Linear conic formulations for two-party correlations and values of nonlocal games

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September 11, 2015

Abstract

In this work we study the sets of two-party correlations generated from a Bell scenario involving two spatially separated systems with respect to various physical models. We show that the sets of classical, quantum, no-signaling and unrestricted correlations can be expressed as projections of affine sections of appropriate convex cones. As a by-product, we identify a spectrahedral outer approximation to the set of quantum correlations which is contained in the first level of the Navascués, Pironio and Acín (NPA) hierarchy and also a sufficient condition for the set of quantum correlations to be closed. Furthermore, by our conic formulations, the value of a nonlocal game over the sets of classical, quantum, no-signaling and unrestricted correlations can be cast as a linear conic program. This allows us to show that a semidefinite programming upper bound to the classical value of a nonlocal game introduced by Feige and Lovász is in fact an upper bound to the quantum value of the game and moreover, it is at least as strong as optimizing over the first level of the NPA hierarchy. Lastly, we show that deciding the existence of a perfect quantum (resp. classical) strategy is equivalent to deciding the feasibility of a linear conic program over the cone of completely positive semidefinite matrices (resp. completely positive matrices). By specializing the results to synchronous nonlocal games, we recover the conic formulations for various quantum and classical graph parameters that were recently derived in the literature.

Keywords. Quantum correlations, nonlocal games, completely positive semidefinite cone, conic programming, quantum graph parameters, semidefinite programming relaxations.

1 Introduction

In one of the most celebrated discoveries of modern physics John Bell showed that quantum mechanical systems can exhibit correlations that cannot be reproduced within the framework of classical physics [Bel64]. This surprising fact is demonstrated by the violation of Bell inequalities and has received extensive experimental verification, see [FC72, AGR82] for examples. In addition to its theoretical significance, quantum nonlocality and entanglement has been increasingly regarded

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as a valuable resource for distributed tasks such as unconditionally secure cryptography [Eke91] and randomness certification [Col06] among others.

This suggests that a problem of fundamental importance is to characterize the structure of the sets of correlations that can arise between two spatially separated systems under various physical models. Such characterizations are also useful to calculate/approximate the maximum violation of a Bell inequality when restricted to correlations that correspond to specific physical models.

There exists a significant body of work addressing these questions from a mathematical optimization perspective. In the celebrated work [NPA08] Navascués, Pironio and Acín constructed a hierarchy of spectrahedral outer approximations to the set of quantum correlations. Another fundamental result is that the quantum value of a Bell inequality that corresponds to an XOR nonlocal game is given by a semidefinite program [Tsi87, CHTW04]. Furthermore, the quantum value of a Bell inequality that corresponds to unique nonlocal game can be tightly approximated using semidefinite programming [KRT10]. Lastly, mathematical optimization has also proven to be extremely useful for (classical and quantum) parallel repetition results [FL92, CSUU08, KRT10, DS14, DSV14].

Our goal in this work is to address these questions taking the viewpoint of linear conic optimization. Specifically, we introduce the notion of conic correlations and show that the sets of classical, quantum, no-signaling and unconstrained correlations can be expressed as conic correlations over appropriate convex cones. Consequently, conic correlations provide us with a unified framework where we can study the properties of many interesting families of correlations. Furthermore, using our conic characterizations, we can express the classical, quantum, no-signaling and unconstrained value of a nonlocal game as a linear conic program. This allows one to use the arsenal of linear conic programming theory in order to study how the various values of a nonlocal game relate to each other and to better understand their properties.

1.1 Two-party correlations

Consider the following thought experiment: Two spatially separated parties, Alice and Bob, perform measurements on some shared physical system. Alice has a set of possible measurements at her disposal, where each measurement is labelled by some element of a finite set S. The set of possible outcomes of each of Alice's measurements is labelled by the elements of some finite set A. Similarly, Bob has a set T of possible measurements at his disposal each with possible outcomes labeled by the elements of some finite set B. Note that we use the term "measurement" very loosely at this point as the details depend on the underlying physical theory. We refer to a thought experiment as described above as a *Bell scenario*.

At each run of the experiment Alice and Bob without communicating choose measurements $s \in S$ and $t \in T$ respectively which they use to measure their individual systems. Following the measurement they get $a \in A$ and $b \in B$ as outcomes. Since the measurement process is probabilistic, each time the experiment is conducted Alice and Bob might generate different outcomes. The Bell scenario is completely described by the joint conditional probability distribution $p = (p(a,b|s,t))_{a,b,s,t}$, where $p(a,b|s,t)$ denotes the conditional probability that upon performing measurements $s \in S$ and $t \in T$, Alice and Bob get outcomes $a \in A$ and $b \in B$, respectively.

For any Bell scenario, the measurement statistics $p = (p(a,b|s,t))$ satisfy the obvious nonnegativity and normalization constraints summarized below.

Definition 1.1. *The set of all valid joint probability distributions, denoted by $\mathcal{P}$, consists of all vectors $p = (p(a,b|s,t)) \in \mathbb{R}^{|A\times B\times S\times T|}$ that satisfy*

$$\begin{aligned} p(a,b|s,t) &\geq 0, \text{ for all } a \in A, b \in B, s \in S, t \in T, \text{ and} \\ \sum_{a\in A, b\in B} p(a,b|s,t) &= 1, \text{ for all } s \in S, t \in T. \end{aligned}$$

The elements of $\mathcal{P}$ are called correlation vectors *or simply* correlations*.*

A question of fundamental theoretical interest is to describe the correlations that can arise within a Bell scenario as described above with respect to various physical models. We now briefly introduce the models and the corresponding sets of correlations that are relevant to this work. For additional details the reader is referred to the extensive survey [BCP+14] and references therein.

Classical correlations. A *classical strategy* allows Alice and Bob to determine their outputs by employing both private and shared randomness. Formally, a classical strategy is given by:

(i) A shared random variable with domain $[n]$, each sample occurring with probability k_i.

(ii) For each $i \in [n]$ and $s \in S$ a probability distribution $\{x_a^{s,i} : a \in A\}$.

(iii) For each $i \in [n]$ and $t \in T$ a probability distribution $\{y_b^{t,i} : b \in B\}$.

If the value of the shared randomness is $i \in [n]$ and Alice chooses measurement $s \in S$ she determines her output $a \in A$ by sampling from the distribution $\{x_a^{s,i} : a \in A\}$. Analogously, given that the value of the shared randomness is $i \in [n]$, if Bob chooses measurement $t \in T$ he samples from the distribution $\{y_b^{t,i} : b \in B\}$ to determine his output $b \in B$.

Notice that if the players select the measurement pair $(s,t) \in S \times T$, the probability of getting outcome $(a,b) \in A \times B$ is given by $\sum_{i=1}^n k_i x_a^{s,i} y_b^{t,i}$. This leads to the following definition.

Definition 1.2. *A correlation $p \in \mathcal{P}$ is called* classical *if there exist scalars $k_i \geq 0,\ x_a^{s,i} \geq 0,\ y_b^{t,i} \geq 0$ satisfying $\sum_{i\in[n]} k_i = 1$, and $\sum_{a\in A} x_a^{s,i} = \sum_{b\in B} y_b^{t,i} = 1$ for all $s \in S, t \in T, i \in [n]$ such that*

$$p(a,b|s,t) = \sum_{i=1}^n k_i x_a^{s,i} y_b^{t,i}, \text{ for all } a \in A, b \in B, s \in S, t \in T. \tag{1}$$

We denote the set of classical correlations by $\mathcal{C}$.

The set of classical correlations forms a convex polytope in $\mathbb{R}^{|A\times B\times S\times T|}$. Its vertices correspond to deterministic strategies, i.e., correlations of the form $p(a,b|s,t) = \delta_{a,\alpha(s)}\delta_{b,\beta(t)}$ for some pair of functions $\alpha : S \to A$ and $\beta : T \to B$, where $\delta_{i,j}$ denotes the Kronecker delta function.

Quantum correlations. A *quantum strategy* for a Bell scenario allows Alice and Bob to determine their outputs by performing measurements on a shared quantum state. We now describe the structure of a general quantum strategy for a Bell scenario. For additional details on the mathematical formalism of quantum mechanics the reader is referred to Section 2.2.

Alice and Bob possess physical systems X and Y whose state spaces are given by the set of unit vectors in some finite dimensional complex Euclidean spaces $\mathcal{X}$ and $\mathcal{Y}$, respectively. A quantum strategy is given by:

(i) A complex unit vector ψ in $\mathcal{X} \otimes \mathcal{Y}$.

(ii) For each $s \in S$, Alice has positive semidefinite matrices $\{X_a^s : a \in A\}$ acting on $\mathcal{X}$ satisfying

$$\sum_{a \in A} X_a^s = \mathbb{I}_{\mathcal{X}}, \text{ for all } s \in S.$$

(iii) For each $t \in T$, Bob has positive semidefinite matrices $\{Y_b^t : b \in B\}$ acting on $\mathcal{Y}$ satisfying

$$\sum_{b \in B} Y_b^t = \mathbb{I}_{\mathcal{Y}}, \text{ for all } t \in T.$$

According to the postulates of quantum mechanics, if the players select the measurement pair $(s,t) \in S \times T$, the probability of getting the outcome $(a,b) \in A \times B$ is given by $\psi^*(X_a^s \otimes Y_b^t)\psi$. This leads to the following definition.

