex conjugate of the proximal mapping \( \Psi^* \).) From Chapter 1 of [1], we see that \( \prox_{\Psi}(\cdot) \) and \( \prox_{\Psi^*}(\cdot) \) are nonexpansive:

\[
\|\prox_{\Psi}(y) - \prox_{\Psi}(x)\| \leq \|y - x\|, \quad \text{and} \quad \|\prox_{\Psi^*}(y) - \prox_{\Psi^*}(x)\| \leq \|y - x\|.
\] (5)

2.4 Coordinate decomposition of \( \mathbb{R}^N \)

Let \( U \in \mathbb{R}^{N \times N} \) be a column permutation of the \( N \times N \) identity matrix and further let \( U = [U_1, U_2, \ldots, U_N] \) be a decomposition of \( U \) into \( N \) submatrices (column vectors), where \( U_i \in \mathbb{R}^{N \times 1} \). It is clear that any vector \( x \in \mathbb{R}^N \) can be written uniquely as \( x = \sum_{i=1}^{N} U_i x_i \), where \( x_i \in \mathbb{R} \).

Define \([N] := \{1, \ldots, N\}\). Then we let \( S \subseteq [N] \) and \( U_S \in \mathbb{R}^{N \times |S|} \) be the collection of columns from matrix \( U \) which have column indices in set \( S \). We denote the vector

\[
x_S := \sum_{i \in S} U_i^T x = U_S^T x.
\] (6)

2.5 Coordinate decomposition of \( \Psi \)

The function \( \Psi : \mathbb{R}^N \to \mathbb{R} \) is assumed to be coordinate separable. That is, we assume that \( \Psi(x) \) can be decomposed as:

\[
\Psi(x) = \sum_{i=1}^{N} \Psi_i(x^i),
\] (7)

where the functions \( \Psi_i : \mathbb{R} \to \mathbb{R} \) are convex and \( x^i \) is the \( i \)-th component of \( x \). We denote by \( \Psi_S : \mathbb{R}^{|S|} \to \mathbb{R} \), where \( S \subseteq [N] \), the function

\[
\Psi_S(x_S) := \sum_{i \in S} \Psi_i(x_i),
\] (8)

where \( x_S \) is the collection of components from \( x \), which have indices in set \( S \). The following relationship will be used repeatedly in this work:

\[
\Psi(x + U_S t^S) - \Psi(x) \overset{(7)+(8)}{=} \Psi_S(x^S + t^S) - \Psi_S(x^S).
\] (9)

2.6 Subset Lipschitz continuity of \( f \)

Throughout the paper we assume that the gradient of \( f \) is Lipschitz, uniformly in \( x \), for all subsets \( S \subseteq [N] \). This means that, for all \( x \in \mathbb{R}^N \), \( S \subseteq [N] \) and \( t^S \in \mathbb{R}^{|S|} \) we have

\[
\|\nabla_S f(x + U_S t^S) - \nabla_S f(x)\| \leq L_S \|t^S\|,
\] (10)

where \( \nabla_S f(x) \overset{(6)}{=} U_S^T \nabla f(x) \). An important consequence of (10) is the following standard inequality [23, p.57]:

\[
f(x + U_S t^S) \leq f(x) + \langle \nabla_S f(x), t^S \rangle + \frac{L_S}{2} \|t^S\|^2.
\] (11)

2.7 Radius of Levelset

Let \( X_* \) be the set of optimal solutions. We define

\[
\mathcal{R}(x) = \max_{y \in X_*} \{\|y - x\| : F(y) \leq F(x)\},
\] (12)
which is a measure of the size of the level set of $F$ given by $x$. In this work we make the standard assumption that $\mathcal{R}(x_0)$ is bounded at the initial iterate $x_0$. We also define a scaled version of (12)

$$\mathcal{R}_w(x) = \max_y \max_{x, z \in X} \{\|y - x_w\|_w : F(y) \leq F(x)\}, \quad (13)$$

where $w \in \mathbb{R}^N$ with positive components and $\|u\|_w = \left(\sum_{i=1}^{N} u_i^2\right)^{1/2}$.

### 2.8 Piecewise Quadratic Model

For fixed $x \in \mathbb{R}^N$, we define a piecewise quadratic approximation of $F$ around the point $(x + t) \in \mathbb{R}^N$ as follows: $F(x + t) \approx f(x) + Q(x; t)$, where

$$Q(x; t) := \langle \nabla f(x), t \rangle + \frac{1}{2} \|t\|^2_{H(x)} + \Psi(x + t) \quad (14)$$

and $H(x)$ is any positive definite matrix, which possibly depends on $x$. We also define

$$Q_S(x, t^S) := \langle \nabla_S f(x), t^S \rangle + \frac{1}{2} \|t^S\|^2_{H^S(x)} + \Psi_S(x^S + t^S), \quad (15)$$

and $H^S(x) \in \mathbb{R}^{|S| \times |S|}$ is a square submatrix of $H(x)$ which corresponds to the selection of columns and rows from $H(x)$ with column and row indices in set $S$. Notice that $Q_S(x, t^S)$ is the quadratic model for the collection of coordinates in $S$.

### 2.9 Stationarity conditions

The following theorem gives the equivalence of some stationarity conditions of problem (1).

**Theorem 3** ([2]). *The following are equivalent first order optimality conditions for problem (1).*

(i) $\nabla f(x) + s = 0$ and $s \in \partial \Psi(x)$,

(ii) $\nabla f(x) \in -\partial \Psi(x)$,

(iii) $\nabla f(x) + \text{prox}_{f^*} (x - \nabla f(x)) = 0$,

(iv) $x = \text{prox}_{f^*} (x - \nabla f(x))$.

Let us define the continuous function

$$g(x; t) := \nabla f(x) + H(x)t + \text{prox}_{f^*} (x + t - \nabla f(x) - H(x)t). \quad (16)$$

By Theorem 3, the points that satisfy $g(x; 0) = \nabla f(x) + \text{prox}_{f^*} (x - \nabla f(x)) = 0$ are stationary points for problem (1). Hence, $g(x; 0)$ is a continuous measure of the distance from the set of stationary points of problem (1). Furthermore, let us define

$$g_S(x; t^S) := \nabla_S f(x) + H^S(x)t^S + \text{prox}_{f^*_S} (x^S + t^S - \nabla_S f(x) - H^S(x)t^S), \quad (17)$$

which will be used as a continuous measure for the distance from stationarity of the quadratic function $Q_S(x_k; t^S)$.

### 3 The Algorithm

In this section we present the Flexible Coordinate Descent (FCD) algorithm for solving problems of the form (1). There are three key steps in the algorithm: (Step 4) the coordinates are sampled randomly; (Step 5) the quadratic model (15) is solved approximately until rigorously defined stopping conditions are satisfied; (Step 6) a line search is performed to find a step size that ensures a sufficient reduction in the objective value. Once these key steps have been performed, the current point $x_k$ is updated to give a new point $x_{k+1}$, and the process is repeated.
In the algorithm description, and later in the paper, we will use the following definition for the set $\Omega$:

$$\Omega = \nabla S f(x_k) + \partial \Psi_S(x_k^0 + t_k^S).$$  (18)

We are now ready to present pseudocode for the FCD algorithm, while a thorough description of each of the key steps in the algorithm will follow in the rest of this section.

**Algorithm 1** Flexible Coordinate Descent (FCD)

1: **Input** Choose $x_0 \in \mathbb{R}^N$, $\theta \in (0, 1)$ and $1 \leq \tau \leq N$.
2: **for** $k = 1, 2, \ldots$ **do**
3: Sample a subset of coordinates $S \subseteq [N]$ with probability $P(S) > 0$
4: If $g_S(x_k; 0) = 0$ go to Step 3; else select $\eta^S_k \in [0, 1)$ and approximately solve

$$t_k^S := \arg\min_{t^S} Q_S(x_k; t^S),$$  (19)

until the following stopping conditions hold:

$$Q(x_k; U St_k^S) < Q(x_k; 0),$$  (20)

and

$$\inf_{u \in \Omega} \| u - g_S(x_k; t_k^S) \|^2 \leq (\eta^S_k \|g_S(x_k; 0)\|^2) - \|g_S(x_k; t_k^S)\|^2.$$

5: Perform a backtracking line search along the direction $t_k^S$ starting from $\alpha = 1$. i.e., find $\alpha \in (0, 1]$ such that

$$F(x_k) - F(x_k + \alpha U St_k^S) \geq \theta \left( \ell(x_k; 0) - \ell(x_k; \alpha U St_k^S) \right),$$

where

$$\ell(x_k; t) := f(x_k) + \langle \nabla S f(x_k), t \rangle + \Psi(x_k + t).$$  (23)

6: Update $x_{k+1} = x_k + \alpha U St_k^S$
7: **end for**

### 3.1 Selection of coordinates (Step 3)

One of the crucial ideas of this algorithm is that a selection of coordinates to be updated at each iteration is chosen randomly. This allows the coordinates to be selected very quickly. In this section we explain in detail, how the coordinates are selected/sampled at each iteration.

We use $2^{|N|}$ to denote the set of all subsets of $[N]$, which includes the empty set $\emptyset$ and $[N]$ itself. At every iteration of the proposed method we sample a subset from $2^{|N|}$. We denote the probability of a set $S \in 2^{|N|}$ being selected with $P(S)$. We are interested in probability distributions with $P(S) > 0 \ \forall S \in 2^{|N|} \setminus \emptyset$ and $P(\emptyset) = 0$. Therefore, all non-empty sets have positive probability to be selected. This is necessary to make sure that a subset of indices in $[N]$ will be selected at every iteration, otherwise convergence of the proposed method cannot be guaranteed. In what follows we discuss a uniform probability distribution for the subsets $P(S) \forall S \in 2^{|N|} \setminus \emptyset$. All the theoretical results in this paper use uniform probabilities.

