When are static and adjustable robust optimization with constraint-wise uncertainty equivalent?

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Abstract

Adjustable Robust Optimization (ARO) yields, in general, better worst-case solutions than static Robust Optimization (RO). However, ARO is computationally more difficult than RO. In this paper, we derive conditions under which the worst-case objective values of ARO and RO problems are equal. We prove that if the uncertainty is constraint-wise and the adjustable variables lie in a compact set, then under one of the following sets of conditions robust solutions are optimal for the corresponding (ARO) problem: (i) the problem is fixed recourse and the uncertainty set is compact, (ii) the problem is convex with respect to the adjustable variables and concave with respect to the parameters defining constraint-wise uncertainty. Furthermore, if we have both constraint-wise and nonconstraint-wise uncertainty, under similar sets of assumptions we prove that there is an optimal decision rule for the Adjustable Robust Optimization problem that does not depend on the parameters defining constraint-wise uncertainty. Also, we show that for a class of problems, using affine decision rules that depend on both types of uncertain parameters yields the same optimal value as ones depending solely on the nonconstraint-wise uncertain parameter. Additionally, we provide several examples not only to illustrate our results, but also to show that the assumptions are crucial and omitting one of them can make the optimal worst-case objective values different.

Keywords Robust Optimization, Adjustable Robust Optimization, Constraint-wise Uncertainty, Hybrid Uncertainty

1 Introduction

Many of the real life optimization problems have parameters that are not exact. One way of dealing with parameter uncertainty is Robust Optimization (RO), which enforces the constraints to hold for all uncertain parameter values in a user specified uncertainty region. All decision variables in RO represent "here and now" decisions, which means they should get specific numerical values as a result of solving the problem and before the actual uncertain parameter values "reveal themselves".

In 2004, Ben-Tal et al. [2] introduced Adjustable Robust Optimization (ARO) as another approach concerning uncertainties. In ARO, only part of the decision variables are "here and now" ones, while the remaining variables represent "wait and see" decisions.

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These last mentioned decisions are assigned numerical values when a part of the uncertain parameters have become known.

The advantage of using ARO lies in the fact that its worst-case objective value is not worse and most of the time better than the corresponding static RO. The main difficulty in using this approach is that solving an ARO problem can be NP-hard even for linear cases [9]. Hence, many methods have been introduced in order to find a close approximation for an ARO problem by using a deterministic optimization one. Ben-Tal et al. [2] mention that if the uncertainty is constraint-wise, under a few assumptions, RO and ARO have the same optimal objective values for linear problems with linear uncertainty. Bertsimas et al. [6] show that even for specific nonconstraint-wise uncertainty, the same result holds.

Besides, using affine decision rules [2] appears to be very effective for many optimization problems. In this method, ”wait and see” decision variables in ARO, which are functions of a part of the uncertain parameters, are restricted to be affine. Bertsimas et al. [4] prove that for a class of problems this method yields the same objective value as the ARO optimal one. Also, in [7], the optimality of an affine decision rule is proved for another class of problems. Although substituting ”wait and see” decision variables with affine functions seems to be really effective, this method needs adding many new variables. Bertsimas et al. [6] provide a tight bound for the worst-case optimal objective value of a two-stage adjustable robust linear optimization problem with a specific type of uncertainty. This bound depends on the optimal objective value of the corresponding RO problem. Also, Bertsimas et al. [5] provide another bound for a specific class of convex ARO problems, depending on the optimal objective value of the corresponding RO one.

The contribution of our paper is fourfold. First, we extend the results in [2], which are for linear problems, to a nonlinear fixed recourse case with nonlinear uncertainty. We reach a set of conditions under which the optimal objective value of a constraint-wise uncertain nonlinear RO problem and the corresponding ARO one are the same. This set of conditions does not contain any convexity assumption, neither for the feasible and uncertainty set, nor for the objective and constraint functions.

Second, we prove that equality of ARO and RO optimal objective values is reached for another set of conditions in which the fixed recourse assumption is not needed. The first two contributions mean that for such problems, there is no need to solve ARO ones. This has two outstanding merits:

i) As it is mentioned before, solving an RO problem is computationally easier than solving an ARO one.

ii) Since ARO is an online approach, parts of the solution can only be implemented after knowing the values of the uncertain parameters. However, the RO approach is an offline one, and all preparations for implementing the solution can start promptly after solving the RO problem.

Third, we study uncertain nonlinear problems containing both constraint-wise and nonconstraint-wise uncertainties. In particular, we prove that for an ARO problem, under similar sets of conditions, there exists an optimal decision rule that depends only on nonconstraint-wise uncertain parameters. Moreover, we show that for a specific class of problems there exists a linear decision rule that is only a function of the nonconstraint-wise uncertain parameters and yields the same objective value as using a linear decision rule that is a function of all uncertain parameters for this class of problems. The merit of this contribution is that it reduces the number of variables in the approximation problem using affine decision rules, because we know that the coefficient of the constraint-wise uncertain parameters are zero beforehand.
As the fourth and last contribution in this paper, we provide several examples to illustrate our results and show that the assumptions are critical. More precisely, we show that by removing an assumption from one of the sets of assumptions, there exists an instance with different RO and ARO optimal objective values.

We emphasize that the results obtained in this paper are with respect to the worst-case objective value of an ARO problem. We provide conditions under which the optimal RO solution is also optimal for the ARO problem. However, in such cases another ARO solution may yield a better objective value than the RO one with respect to the average-case.

The rest of the paper is organized as follows: Section 2 contains our main results. We give two sets of conditions under which constraint-wise RO and ARO problems are equivalent with respect to their optimal objective values. We discuss problems with both constraint-wise and nonconstraint-wise uncertain parameters. Specially, we show that under similar sets of conditions, there exists an optimal decision rule for an ARO problem, that is independent of the constraint-wise uncertain parameters. Section 3 contains several examples to elucidate our main results. We also provide many instances to show that removing one of the assumptions may make the optimal objective value of ARO and RO different.

2 Main results

Consider the following uncertain nonlinear optimization problem

\[
\inf_{x \in X} \inf_{y \in \mathcal{Y}(x)} f(\zeta, x, y) \quad \text{s.t.} \quad g_i(\zeta, x, y) \leq 0, \quad i = 1, \ldots, m,
\]

where $\zeta \in \mathcal{Z} \subseteq \mathbb{R}^l$ is an uncertain parameter and $\mathcal{Z}$ is a nonempty uncertainty set, $x \in \mathcal{X} \subseteq \mathbb{R}^r$ is a non-adjustable variable and $\mathcal{X}$ is a nonempty set defined by constraints depending only on $x$, $y \in \mathcal{Y}(x) \subseteq \mathbb{R}^n$ is an adjustable variable and $\mathcal{Y}(x)$ is a nonempty set defined by constraints independent of $\zeta$. Also, we assume that $f(\zeta, x, y)$ and $g_i(\zeta, x, y)$, $i = 1, \ldots, m$, are continuous. Notice that problem (1) does not contain any equality constraint dependent on $\zeta$.

Corresponding to uncertain problem (1), we can define static and adjustable robust optimization problems.

Definition 1. (Static Robust Optimization) For problem (1), the Static Robust Optimization problem is defined by

\[
(RC) \quad \inf_{x \in \mathcal{X}} \inf_{y \in \mathcal{Y}(x)} f(\zeta, x, y) \quad \text{s.t.} \quad f(\zeta, x, y) \leq t, \quad \forall \zeta \in \mathcal{Z},
\]

\[
g_i(\zeta, x, y) \leq 0, \quad \forall \zeta \in \mathcal{Z}, \quad i = 1, \ldots, m.
\]

Definition 2. (Adjustable Robust Optimization) For problem (1), there are two different definitions for the Adjustable Robust Optimization problem:

\[
\inf \left\{ t \mid \exists x \in \mathcal{X} \quad \forall \zeta \in \mathcal{Z} \quad \exists y \in \mathcal{Y}(x) : \quad f(\zeta, x, y) \leq t, \quad g_i(\zeta, x, y) \leq 0, \quad i = 1, \ldots, m \right\}
\]
and,
\[
\begin{align*}
\text{(ARC)} & \quad \inf_{x \in \mathcal{X}} \sup_{\zeta \in \mathcal{Z}} \inf_{y(\zeta) \in \mathcal{Y}(x), t} t \\
& \quad \text{s.t.} \quad f(\zeta, x, y(\zeta)) \leq t \\
& \quad \quad \quad g_i(\zeta, x, y(\zeta)) \leq 0, \quad i = 1, \ldots, m.
\end{align*}
\]

Takeda et al. [13] prove equivalency of problems (2) and (ARC). In this paper, we denote by \(\text{Opt}(RC)\) and \(\text{Opt}(ARC)\) the objective values of problems (RC) and (ARC), respectively.