Definition 1.3. *A correlation $p \in \mathcal{P}$ is called* quantum *if there exists finite dimensional complex Euclidean spaces $\mathcal{X}$ and $\mathcal{Y}$, a unit vector $\psi \in \mathcal{X} \otimes \mathcal{Y}$, psd operators $\{X_a^s : a \in A\}$ satisfying $\sum_{a \in A} X_a^s = \mathbb{I}_{\mathcal{X}}$ for each $s \in S$ and psd operators $\{Y_b^t : b \in B\}$ satisfying $\sum_{b \in B} Y_b^t = \mathbb{I}_{\mathcal{Y}}$, for each $t \in T$, such that:*

$$p(a,b|s,t) = \psi^*(X_a^s \otimes Y_b^t)\psi, \text{ for all } a \in A, s \in S, b \in B, t \in T. \tag{2}$$

We denote the set of quantum correlations by $\mathcal{Q}$.

The set of quantum correlations is a non-polyhedral set with an infinite number of extreme points. Its structure has been extensively studied but is nevertheless not well understood (e.g. see [BCP+14]). In particular, it is not even known whether $\mathcal{Q}$ forms a closed set. On the positive side, Navascués, Pironio, and Acín in the groundbreaking work [NPA08] identified a hierarchy of spectrahedral outer approximations to the set of quantum correlations. It is still an open problem to determine whether this hierarchy converges to the set of quantum correlations.

No-signaling correlations. A correlation $p \in \mathcal{P}$ is called *no-signaling* if Alice's local marginal probabilities are independent of Bob's choice of measurement and, symmetrically, Bob's local marginal probabilities are independent of Alice's choice of measurement. The formal definition follows.

Definition 1.4. *A correlation $p \in \mathcal{P}$ is called* no-signaling *if it satisfies:*

$$\sum_{b \in B} p(a,b|s,t) = \sum_{b \in B} p(a,b|s,t'), \text{ for all } s \in S, t \neq t' \in T, \tag{3}$$

and

$$\sum_{a \in A} p(a,b|s,t) = \sum_{a \in A} p(a,b|s',t), \text{ for all } t \in T, s \neq s' \in S. \tag{4}$$

We denote the set of no-signaling correlations by $\mathcal{NS}$.

The no-signaling conditions (3) and (4) are a natural physical requirement since if they are violated at least one party can receive information about the other party's input *instantaneously*, contradicting the fact that information cannot travel faster than the speed of light. Lastly, notice that unlike classical and quantum correlations, no-signaling correlations do not admit a natural operational interpretation.

For a correlation $p = (p(a, b|s, t)) \in \mathcal{NS}$ we denote by $p_A(a|s)$ Alice's local marginal probabilities for all $a \in A, s \in S$ and by $p_B(b|t)$ Bob's local marginal probabilities for all $b \in B, t \in T$. It is useful to arrange the marginal probabilities in a vector as follows:

$$(5)\quad p_A(s) := \sum_{a \in A} p_A(a|s) e_a \in \mathbb{R}_+^{|A|}, \text{ for all } s \in S \text{ and } p_B(t) := \sum_{b \in B} p_B(b|t) e_b \in \mathbb{R}_+^{|B|}, \text{ for all } t \in T,$$

and

$$(6)\quad p_A := \sum_{s \in S, a \in A} e_s \otimes p_A(s) \in \mathbb{R}_+^{|S \times A|} \text{ and } p_B := \sum_{t \in T, b \in B} e_t \otimes p_B(t) \in \mathbb{R}_+^{|T \times B|},$$

where e_i denotes the i-th standard basis vector.

It is immediate from physical context that every classical correlation is also quantum (cf. Theorem 3.5). Furthermore, it is easy to verify that every quantum correlation is no-signaling (cf. Theorem 3.8). On the other hand, it is well-known that there exist quantum correlations that are not classical and no-signaling correlations that are not quantum. In other words, we have that

$$(7)\quad \mathcal{C} \subsetneq \mathcal{Q} \subsetneq \mathcal{NS} \subsetneq \mathcal{P},$$

and in this paper we give (alternative) algebraic proofs of these containments.

1.2 Two-player one-round nonlocal games

We have already mentioned that the set of quantum correlations is a strict superset of the set of classical correlations. But how does one go about identifying quantum correlations that are not classical? One approach is within the framework of nonlocal games which we now introduce.

A nonlocal game is a thought experiment between two spatially separated parties, Alice and Bob, who can only communicate with a third party, a referee, who ultimately decides whether they win or lose. Formally, a (two-player one-round) *nonlocal game* is specified by four finite sets A, B, S, T, a probability distribution π on $S \times T$ and a Boolean predicate $V : A \times B \times S \times T \to \{0, 1\}$. We denote such a game by $\mathcal{G}(\pi, V)$ or simply $\mathcal{G}$ when it is not necessary to specify π and V.

The game $\mathcal{G}(\pi, V)$ proceeds as follows: The referee using a probability distribution π samples a pair of questions $(s, t) \in S \times T$ and sends s to Alice and t to Bob. After receiving their questions, Alice and Bob use some strategy to determine their respective answers $a \in A$ and $b \in B$ which they send back to the referee. The players *win* the game if $V(a, b|s, t) = 1$ and they lose otherwise.

The objective of the players is to maximize their probability of winning the game. To do this the players are not allowed to communicate after they receive their questions but they can agree on some common strategy before the start of the game using their knowledge of V and π.

Fix a particular strategy for the game that gives rise to the correlation $p = (p(a, b|s, t)) \in \mathcal{P}$. The probability that Alice and Bob win the game using this strategy is given by

$$\sum_{s \in S} \sum_{t \in T} \pi(s, t) \sum_{a \in A} \sum_{b \in B} V(a, b|s, t) p(a, b|s, t).$$

For a fixed set of correlations $\mathcal{S} \subseteq \mathcal{P}$ we denote by $\omega_{\mathcal{S}}(\mathcal{G})$ the maximum probability Alice and Bob can win the game $\mathcal{G}$ when they use strategies that generate correlations that lie in $\mathcal{S}$. Formally:

$$(8)\quad \omega_{\mathcal{S}}(\mathcal{G}) := \sup \left\{ \sum_{s \in S} \sum_{t \in T} \pi(s, t) \sum_{a \in A} \sum_{b \in B} V(a, b|s, t) p(a, b|s, t) : p \in \mathcal{S} \right\}.$$

In this paper we restrict our attention to *(i)* the *classical value* denoted $\omega_{\mathcal{C}}(\mathcal{G})$, *(ii)* the *quantum value* denoted $\omega_{\mathcal{Q}}(\mathcal{G})$, *(iii)* the *no-signaling value* denoted $\omega_{\mathcal{NS}}(\mathcal{G})$ and *(iv)* the *unrestricted value* denoted $\omega_{\mathcal{P}}(\mathcal{G})$. As an immediate consequence of the set inclusions given in Equation (7) we have

$$\omega_{\mathcal{C}}(\mathcal{G}) \leq \omega_{\mathcal{Q}}(\mathcal{G}) \leq \omega_{\mathcal{NS}}(\mathcal{G}) \leq \omega_{\mathcal{P}}(\mathcal{G}),$$

for any nonlocal game $\mathcal{G}$.

The CHSH game. As a concrete example of the above definitions we now describe the CHSH game that was introduced by Clauser, Horne, Shimony, and Holt [CHSH69]. In the CHSH game we have $A = B = S = T = \{0,1\}$, $\pi(s,t) = \frac{1}{4}$ for all $s,t \in \{0,1\}$ and $V(a,b|s,t) = 1$ if and only if $a \oplus b = s \cdot t$, where $\oplus$ denotes addition modulo 2. Informally, in the CHSH game the referee selects a pair of questions $(s,t) \in \{(0,0),(0,1),(1,0),(1,1)\}$ each with probability 1/4 and sends s to Alice and t to Bob. Each player responds to the referee with a single bit $a \in \{0,1\}$ and $b \in \{0,1\}$, respectively. The players win if $a \oplus b$ is equal to the logical AND of their questions.

Consider the following strategy: On input $(1,1)$ the players output $(0,1)$ with probability $1/2$ and $(1,0)$ with probability $1/2$, and on any other input they respond with $(0,0)$ with probability $1/2$ and $(1,1)$ with probability $1/2$. This strategy wins CHSH with probability 1 and is no-signaling. This shows that $\omega_{\mathcal{NS}}(\text{CHSH}) = 1$ and therefore $\omega_{\mathcal{P}}(\text{CHSH}) = 1$ as well. Using quantum strategies the best the players can do is $\omega_{\mathcal{Q}}(\text{CHSH}) = \cos^2(\pi/8) \approx 0.85$. On the other hand, by examining all possible deterministic strategies we see that $\omega_{\mathcal{C}}(\text{CHSH}) = 3/4$.

1.3 Convex cones of interest

Consider a vector space $\mathcal{V}$ endowed with inner product $\langle \cdot, \cdot \rangle$. The *Gram matrix* of a family of vectors $x_1, \ldots, x_n \in \mathcal{V}$, denoted by $\mathrm{Gram}(x_1, \ldots, x_n)$, is defined as the $n \times n$ matrix whose (i,j) entry is given by $\langle x_i, x_j \rangle$. For a matrix $X = \mathrm{Gram}(x_1, \ldots, x_n)$ we say that the vectors $x_1, \ldots, x_n$ form a Gram representation for X.