#### 3.1.1 Construction of probability distribution

We construct a probability distribution such that subsets in $2^{|N|}$ with the same cardinality have equal probability to be selected. Below we formalize this construction.

**Assumption 4.** $P(S) = P(S') > 0 \ \forall S, S' \in 2^{|N|}$ with $|S| = |S'|$.

This construction of uniform probabilities has been proposed in [29] for a randomized coordinate descent method. In this subsection we present it again for completeness. Let

$$S_j := \left\{ \bigcup_{S \in 2^{|N|}, |S| = j} S \right\}$$
be the set of all subsets \( S \in 2^{[N]} \) with cardinality \(|S| = j\). The probability of a subset \( S \) with cardinality \( j \) being selected is denoted with \( P(S_j) \). By Assumption 4 we have that \( P(S) := q_j > 0 \ \forall S \in 2^{[N]} \) with \(|S| = j\). Therefore we have that

\[
P(S_j) = \sum_{S \in 2^{[N]}: |S| = j} P(S) = q_j \binom{N}{j}.
\]

Using the previous we obtain that the probability distribution \( P(S) \) depends uniquely on \( P(S_{|S|}) \):

\[
P(S) = \frac{P(S_{|S|})}{\binom{N}{|S|}} \ \forall S \in 2^{[N]}.
\] (24)

Therefore, by defining the probability distribution \( P(S_j) \ \forall j = 1, 2, \ldots, N \) we obtain the probability distribution for all \( S \in 2^{[N]} \).

### 3.1.2 Sampling sets with fixed cardinality

In this section we describe a special case of the probability distribution \( P(S_j) \ \forall j = 1, 2, \ldots, N \) in (24). We have the following definition.

**Definition 5** (Sampling). Given an integer \( 1 \leq \tau \leq N \), a set \( S \) with \(|S| = \tau\) is selected with probability one, i.e., \( P(S_{\tau}) = q_{\tau} = 1 \).

Notice that the probability distribution for the subsets in \( 2^{[N]} \), associated with Definition 5, is

\[
P(S) = \frac{1}{\binom{N}{\tau}} \forall S \in 2^{[N]} \text{ with } |S| = \tau, \quad P(S) = 0 \ \forall S \in 2^{[N]} \text{ with } |S| \neq \tau.
\] (25)

In all the theoretical result in this paper we will use the sampling described in Definition 5. We state this formally in the following assumption.

**Assumption 6.** In Step 3 of FCD (Algorithm 1), we assume that the sampling procedure used to generate the subset of coordinates \( S \) in that given in Definition 5.

Below we present a technical result which uses the probability distribution (25) and is used in the worst-case iteration complexity results of this paper. Let \( \theta_i \) with \( i = 1, 2, \ldots, N \) be some constants, then

\[
\mathbb{E}
\left[
\sum_{i \in S} \theta_i
\right]
\]

\[
= \sum_{S \in 2^{[N]}: |S| = \tau} P(S) \sum_{i \in S} \theta_i
\]

\[
= \sum_{i = 1}^{N} \theta_i \sum_{S \in 2^{[N]}: |S| = \tau, i \in S} \frac{1}{\binom{N}{\tau}} = \frac{1}{\binom{N}{\tau}} \sum_{i = 1}^{N} \theta_i \frac{(N - 1)!}{(\tau - 1)!}
\]

\[
= \frac{\tau}{N} \sum_{i = 1}^{N} \theta_i.
\] (26)

There exists a simple and efficient way to simulate the selection of subsets \( S \) from distribution (25) in \( O(N) \) time. Before we describe the process we discuss some preliminaries.

Let \( v \) be a vector which has the components of \([N]\) as its components. For example, if \( N = 3 \), then \( v = [1, 2, 3] \). Let \( D \) with \(|D| = N!\) be the set of all permutations of vector \( v \). Let \( \mathcal{D}_\tau := \{u \in \mathbb{R}^\tau \mid u_i = \tilde{v}_i \ \forall i = 1, 2, \ldots, \tau \ \forall \tilde{v} \in D\} \), which has cardinality \( N!/(N - \tau)! \). The previous is obtained by noticing that each component in \( \mathcal{D}_\tau \) corresponds to \((N - \tau)!\) components in \( D \). By ignoring ordering in set \( \mathcal{D}_\tau \) we obtain the set \( \mathcal{D}_\tau := \{u \in \mathbb{R}^\tau \mid u \in \mathcal{D}_\tau \text{ where ordering matters}\} \), which has cardinality \(|\mathcal{D}_\tau| = \binom{N}{\tau}\), since if we ignore ordering then every set in \( \mathcal{D}_\tau \) appears \( \tau! \) times. Notice that \( \mathcal{D}_\tau \) is the set of all subsets of \( 2^{[N]} \) with cardinality equal to \( \tau \). Moreover, let \( h_\tau(\tilde{v}) = [\tilde{v}_1, \tilde{v}_2, \ldots, \tilde{v}_\tau] \) be the function that maintains only the first \( \tau \) components of the input vector.

Let us assume that we use a procedure which generates uniformly at random permutations of \( v \); each random permutation \( \tilde{v} \in D \) has \( 1/N! \) probability to be selected. Then we use the indices in \( h_\tau(\tilde{v}) \) as the
coordinates to work on for the current iteration in Step 4 of FCD Algorithm. It is easy to show that for all \( \tilde{v} \in \mathcal{D}, h_r(\tilde{v}) \in \tilde{D}_r \) has probability \( 1/(N_r) \) to be selected. Notice that this procedure generates subsets of indices/coordinates with cardinality \( \tau \) in \( v \) and (21) are satisfied, giving an ‘inexact’ search direction.

Fortunately there are many algorithms that calculate uniformly at random permutations of vector \( v \) in \( O(N) \) time, which also have very low memory requirements. For example the Knuth shuffling algorithm in page 145 in [18].

### 3.2 The search direction and Hessian approximation (Step 4)

In this section we describe how FCD determines the search direction. In particular, FCD forms a quadratic model for subset \( S \), and minimizes the model approximately until the stopping conditions (20) and (21) are satisfied, giving an ‘inexact’ search direction.

We also describe the importance of the choice of matrix \( H \), which is an approximate second order information term. From now on, we will often use the shorthand \( H_k^S \equiv H^S(x_k) \).

#### 3.2.1 The search direction

At each iteration the update/search direction is found as follows. The subproblem (19), (where \( Q_S(x_k; t^S) \) is defined in (15)) is approximately solved, and the search direction \( t_k^S \) is accepted when the stopping conditions (20) and (21) are satisfied, for some \( \eta_k^S \in [0,1] \). Notice that

\[
Q(x; U_k t^S) - Q(x; 0) \overset{(14)}{=} \langle \nabla_S f(x), t^S \rangle + \frac{1}{2} \| t^S \|^2_{H^S} + \Psi(x + U_k t^S) - \Psi(x)
\]

\[
\overset{(9)}{=} \langle \nabla_S f(x), t^S \rangle + \frac{1}{2} \| t^S \|^2_{H^S} + \Psi_S(x^S + t^S) - \Psi_S(x^S).
\]

Hence, from (27), the stopping conditions (20) and (21) depend on subset \( S \) only, and are therefore inexpensive to verify, meaning that they are implementable.

**Remark 7.**

(i) At some iteration \( k \), it is possible that \( g_S(x_k; 0) = 0 \). In this case, it is easy to verify that the optimal solution of subproblem (19) is \( t_k^S = 0 \). Therefore, before calculating \( t_k^S \) we check a-priori if condition \( g_S(x_k; 0) = 0 \) is satisfied.

(ii) Notice that, unless at optimality (i.e., \( g(x_k; 0) = 0 \)), there will always be a subset \( S \) such that \( g_S(x_k; 0) \neq 0 \), which implies that \( t_k^S \neq 0 \). Hence, FCD will not stagnate.

#### 3.2.2 The Hessian approximation

Arguably, the most important feature of this method is that the quadratic model (15) incorporates second order information in the form of a positive definite matrix \( H_k^S \). This is key because, depending upon the choice of \( H_k^S \), it makes the method robust. Moreover, at each iteration, the user has complete freedom over the choice of \( H_k^S \).

We now provide a few suggestions for the choice of \( H_k^S \). (This list is not intended to be exhaustive.) Notice that in each case there is a trade off between a matrix that is inexpensive to work with, and one that is a more accurate representation of the true block Hessian.

1. Clearly, the simplest option is to set \( H_k^S = I \) for all \( i \) and \( k \). In this case no second order information is employed by the method.

2. A second option is to let \( H_k^S = \text{diag}(\nabla^2_S f(x_k)) \). In this case \( H_k^S \) and it’s inverse are inexpensive to work with. Moreover, if \( f \) is quadratic, then \( \nabla^2 f(x_k) \) is constant for all \( k \), so \( H = \text{diag}(\nabla^2 f(x)) \) can be computed and stored at the start of the algorithm and elements can be accessed throughout the algorithm as necessary. This is very effective if \( \text{diag}(\nabla^2 f(x)) \) is a good approximation to \( \nabla^2 f(x) \).