We extend the definition of fixed recourse (ARC) for a linear problem with linear uncertainty in [2] to the nonlinear case (nonlinear problem with nonlinear uncertainty) in the following definition.

**Definition 3. (Fixed Recourse Problem)** (ARC) is called fixed recourse, whenever there are continuous functions \(\tilde{f}, \tilde{g}_i : \mathbb{R}^{n+r} \to \mathbb{R}, \bar{f}, \bar{g}_i : \mathbb{R}^{t+r} \to \mathbb{R}, \) for \(i = 1, \ldots, m,\) such that for all \(\zeta \in \mathcal{Z} \subset \mathbb{R}^t, x \in \mathcal{X} \subset \mathbb{R}^r\) and \(y \in \mathcal{Y}(x) \subset \mathbb{R}^n,

\[
\begin{align*}
f(\zeta, x, y) &= \tilde{f}(x, y) + \bar{f}(\zeta, x), \\
g_i(\zeta, x, y) &= \tilde{g}_i(x, y) + \bar{g}_i(\zeta, x), \quad i = 1, \ldots, m.
\end{align*}
\]

In this paper, we mainly work with constraint-wise uncertainty, which is defined as follows.

**Definition 4. (Constraint-wise Uncertainty)** For problem (1), the uncertainty is constraint-wise when each uncertain parameter \(\zeta\) can be split into blocks \(\zeta = [\zeta_0, \ldots, \zeta_m]\) in such a way that the data of the objective depends only on \(\zeta_0,\) the data of the \(i\)-th constraint depends solely on \(\zeta_i,\) and the uncertainty set \(\mathcal{Z} = \mathcal{Z}_0 \times \mathcal{Z}_1 \times \ldots \times \mathcal{Z}_m,\) where \(\mathcal{Z}_j\) is the uncertainty region of \(\zeta_j,\) for \(j = 0, \ldots, m.\)

Intending to express conditions under which \(\text{Opt}(RC) = \text{Opt}(ARC)\), we mention a collection of assumptions here and we use a subset of this collection in each theorem.

**Assumption.** For problem (1):

1) The uncertainty is constraint-wise.

2) The uncertainty set \(\mathcal{Z} \subset \mathbb{R}^t\) is convex.

3) The uncertainty set \(\mathcal{Z} \subset \mathbb{R}^t\) is closed.

4) The uncertainty set \(\mathcal{Z} \subset \mathbb{R}^t\) is compact.

5) \(\mathcal{Y}(x)\) is a closed set, for each \(x \in \mathcal{X}.\)

6) \(\mathcal{Y}(x)\) is a compact set, for each \(x \in \mathcal{X}.\)

7) \(\mathcal{Y}(x)\) is a convex set, for each \(x \in \mathcal{X}.\)

8) (RC) is feasible.

9) (ARC) is fixed recourse.
The following theorem states some conditions under which static and adjustable problems are equivalent, when \((ARC)\) is fixed recourse. This theorem extends the result in [2] even for the linear problems with linear uncertainty.

**Theorem 1.** If Assumption 1, 3, 5, 8, 9 hold, then \(Opt(ARC) = Opt(ARC)\).

**Proof.** First we suppose problem (1) does not contain any non-adjustable variable. According to the definitions of \((RC)\) and \((ARC)\), we have \(Opt(ARC) \leq Opt(RC)\). It means that if \((RC)\) is unbounded, then \(Opt(RC) = Opt(ARC) = -\infty\). Now, we suppose that \((RC)\) is bounded. We show that \(Opt(ARC) \geq Opt(RC)\). Since Assumption 9 (i.e. \((ARC)\) is fixed recourse) holds, we can simplify \((RC)\) to the following problem:

\[
\begin{align*}
\min_{y \in \mathcal{Y}} \quad & t \\
\text{s.t.} \quad & \tilde{f}(y) + f(\bar{\zeta}) \leq t & \forall \bar{\zeta} \in \mathcal{Z}_0, \\
& \tilde{g}_i(y) + \tilde{g}_i(\bar{\zeta}_i) \leq 0 & \forall \bar{\zeta}_i \in \mathcal{Z}_i, \quad i = 1, \ldots, m,
\end{align*}
\]

which is equivalent to,

\[
\begin{align*}
\min_{y \in \mathcal{Y}} \quad & t \\
\text{s.t.} \quad & \tilde{f}(y) + \max_{\bar{\zeta}_0 \in \mathcal{Z}_0} f(\bar{\zeta}_0) \leq t \\
& \tilde{g}_i(y) + \max_{\bar{\zeta}_i \in \mathcal{Z}_i} \tilde{g}_i(\bar{\zeta}_i) \leq 0, \quad i = 1, \ldots, m.
\end{align*}
\]

According to Assumption 1, 3 and 8 (i.e. uncertainty is constraint-wise, \(\mathcal{Z}\) is closed and \((RC)\) is feasible), there is a point \(\bar{\zeta} = [\bar{\zeta}_0, \ldots, \bar{\zeta}_m] \in \mathcal{Z}\) where \(\bar{\zeta}_0\) is an optimal solution of \(\max_{\bar{\zeta}_0 \in \mathcal{Z}_0} f(\bar{\zeta}_0)\), and \(\bar{\zeta}_i\) is an optimal solution of \(\max_{\bar{\zeta}_i \in \mathcal{Z}_i} \tilde{g}_i(\bar{\zeta}_i)\), for all \(i = 1, \ldots, m\). Besides, since Assumption 5 (i.e. \(\mathcal{Y}\) is closed) holds, there is a point \((\bar{y}, \bar{t})\) that is an optimal solution of (4) for the scenario \(\bar{\zeta}\). By \((ARC)\) definition, \(Opt(ARC)\) is not less than the objective value of

\[
\begin{align*}
\min_{y(\bar{\zeta}) \in \mathcal{Y}} \quad & t \\
& \tilde{f}(y(\bar{\zeta})) + f(\bar{\zeta}) \leq t \\
& \tilde{g}_i(y(\bar{\zeta})) + \tilde{g}_i(\bar{\zeta}_i) \leq 0, \quad i = 1, \ldots, m,
\end{align*}
\]

which is equal to \(\bar{t}\). Hence, due to the fact that \(\bar{t} = Opt(RC)\), the equality of the optimal objective values of \((ARC)\) and \((RC)\) is proved.