We denote by $\mathcal{S}^n$ the set of $n \times n$ real symmetric matrices which we equip with the Hilbert-Schmidt inner product $\langle X, Y \rangle := \mathrm{Tr}(XY)$. A matrix $X \in \mathcal{S}^n$ is called *positive semidefinite* (psd) if $X = \mathrm{Gram}(x_1, \ldots, x_n)$ for some family of real vectors $x_1, \ldots, x_n \in \mathbb{R}^d$ (for some $d \geq 1$). A matrix $X \in \mathcal{S}^n_+$ that has full rank is called *positive definite*. We denote by $\mathcal{S}^n_+$ (resp. $\mathcal{S}^n_{++}$) the set of $n \times n$ positive semidefinite matrices (resp. positive definite matrices). The set of positive semidefinite matrices forms a closed, convex, self-dual cone whose structure is well understood (e.g. see [Bar02] and references therein). Linear optimization over $\mathcal{S}^n_+$ is called *semidefinite programming* (SDP) and its optimal value can be approximated within arbitrary precision in polynomial time using the ellipsoid method, under reasonable assumptions (e.g. see [BTN01]).

The *nonnegative cone*, denoted by $\mathcal{N}^n$, consists of the entrywise nonnegative matrices in $\mathcal{S}^n$ i.e.,

$$\mathcal{N}^n := \{X \in \mathcal{S}^n : X_{i,j} \geq 0 \ \text{ for all } \ i,j \in \{1, \ldots, n\}\}. \tag{9}$$

A matrix $X \in \mathcal{S}^n$ is called *doubly nonnegative* if $X \in \mathcal{N}^n \cap \mathcal{S}^n_+$. The set of $n \times n$ doubly nonnegative matrices forms a closed convex cone denoted by $\mathcal{DNN}^n$.

A matrix $X \in \mathcal{S}^n$ is called *completely positive* if $X = \mathrm{Gram}(x_1, \ldots, x_n)$ for some family of entrywise nonnegative vectors $x_1, \ldots, x_n \in \mathbb{R}^d_+$ (for some $d \geq 1$). The set of $n \times n$ completely positive matrices forms a full-dimensional closed convex cone, denoted by $\mathcal{CP}^n$, whose structure has been extensively studied, e.g. see [BSM03]. The dual of $\mathcal{CP}^n$ is the cone of *copositive matrices*, i.e., the matrices $X \in \mathcal{S}^n$ that satisfy $x^\mathsf{T} X x \geq 0$ for all $x \in \mathbb{R}^n_+$. Optimization over $\mathcal{CP}$ is intractable since there exist NP-hard combinatorial optimization problems that can be formulated as linear optimization problems over $\mathcal{CP}$ [dKP02] (see also Section 4.4). On the positive side, there exist semidefinite programming hierarchies that can be used to approximate $\mathcal{CP}$ from the interior [Las13] and from the exterior [Par00].

Thinking of nonnegative vectors as diagonal psd matrices suggests a natural generalization of the $\mathcal{CP}$ cone. A matrix $X \in \mathcal{S}^n$ is called *completely positive semidefinite* (completely psd for short) if $X = \mathrm{Gram}(X_1, \dots, X_n)$ for some family of psd matrices $X_1, \dots, X_n \in \mathcal{S}^d_+$ (for some $d \geq 1$). The set of $n \times n$ completely positive semidefinite matrices forms a full-dimensional convex cone denoted by $\mathcal{CS}^n_+$. The $\mathcal{CS}_+$ cone was introduced recently as a tool to provide conic programming formulations for the quantum chromatic number of a graph [LP14] and quantum graph homomorphisms [Rob14] (cf. Section 5.3). Nevertheless, the structure of the $\mathcal{CS}_+$ cone appears to be very complicated. In particular it is not known whether $\mathcal{CS}_+$ forms a closed set [BLP15]. Furthermore, given a matrix $X \in \mathcal{CS}^n_+$, no upper bound is presently known on the size of the psd matrices in a Gram representation for X. Lastly, combining results from [LP14] and [Ji13] it follows that linear optimization over $\mathcal{CS}_+$ is NP-hard.

It follows immediately from the definitions given above that for every $n \geq 1$ we have

$$\mathcal{CP}^n \subseteq \mathcal{CS}^n_+ \subseteq \mathcal{DNN}^n. \tag{10}$$

For $n \leq 4$ it is known that $\mathcal{CP}^n = \mathcal{DNN}^n$ [MM62]. On the other hand, for $n \geq 5$ all the inclusions in (10) are known to be strict [LP14].

Since the mathematical formulation of quantum mechanics is stated in terms of psd matrices with complex entries, in some parts of this work we consider matrices with complex entries. We denote by $\mathcal{H}^n$ the set of $n \times n$ Hermitian matrices, by $\mathcal{H}^n_+$ the set of $n \times n$ Hermitian positive semidefinite matrices and by $\mathcal{H}^n_{++}$ the set of $n \times n$ Hermitian positive definite matrices. Occasionally, we also use the notation $\mathcal{H}_+(\mathcal{X})$ to denote the positive operators acting on a finite dimensional complex Euclidean space $\mathcal{X}$. For a matrix $X \in \mathcal{H}^n$ we can write $X = \mathcal{R}(X) + i\mathcal{I}(X)$ where $R(X)$ is the real part and $\mathcal{I}(X)$ is the imaginary part of X. If X is Hermitian we get that $\mathcal{R}(X)$ is real symmetric and $\mathcal{I}(X)$ is real skew-symmetric. Moreover, for $X, Y \in \mathcal{H}^n$ we have that $\langle X, Y \rangle = \mathrm{Tr}\big(\mathcal{R}(X)\mathcal{R}(Y) - \mathcal{I}(X)\mathcal{I}(Y)\big)$. For a complex $d \times d$ matrix X, set

$$T(X) := \frac{1}{\sqrt{2}} \begin{pmatrix} \mathcal{R}(X) & -\mathcal{I}(X) \\ \mathcal{I}(X) & \mathcal{R}(X) \end{pmatrix}, \tag{11}$$

and notice that T is a bijection between complex $d \times d$ matrices and real $2d \times 2d$ matrices. More importantly, we have that $X \in \mathcal{H}^n_+$ if and only if $T(X) \in \mathcal{S}^{2n}_+$ and moreover $\langle X, Y \rangle = \langle T(X), T(Y) \rangle$ for all $X, Y \in \mathcal{H}^n_+$. This shows that the set of completely psd matrices does not change if we allow the psd matrices in the Gram decompositions to be Hermitian psd instead of just real psd.

Lastly, a symmetric matrix $X \in \mathcal{N}^N$, where $N := |(S \times A) \cup (T \times B)|$ for finite sets A, B, S and T, is called *no-signaling* if it satisfies

$$\sum_{a \in A} X[(s,a),(t,b)] = \sum_{a \in A} X[(s',a),(t,b)], \text{ for all } b \in B, t \in T, s \neq s' \in S,$$

and

$$\sum_{b \in B} X[(s,a),(t,b)] = \sum_{b \in B} X[(s,a),(t',b)], \text{ for all } a \in A, s \in S, t \neq t' \in T.$$

The set of no-signaling matrices forms a convex cone we denote by $\mathcal{NSO}$.

1.4 Summary of results

Consider a Bell scenario with questions sets S, T and answer sets A, B. Without loss of generality we may assume that the sets A, B and S, T are disjoint. For brevity define $N := |(S \times A) \cup (T \times B)|$. In this work we mostly consider symmetric matrices of size $N \times N$. The rows and columns of a matrix $X \in \mathcal{S}^N$ are indexed by $(S \times A) \cup (T \times B)$ and it is useful to think of X as being partitioned into blocks $X_{i,j}$, where each block is indexed by a pair of questions $i, j \in S \cup T$. The size of each block is equal to: (i) $|A| \times |A|$ if $i, j \in S$, (ii) $|A| \times |B|$ if $i \in S, j \in T$, (iii) $|B| \times |A|$ if $i \in T, j \in S$ and (iv) $|B| \times |B|$ if $i, j \in T$. For $i, j \in S \cup T$, we define $J'_{i,j} \in \mathcal{S}^N$ to be the matrix with entries:

$$J'_{i,j}[(i', k), (j', l)] = \begin{cases} 1, & \text{if } i = i' \text{ and } j = j', \\ 0, & \text{otherwise.} \end{cases} \tag{12}$$

Furthermore, for $i, j \in S \cup T$ we define $J_{i,j}$ to be the symmetric version of $J'_{i,j}$, i.e.,

$$J_{i,j} := \frac{1}{2} J'_{i,j} + \frac{1}{2} (J'_{i,j})^{\mathsf{T}}.$$

Notice that $J_{i,j}$ acts on a matrix $X \in \mathcal{S}^N$ by summing all the entries in block $X_{i,j}$. Moreover, we denote by $\mathcal{A}$ the affine subspace of $\mathcal{S}^N$ consisting of all matrices X for which the sum of the entries in each block $X_{i,j}$ is equal to one, i.e.,

$$\mathcal{A} = \{X \in \mathcal{S}^N : \langle J_{i,j}, X \rangle = 1, \text{ for all } i, j \in S \cup T\}. \tag{13}$$

Lastly, we denote $\Pi : \mathcal{S}^N \to \mathbb{R}^{|A \times B \times S \times T|}$ the projection operator on the subspace of $\mathcal{S}^N$ consisting of the blocks that are indexed by $S \times T$, where we arrange the entries of any element in the image of Π as a vector in $\mathbb{R}^{|A \times B \times S \times T|}$.