3. A third option is to let \( H_k^S = \nabla^2_S f(x_k) \) (i.e., \( H_k^S \) is a principal minor of the Hessian). In this case, \( H_k^S \) provides the most accurate second order information, but it is (potentially) more computationally expensive to work with. In practice the matrix \( \nabla^2_S f(x_k) \) is used in a matrix-free way and is not
We now proceed to show that the conditions are mathematically sound. Let us start by rewriting (21) for search directions.

η hold.) On the other hand, if η in FCD, then the quadratic model (19) is minimized exactly, leading to exact search directions. (i.e., ∇ projection onto an inner product space Ω :=

The left hand side in (28) is a continuous function. This is because the inf operator is an orthonormal projection onto an inner product space Ω := ∇sf(xk) + ∂ΨS(xk; tS2k), see (18), and ∥gs(xk; tS2k)∥^2 is continuous as a function of tS2k. Furthermore, there exists a point tS2k such that (28) is satisfied, and this is the minimizer of (19). In particular, if tS2k is the minimizer of (19), then from Theorem 3 we have that gs(xk; tS2k) = 0 ∈ Ω. Hence, the left hand side in (28) is zero and the condition is satisfied for any η ∈ [0, 1] and gs(xk; 0). Since the right hand side in (28) is a non-negative constant, the left hand side is continuous and the minimizer of (19) satisfies (28), then there exists a closed ball centered at the minimizer such that within this ball (28) is satisfied. Therefore, any convergent method which can be used to minimize (19) will eventually satisfy the termination condition (28) without solving (19) exactly, unless the right hand side in (28) is zero.

We stress that this projection operation is inexpensive for the applications that we consider, e.g., when Ψ is the ℓ1 norm or the ℓ2 norm. (See Section 7 for concrete examples.)

4. Another option is to use a quasi-Newton type approach where HkS is an approximation to ∇^2f(xk) based on the limited-memory BFGS update scheme, see [26, Section 8]. This approach might be more suitable in cases that the problem is not very ill-conditioned and additionally performing matrix-vector products with ∇^2f(xk) is expensive.

Remark 8. We have the following remarks regarding the choice of HkS.

1. If any of the matrices above are not positive definite, then they can be altered to make them so. For example, if HkS is diagonal, any zero that appears on the diagonal can be replaced with a positive number. Moreover, if ∇^2f(xk) is not positive definite, a multiple of the identity can be added to it.

2. If HkS is chosen to be a diagonal matrix, then the quadratic model (15) is separable, so the subproblems for each coordinate in S are independent. Moreover, in some cases this gives rise to a closed form solution for the search direction tS2k. (For example, if Ψ(x) ≡ ||x||1, then soft thresholding may be used.)

An advantage of the FCD algorithm (if Option 3 is used for HkS) is that all elements of the Hessian can be accessed. This is because the set of coordinates can change at every iteration, and so too can the Hessian. The importance of incorporation of information from off-diagonal elements is demonstrated in Section 7.

3.2.3 Termination criteria for model minimization

In Step 4 of FCD, the termination criteria (20) and (21) are used to determine whether an acceptable solution is demonstrated in Section 7.

Elements for each coordinate in S are independent. Moreover, in some cases this gives rise to a closed form solution for the search direction tS2k. (For example, if Ψ(x) ≡ ||x||1, then soft thresholding may be used.)

An advantage of the FCD algorithm (if Option 3 is used for HkS) is that all elements of the Hessian can be accessed. This is because the set of coordinates can change at every iteration, and so too can matrix HkS. This makes FCD extremely flexible and is particularly advantageous when there are large off diagonal elements in the Hessian. The importance of incorporation of information from off-diagonal elements is demonstrated in Section 7.

In Step 4 of FCD, the termination criteria (20) and (21) are used to determine whether an acceptable solution is demonstrated in Section 7.
3.3 The line search (Step 5)

The stopping conditions (20) and (21) ensure that $t^S_k$ is a descent direction, but if the full step $x_k + USt_k^S$ is taken, a reduction in the function value (1) is not guaranteed. To this end, we include a line search step in our algorithm in order to guarantee monotonic decrease of function $F$. Essentially, the line search guarantees the sufficient decrease of $F$ at every iteration, where sufficient decrease is measured by the loss function (23).

In particular, for fixed $\theta \in (0, 1)$, we require that for some $\alpha \in (0, 1]$, (22) is satisfied. (In Lemma 11 we prove the existence of such and $\alpha$.) Notice that

$$
\ell(x; USt^S) - \ell(x; 0) \overset{(23)}{=} \langle \nabla_S f(x), t^S \rangle + \Psi(x + USt^S) - \Psi(x),
$$

where $\Psi(\cdot)$ is taken, a reduction in the function value (1) is not guaranteed. To this end, we include a line search step in our algorithm in order to guarantee monotonic decrease of function $F$. Essentially, the line search guarantees the sufficient decrease of $F$ at every iteration, where sufficient decrease is measured by the loss function (23).

3.4 Concrete examples

3.4.1 Quadratic loss plus regularization example

Suppose that $f(x) = \frac{1}{2} \|Ax - b\|^2$ and $\Psi(x) \neq 0$, where $A \in \mathbb{R}^{m \times N}$, $b \in \mathbb{R}^m$ and $x \in \mathbb{R}^N$. Then

$$
F(x_k) - F(x_k + \alpha USt^S_k) \overset{(9)}{=} f(x_k) + \Psi_S(x^S) - f(x_k + \alpha USt^S_k) - \Psi_S(x^S_k + \alpha t^S_k)
$$

$$
= \Psi_S(x^S) - \alpha \langle \nabla_S f(x), t^S_k \rangle - \frac{\alpha^2}{2} \|A_1 t^S_k\|^2 - \Psi_S(x^S_k + \alpha t^S_k).
$$

Notice that calculation of $F(x_k) - F(x_k + \alpha USt^S_k)$ as a function of $\alpha$ only depends on subset $S$, hence, it is inexpensive. Moreover, in some cases some of the quantities in (30) are already needed in the computation of the search direction $t$, so regarding the line search step, they essentially come “for free”.

3.4.2 Logistic regression example

Suppose that

$$
f(x) = \sum_{j=1}^{m} \log(1 + e^{-b_j a_j^T x}) \quad \text{and} \quad \Psi(x) \neq 0,
$$

where $a_j^T$ is the $j$th row of a matrix $A \in \mathbb{R}^{m \times n}$ and $b_j$ is the $j$th component of vector $b \in \mathbb{R}^m$. As before, we need to evaluate (30). Let us split calculation of $F(x_k) - F(x_k + \alpha USt^S_k)$ in parts. The first part $\Psi_S(x^S) - \Psi_S(x^S_k + \alpha t^S_k)$ is inexpensive, since it depends only on subset $S$. The second part $f(x_k) - f(x_k + \alpha USt^S_k)$ is more expensive because is depends upon the logarithm. In this case, one can calculate $f(x_0)$ once at the beginning of the algorithm and then update $f(x_k + \alpha USt^S_k)$ for $k \geq 1$ less expensively. In particular, let us assume that the following terms:

$$
e^{-b_j a_j^T x_0} \quad \forall j \quad \text{and} \quad f(x_0) = \sum_{j=1}^{m} \log(1 + e^{-b_j a_j^T x_0}),
$$

are calculated once and stored in memory. Then, at iteration $1$, the calculation of $f(x_0 + \alpha USt^S_0) = \sum_{j=1}^{m} \log(1 + e^{-b_j a_j^T x_0} e^{-a_j^T (USt^S_0)})$ is required for different values of $\alpha$ by the backtracking line search algorithm. The most demanding task in calculating $f(x_0 + \alpha USt^S_0)$ is the calculation of the products $b_j a_j^T (USt^S_0) \forall j$ once, which is inexpensive since this operation depends only on subset $S$. Having $b_j a_j^T (USt^S_0) \forall j$ and (31) calculation of $f(x_0) - f(x_0 + \alpha USt^S_0)$ for different values of $\alpha$ is inexpensive. At the end of the process, $f(x_1)$ and $e^{-b_j a_j^T x_1} \forall j$ will be given for free, and the same process can be followed for the calculation of $f(x_1) - f(x_1 + \alpha USt^S_1)$ etc.
4 Bounded step-size and monotonic decrease of $F$

Throughout this section we denote $H^S_k \equiv H^S(x_k)$. The following assumptions are made about $H^S_k$ and $f$. Assumption 9 explains that the Hessian approximation $H^S_k$ must be positive definite for all subsets of coordinates $S$ at all iterations $k$. Assumption 10 explains that $f$ must be Lipschitz for all subsets $S$.

**Assumption 9.** There exist $0 < \lambda_S \leq \Lambda_S$, such that the sequence $\{H^S_k\}_{k \geq 0}$ satisfies:

$$0 < \lambda_S \leq \lambda_{\min}(H^S_k) \quad \text{and} \quad \lambda_{\max}(H^S_k) \leq \Lambda_S, \quad \text{for all } S \subseteq [N]. \tag{32}$$

**Assumption 10.** The function $f$ is smooth and satisfies (10) for all $S \subseteq [N]$.

The following lemma shows that if $t^S_k$ is nonzero, then $F$ is decreased.