Now, for the general case that \((ARC)\) contains a nonadjustable variable, we have to solve:

\[
\begin{align*}
\min_{x \in \mathcal{X}} \quad & \max_{\zeta \in \mathcal{Z}} \min_{y(\zeta) \in \mathcal{Y}(x)} f(\zeta, x, y(\zeta)) \\
& g_i(\zeta, x, y(\zeta)) \leq 0, \quad i = 1, \ldots, m.
\end{align*}
\]

According to the first part of the proof, for each \(x \in \mathcal{X}\) that is feasible for \((RC)\), the objective value of

\[
\begin{align*}
\max_{\zeta \in \mathcal{Z}} \quad & \min_{y(\zeta) \in \mathcal{Y}(x)} f(\zeta, x, y(\zeta)) \\
\text{s.t.} \quad & g_i(\zeta, x, y(\zeta)) \leq 0, \quad i = 1, \ldots, m,
\end{align*}
\]
is equal to the objective value of
\[
\min_{y \in \mathcal{Y}(x)} \max_{\zeta \in \mathcal{Z}} \ f(\zeta_0, x, y) \\
\text{s.t.} \quad g_i(\zeta_i, x, y(\zeta)) \leq 0, \quad \forall \zeta_i \in \mathcal{Z}_i, \quad i = 1, \ldots, m.
\] (8)

It follows that the optimal objective value of problem (6) equals that of
\[
\min_{x \in \mathcal{X}, y \in \mathcal{Y}(x)} \max_{\zeta \in \mathcal{Z}} \ f(\zeta_0, x, y) \\
\text{s.t.} \quad g_i(\zeta_i, x, y(\zeta)) \leq 0, \quad \forall \zeta_i \in \mathcal{Z}_i, \quad i = 1, \ldots, m.
\] (9)

Therefore, \( \text{Opt}(ARC) = \text{Opt}(RC) \).

According to the proof of this theorem, we can replace Assumption 8 with Assumption 4 to reach to the following corollary.

**Corollary 1.** If Assumption 1, 4, 5, 9 hold, then \( \text{Opt}(RC) = \text{Opt}(ARC) \).

**Proof.** In the proof of Theorem 1, if \( \mathcal{Z} \) is compact, then existence of \( \bar{\zeta} \) can be concluded directly without using Assumption 8.

In Section 3, we provide several examples illustrating that if Assumption 9 is removed, even if the other assumptions are satisfied, equality of \( \text{Opt}(ARC) \) and \( \text{Opt}(RC) \) does not necessarily hold.

The following theorem shows that \( \text{Opt}(ARC) = \text{Opt}(RC) \) under another set of conditions that does not include Assumption 9.

**Theorem 2.** If Assumption 1, 2, 6, 7, 10, 11 hold, then \( \text{Opt}(RC) = \text{Opt}(ARC) \).

**Proof.** The line of reasoning is the same as in [2]. Similar as it is shown in the proof of Theorem 1, we can assume without loss of generality that problem (1) does not contain any non-adjustable variable. We are going to prove that \( \text{Opt}(ARC) \geq \text{Opt}(RC) \). First, we assume that \( (RC) \) is feasible. So, it is sufficient to show that whenever \( \bar{t} \geq \text{Opt}(ARC) \) then \( \bar{t} \geq \text{Opt}(RC) \) (feasibility of \( (RC) \) implies \( \text{Opt}(ARC) < \infty \)). According to the definitions, we have:

\[
\text{Opt}(ARC) = \\
\inf \left\{ t \mid \forall \zeta \in \mathcal{Z} = \mathcal{Z}_0 \times \ldots \times \mathcal{Z}_m \quad \exists y \in \mathcal{Y} : \begin{array}{l}
f(\zeta_0, y) \leq t, \\
g_i(\zeta_i, y) \leq 0, \quad i = 1, \ldots, m \end{array} \right\} \quad (10)
\]

and,

\[
\text{Opt}(RC) = \\
\inf \left\{ t \mid \exists y \in \mathcal{Y} : \forall \zeta \in \mathcal{Z} = \mathcal{Z}_0 \times \ldots \times \mathcal{Z}_m : f(\zeta_0, y) \leq t, g_i(\zeta_i, y) \leq 0, \quad i = 1, \ldots, m \right\}. \quad (11)
\]

By contradiction, suppose that there is a scalar \( \bar{t} \) such that \( \bar{t} \geq \text{Opt}(ARC) \) and \( \bar{t} < \text{Opt}(RC) \). By (11), Assumption 1 (i.e. uncertainty is constraint-wise) and setting \( \beta = (1, 0, 0, \ldots, 0)^T \), \( G_0(\zeta_0, y) = f(\zeta_0, y) \) and \( G_i(\zeta_i, y) = g_i(\zeta_i, y) \), for \( i = 1, \ldots, m \), it follows that

\[
\forall y \in \mathcal{Y} \quad \exists \zeta^y \in \mathcal{Z} \quad \exists i_y \in \{0, \ldots, m\} : G_{i_y}(\zeta^y_{i_y}, y) - \beta_{i_y} \bar{t} > 0.
\]

Also, continuity implies

\[
\forall y \in \mathcal{Y} \quad \exists \epsilon^y > 0 \quad \exists U_y \quad \forall z \in U_y : G_{i_y}(\zeta^y_{i_y}, z) - \beta_{i_y} \bar{t} > \epsilon^y, \quad (12)
\]
where $U_y$ is a neighbourhood containing $y$. Since Assumption 6 (i.e. $Y$ is compact) holds, there are $\{y^k : k = 1, ..., N\}$ such that $Y \subset \bigcup_{k=1}^{N} U_{y^k}$. So,

$$\forall k = 1, ..., N, \forall z \in Y \max_{i_k} G_{i_k}(\zeta_{y^k}, z) - \beta_{i_k} \bar{t} > \epsilon,$$

where $\epsilon = \min_k e^{y^k}$. As a simplification, we set $\zeta^k = \zeta_{y^k}, i_k = i_{y^k}$ and

$$f_k(z) = G_{i_k}(\zeta^k, z) - \beta_{i_k} \bar{t} \quad \forall z \in Y.$$

Since Assumption 7 (i.e. $Y$ is convex) holds and all $f_k(z)$ are convex and continuous on $Y$ due to Assumption 11, and because $\max_k f_k(z) \geq \epsilon$ for each $z \in Y$, there are nonnegative weights $\lambda_k$ with $\sum_k \lambda_k = 1$ such that:

$$f(z) := \sum_k \lambda_k f_k(z) \geq \epsilon \quad \forall z \in Y.$$

We define

$$w_i = \sum_{k : i_k = i} \lambda_k \quad i = 0, ..., m$$

$$\bar{\zeta}_i = \begin{cases} \sum_{k : i_k = i} \frac{\lambda_k}{w_i} \zeta^k, & w_i \neq 0 \\ \text{an arbitrary point in } Z_i, & w_i = 0 \end{cases}$$

$$\bar{\zeta} = [\bar{\zeta}_0, ..., \bar{\zeta}_m].$$

It is clear by Assumption 2 (i.e. convexity of $Z$) that $\bar{\zeta} \in Z$. Additionally, by the assumption $\bar{t} \geq \text{Opt}(ARC)$, we have:

$$\exists t \leq \bar{t} : \forall \zeta \in Z \exists y \in Y, \quad f(\zeta_0, y) \leq t, \quad g_i(\zeta, y) \leq 0, \quad i = 1, ..., m,$$

which means

$$\exists \bar{y} \in Y : G_i(\bar{\zeta}_i, \bar{y}) - \beta_i \bar{t} \leq 0, \quad i = 0, ..., m.$$
Now, we consider the case where \((RC)\) is not feasible, which means \(\text{opt}(RC) = +\infty\). To prove equality of \((RC)\) and \((ARC)\) with respect to the worst-case objective value, it is sufficient to show that there is no \(\bar{t} \in \mathbb{R}\) such that \(\bar{t} \geq \text{opt}(ARC)\). So, the same argument as the previous part completes the proof.

\(\Box\)

Theorem 2 extends the results for linear problems, obtained by Ben-Tal et al. [2], to nonlinear ones. In the next theorem, we replace Assumption 6 in Theorem 2 to another assumption in order to give a simpler and more direct proof. In this theorem, without loss of generality, we assume that \((RC)\) is:

\[
\min_{x \in X} \min_{y \in Y(x)} c^T y \\
\text{s.t. } g_i(\zeta_i, x, y) \leq 0, \quad i = 0, \ldots, m, \quad \forall \zeta_i \in Z_i,
\]

where \(c \in \mathbb{R}^r\) is certain. Also, we use the following two assumptions in it:

(12) \(r_i(Z_i) \neq \emptyset\), for all \(i = 1, \ldots, m\).