Correlations sets. In the first part of this work we study the sets of classical, quantum, no-signaling and unrestricted correlations and express them as projections of affine slices of appropriate convex cones. For this we use the notion of conic correlations which we now introduce.

Definition 1.5. *For a convex cone $\mathcal{K} \subseteq \mathcal{N}^N$ the set of* $\mathcal{K}$-correlations *is defined as $\Pi(\mathcal{A} \cap \mathcal{K})$, i.e., the set of vectors $p = (p(a, b|s, t)) \in \mathbb{R}^{|A \times B \times S \times T|}$ for which there exists a matrix $X \in \mathcal{K}$ satisfying:*

$$\begin{gathered} \langle J_{i,j}, X \rangle = 1, \text{ for all } i, j \in S \cup T, \text{ and} \\ X[(s, a), (t, b)] = p(a, b|s, t), \text{ for all } a \in A, b \in B, s \in S, t \in T. \end{gathered} \tag{14}$$

We denote the set of $\mathcal{K}$-correlations by $\mathrm{Corr}(\mathcal{K})$.

In Theorem 3.7 we show there exist appropriate choices of convex cones $\mathcal{K}$ for which the sets of $\mathcal{K}$-correlations capture the sets of classical, quantum, no-signaling and unrestricted correlations.

Result 1. *Consider an arbitrary vector $p = (p(a, b|s, t)) \in \mathbb{R}^{|A \times B \times S \times T|}$. Then,*

(i) p is a classical correlation (i.e., $p \in \mathcal{C}$) if and only if $p \in \mathrm{Corr}(\mathcal{CP})$.

(ii) p is quantum correlation (i.e., $p \in \mathcal{Q}$) if and only if $p \in \mathrm{Corr}(\mathcal{CS}_+)$.

(iii) p is a no-signaling correlation (i.e., $p \in \mathcal{NS}$) if and only if $p \in \mathrm{Corr}(\mathcal{NSO})$.

(iv) p is a correlation (i.e., $p \in \mathcal{P}$) if and only if $p \in \mathrm{Corr}(\mathcal{N})$.

We note that upon completion of this work we discovered that a result similar to Result 1 *(ii)* has been derived independently in the unpublished note [MR14][1].

As suggested by Result 1 the notion of conic correlations provides a general framework allowing us to phrase and study the properties of many interesting sets of correlations. Notice that whenever $\mathcal{K}_1 \subseteq \mathcal{K}_2$ we have that $\mathrm{Corr}(\mathcal{K}_1) \subseteq \mathrm{Corr}(\mathcal{K}_2)$. Consequently, the conic inclusions given in (10) combined with Result 1 immediately give that $\mathcal{C} \subseteq \mathcal{Q} \subseteq \mathcal{P}$. As already mentioned in the introduction these inclusions are obvious by physical context but the notion of conic correlations allows us to recover these within a purely mathematical framework.

Furthermore, notice that whenever $\mathcal{K}$ is a *closed* convex cone the set of $\mathcal{K}$-correlations is also closed. Thus Result 1 implies the well-known facts that the sets $\mathcal{C}$, $\mathcal{NS}$, and $\mathcal{P}$ are all closed. On the other hand, the set of quantum correlations is not known to be closed and in fact proving so would have many important consequences. As an immediate consequence of Result 1 *(ii)* we arrive at a sufficient condition for determining the closedness of the set of quantum correlations. We note that the same observation was also made independently in the unpublished note [MR14].

Result 2. *If the cone $\mathcal{CS}_+$ is closed then the set of quantum correlations is closed.*

To the best of our knowledge, the first work where the structure of the closure of the set of quantum correlations was studied is [Fri12b].

Lastly, in Theorem 3.8 we show that for any cone $\mathcal{K} \subseteq \mathcal{DNN}$ we have that $\mathrm{Corr}(\mathcal{K}) \subseteq \mathcal{NS}$. This fact combined with Result 1 and the inclusion $\mathcal{CS}_+ \subseteq \mathcal{DNN}$ implies that

$$\mathcal{Q} \subseteq \mathrm{Corr}(\mathcal{DNN}) \subseteq \mathcal{NS}, \tag{15}$$

i.e., $\mathrm{Corr}(\mathcal{DNN})$ forms a spectrahedral outer approximation for the set of quantum correlations which is contained in the set of no-signaling correlations. In Theorem 3.16 we compare $\mathrm{Corr}(\mathcal{DNN})$ with the first level of the NPA hierarchy, denoted by $\mathrm{NPA}^{(1)}$.

Result 3. *For any Bell scenario we have that* $\mathrm{Corr}(\mathcal{DNN}) \subseteq \mathrm{NPA}^{(1)}$.

The use of convex cones to characterize the sets of quantum, classical, and no-signaling correlations was also an essential ingredient in [Fri12a]. It was shown there that in certain special cases, classical correlations are the polyhedral dual to the set of no-signaling correlations.

Game values. In the second part of this work we study the value of a nonlocal game when the players use strategies that generate classical, quantum, no-signaling or unrestricted correlations. To state our results in a succinct manner we introduce some notation that is used throughout the paper. The *cost matrix* of a game $\mathcal{G}(\pi, V)$ is the $|S \cup A| \times |T \cup B|$ matrix C whose entries are given by

$$C[(s,a),(t,b)] := \pi(s,t)V(a,b|s,t), \ \text{ for all } a \in A, b \in B, s \in S, t \in T. \tag{16}$$

The *symmetric cost matrix* of the game $\mathcal{G}$ is the $N \times N$ matrix

$$\hat{C} := \frac{1}{2}\begin{pmatrix} 0 & C \\ C^{\mathsf{T}} & 0 \end{pmatrix}. \tag{17}$$

Lastly, for a convex cone $\mathcal{K} \subseteq \mathcal{N}^N$ we denote by $\omega(\mathcal{K}, \mathcal{G})$ the maximum success probability of winning the game $\mathcal{G}$ when the players only use strategies that generate $\mathcal{K}$-correlations, i.e.,

$$(\mathcal{P}_{\mathcal{K}}) \qquad \begin{aligned} \omega(\mathcal{K}, \mathcal{G}) := \text{supremum} \quad & \langle \hat{C}, X \rangle \\ \text{subject to} \quad & \langle J_{i,j}, X \rangle = 1, \ \text{for all } i,j \in S \cup T, \\ & X \in \mathcal{K}. \end{aligned}$$

[1] We note that this manuscript has been made publicly available only on the personal webpage of one of the authors.

Notice that $(\mathcal{P}_\mathcal{K})$ is an instance of a linear conic program over the convex cone $\mathcal{K}$. As an immediate consequence of Result 1, the classical, quantum, no-signaling and unrestricted value of a nonlocal game can all be expressed as linear conic programs over appropriate convex cones.

Result 4. *For any nonlocal game G we have that:*

(i) The classical value $\omega_\mathcal{C}(\mathcal{G})$ *is equal to* $\omega(\mathcal{CP}, \mathcal{G})$.

(ii) The quantum value $\omega_\mathcal{Q}(\mathcal{G})$ *is equal to* $\omega(\mathcal{CS}_+, \mathcal{G})$.

(iii) The no-signaling value $\omega_\mathcal{NS}(\mathcal{G})$ *is equal to* $\omega(\mathcal{NSO}, \mathcal{G})$.

(iv) The unrestricted value $\omega_\mathcal{P}(\mathcal{G})$ *is equal to* $\omega(\mathcal{N}, \mathcal{G})$.

Having established conic formulations for the classical and quantum value of a nonlocal game, in Section 4.2 we study the corresponding dual conic programs and their properties. For a brief overview of the duality theory of linear conic programming the reader is referred to Section 2.3.

Definition 1.6. *Consider a nonlocal game $\mathcal{G}(\pi, V)$ with questions sets S, T and answer sets A, B. For any convex cone $\mathcal{K} \subseteq \mathcal{N}^N$ define the following linear conic program:*

$$(\mathcal{D}_\mathcal{K}) \qquad \begin{aligned} \xi(\mathcal{K}, \mathcal{G}) = \text{infimum} \quad & \sum_{i,j \in S \cup T} v_{i,j} \\ \text{subject to} \quad & \sum_{i,j \in S \cup T} v_{i,j} J_{i,j} - \hat{C} \in \mathcal{K}^*, \end{aligned}$$

where $\mathcal{K}^*$ denotes the dual cone of $\mathcal{K}$.

Conic programming duality implies that for any game $\mathcal{G}$ and convex cone $\mathcal{K} \subseteq \mathcal{N}^N$, we have $\omega(\mathcal{K}, \mathcal{G}) \leq \xi(\mathcal{K}, \mathcal{G})$. In Proposition 4.2 we show that if $\mathcal{K}$ is a closed convex cone such that $(\mathcal{P}_\mathcal{K})$ is feasible, then $\omega(\mathcal{K}, \mathcal{G}) = \xi(\mathcal{K}, \mathcal{G})$ and moreover $(\mathcal{P}_\mathcal{K})$ attains its optimum.

Since $\mathcal{CS}_+$ is not known to be closed there has been recent interest in the study of $\omega(\mathrm{cl}(\mathcal{CS}_+), \mathcal{G})$, where $\mathrm{cl}(\mathcal{CS}_+)$ denotes the closure of the cone $\mathcal{CS}_+$ [LP14, BLP15]. In Corollary 4.3 we show that $\omega\left(\mathrm{cl}(\mathcal{CS}_+^N), \mathcal{G}\right) = \xi\left(\mathcal{CS}_+^N, \mathcal{G}\right)$. This leads to a sufficient condition for showing that the $\mathcal{CS}_+$ cone is not closed: identify a nonlocal game $\mathcal{G}$ for which $\omega\left(\mathcal{CS}_+^N, \mathcal{G}\right) < \xi\left(\mathcal{CS}_+^N, \mathcal{G}\right)$.