**Lemma 11.** Let Assumptions 9 and 10 hold. Let $\theta \in (0, 1/2)$ and let $x_k$ and $S \subseteq [N]$ be generated by FCD. Then Step 6 of FCD will accept a step-size $\alpha$ that satisfies

$$\alpha \geq \alpha_S \quad \text{where} \quad \alpha_S := (1 - \theta)\frac{\lambda_S}{2L_S}. \tag{33}$$

Furthermore,

$$F(x_k) - F(x_k + \alpha U_S t^S_k) > \theta(1 - \theta)\frac{\lambda^2_S}{4L^2_S}||t^S_k||^2. \tag{34}$$

**Proof.** The proof closely follows that of [3, Theorem 3.1]. From (20),

$$0 > Q(x_k; U_S t^S_k) - Q(x_k; 0) = \ell(x_k; U_S t^S_k) - \ell(x_k; 0) + \frac{1}{2}||t^S_k||^2_{H^S_k}. \tag{35}$$

Rearranging gives

$$\ell(x_k; 0) - \ell(x_k; U_S t^S_k) > \frac{1}{2}||t^S_k||^2_{H^S_k} \geq \frac{1}{2}\lambda_S||t^S_k||^2. \tag{36}$$

Denote $x_{k+1} = x_k + \alpha U_S t^S_k$. By Assumption 10, for $\alpha \in (0, 1]$ we have

$$F(x_{k+1}) \leq f(x_k) + \alpha \langle \nabla_S f(x_k), t^S_k \rangle + \frac{L^2_S}{2}\alpha^2||t^S_k||^2 + \Psi(x_k + \alpha U_S t^S_k).$$

Adding $\Psi(x_k)$ to both sides of the above and rearranging gives

$$F(x_k) - F(x_{k+1}) \geq -\alpha \langle \nabla_S f(x_k), t^S_k \rangle - \frac{L^2_S}{2}\alpha^2||t^S_k||^2 - \Psi(x_k + \alpha U_S t^S_k) + \Psi(x_k) \tag{37}$$

By convexity of $\Psi(x)$ we have that

$$\ell(x_k; 0) - \ell(x_k; \alpha U_S t^S_k) \geq \alpha(\ell(x_k; 0) - \ell(x_k; U_S t^S_k)). \tag{38}$$

Then

$$F(x_k) - F(x_{k+1}) = \theta(\ell(x_k; 0) - \ell(x_k; \alpha U_S t^S_k)) \tag{37}$$

$$\geq (1 - \theta)(\ell(x_k; 0) - \ell(x_k; \alpha U_S t^S_k)) - \frac{L^2_S}{2}\alpha^2||t^S_k||^2 \tag{38}$$

$$\geq \alpha(1 - \theta)(\ell(x_k; 0) - \ell(x_k; U_S t^S_k)) - \frac{L^2_S}{2}\alpha^2||t^S_k||^2 \tag{36}$$

$$\geq \frac{1}{2}\alpha(1 - \theta)\lambda_S||t^S_k||^2 - L_S\alpha^2||t^S_k||^2 \tag{38}$$

$$= \frac{\alpha}{2}(1 - \theta)\lambda_S - L_S\alpha||t^S_k||^2$$

$$\geq 0, \tag{39}$$

13
if \((1 - \theta)\lambda_S - L_S \alpha \geq 0\). Therefore, we observe that if \(\alpha\) satisfies \(0 \leq \alpha \leq (1 - \theta)\frac{\lambda_S}{2L_S}\), then \(\alpha\) also satisfies the backtracking line search step of FCD (a nonnegative function value reduction is obtained). Suppose that any \(\alpha\) that is rejected by the line search is halved for the next line search trial. Then, it is guaranteed that the \(\alpha\) that is accepted satisfies (33).

We now show that (34) holds. By the previous arguments, the line search condition (22) is guaranteed to be satisfied for some step size \(\alpha\). Then,

\[
F(x_k) - F(x_{k+1}) \geq \theta(\ell(x_k; 0) - \ell(x_k; U_S t_k^S)) \tag{22}
\]

\[
\geq \theta \alpha (\ell(x_k; 0) - \ell(x_k; U_S t_k^S)) \tag{38}
\]

\[
\geq \frac{\alpha \theta}{2} \|t_k^S\|_{H_k^S}^2 \tag{36}
\]

\[
\geq \frac{\theta(1 - \theta)\lambda_S}{4L_S} \|t_k^S\|_{H_k^S}^2. \tag{33}
\]

Finally, using (32) in (40) gives (34).

The following lemma bounds the norm of the direction \(t_k^S\) in terms of \(g_S(x_k, 0)\).

**Lemma 12.** Let Assumptions 9 and 10 hold and suppose that \(x_k, t_k^S, \alpha\) and \(S \subseteq [N]\) are generated by FCD. Then we have

\[
\|t_k^S\| \geq \gamma_S \|g_S(x_k; 0)\|, \quad \text{where} \quad \gamma_S := \frac{1 - \eta_k^S}{1 + 2\Lambda_S} \tag{41}
\]

and \(\eta_k^S \in [0,1)\) is defined in (21). Moreover,

\[
F(x_k) - F(x_k + \alpha U_S t_k^S) > \theta(1 - \theta) \frac{\lambda_S^2}{4L_S} \gamma_S^2 \|g_S(x_k; 0)\|^2. \tag{42}
\]

**Proof.** This proof closely follows that of [3, Theorem 3.1]. From the termination condition (21) we have that:

\[
\|g_S(x_k; t_k^S)\| \leq \eta_k^S \|g_S(x_k; 0)\|. \tag{43}
\]

Using the reverse triangular inequality and the fact that \(\text{prox}_{\eta S} (\cdot)\) is nonexpansive, we have that

\[
(1 - \eta_k^S)\|g_S(x_k; 0)\| \leq \|g_S(x_k; 0)\| - \|g_S(x_k; t_k^S)\| \tag{43}
\]

\[
\leq \|g_S(x_k; t_k^S) - g_S(x_k; 0)\| \tag{44}
\]

\[
(1 - \eta_k^S)\|g_S(x_k; 0)\| \leq \|H_k^S t_k^S + \text{prox}_{\eta S} (x_k^S + t_k^S - (\nabla_S f(x_k) + H_k^S t_k^S)) \|
\]

\[
- \text{prox}_{\eta S} (x_k^S - \nabla_S f(x_k)) \|
\]

\[
\leq \|H_k^S t_k^S\| + \|(I - H_k^S) t_k^S\|
\]

\[
\leq (1 + 2\|H_k^S\|)\|t_k^S\|
\]

\[
\leq (1 + 2\Lambda_S)\|t_k^S\|.
\]

Rearranging gives (41), and combining (41) with (34) gives (42).

\[\square\]

## 5 Technical results

In this section we will present several technical results that will be necessary when establishing our main iteration complexity results for FCD. The first result (Theorem 13) is taken from [30] and will be used to finish all our iteration complexity results for FCD. The second result (Lemma 14) provides an upper bound on the decrease in the quadratic model that is obtained when an inexact search direction is used in FCD. The final two results provide upper bounds on difference of function values \(F(x_{k+1}) - F(x_k)\) (Lemma 16) and the expected distance to the solution \(E[F(x_{k+1}) - F^*]\) (Lemma 18).

The following Theorem will be used at the end of all our proofs to obtain iteration complexity results for FCD.
Theorem 13 (Theorem 1 in [30]). Fix $\xi_0 \in \mathbb{R}^N$ and let $\{\xi_k\}_{k \geq 0}$ be a sequence of random vectors in $\mathbb{R}^N$ with $\xi_{k+1}$ depending on $\xi_k$ only. Let $\{\xi_k\}_{k \geq 0}$ be a nonnegative non increasing sequence of random variables and assume that it has one of the following properties:

(i) $E[\xi_{k+1}|\xi_k] \leq \xi_k - \frac{\xi^2}{c_1}$ for all $k$, where $c_1 > 0$ is a constant,

(ii) $E[\xi_{k+1}|\xi_k] \leq (1 - \frac{1}{c_2})\xi_k$ for all $k$ such that $\xi_k \geq \epsilon$, where $c_2 > 1$ is a constant.

Choose accuracy level $0 < \epsilon < \xi_0$, confidence level $\rho \in (0, 1)$. If (i) holds and we choose $\epsilon < c_1$ and

$$K \geq \frac{c_1}{\epsilon} \left(1 + \ln \frac{1}{\rho}\right) + 2 - \frac{c_1}{\xi_0} \quad (44)$$

or if property (ii) holds and we choose

$$K \geq c_2 \ln \frac{\xi_0}{\epsilon \rho} \quad (45)$$

then $P(\xi_K \leq \epsilon) \geq 1 - \rho$.

The following result gives an upper bound on the decrease in the quadratic model when an inexact update is used in FCD.