(13) For each \(x \in X\) and \(\zeta \in Z\), Slater’s condition holds for

\[
\min_{y(\zeta) \in Y(x)} \quad c^T y(\zeta) \\
\text{s.t. } g_i(\zeta_i, x, y(\zeta)) \leq 0, \quad i = 0, \ldots, m.
\]

Theorem 3. If Assumption 1, 2, 4, 7, 10, 11, 12 and 13 hold, then \(\text{opt}(ARC) = \text{opt}(RC)\).

Proof. Consider the \((ARC)\) corresponding to (19):

\[
\min_{x \in X} \max_{\{v^i\}, v_{m+1}^1} \min_{y(\zeta) \in Y(x)} c^T y(\zeta) \\
\text{s.t. } g_i(\zeta_i, x, y(\zeta)) \leq 0, \quad i = 0, \ldots, m.
\]

(21)

Taking the Fenchel dual of the inner minimization problem, we get

\[
\min_{x \in X} \max_{\zeta = [\zeta_0, \ldots, \zeta_m] \in Z} \max_{u \in \mathbb{R}_{\leq}^{m+1}} \left\{ \sum_{i=0}^{m} u_i g_i^*(\zeta_i, x, v_i^1) + u_{m+1} \delta_{Y(x)}^{*} \left( \frac{v_{m+1}^1}{u_{m+1}} \right) \right\} \\
\text{s.t. } \sum_{i=1}^{m} v_i^1 = c,
\]

where \(\delta_S(.)\) is the indicator function of a general set \(S\), and \(g^*(\cdot)\) is the convex conjugate of a general convex function \(g(\cdot)\). Equality of the optimal objective values of (21) and (22) follows by Fenchel Duality [3], which can be applied due to Assumption 7, 11 and 13. It is easy to check that (22) is equivalent to

\[
\min_{x \in X} \max_{u \in \mathbb{R}_{\leq}^{m+1}} \max_{\zeta = [\zeta_0, \ldots, \zeta_m] \in Z} \sum_{i=0}^{m} u_i g_i^*(\zeta_i, x, v_i^1) + u_{m+1} \delta_{Y(x)}^{*} \left( \frac{v_{m+1}^1}{u_{m+1}} \right) \\
\text{s.t. } \sum_{i=1}^{m} v_i^1 = c,
\]

\[
\delta_{Z_i}(\zeta_i) \leq 0, \quad i = 1, \ldots, m.
\]

(23)
It is clear that (23) has the same optimal objective value as

\[
\min_{x \in X} \max_{u \in \mathbb{R}_+^{m+1}} \max_{v^i, v^m} \sum_{i=0}^{m} u_i g_i^* \left( \frac{w^i}{u_i}, x, \frac{v^i}{u_i} \right) + u_{m+1}^* \gamma(x) \left( \frac{v^{m+1}}{u_{m+1}} \right)
\]

s.t. \[
\sum_{i=1}^{m} v^i = c,
\]

\[-u_i \delta_{Z_i} \left( \frac{w^i}{u_i} \right) \leq 0, \quad i = 1, \ldots, m,
\]

which is the dual of (19) with the same optimal objective values according to [10], because of Assumption 1, 2, 4, 10 and 12. So, \( \text{Opt}(ARC) = \text{Opt}(RC) \).

Till now, we focus on constraint-wise uncertainties. The question is what can be concluded when we have two different types of uncertainties (constraint-wise and nonconstraint-wise). Consider the following problem:

\[\text{(HRC)} \quad \min_{x \in X} \max_{y \in Y(x)} \min_{\alpha \in A} \max_{\zeta \in Z} f(\zeta_0, \alpha, x, y(\zeta, \alpha)) \leq t, \quad \forall \zeta_0 \in Z_0, \quad g_i(\zeta_i, \alpha, x, y(\zeta, \alpha)) \leq 0, \quad i = 1, \ldots, m, \quad \forall \zeta_i \in Z_i, \]

where \( \zeta = (\zeta_0, \ldots, \zeta_m) \in Z = Z_0 \times \ldots \times Z_m \) and \( \alpha \in A \) are uncertain parameters (\( \zeta \) is constraint-wise and \( \alpha \) is nonconstraint-wise). This problem has a hybrid uncertainty, so we cannot use the aforementioned results, since Assumption 1 does not hold. However, Corollary 2 states that if in such a case the same sets of conditions as in Theorem 1 or 2 are satisfied with respect to the constraint-wise uncertain parameters, then there exists an optimal decision rule that is a function of only the nonconstraint-wise uncertain parameters. In other words, the following two problems:

\[\text{(HARC)} \quad \min_{x \in X} \max_{\zeta \in Z} \min_{y(\zeta, \alpha) \in Y(x)} \max_{\alpha \in A} \min_{\zeta_0, \ldots, \zeta_m} f(\zeta_0, \alpha, x, y(\zeta, \alpha)) \leq t, \quad \forall \zeta_0 \in Z_0, \quad g_i(\zeta_i, \alpha, x, y(\zeta, \alpha)) \leq 0, \quad i = 1, \ldots, m, \quad \forall \zeta_i \in Z_i,
\]

and,

\[\text{(HARC}_\alpha) \quad \min_{x \in X} \max_{\zeta \in Z} \min_{\zeta_0, \ldots, \zeta_m} \max_{\alpha \in A} \min_{y(\zeta, \alpha) \in Y(x)} \text{ s.t. } f(\zeta_0, \alpha, x, y(\alpha)) \leq t, \quad \forall \zeta_0 \in Z_0, \quad g_i(\zeta_i, \alpha, x, y(\alpha)) \leq 0, \quad i = 1, \ldots, m, \quad \forall \zeta_i \in Z_i,
\]

have the same optimal objective values. We denote by \( \text{Opt}(HARC) \) and \( \text{Opt}(HARC}_\alpha) \) the optimal objective value of \( \text{(HARC)} \) and \( \text{(HARC}_\alpha) \), respectively.

**Corollary 2.** Consider problem \( \text{(HRC)} \) and suppose that for all \( \alpha \in A \) the assumptions of Theorem 1 or 2 hold with respect to \( \zeta, x, y \). Then, \( \text{Opt}(HARC) = \text{Opt}(HARC}_\alpha) \).
Proof. By fixing $\alpha \in \mathcal{A}$, $x \in \mathcal{X}$ and applying Theorem 1 or 2, the optimal objective value of
\[
\max_{\zeta \in \mathcal{Z}} \min_{y(\zeta, \alpha) \in \mathcal{Y}(x)} \, t
\]
\[
\text{s.t. } f(\zeta_0, \alpha, x, y(\zeta, \alpha)) \leq t,
\]
\[
g_i(\zeta_i, \alpha, x, y(\zeta, \alpha)) \leq 0, \quad i = 1, ..., m,
\]
and
\[
\min_{y(\alpha) \in \mathcal{Y}(x)} \, t
\]
\[
\text{s.t. } f(\zeta_0, \alpha, x, y(\alpha)) \leq t \quad \forall \zeta_0 \in \mathcal{Z}_0,
\]
\[
g_i(\zeta_i, \alpha, x, y(\alpha)) \leq 0, \quad i = 1, ..., m, \quad \forall \zeta_i \in \mathcal{Z}_i,
\]
are equal. By taking the maximum over $\alpha \in \mathcal{A}$ and minimum over $x \in \mathcal{X}$, the result follows.

Corollary 2 may be used to reduce the complexity of solving adjustable robust optimization problems. Because, in order to solve \((HARC)\), one needs to find an optimal decision rule with respect to both $\alpha$ and $\zeta$, but by applying this corollary, we are ensured about existence of an optimal decision rule that only depends on $\alpha$. However, it is important to note that if we restrict ourselves to a class of decision rules, e.g., linear decision rules, as is usually done, then Corollary 2 does not necessarily guarantee that there exists an optimal decision rule that only depends on $\alpha$. The following corollary states, however, that if the problem is fixed recourse with respect to the constraint-wise uncertain parameter and we use a specific class of decision rules that are separable with respect to $\zeta$ and $\alpha$, then the optimal one depends only on $\alpha$.