In Section 4.3 we use our conic formulations to compare the various values of a nonlocal game. Notice that whenever $\mathcal{K}_1 \subseteq \mathcal{K}_2$ we have that $\omega(\mathcal{K}_1, \mathcal{G}) \leq \omega(\mathcal{K}_2, \mathcal{G})$. Combining this fact with the cone inclusions given in (10) we get that $\omega(\mathcal{DNN}, \mathcal{G})$ is a semidefinite programming upper bound to the quantum value of an arbitrary game $\mathcal{G}$. Furthermore, by Result 3 we have that $\omega(\mathcal{DNN}, \mathcal{G})$ is upper bounded by $\mathrm{SDP}^{(1)}(\mathcal{G})$, i.e., the value of the SDP obtained by optimizing over $\mathrm{NPA}^{(1)}$.

Result 5. *For any game $\mathcal{G}$ we have $\omega_\mathcal{Q}(\mathcal{G}) \leq \omega(\mathcal{DNN}, \mathcal{G}) \leq \mathrm{SDP}^{(1)}(\mathcal{G})$.*

Interestingly, $\omega(\mathcal{DNN}, \mathcal{G})$ was already introduced by Feige and Lovász as a semidefinite programming upper bound to the classical value of a nonlocal game [FL92] (cf. Section 4.3).

In a very recent and independent work, a similar observation was also made by creating a new SDP hierarchy approximating (among other quantities) the quantum value of a nonlocal game [BFS15]. The first level of their hierarchy corresponds to $\omega(\mathcal{DNN}, \mathcal{G})$.

Lastly, in Section 4.4 we use our conic formulations for the various game values to study the problem of deciding the existence of a strategy that wins a nonlocal game with certainty.

Definition 1.7. *Consider a nonlocal game $\mathcal{G}(\pi, V)$ and a convex cone $\mathcal{K} \subseteq \mathcal{N}$. We say that $\mathcal{G}$ admits a* perfect $\mathcal{K}$-strategy *if there exists a correlation $p \in \mathrm{Corr}(\mathcal{K})$ that satisfies $\omega(\mathcal{K}, \mathcal{G}) = 1$.*

In Lemma 4.6 we show that deciding the existence of a perfect $\mathcal{K}$-strategy is equivalent to the feasibility of a linear conic program over $\mathcal{K}$. Then, using Result 1 we show in Theorem 4.7 that deciding the existence of a perfect classical, quantum, no-signaling and unrestricted strategy is equivalent to the feasibility of a conic program over the cones $\mathcal{CP}, \mathcal{CS}_+, \mathcal{NSO}$ and $\mathcal{N}$, respectively.

A nonlocal game $\mathcal{G}$ admits a perfect unrestricted strategy as long as there does not exist a question pair $(s,t) \in S \times T$ such that $\pi(s,t) > 0$ and $V(a,b|s,t) = 0$ for all $a \in A, b \in B$. In the classical case, there are many NP-hard combinatorial problems that can be reformulated as deciding whether a certain nonlocal game admits a perfect classical strategy (cf. Section 5.3). This implies that deciding the existence of a perfect classical strategy is NP-hard. Furthermore, it was recently shown by Ji that deciding whether a game admits a perfect quantum strategy is also NP-hard (already when the input is restricted to be a binary constraint system game) [Ji13]. Nevertheless, despite significant efforts this problem is currently not known to be decidable. Unfortunately, our reformulation to a conic feasibility program does not render the problem decidable since no algorithms are currently known for determining the feasibility of a $\mathcal{CS}_+$- program.

Synchronous correlations and game values. In the third part of this work we restrict to Bell scenarios where $S = T$ and $A = B$. We first specialize our conic characterizations to the special class of synchronous correlations (cf. Definition 1.8). Using these simplified formulations we study the value of a game when the players use strategies that generate synchronous correlations. Lastly, we identify necessary and sufficient conditions for determining the existence of perfect strategies to synchronous nonlocal games (cf. Definition 1.9) that generalize recent results on the existence of perfect strategies to graph homomorphism games.

Definition 1.8. *A correlation $p \in \mathcal{P}$ is called* synchronous *if the players always respond with the same answer upon receiving the same question, i.e.,*

$$p(a,a'|s,s) = 0, \text{ for all } s \in S \text{ and } a \neq a' \in A. \tag{18}$$

It is useful to arrange the entries of a synchronous correlation $p = (p(a,a'|s,s'))$ in a square matrix of size $|S \times A|$ as follows:

$$P := \sum_{a,a' \in A, s,s' \in S} p(a,a'|s,s')\ e_s e_{s'}^{\mathsf{T}} \otimes e_a e_{a'}^{\mathsf{T}}. \tag{19}$$

Recently there has been significant interest in the study of synchronous correlations since they are related to perfect strategies for graph homomorphism games and more generally, synchronous nonlocal games (e.g. [PSS+14, MRV14, DP15]). In Section 5.1 (cf. Theorem 5.3) we show that the characterizations of quantum and classical correlations from Result 1 can be further simplified in the special case of synchronous correlations.

Result 6. *Let $p \in \mathcal{P}$ be a synchronous correlation and P as in* (19). *For $\mathcal{K} \in \{\mathcal{CP}, \mathcal{CS}_+\}$ we have that*

$$p \in \mathrm{Corr}(\mathcal{K}) \text{ if and only if } P \in \mathcal{K}^{|S\times A|}. \tag{20}$$

Another characterization of the set of synchronous quantum correlations in terms of the existence of a C^*-algebra with certain properties was identified in [PSS+14]. Furthermore, it was recently shown in [DP15] that Connes' embedding conjecture is equivalent to showing that two families of sets of quantum synchronous correlations coincide.

In Section 5.2 we study the *$\mathcal{K}$-synchronous value* of a game $\mathcal{G}$, defined as the maximum winning probability when the players use strategies that generate synchronous $\mathcal{K}$-correlations (cf. Definition 5.5). Using Result 6 it follows that the synchronous quantum and classical values can be formulated as conic programs with matrix variables of size $|S \times A|$. As $\mathcal{CP}^n = \mathcal{CS}_+^n$ for all $n \leq 4$, we get that for any binary game (i.e., a game with $|A| = |S| = 2$) the classical and quantum synchronous values coincide and, moreover, they can be computed by a semidefinite program. We note that another formulation for the quantum synchronous value of a nonlocal game (where the optimization ranges over all projective measurements) was identified in [DP15].

Many natural classes of nonlocal games considered in the literature have the property that both players share the same question and answer sets and in order to win, whenever the players receive the same question they have to respond with the same answer (e.g. [CMN+07, LP14, Rob14, MR15]). This motivates the following definition.

Definition 1.9. *A nonlocal game $\mathcal{G} = (\pi, V)$ is called* synchronous *if both players share the same question set S and the same answer set A, and moreover:*

(i) $V(a, a'|s, s) = 0$, *for all $s \in S, a \neq a' \in A$, and*

(ii) $\pi(s, s) > 0$, *for all $s \in S$.*

The notion of synchronous games was implicit in [PSS+14] and was formally defined in [MRV14] and [DP15]. Synchronous games enjoy special properties with respect to perfect strategies (e.g. see [MRV14]). In particular, every perfect strategy for a synchronous game generates a synchronous correlation. In Section 5.3 we focus on the problem of deciding whether a synchronous game admits a perfect classical or quantum strategy. Using Result 6 we show in Theorem 5.8 that this problem is equivalent to deciding the feasibility of a conic program with matrix variables of size $|S \times A|$.

Result 7. *Let $\mathcal{G}(\pi, V)$ be a synchronous game and $\mathcal{K} \in \{\mathcal{CP}, \mathcal{CS}_+\}$. The following are equivalent:*

(i) $\mathcal{G}$ admits a perfect $\mathcal{K}$-strategy.

(ii) There exists a matrix $X \in \mathcal{K}^{|S \times A|}$ satisfying:

- $\langle J_{s,s'}, X \rangle = 1$, *for all $s, s' \in S$,*
- $X[(s, a), (s, a')] = 0$, *for all $s \in S, a \neq a' \in A$,*
- $X[(s, a), (s', a')] = 0$, *for all $a, a' \in A, s, s' \in S$, such that $\pi(s, s') > 0$ and $V(a, a'|s, s') = 0$.*

In Sections 5.3.1 and 5.3.2 we specialize Result 7 to graph homomorphism and graph coloring games. This enables us to recover in a uniform manner the conic formulations for quantum graph homomorphisms, the quantum chromatic number and the quantum independence number that were recently derived in the literature [LP14, Rob14]. Lastly, in Section 5.3.3 we introduce the notion of quantum satisfiability for binary constraint satisfaction problems and show that all these combinatorial parameters can be realized as special instances of this general framework.