Lemma 14. Let Assumptions 9 and 10 hold and let $t^S_k$ be the exact minimizer of subproblem (19). Given $x_k \in \mathbb{R}^N$, let $x_{k+1} = x_k + \alpha U_S t^S_k$ be generated by FCD (Algorithm 1). Then

$$Q(x_k; U_S t^S_k) - Q(x_k; 0) \leq Q(x_k; U_S t^S_k) - Q(x_k; U_S t^S_k)$$

The term $Q(x_k; U_S t^S_k) - Q(x_k; U_S t^S_k)$ can be bounded using strong convexity of $Q_S$ (see [2, p.460]), which implies:

$$Q(x_k; U_S t^S_k) - Q(x_k; U_S t^S_k) \leq \frac{1}{2\lambda_S} \|u\|^2 \quad \forall u \in \partial Q(x_k; U_S t^S_k).$$

Hence,

$$Q(x_k; U_S t^S_k) - Q(x_k; U_S t^S_k) \leq \frac{1}{2\lambda_S} \|u + g_S(x_k; t^S_k) - g_S(x_k; t^S_k)\|^2 \leq \frac{1}{2\lambda_S} \left(\|u - g_S(x_k; t^S_k)\|^2 + \|g_S(x_k; t^S_k)\|^2\right).$$

The previous inequality holds for every $u \in \partial Q(x_k; U_S t^S_k)$. Therefore, using termination condition (21), where $\Omega$ is defined in (18), we have that

$$Q(x_k; U_S t^S_k) - Q(x_k; U_S t^S_k) \leq \frac{1}{2\lambda_S} \left(\inf_{u \in \Omega} \|u - g_S(x_k; t^S_k)\|^2 + \|g_S(x_k; t^S_k)\|^2\right) \leq \frac{1}{2\lambda_S} \left(\eta^S \|g_S(x_k; 0)\|^2\right)^2.$$

Using Lemma 12 in the previous inequality we get

$$Q(x_k; U_S t^S_k) - Q(x_k; U_S t^S_k) \leq \frac{2(\eta^S)^2 L_S}{\theta(1 - \theta)\lambda_S^2} (F(x_k) - F(x_{k+1})), $$

which, combined with (46), gives the result. 

\[\Box\]
In the next Lemma we establish an upper bound on the difference between consecutive function values obtained using FCD. First, we require the following assumption.

**Assumption 15.** The relation $\lambda_S \leq L_S$ holds for all $S \subseteq [n]$.

We now ready to present the Lemma.

**Lemma 16.** Let Assumptions 9, 10 and 15 hold and let $t^S_k$ denote the exact minimizer of subproblem (19). Given $x_k$, let $x_{k+1} = x_k + U_S t^S_k$ be generated by FCD (Algorithm 1), and recall that $\theta$ and $\gamma_S$ are defined in (22) and (41), respectively. Fix $\bar{\eta} \in [0,1)$, and let $\eta^S_k \in [0, \bar{\eta}]$ for all $k$ and all $S \in 2^{[N]}$. Then

$$F(x_{k+1}) - F(x_k) \leq \chi(\eta) (Q(x_k; U_S t^S_k) - Q(x_k; 0)), \quad (47)$$

where

$$\chi(\eta) := \min_{S \in 2^{[N]} : |S| = \tau} \frac{\alpha_S}{1 + \frac{\bar{\eta}^2}{\theta \lambda_S \gamma_S^2} + \frac{L_S - \lambda_S}{\lambda_S \theta}}. \quad (48)$$

**Proof.** Using block Lipschitz continuity of $f$ (11), and the definition of $Q$ (14) we get

$$F(x_{k+1}) \leq F(x_k) + (Q(x_k; U_S t^S_k) - Q(x_k; 0)) = \frac{\alpha^2}{2} \| t^S_k \|^2_{H^S_k} + \frac{2L_S}{\lambda_S} \| t^S_k \|^2.$$  

Using convexity of $Q$ with respect to $\alpha$ we get

$$F(x_{k+1}) \leq F(x_k) + \alpha_S (Q(x_k; U_S t^S_k) - Q(x_k; 0)) + \frac{\alpha^2}{2} \left( \frac{L_S}{\lambda_S} - 1 \right) \| t^S_k \|^2_{H^S_k}.$$  

Recall that $Q(x_k; U_S t^S_k) - Q(x_k; 0) \leq 0$ by (20), and $\alpha \geq \alpha_S$ by (33). Hence, we can replace $\alpha$ with $\alpha_S$ to get

$$F(x_{k+1}) \leq F(x_k) + \alpha_S (Q(x_k; U_S t^S_k) - Q(x_k; 0)) + \frac{\alpha_S^2}{2} \left( \frac{L_S}{\lambda_S} - 1 \right) \| t^S_k \|^2_{H^S_k}. \quad (49)$$

By rearranging (40) we get $\frac{\alpha}{2} \| t^S_k \|^2_{H^S_k} \leq \frac{1}{\theta} (F(x_k) - F(x_{k+1})).$ Using this in (49) gives

$$F(x_{k+1}) \leq F(x_k) + \alpha_S (Q(x_k; U_S t^S_k) - Q(x_k; 0)) + \frac{L_S}{\lambda_S} - 1 \frac{1}{\theta} (F(x_k) - F(x_{k+1})).$$

By Assumption 15, $L_S \geq \lambda_S$, so we can replace $\alpha \in (0,1]$ with $1$ to get

$$F(x_{k+1}) \leq F(x_k) + \alpha_S (Q(x_k; U_S t^S_k) - Q(x_k; 0)) + \frac{L_S}{\lambda_S} - 1 \frac{1}{\theta} (F(x_k) - F(x_{k+1})). \quad (50)$$

Using Lemma 14 and the definition of $\bar{\eta}$ we get

$$F(x_{k+1}) \leq F(x_k) + \alpha_S (Q(x_k; U_S t^S_k) - Q(x_k; 0)) + \left( \frac{2 \alpha_S \bar{\eta}^2 L_S}{\theta (1 - \theta) \lambda_S \gamma_S^2} + \frac{L_S}{\lambda_S} - 1 \frac{1}{\theta} \right) (F(x_k) - F(x_{k+1})) \quad (51)$$

$$\leq F(x_k) + \alpha_S (Q(x_k; U_S t^S_k) - Q(x_k; 0)) + \left( \frac{\bar{\eta}^2}{\theta \lambda_S \gamma_S^2} + \frac{L_S}{\lambda_S} - 1 \frac{1}{\theta} \right) (F(x_k) - F(x_{k+1})). \quad (52)$$

Rearranging (51), using the definition of $\chi(\eta)$ in (48), and noticing that $Q(x_k; U_S t^S_k) - Q(x_k; 0) < 0$, gives the result (47). \(\blacksquare\)

**Remark 17.** Notice that $\chi(\eta)$ simplifies to

$$\chi(\eta) = \min_{S \in 2^{[N]} : |S| = \tau} \frac{\theta (1 - \theta) \lambda_S^2 \gamma_S^2}{2L_S \left( \bar{\eta}^2 + \lambda_S \gamma_S^2 (L_S - (1 - \theta) \lambda_S) \right)}. \quad (53)$$
Further, when exact directions are used in FCD (i.e., \( \eta = 0 \)), we have that

\[
\chi(0) = \min_{S \in 2^{[N]} : |S| = \tau} \frac{\theta(1 - \theta)\lambda_S^2}{2L_S^2 - 2(1 - \theta)\lambda\lambda_S L_S}.
\]

(54)

That is, when inexact directions are used, \( \chi(\tilde{\eta}) \) depends cubically on \( \lambda_S \), whereas is inexact directions are used, \( \chi(\tilde{\eta}) \) only depends on \( \lambda_S^2 \).

The final result in this section gives an upper bound on the expected distance between the current function value and the optimal value.

**Lemma 18.** Let Assumptions 9, 10 and 15 hold, let \( \mu_f \geq 0 \) be the strong convexity constant of \( f \) and fix \( \tilde{\eta} \in [0, 1] \). Given \( x_k \), let \( x_{k+1} = x_k + \alpha U_t \tilde{S}_k \) be generated by FCD (Algorithm 1) using the sampling procedure described in Subsection 3.1.2 with \( \eta_k \in [0, \tilde{\eta}] \) and \( 1 \leq \tau \leq N \). Then

\[
\mathbb{E}[F(x_{k+1}) - F^* | x_k] \leq \left(1 - \frac{\tau \chi(\tilde{\eta}) \zeta}{N}\right) (F(x_k) - F^*) - (\mu_f - \Lambda_{\text{max}} \zeta) \frac{\tau \chi(\tilde{\eta}) \zeta}{2N} \|x_k - x^*\|^2,
\]

where \( \zeta \in [0, 1], \chi(\tilde{\eta}) \) is defined in (48) and

\[
\Lambda_{\text{max}} := \max_{S \in 2^{[N]} : |S| = \tau} \Lambda_S.
\]

(55)

**Proof.** Since all the assumptions of Lemma 16 are satisfied, we have

\[
F(x_{k+1}) - F(x_k) \leq \chi(\tilde{\eta})(Q(x_k; U_t \tilde{S}_k) - Q(x_k; 0)).
\]

(56)

Notice that \( t_S \) is the vector that makes the difference \( Q(x_k; U_t \tilde{S}_k) - Q(x_k; 0) \), as small/negative as possible for the subspace defined by the columns of \( U_S \). Therefore, for any vector \( y_k \neq t_S \) we have

\[
Q(x_k; U_t \tilde{S}_k) - Q(x_k; 0) \leq Q(x_k; U_t y_k) - Q(x_k; 0).
\]

Choosing \( y_k = \zeta(x^*_k - x_k^S) \) for any \( \zeta \in [0, 1] \) and replacing \( y_k \) in (56) gives

\[
F(x_{k+1}) - F(x_k) \leq \chi(\tilde{\eta})(Q(x_k; U_S(x^*_k - x_k^S)) - Q(x_k; 0))
\]

\[
\leq \chi(\tilde{\eta}) \left( \zeta (\langle \nabla f(x_k), U_S(x^*_k - x_k^S) \rangle) + \frac{\zeta^2}{2} \|x^*_k - x_k^S\|_{H_k}^2 + \Psi_S(x_k^S + \zeta(x^*_k - x_k^S) - \Psi_S(x_k^S)) \right).
\]

Using convexity of \( \Psi \), (32) and the definition \( \Lambda_{\text{max}} \) (55) we get

\[
F(x_{k+1}) - F(x_k) \leq \chi(\tilde{\eta}) \zeta \left( \langle \nabla f(x_k), U_S(x^*_k - x_k^S) \rangle + \Psi_S(x_k^S) - \Psi_S(x_k^S) \right)
\]

\[
+ \chi(\tilde{\eta}) \Lambda_{\text{max}} \zeta^2 \|x^*_k - x_k^S\|^2.
\]

(57)

Taking expectation of (57) and using convexity of \( f \) gives

\[
\mathbb{E}[F(x_{k+1}) - F(x_k) | x_k] \leq \mathbb{E}[\chi(\tilde{\eta}) \zeta \left( \langle \nabla f(x_k), x^*_k - x_k \rangle + \Psi(x^*_k) - \Psi(x_k) \right) + \chi(\tilde{\eta}) \Lambda_{\text{max}} \zeta^2 \|x^*_k - x_k^S\|^2.
\]

(26)

Rearranging and adding/subtracting \( F^* \) gives the desired result. \( \square \)
6 Iteration complexity results for inexact FCD

In this section we present iteration complexity results for FCD applied to problem (1) when an inexact update is used.