Let us denote by $\tilde{y}_\omega(\alpha) \in \mathcal{Y}(x)$ a function of $\alpha$ that belongs to a specific class parametrized by $\omega$. One of the examples for $\tilde{y}_\omega(\alpha)$ can be a polynomial. In this case, $\omega$ could be the vector of coefficients for the monomials.

**Theorem 4.** Assume that \((HARC)\) is fixed recourse with respect to the constraint-wise uncertain parameter, i.e.
\[
g_i(\zeta_i, \alpha, x, y) = \tilde{g}_i(\zeta_i, x) + \tilde{g}_i(\alpha, x, y), \quad i = 0, ..., m,
\]
where $g_0(\zeta_0, \alpha, x, y) = f(\zeta_0, \alpha, x, y)$, and $\tilde{g}_i(\zeta_i, x)$ and $\tilde{g}_i(\alpha, x, y)$ are continuous, for $i = 0, ..., m$. Also, assume that we restrict the decision rules to be in the form of $y(\zeta) + \tilde{y}_\omega(\alpha)$, where $y(\zeta)$ is a general function. If $\mathcal{A}$ and $\mathcal{Y}(x)$, for each $x \in \mathcal{X}$, are closed, and for each fixed $\alpha \in \mathcal{A}$, \((RC)\) is feasible, then the optimal objective value of \((HRC)\) when using this decision rule is equal to that of using decision rule $y + \tilde{y}_\omega(\alpha)$.

**Proof.** Consider the following problem,
\[
\min_{x \in \mathcal{X}, \omega} \max_{\zeta \in \mathcal{Z}} \min_{y(\zeta)} \, t
\]
\[
\text{s.t. } \tilde{g}_0(\zeta_0, x) + \tilde{g}_0(\alpha, x, y(\zeta) + \tilde{y}_\omega(\alpha)) \leq t, \quad \forall \alpha \in \mathcal{A},
\]
\[
\tilde{g}_i(\zeta_i, x) + \tilde{g}_i(\alpha, x, y(\zeta) + \tilde{y}_\omega(\alpha)) \leq 0, \quad \forall \alpha \in \mathcal{A}, \quad i = 1, ..., m,
\]
\[
y(\zeta) + \tilde{y}_\omega(\alpha) \in \mathcal{Y}(x), \quad \forall \alpha \in \mathcal{A}.
\]
The optimal objective value of (24) is equal to the optimal objective value of
\[
\begin{align*}
\min_{x \in X, \omega} & \quad \max_{\zeta \in \mathcal{Z}} \min_{t} \max_{y(\zeta) \in \mathcal{Y}(x, \omega), t} t \\
\text{s.t.} & \quad \hat{g}_i(\zeta_i, x) + \max_{\alpha \in A} \hat{g}_i(\alpha, x, y(\zeta) + \bar{y}_\omega(\alpha)) \leq t, \\
& \quad \bar{g}_i(\zeta_i, x) + \max_{\alpha \in A} \bar{g}_i(\alpha, x, y(\zeta) + \bar{y}_\omega(\alpha)) \leq 0, \quad i = 1, \ldots, m,
\end{align*}
\]
where \(\mathcal{Y}(x, \omega) = \bigcap_{\alpha \in A} [\mathcal{Y}(x) - \bar{y}_\omega(\alpha)]\). It is clear that \(\mathcal{Y}(x, \omega)\) is a closed set. Setting
\[
\hat{g}_i(x, \omega, y(\zeta)) = \max_{\alpha \in A} \hat{g}_i(\alpha, x, y(\zeta) + \bar{y}_\omega(\alpha)), \quad i = 0, \ldots, m,
\]
implies that (24) is equal to
\[
\begin{align*}
\min_{x \in X, \omega} & \quad \max_{\zeta \in \mathcal{Z}} \min_{t} \max_{y(\zeta) \in \mathcal{Y}(x, \omega), t} t \\
\text{s.t.} & \quad \hat{g}_0(\zeta_0, x) + \hat{g}_0(x, \omega, y(\zeta)) \leq t, \\
& \quad \hat{g}_i(\zeta_i, x) + \hat{g}_i(x, \omega, y(\zeta)) \leq 0, \quad i = 1, \ldots, m,
\end{align*}
\]
which is the (ARC) related to the following (RC) problem:
\[
\begin{align*}
\min_{x \in X, \omega} & \quad \min_{y \in \mathcal{Y}(x, \omega), t} t \\
\text{s.t.} & \quad \hat{g}_0(\zeta_0, x) + \hat{g}_0(x, \omega, y) \leq t, \quad \forall \zeta_0 \in \mathcal{Z}_0, \\
& \quad \hat{g}_i(\zeta_i, x) + \hat{g}_i(x, \omega, y) \leq 0, \quad \forall \zeta_i \in \mathcal{Z}_i, \quad i = 1, \ldots, m.
\end{align*}
\]
Now, because for each \(\alpha \in A\), the assumptions of Theorem 1 hold, (25) and (26) have the same optimal objective value. By looking at functions \(\hat{g}_i(x, \omega, y, i = 0, \ldots, m,\) we can easily see that the optimal objective value of (26) is equal to the optimal objective value of
\[
\begin{align*}
\min_{x \in X, \omega} & \quad \min_{y, t} t \\
\text{s.t.} & \quad \hat{g}_0(\zeta_0, x) + \hat{g}_0(\alpha, x, y + \bar{y}_\omega(\alpha)) \leq t, \quad \forall \alpha \in A, \quad \forall \zeta_0 \in \mathcal{Z}_0, \\
& \quad \hat{g}_i(\zeta_i, x) + \hat{g}_i(\alpha, x, y + \bar{y}_\omega(\alpha)) \leq 0, \quad \forall \alpha \in A, \quad \forall \zeta_i \in \mathcal{Z}_i, \quad i = 1, \ldots, m, \\
& \quad y + \bar{y}_\omega(\alpha) \in \mathcal{Y}(x), \quad \forall \alpha \in A.
\end{align*}
\]
So, we prove that the optimal objective value of (24) and (27) are the same. It means that using \(y(\zeta) + \bar{y}_\omega(\alpha)\), and \(y + \bar{y}_\omega(\alpha)\), as the form of decision rules, yields the same approximation values.

In Theorem 4, \(y(\zeta)\) is a general function. For instance, if we assume that \(\bar{y}_\omega(\alpha)\) lies in the class of linear functions, even for a general \(y(\zeta)\), the optimal objective value is independent from \(\zeta\). The other example is when both \(y(\zeta)\) and \(\bar{y}_\omega(\alpha)\) are linear, which means that the decision rule is affine. We consider this case in the next corollary.

**Corollary 3.** Suppose that problem (HARC) is fixed recourse with respect to constraint-wise uncertain parameter \(\zeta\), and the assumptions of Theorem 1 hold for each \(\alpha \in A\). Then in order to approximate (HARC) by an affine decision rule, \(y(\alpha) = u + Wx\) or \(y(\zeta, \alpha) = u + V\zeta + Wx\) yields the same approximate optimal value. 

\[\square\]
Another application of Corollary 2 is for specific two-stage linear optimization problems with uncertainty in both the constraints and the objective. In [6] a bound is derived for this class of problems and the authors show that if the uncertainty in the objective is independent of that in the constraints then this bound does not depend on the objective uncertainty. In the Appendix we show that this result is a consequence of Corollary 2.

In Corollary 3, two different problems are mentioned to approximate \( HARC \), one is considering \( y(\alpha) = u + W\alpha \) as the form of decision rules and the other is considering \( y(\zeta, \alpha) = u + V\zeta + W\alpha \). Let us denote by \( \text{Opt}(AARC_\alpha) \) and \( \text{Opt}(AARC_{\zeta,\alpha}) \) their optimal objective value, respectively. Then, in general for problem \( (HRC) \), we have

\[
\text{Opt}(HARC) \leq \text{Opt}(AARC_{\zeta,\alpha}) \leq \text{Opt}(AARC_\alpha) \leq \text{Opt}(HRC).
\]

In this section, we discuss different conditions that turn inequalities in (28) into equalities. Theorem 1 and 2 provide sets of conditions under which all of the inequalities can be replaced by equalities. Besides, under similar sets of conditions Corollary 3 ensures us that the middle inequality in (28) turns into equality. Moreover, another set of conditions for which \( \text{Opt}(HARC) = \text{Opt}(AARC_{\zeta,\alpha}) \) is proposed in [4]. In the next section, we provide some examples to show that these inequalities can be strict.