1.5 Organization of the paper

In Section 2 we introduce the notation and background on linear algebra, quantum mechanics, and linear conic programming needed for this work. In Section 3 we discuss how correlations corresponding to various physical models can be represented as projections of affine slices of appropriate convex cones and identify a spectrahedral outer approximation for the set of quantum correlations. In Section 4 we show that values of nonlocal games can be formulated as conic programming problems and we further discuss the Feige-Lovász SDP relaxation for the value of a nonlocal game. Additionally, we show that deciding the existence of a perfect strategy is equivalent to a conic feasibility problem. Finally, in Section 5 we specialize our characterizations to synchronous correlations and synchronous game values. This allows us to recover the conic formulations for various quantum graph parameters that were recently obtained in the literature.

2 Notation and background

2.1 Linear algebra

Throughout this work, a *finite dimensional complex Euclidean space* refers to the vector space $\mathbb{C}^n$ (for some $n \geq 1$) equipped with the canonical inner product on $\mathbb{C}^n$. We denote by $\{e_i\}_{i=1}^n$ the standard orthonormal basis of $\mathbb{C}^n$. Given two complex Euclidean spaces $\mathcal{X}, \mathcal{Y}$ we denote by $\mathrm{L}(\mathcal{X}, \mathcal{Y})$ the space of linear operators from $\mathcal{X}$ to $\mathcal{Y}$ which we endow with the Hilbert-Schmidt inner product $\langle X, Y \rangle := \mathrm{Tr}(X^* Y)$ for $X, Y \in \mathrm{L}(\mathcal{X}, \mathcal{Y})$. For an operator $X \in \mathrm{L}(\mathcal{X}, \mathcal{Y})$ we denote its *adjoint* operator by $X^* \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$ and its *transpose* by $X^\mathsf{T} \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$. We make extensive use of the correspondence between $\mathrm{L}(\mathcal{X}, \mathcal{Y})$ and $\mathcal{Y} \otimes \mathcal{X}$ given by the map $\mathrm{vec} : \mathrm{L}(\mathcal{X}, \mathcal{Y}) \to \mathcal{Y} \otimes \mathcal{X}$, which is given by

$$\mathrm{vec}(e_i e_j^*) = e_i \otimes e_j,$$

on basis vectors and is extended linearly. The $\mathrm{vec}(\cdot)$ map is a linear bijection between $\mathrm{L}(\mathcal{X}, \mathcal{Y})$ and $\mathcal{Y} \otimes \mathcal{X}$ and furthermore it is an isometry, i.e., $\langle X, Y \rangle = \langle \mathrm{vec}(Y), \mathrm{vec}(X) \rangle$ for all $X, Y \in \mathrm{L}(\mathcal{X}, \mathcal{Y})$. Notice that

$$(X \otimes Y)\mathrm{vec}(Z) = \mathrm{vec}(XZY^\mathsf{T}), \tag{21}$$

for operators X, Y, Z of the appropriate size. Using (21) it follows that

$$\mathrm{vec}(W)^*(X \otimes Y)\mathrm{vec}(Z) = \mathrm{vec}(W)^*\mathrm{vec}(XZY^\mathsf{T}) = \langle W, XZY^\mathsf{T} \rangle, \tag{22}$$

an identity that we use repeatedly in this paper.

Any vector $\psi \in \mathcal{Y} \otimes \mathcal{X}$ can be expressed as $\psi = \sum_{i=1}^d \lambda_i \, y_i \otimes x_i$ for some integer $d \geq 1$, positive scalars $\{\lambda_i : i \in [d]\}$, and orthonormal sets $\{y_i : i \in [d]\} \subseteq \mathcal{Y}$ and $\{x_i : i \in [d]\} \subseteq \mathcal{X}$. An expression of this form is known as a *Schmidt decomposition* for ψ and is derived by the singular value decomposition of $\mathrm{vec}^{-1}(\psi)$. The scalars $\{\lambda_i\}_{i=1}^d$ and the integer d are uniquely defined and are called the *Schmidt coefficients* and the *Schmidt rank* of ψ, respectively. If $\psi = \sum_{i=1}^d \lambda_i \, y_i \otimes x_i$ is a Schmidt decomposition for ψ notice that $\|\psi\|_2^2 = \sum_{i=1}^d \lambda_i^2$. Lastly, given a vector $x \in \mathbb{C}^n$ we define $\mathrm{Diag}(x) := \sum_{i=1}^n x_i e_i e_i^*$.

2.2 Quantum mechanics

In this section, we give a brief overview of the mathematical formulation of quantum mechanics. The reader is referred to [NC00] and [Wat11] for a more thorough introduction.

According to the axioms of quantum mechanics, associated to any physical system X is a finite dimensional complex Euclidean space $\mathcal{X}$. The *state space* of X is identified with the set of unit vectors in $\mathcal{X}$. A *measurement* on a system X is specified by a family of positive semidefinite operators $\{X_i : i \in \mathcal{I}\} \subseteq \mathcal{H}_+(\mathcal{X})$ with the property that $\sum_{i \in \mathcal{I}} X_i = \mathbb{I}_\mathcal{X}$. The set $\mathcal{I}$ labels the set of possible outcomes of the measurement. According to the axioms of quantum mechanics, when the measurement $\{X_i : i \in \mathcal{I}\}$ is performed on a system X which is in state $\psi \in \mathcal{X}$ the outcome $i \in \mathcal{I}$ occurs with probability $p(i) = \psi^* X_i \psi$. Notice that $\{p(i) : i \in \mathcal{I}\}$ forms a valid probability distribution since by the definition of a measurement we have that $p(i) \geq 0$ for all $i \in \mathcal{I}$ and $\sum_{i \in \mathcal{I}} p(i) = 1$.

Consider two quantum systems X and Y with corresponding state spaces $\mathcal{X}$ and $\mathcal{Y}$ respectively. According to the axioms of quantum mechanics the Euclidean space that corresponds to the *joint* system (X, Y) is given by the tensor product $\mathcal{X} \otimes \mathcal{Y}$ of the individual spaces. Furthermore, if the systems X and Y are independently prepared in states $\psi_1 \in \mathcal{X}$ and $\psi_2 \in \mathcal{Y}$ then the state of the joint system is given by the vector $\psi_1 \otimes \psi_2 \in \mathcal{X} \otimes \mathcal{Y}$. A state in $\mathcal{X} \otimes \mathcal{Y}$ of the form $\psi_1 \otimes \psi_2$ for some

$\psi_1 \in \mathcal{X}$, $\psi_2 \in \mathcal{Y}$ is called a *product state*. Quantum states that are not in product form are called *entangled*. Lastly, any two measurements $\{X_i : i \in \mathcal{I}\} \subseteq \mathcal{H}_+(\mathcal{X})$ and $\{Y_j : j \in \mathcal{J}\} \subseteq \mathcal{H}_+(\mathcal{Y})$ on the individual systems X and Y define a *product measurement* on the joint system with outcomes $\{(i,j) : i \in \mathcal{I}, j \in \mathcal{J}\}$. The corresponding measurement operators are given by $\{X_i \otimes Y_j : i \in \mathcal{I}, j \in \mathcal{J}\} \subseteq \mathcal{H}_+(\mathcal{X} \otimes \mathcal{Y})$ and the probability of getting outcome $(i,j) \in \mathcal{I} \times \mathcal{J}$ is equal to $\psi^*(X_i \otimes Y_j)\psi$.

2.3 Convex analysis and linear conic programming

In this section, we briefly introduce conic programming and state the duality results that are relevant to this work. For additional details, the reader is referred to Ben-Tal and Nemirovski [BTN01].

Let $\mathcal{V}$ be a finite dimensional vector space equipped with inner product $\langle \cdot, \cdot \rangle$. Given a subset $A \subseteq \mathcal{V}$ we denote by $\mathrm{cl}(A)$ the *closure* of A and by $\mathrm{int}(A)$ the *interior* of A with respect to the topology induced by the inner product. A subset $\mathcal{K} \subseteq \mathcal{V}$ is called a *cone* if $X \in \mathcal{K}$ implies that $\lambda X \in \mathcal{K}$ for all $\lambda \geq 0$. A cone $\mathcal{K}$ is *convex* if $X, Y \in \mathcal{K}$ implies that $X + Y \in \mathcal{K}$. For any cone $\mathcal{K}$ we can define its *dual cone*, denoted by $\mathcal{K}^*$, given by

$$\mathcal{K}^* := \{S \in \mathcal{V} : \langle X, S \rangle \geq 0 \text{ for all } X \in \mathcal{K}\}.$$

The dual cone $\mathcal{K}^*$ is always closed. A cone $\mathcal{K}$ is called *self-dual* if $\mathcal{K} = \mathcal{K}^*$. For every convex cone $\mathcal{K}$ we have that $(\mathcal{K}^*)^* = \mathrm{cl}(\mathcal{K})$. As a consequence a cone $\mathcal{K}$ is closed if and only if $\mathcal{K} = (\mathcal{K}^*)^*$.

Consider two finite dimensional inner-product spaces $\mathcal{V}$ and $\mathcal{W}$ and a convex cone $\mathcal{K} \subseteq \mathcal{V}$. A *linear conic program* (over the cone $\mathcal{K}$) is specified by a triple $(C, \mathcal{L}, B)$ where $C \in \mathcal{V}$, $B \in \mathcal{W}$ and $\mathcal{L} : \mathcal{V} \to \mathcal{W}$ is a linear transformation. To such a triple we associate two optimization problems:

Primal problem (P)	Dual problem (D)
$p :=$ supremum $\langle C, X \rangle$	$d :=$ infimum $\langle B, Y \rangle$
subject to $\mathcal{L}(X) = B$,	subject to $\mathcal{L}^*(Y) - C \in \mathcal{K}^*$,
$X \in \mathcal{K}$,	$Y \in \mathcal{W}$,

referred to as the *primal* and the *dual*, respectively. For brevity, sometimes we drop the "linear" and just refer to them as *conic programs*. We call p the *primal value* and d the *dual value* of $(C, \mathcal{L}, B)$.