6.1 Iteration complexity in the convex case

In this section we present iteration complexity results for FCD in the convex case.

Theorem 19. Let $F$ be the convex function defined in (1) and let Assumptions 9, 10 and 15 hold. Choose an initial point $x_0 \in \mathbb{R}^N$, a target confidence $\rho \in (0, 1)$, fix $\bar{\eta} \in [0, 1)$, and recall that $\chi(\bar{\eta})$ is defined in (48). Let the target accuracy $\epsilon$ and iteration counter $K$ be chosen in either of the following two ways:

(i) Let $m_1 = \max\{\mathcal{R}^2(x_0), F(x_0) - F^*\}$, $\epsilon < F(x_0) - F^*$ and

\[
K \geq \frac{2N}{\tau\chi(\bar{\eta})} \frac{m_1}{\epsilon} \left(1 + \ln \frac{1}{\rho}\right) + 2 - \frac{2N}{\tau\chi(\bar{\eta})}(F(x_0) - F^*)^2,
\]  

(58)

(ii) Let $\epsilon < \min\{\mathcal{R}^2(x_0), F(x_0) - F^*\}$ and

\[
K \geq \frac{2N}{\tau\chi(\bar{\eta})} \frac{\mathcal{R}^2(x_0)}{\epsilon} \ln \frac{F(x_0) - F^*}{\epsilon \rho}.
\]  

(59)

If $\{x_k\}_{k \geq 0}$ are the random points generated by FCD (Algorithm 1) using the sampling procedure described in Subsection 3.1.2, with $\bar{\eta}^S_k \in [0, \bar{\eta}]$ for all $k$ and all $S \in 2^{[N]}$, then $P(F(x_K) - F^* \leq \epsilon) \geq 1 - \rho$.

Proof. By Lemma 11 we have that $F(x_k) \leq F(x_0)$ for all $k$, which means that $\|x_k - x_*\| \leq \mathcal{R}(x_0)$ for all $x_* \in X^*$, where $\mathcal{R}(x_0)$ is defined in (12). Then, using Lemma 18 with $\mu_f = 0$ gives

\[
E[F(x_{k+1}) - F^*|x_k] \leq \left(1 - \frac{\tau\chi(\bar{\eta})}{N}\right)(F(x_k) - F^*) + \frac{\tau\chi(\bar{\eta})\Lambda_{\max}\xi^2}{2N}\mathcal{R}^2(x_0).
\]  

(60)

Minimizing the right hand side of the previous with respect to $\xi$ gives:

\[
\xi = \min \left\{ \frac{F(x_k) - F^*}{\Lambda_{\max}\mathcal{R}^2(x_0)^2}, 1 \right\},
\]

(61)

so using $\xi$ in (61), (60) becomes

\[
E[F(x_{k+1}) - F^*|x_k] \leq \left\{ \begin{array}{l l}
1 - \frac{\tau\chi(\bar{\eta})}{2N}\frac{F(x_k) - F^*}{\Lambda_{\max}\mathcal{R}^2(x_0)} & \text{if } F(x_k) - F^* \leq \Lambda_{\max}\mathcal{R}^2(x_0) \\
1 - \frac{\tau\chi(\bar{\eta})}{2N}(F(x_k) - F^*) & \text{otherwise.}
\end{array} \right.
\]

We can combine this into a single equation:

\[
E[F(x_{k+1}) - F^*|x_k] \leq \left(1 - \frac{\tau\chi(\bar{\eta})}{2N}\min \left\{ \frac{F(x_k) - F^*}{\Lambda_{\max}\mathcal{R}^2(x_0)^2}, 1 \right\} \right)(F(x_k) - F^*).
\]  

(62)

So, at any particular iteration of FCD, either of the two options inside the ‘min’ in (62) could be the smallest. However, notice that $\Lambda_{\max}\mathcal{R}^2(x_0)$ is fixed throughout the iterations, and $F(x_k) - F^*$ is becoming smaller as the iterations progress (as was established in Lemma 11). Therefore, eventually it will be the case that the first term inside the ‘min’ in (62) will be the smallest. Thus, it is enough to study (62) under the condition that $F(x_k) - F^* \leq \Lambda_{\max}\mathcal{R}^2(x_0)$. Hence, we now study the following two cases.

(i) Letting $c_1 = \frac{2N}{\tau\chi(\bar{\eta})}\max\{\Lambda_{\max}\mathcal{R}^2(x_0), F(x_0) - F^*\}$, we see that (62) becomes

\[
E[F(x_{k+1}) - F^*|x_k] \leq \left(1 - \frac{\tau\chi(\bar{\eta})}{2N}\frac{F(x_k) - F^*}{\Lambda_{\max}\mathcal{R}^2(x_0)} \right)(F(x_k) - F^*)
\]

\[
\leq \left(1 - \frac{F(x_k) - F^*}{c_1} \right)(F(x_k) - F^*).
\]

Notice that for this choice of $c_1$, we have $\epsilon < F(x_0) - F^* < c_1$, so it suffices to apply Theorem 13(i) and the result follows.
Assumption 22. We assume that $\nabla \eta$ in Subsection 3.1.2, with
\[
\{ \mu \text{ if } \mu \text{ because } \}
\]
It is straightforward to show that $\mu f$, are strongly convex. In this case the strong convexity parameters of $f$ and $F$ satisfy $\mu f > 0$ and $\mu F > 0$. Moreover, $\mu f \leq \mu F$.

Before presenting our iteration complexity results, we will make the following assumption.

**Assumption 22.** We assume that $\Lambda_{\text{max}} > \mu f > 0$.

We will also make use of the following definition,
\[
\delta := \begin{cases} 
\frac{\mu f + \mu F}{\Lambda_{\text{max}}} \left(1 + \frac{\mu f}{\mu F}\right) & \text{if } \mu F + \mu f < 2\Lambda_{\text{max}} \\
1 - \frac{\Lambda_{\text{max}} - \mu f}{\mu F} & \text{otherwise.}
\end{cases}
\]

It is straightforward to show that $\delta \in (0, 1)$. In particular, if $\mu F + \mu f < 2\Lambda_{\text{max}}$, the conclusion follows because $\mu F \geq \mu f \Rightarrow 1 + \frac{\mu f}{\mu F} \leq 2$. In the other case, it is enough to show that $\frac{\Lambda_{\text{max}} - \mu f}{\mu F} \in (0, 1)$. The lower bound holds by Assumption 22. Then
\[
0 < \frac{\Lambda_{\text{max}} - \mu f}{\mu F} \leq \frac{\Lambda_{\text{max}} - \mu f}{2\Lambda_{\text{max}} - \mu f} = \frac{\Lambda_{\text{max}} - \mu F}{\Lambda_{\text{max}} + (\Lambda_{\text{max}} - \mu f)} = \frac{1}{\frac{\Lambda_{\text{max}} - \mu f}{\mu F}} < 1.
\]

We are now ready to present our main result.

**Theorem 23.** Let $f$ and $F$ be the strongly convex functions defined in (1) with $\mu f > 0$ and $\mu F > 0$ and let Assumptions 9, 10, 15 and 22 hold. Choose an initial point $x_0 \in \mathbb{R}^N$, a target confidence $\rho \in (0, 1)$, a target accuracy $\epsilon > 0$, and fix $\bar{\eta} \in [0, 1]$. Let $\chi(\bar{\eta})$ and $\delta$ be defined in (48) and (64) respectively, and let
\[
K \geq \frac{N}{\tau \chi(\bar{\eta}) \delta} \ln \left( \frac{F(x_0) - F^*}{\epsilon \rho} \right).
\]

If $\{x_k\}_{k \geq 0}$ are the random points generated by FCD (Algorithm 1) using the sampling procedure described in Subsection 3.1.2, with $\eta^k_S \in [0, \bar{\eta}]$ for all $k$ and all $S \in 2^{[N]}$, then $\mathbb{P}(F(x_K) - F^* \leq \epsilon) \geq 1 - \rho$. 

The results of the previous theorem simplify when we assume that FCD solves the subproblems exactly. This is shown in the following result.