3 Examples

This section is divided into two parts. In the first part, examples are illustrating the results mentioned in the previous section. The second part includes several examples to show that the assumptions in Theorem 1 and 2 are essential.

3.1 Illustrative Examples

**Example 1. (Illustrating Theorem 1)** Consider the uncertain problem

\[
\begin{align*}
\min & \quad y^2 + x^3 \\
\text{s.t.} & \quad y^3 + \zeta^3 x \leq 0, \\
& \quad y^2 + x^2 \leq 8, \\
& \quad |x| \leq 1,
\end{align*}
\]

where \( \zeta \in \mathcal{Z} = [-2, 2] \) is an uncertain parameter, \( y \) is an adjustable and \( x \) is a nonadjustable variable. For this problem, \( \mathcal{X} = [-1, 1] \) and \( \mathcal{Y}(x) = \{ y \mid y^2 + x^2 \leq 8 \} \) for each \( x \in \mathcal{X} \).

First, we use Theorem 1 to calculate \( \text{Opt}(ARC) \) because its assumptions hold for this problem. According to this theorem, \( \text{Opt}(ARC) = \text{Opt}(RC) \). Since \( \zeta^3 \) is an increasing function, \( (RC) \) is equivalent to

\[
\begin{align*}
\min & \quad y^2 + x^3 \\
\text{s.t.} & \quad y^3 + 8x \leq 0, \\
& \quad y^3 - 8x \leq 0, \\
& \quad y^2 + x^2 \leq 8, \\
& \quad |x| \leq 1.
\end{align*}
\]
It is easy to verify that $\text{Opt}(RC) = 0$. Now, we solve the (ARC) problem directly:

$$\min_{x \in \mathcal{X}} \max_{\zeta \in \mathcal{Z}} \min_{y(\zeta)} y(\zeta)^2 + x^3$$

subject to

$$y(\zeta)^3 + \zeta^3 x \leq 0,$$
$$y(\zeta)^2 + x^2 \leq 8.$$

First, we solve

$$z^*(\zeta, x) := \min_{y(\zeta)} y(\zeta)^2$$

subject to

$$y(\zeta)^3 + \zeta^3 x \leq 0,$$
$$y(\zeta)^2 + x^2 \leq 8,$$

for each $\zeta \in \mathcal{Z}$ and $x \in \mathcal{X}$. It is clear that $z^*(\zeta, x) = \begin{cases} \left(\sqrt[3]{-\zeta^3} x\right)^2, & \zeta x \geq 0, \\ 0, & \text{o.w.} \end{cases}$

Hence, $\text{Opt}(ARC) = \min_{x \in \mathcal{X}} x^3 + \max_{\zeta \in \mathcal{Z}} z^*(\zeta, x)$ for each $x \in \mathcal{X}$. By checking the different values of $x$, we find $4\sqrt[3]{x^2}$ as its optimal objective value. Hence, $\text{Opt}(ARC) = \min_{x \in [-1,1]} x^3 + 4\sqrt[3]{x^2} = 0$.

After considering a fixed recourse (ARC), we are looking into a convex optimization problem.

**Example 2. (Illustrating Theorem 2)** Consider the following problem:

$$\min y_1 + y_2$$

subject to

$$\ln(\zeta)y_1^2 + y_2^2 \leq 3,$$
$$y_1^2 + y_2^2 \leq 4,$$

where $\zeta \in \mathcal{Z} = [1,4]$ is an uncertain parameter, and $y = (y_1, y_2)$ is an adjustable variable. First, we use the symmetry bound introduced in [5], which is $(1 + \rho) \text{Opt}(RC) \leq \text{Opt}(ARC) \leq \text{Opt}(RC)$ where

$$\rho = \min \left\{ \alpha \geq 0 \left| Z - (1 - \alpha)\frac{5}{2} \subset \mathbb{R}_+ \right. \right\} = \frac{3}{5}.$$

Hence, $\left(\frac{8}{5}\right) \text{Opt}(RC) \leq \text{Opt}(ARC) \leq \text{Opt}(RC)$.

Now, by defining $\mathcal{Y} = \{y \left| y_1^2 + y_2^2 \leq 4 \right. \}$ and the fact that assumptions of Theorem 2 are satisfied for this problem, we have $\text{Opt}(RC) = \text{Opt}(ARC)$. Since $\ln(\zeta)$ is an increasing function, (RC) is as follows:

$$\min y_1 + y_2$$

subject to

$$\ln(4)y_1^2 + y_2^2 \leq 3,$$
$$y_1^2 + y_2^2 \leq 4.$$

The optimal objective value of $-2.2725$ is obtained by CVX, a package for specifying and solving convex programs [11], [12].
Example 2 shows that the symmetry bound is not tight in the presence of constraint-wise uncertainty, even when there is only one uncertain parameter in the problem.

Hitherto, we study examples regarding constraint-wise uncertainty. Now, we consider an example possessing both constraint-wise and nonconstraint-wise uncertainties.

**Example 3. (Hybrid uncertainty)** Consider the following uncertain problem,

$$
\begin{align*}
\min_{y,x} & \quad -x \\
\text{s.t.} & \quad (1 - 2\alpha)x + y \geq \zeta, \\
& \quad \alpha x - y \geq 0, \\
& \quad x \leq 1,
\end{align*}
$$

(29)

where $\alpha \in [0,1]$ is a nonconstraint-wise and $\zeta \in [-1,0]$ a constraint-wise uncertain parameter, $y$ is an adjustable and $x$ is a nonadjustable variable.

Corollary 2 predicts that there exists an optimal decision rule for the (HARC) problem that is independent of $\zeta$. In this example, we check the inequalities in (28). First, we find the optimal objective value of (HRC) and (HARC), and after that we discuss the (HARC) optimal decision rules.

To calculate $\text{Opt}(\text{HRC})$, it is sufficient to solve the following problem,

$$
\begin{align*}
\min_{y,x} & \quad -x \\
\text{s.t.} & \quad x + y \geq 0, \\
& \quad -x + y \geq 0, \\
& \quad -y \geq 0, \\
& \quad x - y \geq 0, \\
& \quad x \leq 1,
\end{align*}
$$

because the constraints are linear with respect to the uncertain parameters $\alpha$ and $\zeta$. It means that $(0,0)$ is the only robust feasible solution of (29). Hence, $\text{Opt}(\text{HRC}) = 0$.

(HARC) problem is as follows:

$$
\begin{align*}
\min_x & \quad \max_{(\alpha,\zeta) \in Z} \min_{y(\alpha,\zeta)} -x \\
\text{s.t.} & \quad (1 - 2\alpha)x + y(\alpha,\zeta) \geq \zeta, \\
& \quad \alpha x - y(\alpha,\zeta) \geq 0, \\
& \quad x \leq 1,
\end{align*}
$$

where $(\alpha,\zeta)$ is the uncertain parameter and $Z = [0,1] \times [-1,0]$ is the uncertainty set. According to the last constraint, $\text{Opt}(\text{HARC}) \geq -1$. Fixing $x = 1$, we have

$$
\zeta + 2\alpha - 1 \leq y(\alpha,\zeta) \leq \alpha,
$$

(30)

which means that $\text{Opt}(\text{HARC}) = -1$ by choosing $y^*(\alpha,\zeta) = \zeta + 2\alpha - 1$ as the optimal decision rule, which depends on both $\alpha$ and $\zeta$. However, $y^{**}(\alpha,\zeta) = \alpha$ is another optimal decision rule for the (HARC), which is independent of $\zeta$. By this discussion, we show that:

$$
-1 = \text{Opt}(\text{HARC}) = \text{Opt}(\text{AARC}_{\zeta,\alpha}) = \text{Opt}(\text{AARC}_\alpha) < \text{Opt}(\text{HRC}) = 0.
$$

$\square$
3.2 Counterexamples

After considering examples that demonstrate the results in Section 2, we start looking into examples in which all of the assumptions of Theorem 1 or 2 are satisfied except one. We name each example to the assumption that is not satisfied.