Linear conic programming constitutes a wide generalization of several well-studied models of mathematical optimization. For example, setting $\mathcal{V} = \mathbb{R}^n$, $\mathcal{W} = \mathbb{R}^m$ (equipped with the canonical inner-product) and $\mathcal{K} = \mathbb{R}^n_+$ then (P) and (D) form a pair of primal-dual *linear programs*. Furthermore, setting $\mathcal{V} = S^n$, $\mathcal{W} = \mathcal{S}^m$ (equipped with the Hilbert-Schmidt inner product) and $\mathcal{K} = \mathcal{S}^n_+$ then (P) and (D) form a pair of primal-dual *semidefinite programs*.

We say that the conic program $(C, \mathcal{L}, B)$ is *primal feasible* if $\{X \in \mathcal{V} : \mathcal{L}(X) = B\} \cap \mathcal{K} \neq \emptyset$ and *primal strictly feasible* if $\{X \in \mathcal{V} : \mathcal{L}(X) = B\} \cap \mathrm{int}(\mathcal{K}) \neq \emptyset$. Analogously, the conic program $(C, \mathcal{L}, B)$ is called *dual feasible* if there exists $Y \in \mathcal{W}$ such that $\mathcal{L}^*(Y) - C \in \mathcal{K}^*$ and *dual strictly feasible* if there exists $Y \in \mathcal{W}$ such that $\mathcal{L}^*(Y) - C \in \mathrm{int}(\mathcal{K}^*)$. The set of feasible solutions of a linear programming problem is called a *polyhedron* and the set of feasible solutions of a semidefinite programming problem is usually called a *spectrahedron*.

Conic programs share some of the duality theory available for linear and semidefinite programs. In particular, the dual value is always an upper bound on the primal value and, moreover, equality and attainment hold assuming appropriate constraint qualifications.

Theorem 2.1. *Let $\mathcal{K}$ be a convex cone and $(C, \mathcal{L}, B)$ be a linear conic program over $\mathcal{K}$.*

(i) (Weak duality) If X is primal feasible and Y is dual feasible then $\langle C, X \rangle \leq \langle B, Y \rangle$.

(ii) *(Strong duality) Suppose $\mathcal{K}$ is a closed convex cone. If the primal is strictly feasible and $p < +\infty$ we have that $p = d$ and moreover the dual value is attained. Symmetrically, if the dual program is strictly feasible and $d > -\infty$ then $p = d$ and the primal value is attained.*

Strong duality results in the conic programming setting are stated for *closed* convex cones. For a closed convex cone $\mathcal{K}$ we have $\mathcal{K} = (\mathcal{K}^*)^*$ so the duality results are symmetric with respect to the primal and the dual problem. Since the $\mathcal{CS}_+$ cone is not known to be closed we cannot apply Theorem 2.1 *(ii)* for $\mathcal{K} = \mathcal{CS}_+$. In Section 4.2 we apply Theorem 2.1 *(ii)* to $\mathrm{cl}(\mathcal{CS}_+)$ and $\mathcal{CS}_+^*$.

3 Correlations as projections of affine slices of convex cones

In this section we study the sets of classical, quantum, no-signaling and unrestricted correlations and express them in a uniform manner as projections of affine slices of appropriate convex cones. Using these characterizations we identify a spectrahedral outer approximation to the set of quantum correlations which is contained in the first level of the NPA hierarchy and a sufficient condition for showing that the set of quantum correlations is closed.

3.1 An algebraic characterization of quantum correlations

In our first result in this section we investigate the structure of the quantum states that can be used to generate a quantum correlation and show they can taken to have a specific form.

Lemma 3.1. *Any quantum correlation $p = (p(a, b|s, t)) \in \mathcal{Q}$ can be generated by a quantum state of the form $\psi = \sum_{i=1}^d \sqrt{\lambda_i}\, e_i \otimes e_i \in \mathbb{C}^d \otimes \mathbb{C}^d$, where $\sum_{i=1}^d \lambda_i = 1$ and $\{e_i : i \in [d]\}$ is the standard basis for $\mathbb{C}^d$.*

Proof. Since $p \in \mathcal{Q}$ there exists a quantum state $\psi \in \mathcal{X} \otimes \mathcal{Y}$ and quantum measurement operators $\{X_a^s : a \in A\} \subseteq \mathcal{H}_+(\mathcal{X})$ and $\{Y_b^t : b \in B\} \subseteq \mathcal{H}_+(\mathcal{Y})$ satisfying $p(a, b|s, t) = \psi^*(X_a^s \otimes Y_b^t)\psi$, for all $a \in A, b \in B, s \in S, t \in T$. By the Schmidt decomposition, the vector $\psi \in \mathcal{X} \otimes \mathcal{Y}$ can be expressed as $\psi = \sum_{i=1}^d \sqrt{\lambda_i}\, x_i \otimes y_i$ where $\sum_{i=1}^d \lambda_i = 1$ and $\{x_i : i \in [d]\} \subseteq \mathcal{X}$, $\{y_i : i \in [d]\} \subseteq \mathcal{Y}$ are orthonormal sets of vectors. Define the operators $U := \sum_{i=1}^d e_i x_i^*$ and $U' := \sum_{i=1}^d e_i {y_i}^*$. Notice that

- $\tilde{\psi} := (U \otimes U')\psi = \sum_{i=1}^d \sqrt{\lambda_i}\, e_i \otimes e_i \in \mathbb{C}^d \otimes \mathbb{C}^d$ is a valid quantum state.
- For all $s \in S$, the operators $\{\tilde{X}_a^s := U X_a^s U^* : a \in A\}$ form a valid measurement on $\mathbb{C}^d \otimes \mathbb{C}^d$.
- For all $t \in T$, the operators $\{\tilde{Y}_b^t := U' Y_b^t (U')^* : b \in B\}$ form a valid measurement on $\mathbb{C}^d \otimes \mathbb{C}^d$.
- $\tilde{\psi}^*(\tilde{X}_a^s \otimes \tilde{Y}_b^t)\tilde{\psi} = \psi^*(X_a^s \otimes Y_b^t)\psi$, for all $a \in A, b \in B, s \in S, t \in T$.

Thus, the strategy given by the quantum state $\tilde{\psi}$ and the quantum measurements $\{\tilde{X}_a^s : a \in A\}$ and $\{\tilde{Y}_b^t : b \in B\}$ also generates the correlation p and has the desired properties. □

Based on Lemma 3.1 we arrive at a new algebraic characterization of the set of quantum correlations that is of central importance in Section 3.3.

Theorem 3.2. *Consider a vector $p = (p(a, b|s, t)) \in \mathbb{R}^{|A \times B \times S \times T|}$. The following are equivalent:*

(i) *p is a quantum correlation.*

(ii) There exists an integer $d \geq 1$ such that the following system is feasible:

$$
\begin{aligned}
&\langle K, K\rangle = 1, \\
&\sum_{a\in A} X_a^s = K, \text{ for all } s \in S, \\
(23) \qquad &\sum_{b\in B} Y_b^t = K, \text{ for all } t \in T, \\
&p(a,b|s,t) = \langle X_a^s, Y_b^t\rangle, \text{ for all } a \in A, b \in B, s \in S, t \in T, \\
&K, X_a^s, Y_b^t \in \mathcal{H}_+^d, \text{ for all } a \in A, b \in B, s \in S, t \in T.
\end{aligned}
$$

Proof. Let $p = (p(a,b|s,t)) \in \mathcal{Q}$ be a quantum correlation. By Definition 1.3 there exists a quantum state $\psi \in \mathcal{X} \otimes \mathcal{Y}$ and measurements $\{\tilde{X}_a^s : a \in A\} \subseteq \mathcal{H}_+(\mathcal{X})$ and $\{\tilde{Y}_b^t : b \in B\} \subseteq \mathcal{H}_+(\mathcal{Y})$ such that

$$(24) \qquad p(a,b|s,t) = \psi^*(\tilde{X}_a^s \otimes \tilde{Y}_b^t)\psi, \text{ for all } a,b,s,t.$$

By Lemma 3.1 we may assume $\mathcal{X} = \mathcal{Y} = \mathbb{C}^d$, for some integer $d \geq 1$ and that $\psi = \sum_{i=1}^d \sqrt{\lambda_i}\, e_i \otimes e_i$, where $\sum_{i=1}^d \lambda_i = 1$, and $\{e_i : i \in [d]\}$ is the standard basis in $\mathbb{C}^d$. Define

- $K := \sum_{i=1}^d \sqrt{\lambda_i}\, e_i e_i^* \in \mathcal{S}_+^d$. Notice that $\mathrm{vec}(K) = \psi$ and $\langle K, K\rangle = 1$.
- $X_a^s := K^{1/2}(\tilde{X}_a^s)K^{1/2} \in \mathcal{S}_+^d$, for all a, s. These operators satisfy $\sum_{a\in A} X_a^s = K$, for all $s \in S$.
- $Y_b^t := K^{1/2}(\tilde{Y}_b^t)^{\mathsf{T}} K^{1/2} \in \mathcal{S}_+^d$, for all b, t. These operators satisfy $\sum_{b\in B} Y_b^t = K$, for all $t \in T$.