**Corollary 20.** Let the conditions of Theorem 19 hold, and let the target accuracy $\epsilon$, and the iteration counter $K$ be chosen in either of the following two ways:
\[
(i) \epsilon < F(x_0) - F^* \text{ and } K \geq \frac{2N}{\tau \chi(0)} \frac{m_i}{\epsilon} \left(1 + \ln \frac{1}{\rho}\right) + 2 - \frac{2N}{\tau \chi(0)} \frac{m_i}{\epsilon^2} \left(F(x_0) - F^*\right),
\]
\[
(ii) \epsilon < \min\{\mathcal{R}^2(x_0), F(x_0) - F^*\} \text{ and } K \geq \frac{2N}{\tau \chi(0)} \frac{\mathcal{R}^2(x_0)}{\epsilon} \ln \frac{F(x_0) - F^*}{\epsilon \rho}.
\]

If $\{x_k\}_{k \geq 0}$ are the random points generated by FCD (Algorithm 1) using the sampling procedure described in Subsection 3.1.2, with $\eta^k_S = 0$ for all $k$ and all $S \in 2^{[N]}$, then $\mathbb{P}(F(x_K) - F^* \leq \epsilon) \geq 1 - \rho$.

**Remark 21.** Notice that the exact case (Corollary 20) is simply a special case of the inexact case (Theorem 19) with $\eta^k_S = \bar{\eta} = 0$.

### 6.2 Iteration complexity in the strongly convex case

In this section we establish iteration complexity results for FCD when the objective function $F$, and the smooth function $f$, are strongly convex. In this case the strong convexity parameters of $f$ and $F$ satisfy $\mu f > 0$ and $\mu F > 0$. Moreover, $\mu f \leq \mu F$.

Before presenting our iteration complexity results, we will make the following assumption.

**Assumption 22.** We assume that $\Lambda_{\text{max}} > \mu f > 0$.

We will also make use of the following definition,
\[
\delta := \begin{cases} 
\frac{\mu f + \mu F}{\Lambda_{\text{max}}} \left(1 + \frac{\mu f}{\mu F}\right) & \text{if } \mu F + \mu f < 2\Lambda_{\text{max}} \\
1 - \frac{\Lambda_{\text{max}} - \mu f}{\mu F} & \text{otherwise.}
\end{cases}
\]

It is straightforward to show that $\delta \in (0, 1)$. In particular, if $\mu F + \mu f < 2\Lambda_{\text{max}}$, the conclusion follows because $\mu F \geq \mu f \Rightarrow 1 + \frac{\mu f}{\mu F} \leq 2$. In the other case, it is enough to show that $\frac{\Lambda_{\text{max}} - \mu f}{\mu F} \in (0, 1)$. The lower bound holds by Assumption 22. Then
\[
0 < \frac{\Lambda_{\text{max}} - \mu f}{\mu F} \leq \frac{\Lambda_{\text{max}} - \mu f}{2\Lambda_{\text{max}} - \mu f} = \frac{\Lambda_{\text{max}} - \mu f}{\Lambda_{\text{max}} + (\Lambda_{\text{max}} - \mu f)} = \frac{1}{\frac{\Lambda_{\text{max}} - \mu f}{\mu F}} < 1.
\]

We are now ready to present our main result.

**Theorem 23.** Let $f$ and $F$ be the strongly convex functions defined in (1) with $\mu f > 0$ and $\mu F > 0$ and let Assumptions 9, 10, 15 and 22 hold. Choose an initial point $x_0 \in \mathbb{R}^N$, a target confidence $\rho \in (0, 1)$, a target accuracy $\epsilon > 0$, and fix $\bar{\eta} \in [0, 1]$. Let $\chi(\bar{\eta})$ and $\delta$ be defined in (48) and (64) respectively, and let
\[
K \geq \frac{N}{\tau \chi(\bar{\eta}) \delta} \ln \left( \frac{F(x_0) - F^*}{\epsilon \rho} \right).
\]

If $\{x_k\}_{k \geq 0}$ are the random points generated by FCD (Algorithm 1) using the sampling procedure described in Subsection 3.1.2, with $\eta^k_S \in [0, \bar{\eta}]$ for all $k$ and all $S \in 2^{[N]}$, then $\mathbb{P}(F(x_K) - F^* \leq \epsilon) \geq 1 - \rho$. 

19
Proof. From Lemma 18 we have that

\[
E[F(x_{k+1}) - F^*|x_k] \leq \left(1 - \frac{\tau x(0)k}{N}\right) (F(x_k) - F^*) - (\mu_f - \Lambda_{\max} \zeta) \frac{\tau x(0)k}{2N^2} \|x_k - x^*\|^2 \\
\leq \left(1 - \frac{\tau x(0)k}{N}\right) (F(x_k) - F^*) - (\mu_f - \Lambda_{\max} \zeta) \frac{\tau x(0)k}{N\mu_f} (F(x_k) - F^*).
\]

Notice that (67) will only hold if \(\mu_f < \Lambda_{\max} \zeta\). But, differentiating the right hand side of (67) with respect to \(\zeta\) gives \(\zeta = \min\{\mu_F + \mu_f\}/2\Lambda_{\max}, 1\}, so the condition is satisfied.\(^3\) Then we have that

\[
E[F(x_{k+1}) - F^*|x_k] \leq \left\{ \begin{array}{ll} 
1 - \frac{\tau x(0)k(\mu_F + \mu_f)}{4N\Lambda_{\max}} \left(1 + \frac{\mu_f}{\mu_F}\right) & \text{if } \mu_F + \mu_f < 2\Lambda_{\max} \\
1 - \frac{\tau x(0)k}{N} \left(1 - \frac{\Lambda_{\max} - \mu_f}{\mu_F}\right) & \text{otherwise.}
\end{array} \right.
\]

Now, using (64), we can write

\[
E[F(x_{k+1}) - F^*|x_k] \leq \left(1 - \frac{\tau x(0)k}{N}(\mu_f - \mu_f)\right) (F(x_k) - F^*).\]

The result follows by applying Theorem 13(ii) with \(c_2 = \frac{N}{\tau x(0)k} > 1\).

Again, the result simplifies when we assume that the subproblems are solved exactly.

Corollary 24. Let the conditions of Theorem 23 hold, and let

\[
K \geq \frac{N}{\tau x(0)k} \ln \left(\frac{F(x_0) - F^*}{\epsilon \rho}\right).
\]

If \(\{x_k\}_{k \geq 0}\) are the random points generated by FCD (Algorithm 1) using the sampling procedure described in Subsection 3.1.2, with \(\eta_k = 0\) for all \(k\) and all \(S \in 2^{|N|}\), then \(P(F(x_K) - F^* \leq \epsilon) \geq 1 - \rho\).

7 Numerical Experiments

In this section we examine the performance of two versions of FCD and two versions of a Uniform Coordinate Descent method (UCDC) [30] on two common optimization problems. The first problem is an \(\ell_1\)-regularized least squares problem of the form (1) with

\[
f(x) = \frac{1}{2} \|Ax - b\|^2 \quad \text{and} \quad \Psi(x) = c\|x\|_1,
\]

where \(c > 0\), \(x \in \mathbb{R}^N\), \(A \in \mathbb{R}^{m \times N}\) and \(b \in \mathbb{R}^m\). The second problem is an \(\ell_1\)-regularized logistic regression problems of the form (1) with

\[
f(x) = \sum_{j=1}^m \log(1 + e^{-b_jx^Ta_j}) \quad \text{and} \quad \Psi(x) = c\|x\|_1,
\]

where \(c > 0\), \(a_j \in \mathbb{R}^N\) \(\forall j = 1, 2, \ldots, m\) are the training samples and \(b_j \in \{-1, +1\}\) are the corresponding labels.

For (69) a synthetic sparse large scale experiment is performed and for (70) we compare the methods on two real world large scale problems from machine learning. Notice that for both (69) and (70), \(\Psi(x) = c\|x\|_1\), which is fully separable into coordinates. This means that, for FCD, we have complete control over the block decomposition, and the indices making up each block can change at every iteration.

All algorithms are coded in MATLAB, and for fairness, MATLAB is limited to a single computational thread for each test run. All experiments are performed on a Dell PowerEdge R920 running Redhat Enterprise Linux with four Intel Xeon E7-4830 v2 2.2GHz, 20M Cache, 7.2 GT/s QPI, Turbo (4x10Cores).

\(^3\)We remark that we could simply have set \(\zeta = 1\) from the beginning, so that \(\mu_f \leq \Lambda_{\max} \zeta\) is satisfied by Assumption 22. However, we obtain a better complexity result by taking a smaller \(\zeta\).
7.1 Implementations of FCD and UCDC

In this section we discuss some details of the implementations of methods FCD and UCDC.

7.1.1 FCD

For the FCD method, we fix the size of blocks $\tau > 1$ (to be given in the numerical experiments subsections), and at every iteration of FCD, $\tau$ coordinates are sampled uniformly at random without replacement.

We implement two versions of FCD, which we denote by FCD v.1 and FCD v.2. The two versions only differ in how matrix $H^S_k$ is chosen. In particular, for FCD v.1 we set $H^S_k := \text{diag}(\nabla^2 f(x_k))$ for all $i$ and $k$. In this case subproblem (19) is separable and it has a closed form solution

$$t^S_k = S(x^S_k - (H^S_k)^{-1}\nabla_S f(x^S_k), c\text{diag}((H^S_k)^{-1})),$$

where

$$S(u, v) = \text{sign}(u) \max(|u| - v, 0)$$

is the well-known soft-thresholding operator which is applied component wise when $u$ and $v$ are vectors. Notice that since the subproblem is solved exactly there is no need to verify the stopping conditions (21).