Example 4. (Problem without equality constraints) One of the assumptions we use in this paper is having no equality constraint that is dependent on \( \zeta \). If a problem contains such a constraint, turning it into inequalities destroys the constraint-wise assumption patently. So, we cannot apply our result to it. In this example, we consider the following problem, which has an equality constraint,

\[
\min -y_1 \\
\text{s.t. } \zeta y_1 + y_2 = 1, \\
0 \leq y \leq 10,
\]

where \( \zeta \in [1, 2] \). It is clear that \( \text{Opt}(RC) = 0 \), since \((0, 1)\) is the only robust feasible solution. However, by defining

\[
y_2(\zeta) = 0, \quad y_1(\zeta) = \frac{1}{\zeta},
\]

as a feasible decision rule for the corresponding \( \text{(ARC)} \), we get \( \text{Opt}(ARC) \leq -\frac{1}{2} \). Hence, \( \text{Opt}(RC) > \text{Opt}(ARC) \). \( \Box \)

Example 5. (Constraint-wise Uncertainty, Assumption 1 in Theorem 1) Ben-Tal et al. [2] consider the following uncertain problem:

\[
\min -x \\
\text{s.t. } (1 - 2\zeta)x + y \geq 0, \\
\zeta x - y \geq 0, \\
0 \leq x \leq 1, \\
|y| \leq 2,
\]

where \( \zeta \in [0, 1] \) is an uncertain parameter, \( y \) is an adjustable variable and \( x \) is a nonadjustable variable. It is easy to check that Assumption 2,..., 11 hold. The corresponding \( \text{(RC)} \) can be reformulated as

\[
\min -x \\
\text{s.t. } x + y \geq 0, \\
x - y \geq 0, \\
-x + y \geq 0, \\
-y \geq 0, \\
0 \leq x \leq 1, \\
|y| \leq 2.
\]

It can easily be checked that \( \text{Opt}(RC) = 0 \). The corresponding \( \text{(ARC)} \) is as follows:

\[
\min \max_{x} \min_{\zeta} -x \\
\text{s.t. } (1 - 2\zeta)x + y(\zeta) \geq 0, \\
\zeta x - y(\zeta) \geq 0, \\
0 \leq x \leq 1, \\
|y(\zeta)| \leq 2.
\]
Similar as the discussion in Example 3, we can verify that $\text{Opt}(\text{ARC}) = -1$, which means $\text{Opt}(\text{ARC}) < \text{Opt}(\text{RC})$.

**Example 6. (Feasible (RC), Assumption 8 in Theorem 1)** Consider the following problem

$$
\begin{align*}
\min \quad & -y^2 \\
\text{s.t.} \quad & y \leq \zeta,
\end{align*}
$$

where $\zeta \leq 0$. It is clear that (RC) is infeasible, the problem is fixed recourse, $\mathcal{Z}$ and $\mathcal{Y}$ are closed. However, (ARC) is feasible and $\text{Opt}(\text{ARC}) = -\infty$.

**Example 7. (Convexity of $\mathcal{Z}$, Assumption 2 in Theorem 2)** Consider the problem

$$
\begin{align*}
\min \quad & \zeta y \\
\text{s.t.} \quad & \zeta \in \mathcal{Z}
\end{align*}
$$

where $\zeta \in \{-1, 2\}$ is the uncertain parameter, $\mathcal{Z} = \{-1, 2\}$ is the uncertainty set, and $\mathcal{Y} = [-1, 1]$. It is clear that all the assumptions of Theorem 2 hold except Assumption 2. It is straightforward that

$$
\text{Opt}(\text{ARC}) = \max \min \{\min \{\min \{\min y \in [0, 1] \max y \in [-1, 0] \zeta y\} - y, \min y \in [-1, 0] 2y\} = \max \{-1, -2\} = -1.
$$

So, $\text{Opt}(\text{ARC}) < \text{Opt}(\text{RC})$. However, if we replace $\mathcal{Z}$ with $\text{Conv}(\mathcal{Z})$ then $\text{Opt}(\text{RC})$ remains the same but $\text{Opt}(\text{ARC})$ becomes zero which shows that convexity of $\mathcal{Z}$ is crucial to get $\text{Opt}(\text{ARC}) = \text{Opt}(\text{RC})$.

**Example 8. (Convexity of $\mathcal{Y}(x)$, Assumption 7 in Theorem 2)** As a counterexample for the case that Assumption 7 is not satisfied, we can use the problem in Example 7 with $\mathcal{Z} = [-1, 2]$ and $\mathcal{Y} = \{-1, 1\}$. Then, $\text{Opt}(\text{ARC}) = 0 < \text{Opt}(\text{RC}) = 1$.

We emphasize that for the next examples, Assumption 9 does not hold.

**Example 9. (Concavity of Functions in $\zeta$, Assumption 10 in Theorem 2)** Consider the problem

$$
\begin{align*}
\min \quad & -y_1 - y_2 \\
\text{s.t.} \quad & \zeta^2 + (1 - \zeta)y_1 + (1 + \zeta)y_2 \leq 3, \\
& |y_i| \leq 3, \quad i = 1, 2,
\end{align*}
$$

where $\zeta \in [-1, 1]$ is an uncertain parameter, $y = (y_1, y_2)$ is an adjustable variable. It is clear that (35) is not concave in $\zeta$, but convex in $y_1$ and $y_2$. Also, $\mathcal{Z} = [-1, 1]$ and $\mathcal{Y} = \{(y_1, y_2) : |y_i| \leq 3, \quad i = 1, 2\}$ are compact and convex, and the uncertainty is constraint-wise. The (RC) corresponding to (35) is as follows:

$$
\begin{align*}
\min \quad & -y_1 - y_2 \\
\text{s.t.} \quad & \max_{\zeta \in [-1, 1]} [\zeta^2 + (1 - \zeta)y_1 + (1 + \zeta)y_2] \leq 3, \\
& |y_i| \leq 3 \quad i = 1, 2.
\end{align*}
$$
Due to the fact that the maximum value of a convex function over a convex set is attained at one of the extreme points \([1]\), (RC) is equivalent to the following problem whose optimal objective value is \(-2\),

\[
\begin{align*}
\min & \quad -y_1 - y_2 \\
\text{s.t.} & \quad y_1 \leq 1, \\
& \quad y_2 \leq 1, \\
& \quad |y_i| \leq 3, \quad i = 1, 2.
\end{align*}
\]

To get an upper bound for Opt(ARC), we choose \(y_1(\zeta) = \frac{3}{2}(1 + \zeta)\) and \(y_2(\zeta) = \frac{3}{2}(1 - \zeta)\) as a decision rule and it is easy to check feasibility of \((y_1, y_2)\). Hence, an upper bound for Opt(ARC) is

\[
\max_{\zeta \in [-1, 1]} -y_1(\zeta) - y_2(\zeta) = \max_{\zeta \in [-1, 1]} -3 = -3.
\]

So, \(\text{Opt}(\text{ARC}) \leq -3 < -2 = \text{Opt}(\text{RC})\).

**Example 10. (Convexity of Functions in \(y\), Assumption 11 in Theorem 2)**

Consider the problem

\[
\begin{align*}
\min & \quad t \\
\text{s.t.} & \quad |y_1| \leq t, \\
& \quad |y_2| \leq t, \\
& \quad -(y_1 - \zeta_1)^2 - (y_2 - \zeta_2)^2 \leq -4 - 2\zeta_1^2, \\
& \quad -(y_1 - \zeta_2)^2 - (y_2 - \zeta_2)^2 \leq -4 - 2\zeta_2^2, \\
& \quad |y_i| \leq 5, \quad i = 1, 2, \\
\end{align*}
\]

where \(\zeta_1 \in [-1, 2], \zeta_2 \in [-2, 1], \zeta = (\zeta_1, \zeta_2)\) is an uncertain parameter, and \(y = (y_1, y_2)\) is an adjustable variable. It is easy to check that (36) is concave (and more precisely it is linear) in the uncertain parameter \(\zeta\), and the uncertainty is constraint-wise. Also, \(X = [-1, 2] \times [-2, 1]\) and \(Y = [-5, 5] \times [-5, 5]\) are convex and compact. However, the problem is not convex in the adjustable variable \(y = (y_1, y_2)\). The (RC) corresponding to (36) is equivalent to

\[
\begin{align*}
\min & \quad \|y\|_\infty \\
\text{s.t.} & \quad (y_1 + 1)^2 + (y_2 + 1)^2 \geq 6, \\
& \quad (y_1 - 2)^2 + (y_2 - 2)^2 \geq 12, \\
& \quad (y_1 - 2)^2 + (y_2 + 2)^2 \geq 12, \\
& \quad (y_1 + 1)^2 + (y_2 - 1)^2 \geq 6, \\
& \quad |y_i| \leq 5, \quad i = 1, 2.
\end{align*}
\]

It is not difficult to find \(y_1 = -\frac{2 + \sqrt{13}}{5} \approx -1.15\) and \(y_2 = \frac{5 + \sqrt{127 + 6\sqrt{13}}}{5} \approx 3.44\) as an optimal solution with the approximated objective value 3.44 for the problem. We choose

\[
y_1(\zeta) = \begin{cases} 
-1.7, & \zeta_2 \leq 0.3 \\
1.6, & \text{otherwise}
\end{cases}, \quad y_2(\zeta) = \begin{cases} 
2.2, & \zeta_2 \leq 0.3 \\
-1.6, & \text{otherwise}
\end{cases}
\]

as a decision rule to find an upper bound for Opt(ARC). The feasibility of the decision rule can easily be checked and it implies that \(\text{Opt}(\text{ARC}) \leq 2.2 < 3.44 \approx \text{Opt}(\text{RC})\).
4 Conclusion

In this paper, we study optimality of robust solutions for adjustable robust constraint-wise problems. We show that for two classes of constraint-wise uncertain problems, the robust optimal solution is also optimal for the adjustable robust problem: i) Fixed recourse problems, for which the adjustable variables lie in a compact set, and the uncertainty set is compact and convex, ii) Problems that are convex with respect to the adjustable variables and concave with respect to the uncertain parameters, and that have a convex uncertainty set and adjustable variables lie in a compact set. These results do not hold when a problem has both constraint-wise and nonconstraint-wise uncertainties, but under similar sets of assumptions we can prove that there exists an optimal decision rule that does not depend on the constraint-wise uncertain parameters. Also, we show that for a class of problems, restricting decision rules to be linear and independent of constraint-wise uncertain parameter, yields the same objective value as the case in which the decision rules are linear and depending on both the constraint-wise and nonconstraint-wise uncertain parameters. Lastly, we provide several examples not only to illustrate our results, but also to show that the assumptions we have in our theorems are not redundant.

Appendix: Two-stage linear optimization problems

As in [6], we consider the following problem

\begin{align*}
(LARC) & \quad \min c^Tx + \max_{(B,d) \in Z} \min_{y(B,d)} d^Ty(B,d) \\
\text{s.t.} & \quad Ax + By(B,d) \leq h, \\
& \quad x \in \mathbb{R}^r_+, \\
& \quad y(B,d) \in \mathcal{Y}(x),
\end{align*}

where \(A \in \mathbb{R}^{m \times r}, c \in \mathbb{R}^r_+, h \in \mathbb{R}^m, \mathcal{Y}(x) \subset \mathbb{R}^n_+\) is a polytope, \(B\) is an uncertain second-stage constraint matrix, and \(d\) is an uncertain objective coefficient vector. Also, let \(Z = Z^B \times Z^d \subseteq \mathbb{R}^{m \times n}_+ \times \mathbb{R}^n_+\) be a convex compact uncertainty set. In addition, we suppose that \(Z^d\) is a polytope, as well. In [6], for problems with deterministic objective coefficient \(d\), it is shown that

\[\text{Opt}(LARC) \geq \rho(Z)\text{Opt}(LRC),\]

where

\[\rho(Z) = \max \{\kappa(T(Z,h)) \mid h > 0\},\]

\[T(Z,h) = \{B^T \mu \mid h^T \mu = 1, \ B \in Z, \ \mu \geq 0\},\]

\[\kappa(T(Z,h)) = \min \{\alpha \mid \text{conv}(T(Z,h)) \subseteq \alpha T(Z,h)\},\]

and \((LRC)\) is the Robust Optimization problem corresponding to \((LARC)\). Then, they in [6] show separately that for problem \((LARC)\), which has uncertainty on objective coefficient \(d\) and second-stage matrix coefficient \(B\), the lower bound is independent of the objective uncertainty, i.e. they show that \(\text{Opt}(LARC) \geq \rho(Z^B)\text{Opt}(LRC)\).

Here, we show that the latter result is a direct consequence of Corollary 2 and (38).
To see that, consider problem

\[
\min c^T x + \max_{B \in \mathcal{Z}^B} \min_{y(B)} t \\
\text{s.t. } \max_{d \in \mathcal{Z}^d} d^T y(B) \leq t, \\
Ax + By(B) \leq h, \\
x \in \mathbb{R}^r_+, \\
y(B) \in \mathcal{Y}(x),
\]

(39)

It is clear that all assumptions of Corollary 2 hold. So, applying the result of this corollary to (LARC) implies that (LARC) and (39) have the same optimal objective values. Now, assume that \(d^j \in \mathcal{Z}^d\), for \(j = 1, ..., K\), are the extreme points of the polytope \(\mathcal{Z}^d\). Then, the optimal objective value of (39) is equal to that of

\[
\min c^T x + \max_{B \in \mathcal{Z}^B} \min_{y(B)} t \\
\text{s.t. } d^j^T y(B) \leq t, \ j = 1, ..., K, \\
Ax + By(B) \leq h, \\
x \in \mathbb{R}^r_+, \\
y(B) \in \mathcal{Y}(x),
\]

(40)

Defining

\[
\bar{B} = \begin{pmatrix} d^1^T \\ \vdots \\ d^K^T \\ B \end{pmatrix}, \quad \bar{A} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ A \end{pmatrix}, \quad \bar{h} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ h \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix},
\]

we rewrite (40) as

\[
(\text{LARC}) \quad \min_{x,t} c^T x + \max_{B \in \mathcal{Z}} \min_{y(B)} t \\
\text{s.t. } \bar{A}x - \beta t + \bar{B}y(B) \leq \bar{h}, \\
x \in \mathbb{R}^r_+, \\
y(B) \in \mathcal{Y}(x),
\]

where \(\mathcal{Z} = [d^1^T, ..., d^K^T]^T \times \mathcal{Z}^B\). Consequently, we get \(\text{Opt}(\text{LARC}) = \text{Opt}(\overline{\text{LARC}}) \geq \rho(\mathcal{Z}^B)\text{Opt}(\text{LRC})\) by applying (38) and the fact that \(\rho(\mathcal{Z}) = \rho(\mathcal{Z}^B)\).

It is worth mentioning that if the uncertainty set \(\mathcal{Z}^B\) in (LARC) is a Cartesian product of the uncertainty region of \(B^j\) with another set, where \(B^j\) is the \(j\)-th row of \(B\), then it is proved analogously that the bound is independent of the uncertainty in \(B^j\). Even though this is not an extension of the results in [6], it gives an intuition behind why the bound is independent of constraint-wise uncertainty.

We emphasize that the proofs in [6] are for polytopal uncertainty sets. However, the authors provide us additional proofs for general uncertainty sets (private communications).

References