For FCD v.2, we set

$$H^S_k := \nabla^2 f(x_k) + \rho I_{N_i}, \text{ for all } i \text{ and } k,$$

where $\rho > 0$ guarantees that $H^S_k$ is positive definite for all $i, k$. Hence, the subproblem (19) is well defined. The larger $\rho$ is the smaller the condition number of matrix $H_k$ becomes, hence, the faster subproblem (19) will be solved by an iterative solver. However, we do not want $\rho$ to dominate matrix $H_k$ because the essential second order information from $\nabla^2 f(x_k)$ will be lost.

In this setting of matrix $H^S_k$ we solve subproblems (19) iteratively using an Orthant Wise Limited-memory Quasi-Newton (OWL) method, which can be downloaded from [http://www.di.ens.fr/~mschmidt/Software/L1General.html](http://www.di.ens.fr/~mschmidt/Software/L1General.html). We chose OWL because it has been shown in [3] to result in a robust and efficient deterministic version of FCD, i.e. $\tau = N$ (one block of size $N$). Note that we never explicitly form matrix $H^S_k$, we only perform matrix-vector products with it in a matrix-free manner.

7.1.2 UCDC

We also implement two versions of a uniform coordinate descent method as it is described in Algorithm 2 in [30]. For both versions the size $\tau$ of the blocks and the decomposition of $\mathbb{R}^N$ into $\lceil N/\tau \rceil$ blocks are fixed a-priori and all blocks are selected by UCDC with uniform probability. We compare two versions of UCDC, denoted by UCDC v.1 and UCDC v.2 respectively. For UCDC v.1 we set $\tau = 1$ and for UCDC v.2 we set $\tau > 1$ (the exact $\tau$ is given later).

One of the key ingredients of UCDC are the block Lipschitz constants, which are explicitly required in the algorithm. For single coordinate blocks, the Lipschitz constants can be computed with relative ease. However, for blocks of size greater than 1, the block Lipschitz constants can be far more expensive to compute. (For example, for problem (69), the block Lipschitz constants correspond to the maximum eigenvalue of $A^T_i A_i$, where $A_i := AU_S$.) For this reason, we do not compute the actual block Lipschitz constants, rather, we use an overapproximation.

To this end, let $L_j > 0 \forall j = 1, 2, \ldots, N$ denote the coordinate Lipschitz constants of function $f$. Then the direction $t^S$ at every iteration is obtained by solving exactly subproblem (19) with

$$H^S_k := \left(\sum_{j \in i} L_j\right) I_{\tau},$$

using operator (71). Notice that for problem (69), (73) is equivalent to $H^S_k = \text{trace}(A^T_i A_i) I_{\tau}$, where $\text{trace}(A^T_i A_i)$ is an overapproximation of the maximum eigenvalue of $A^T_i A_i$.

Moreover, notice that Algorithm 2 in [30] is a special case of FCD where the subproblem (19) is solved exactly and line search is unnecessary. This is because, by setting $H^S_k$ as in (73), then subproblem (19) is an over estimator of function $F$ along block coordinate direction $t^S$ (for details we refer the reader to [30]).
7.2 Termination Criteria and Parameter Tuning

The only termination criteria that FCD and UCDC should have are maximum number of iterations or maximum running time. This is because using subgradients as a measure of distance from optimality or any other operation of similar cost are considered as expensive tasks for large scale problems. In our experiments FCD and UCDC are terminated when their running time exceeds the maximum allowed running time. Furthermore, for FCD we set parameter $\eta_k$ in (21) equal to 0.9 $\forall i, k$ and $\rho = 10^{-6}$ in (72). The maximum number of backtracking line search iterations is set to 10 and $\theta = 10^{-3}$. For UCDC the coordinate Lipschitz constants $L_j$ $\forall j$ are calculated once at the beginning of the algorithm and this task is included in the overall running time. Finally, all methods are initialized with the zero solution.

7.3 $\ell_1$-Regularized Least Squares

In this subsection we present the performance of FCD and UCDC on the $\ell_1$-regularized least squares problem (69). For this problem the data $A$ and $b$ were synthetically constructed using a generator proposed in [25, Section 6], and we set $c = 1$. The advantage of this generator is that it produces data $A$ and $b$ with a known minimizer $x^*$. We slightly modified the generator so that we could control the density of $A$.

The dimensions of the problem are $N = 2^{21}$ and $m = N/4$ and the generated matrix $A$ is full rank (with at least one non zero component per column) and a density of $\approx 10^{-4} mN$. The optimal solution is set to have $\lceil 0.01N \rceil$ non zero components with values uniformly at random in the interval $[-1, 1]$. For UCDC, the coordinate Lipschitz constants are $L_j := \|A_j\|_2^2$ $j = 1, 2, \ldots, N$, and for FCD v.1, FCD v.2 and UCDC v.2, we set $\tau = \lfloor 0.01N \rfloor$.

The result of this experiment is shown in Figure 1. In this figure notice that all methods were terminated after $10^4$ seconds. For practical purposes, for UCDC v.1 results are printed every ten thousand iterations. Calculation of $F(x)$ is not included in running time of the methods. Fig.1a shows how the objective function $F(x)$ decreases as a function of the number of iterations. Fig.1b shows how the objective function $F(x)$ decreases as a function of wall-clock time measured in seconds.

![Figure 1](image)

Figure 1: Performance of all four methods FCD v.1 and v.2 and UCDC v.1 and v.2 on a sparse large scale $\ell_1$-regularized least squares problem. For practical purposes, for UCDC v.1 results are printed every ten thousand iterations. Calculation of $F(x)$ is not included in running time of the methods. Fig.1a shows how the objective function $F(x)$ decreases as a function of the number of iterations. Fig.1b shows how the objective function $F(x)$ decreases as a function of wall-clock time measured in seconds.
accuracy within the required maximum time. Moreover, observe in sub Figure 1b that for purely practical purposes it might be better to have a combination of methods FCD v.1 and v.2. The former could be used at the beginning of the process while the latter could be used at later stages in order to guarantee robustness and speed closer to the optimal solution. Finally, it is important to mention that on this problem for both FCD versions unit step sizes $\alpha$ were accepted by the backtracking line search for a major part of the process. Hence, backtracking line search was inexpensive.

7.4 $\ell_1$-Regularized Logistic Regression

In this section we present the performance of FCD and UCDC on the $\ell_1$-regularized logistic regression problems (70).

Such problems are important in machine learning and are used for training a linear classifier $x \in \mathbb{R}^N$ that separates input data into two distinct clusters, for example, see [42] for further details.

We present the performance of the methods on two sparse large scale data sets. Problem details are given in Table 1, where $A \in \mathbb{R}^{m \times N}$ is a matrix whose rows are training samples.

Table 1: Properties of two $\ell_1$-regularized logistic regression problems. The second and third columns show the number of training samples and features, respectively. The fourth column shows the sparsity of matrix $A$.

<table>
<thead>
<tr>
<th>Problem</th>
<th>$m$</th>
<th>$N$</th>
<th>nnz($A$)/(mN)</th>
</tr>
</thead>
<tbody>
<tr>
<td>webspam</td>
<td>350,000</td>
<td>16,609,143</td>
<td>2.24e-4</td>
</tr>
<tr>
<td>kdd2010 (algebra)</td>
<td>8,407,752</td>
<td>20,216,830</td>
<td>1.79e-6</td>
</tr>
</tbody>
</table>

The data sets can be downloaded from the collection of LSVM problems in http://www.csie.ntu.edu.tw/~cjlin/libsvmtools/datasets/. For both experiments we set $c = 10$, which resulted in more than 99% classification accuracy of the used data sets.

By [30, Table 10], the coordinate Lipschitz constants for UCDC are

$$L_j := \frac{1}{4} \sum_{q=1}^{m} (A_{qj} y_q)^2 \quad \forall j = 1, 2, \ldots, N,$$

where $A_{qj}$ is the component of matrix $A$ at $q$th row and $j$th column. For block versions FCD v.1, FCD v.2 and UCDC v.2, we set $\tau = \lceil 0.001N \rceil$.

The result of this experiment is shown in Figure 2. In this experiment all methods were terminated after one hour of running time. Notice that FCD versions were more efficient than both UCDC versions, with FCD v.1 being the fastest among all. An interesting observation in Figures 2a and 2c is that FCD versions had similar per iteration computational complexity since they performed similar number of iterations within the maximum allowed running time. However, for FCD v.1, it seems that diagonal information from the second order derivatives of $f$ was enough to decrease faster the objective function for all iterations compared to FCD v.2. Finally, in this experiment we observed that both FCD versions accepted unit step sizes for a major part of the process.

8 Conclusion

In this work we have presented a flexible randomized block coordinate descent method (FCD) that can be applied to convex composite functions of the form (1).

The proposed method can vary from first- to second-order; depending on how large the block updates are set, how accurate second-order information are used and how inexactly the arising subproblems are solved. Although the per iteration computational complexity might be higher for FCD, we present synthetic and real world large scale examples where the number of iterations substantially decreases, as well as the overall time.

We have also presented high probability iteration complexity guarantees to show that FCD converges in expectation.
Figure 2: Performance of all four methods FCD v.1 and v.2 and UCDC v.1 and v.2 on two large scale \(\ell_1\)-regularized logistic regression problems. The first and second rows of figures show the results for problems webspam and kdda, respectively. Calculation of \(F(x)\) is not included in running time of the methods.

References