From the definitions above and the properties of the vec operator (cf. (22)) it follows that

$$(25) \qquad \langle X_a^s, Y_b^t\rangle = \mathrm{Tr}(K\tilde{X}_a^s K(\tilde{Y}_b^t)^{\mathsf{T}}) = \mathrm{vec}(K)^*(\tilde{X}_a^s \otimes \tilde{Y}_b^t)\mathrm{vec}(K) = \psi^*(\tilde{X}_a^s \otimes \tilde{Y}_b^t)\psi = p(a,b|s,t),$$

for all $a \in A, b \in B, s \in S, t \in T$ and thus (23) is feasible.

Conversely let $K, \{X_a^s\}_{s\in S, a\in A}, \{Y_b^t\}_{t\in T, b\in B}$ be feasible for (23). Without loss of generality, we may assume K has full rank. Define:

- $\psi := \mathrm{vec}(K)$. Notice that $\|\psi\|_2 = 1$.
- $\tilde{X}_a^s := K^{-1/2} X_a^s K^{-1/2} \in \mathcal{S}_+^d$, for $a \in A$, $s \in S$. These operators satisfy $\sum_{a\in A} \tilde{X}_a^s = \mathbb{I}_d$ for all s.
- $\tilde{Y}_b^t := \left(K^{-1/2} Y_b^t K^{-1/2}\right)^{\mathsf{T}} \in \mathcal{S}_+^d$ for $b \in B$, $t \in T$. These operators satisfy $\sum_{b\in B} \tilde{Y}_b^t = \mathbb{I}_d$ for all t.

Reversing the calculation in (25) we get that $p(a,b|s,t) = \langle X_a^s, Y_b^t\rangle = \psi^*(\tilde{X}_a^s \otimes \tilde{Y}_b^t)\psi$, for all a,b,s,t which shows that $p \in \mathcal{Q}$. □

Remark 3.3. *A close inspection of the proof of Theorem 3.2 allows us to explicitly work out the dependency of the parameter d on the dimension of the underlying quantum system. Specifically, for a correlation $p \in \mathcal{Q}$ that is generated by a state $\psi \in \mathcal{X} \otimes \mathcal{Y}$ there exist Hermitian psd matrices of size $d \leq \min\{\dim(\mathcal{X}), \dim(\mathcal{Y})\}$ that satisfy* (23). *Conversely, if* (23) *has a feasible solution with matrices (real or complex) of size $d \geq 1$ then the correlation $p = (p(a,b|s,t))$ can be generated by a state in $\mathbb{C}^d \otimes \mathbb{C}^d$. This observation can be used to derive a lower bound on the dimension of a Hilbert space needed to generate an arbitrary quantum correlation [SVW15].*

Remark 3.4. *As another by-product of Theorem 3.2 we obtain an algebraic characterization of the quantum correlations that can be generated using a maximally entangled state. Specifically, it follows easily from the proof of Theorem 3.2 that a quantum correlation $p \in \mathcal{Q}$ can be generated using the d-*dimensional maximally entangled state $\psi_d := \mathrm{vec}\left(\frac{1}{\sqrt{d}} I_d\right)$ *if and only if* (23) *is feasible with $K = \frac{1}{\sqrt{d}}\mathbb{I}_d$.*

3.2 An algebraic characterization of classical correlations

It is clear from physical context that every classical correlation is also quantum. In particular, any classical correlation admits a representation as a quantum correlation for some appropriate choice of quantum state and measurement operators. In the next result we show that a correlation vector is classical if and only if (23) admits a solution with diagonal psd matrices.

Theorem 3.5. *Consider a vector $p = (p(a, b|s, t)) \in \mathbb{R}^{|A\times B\times S\times T|}$. The following are equivalent:*

(i) p is a classical correlation.

(ii) p can be generated by a state of the form

$$\psi = \sum_{i=1}^{n} \sqrt{\lambda_i}\; e_i \otimes e_i, \textit{ where } \sum_{i=1}^{n} \lambda_i = 1,\; \lambda_i \geq 0 \textit{ for } i \in [n],$$

and diagonal *measurement operators.*

(iii) There exists a solution to (23) *with diagonal matrices.*

Proof. *(i) $\implies$ (ii)*: Let $p = (p(a, b|s, t)) \in \mathcal{P}$ be a classical correlation. By Definition 1.2 there exist nonnegative scalars $k_i \geq 0,\ x_a^{s,i} \geq 0$, and $y_b^{t,i} \geq 0$ satisfying $p(a, b|s, t) = \sum_{i=1}^{n} k_i x_a^{s,i} y_b^{t,i}$, for all $a \in A, b \in B, s \in S, t \in T$, where $\sum_{i=1}^{n} k_i = 1$, $\sum_{a\in A} x_a^{s,i} = 1$ for all $i \in [n], s \in S$ and $\sum_{b\in B} y_b^{t,i} = 1$ for all $i \in [n], t \in T$. Define

- $\psi := \sum_{i=1}^{n} \sqrt{k_i}\; e_i \otimes e_i$, which is a quantum state of the form in Lemma 3.1,
- $X_a^s := \sum_{i=1}^{n} x_a^{s,i} e_i e_i^*$, for $s \in S$, $a \in A$, which are diagonal,
- $Y_b^t := \sum_{i=1}^{n} y_b^{t,i} e_i e_i^*$, for $t \in T$, $b \in B$, which are diagonal.

A straightforward calculation shows that the state ψ and the measurements $\{X_a^s\}_{s\in S, a\in A}$ and $\{Y_b^t\}_{t\in T, b\in B}$ generate p.

(ii) $\implies$ (i): Assume that p can be generated by a quantum state $\psi = \sum_{i=1}^{n} \sqrt{\lambda_i}\, e_i \otimes e_i$ and diagonal measurement operators $\{X_a^s\}_{s\in S, a\in A}$ and $\{Y_b^t\}_{t\in T, b\in B}$. Define

- $k_i := \lambda_i,$ for all $i \in [n]$,
- $x_a^{s,i} := X_a^s[i, i],$ for all $i \in [n], s \in S$ and $a \in A$,
- $y_b^{t,i} := Y_b^t[i, i],$ for all $i \in [n], t \in T$ and $b \in B$,

and notice that this defines a classical strategy that generates the correlation p.

(ii) $\iff$ (iii): This is clear from the proof of Theorem 3.2 noting that K being diagonal and satisfying $\langle K, K\rangle = 1$, implies that the quantum state $\psi := \mathrm{vec}(K)$ is of the required form. □

3.3 Conic characterization of correlations

For a convex cone $\mathcal{K} \subseteq \mathcal{N}$ we defined the set of $\mathcal{K}$-*correlations* as $\mathrm{Corr}(\mathcal{K}) = \Pi(\mathcal{A} \cap \mathcal{K})$ (recall Definition 1.5). In this section we show that the sets of classical, quantum, no-signaling and unrestricted correlations can be expressed as the sets of conic correlations for appropriate choices of convex cones $\mathcal{K}$. The characterization for the quantum and classical case relies on the algebraic characterizations that were derived in Theorem 3.2 and Theorem 3.5 respectively.

We start with a geometric lemma of central importance in this section.

Lemma 3.6. *Consider two families of vectors $\{x_a^s\}_{s\in S,a\in A}$ and $\{y_b^t\}_{t\in T,b\in B}$ in some Euclidean space $\mathcal{X}$.*

(a) For $X := \mathrm{Gram}(\{x_a^s\}_{s\in S,a\in A}, \{y_b^t\}_{t\in T,b\in B})$ the following are equivalent:

(i) There exists $k \in \mathcal{X}$ such that $\sum_{a\in A} x_a^s = \sum_{b\in B} y_b^t = k,$ for all $s \in S$, $t \in T$, and $\langle k, k\rangle = 1$.

(ii) $\langle J_{i,j}, X\rangle = 1,$ for all $i, j \in S \cup T$.

(b) Set $\tilde{X} := \mathrm{Gram}(k, \{x_a^s\}_{s\in S,a\in A}, \{y_b^t\}_{t\in T,b\in B})$ where $k \in \mathcal{X}$ with $\langle k, k\rangle = 1$. The following are equivalent:

(i) $\sum_{a\in A} x_a^s = \sum_{b\in B} y_b^t = k,$ for all $s \in S, t \in T$.

(ii) $\langle J_{i,j}, \tilde{X}\rangle = 1,$ for all $i, j \in \{0\} \cup S \cup T$.

Proof. We start with part *(a)*. To show *(i)* implies *(ii)*, consider $i \in S$ and $j \in T$ and notice that

$$\langle J_{i,j}, X\rangle = \sum_{a\in A}\sum_{b\in B} X[(i,a),(j,b)] = \sum_{a\in A}\sum_{b\in B}\langle x_a^i, y_b^j\rangle = \langle k, k\rangle = 1.$$

For the other direction, define $x_i := \sum_{a\in A} x_a^i$ for all $i \in S$ and $y_i := \sum_{b\in B} y_b^i$ for all $i \in T$. Notice that for any $i, j \in S$ the equation $\langle J_{i,j}, X\rangle = 1$ is equivalent to $\langle x_i, x_j\rangle = 1$. This implies that $\langle x_i - x_j, x_i - x_j\rangle = 0$ for all $i, j \in S$ and thus $x_i = x_j$ for all $i, j \in S$. Similarly we have that $y_i = y_j$ for all $i, j \in T$. Lastly, fix any $i \in S$ and $j \in T$ and notice that $\langle J_{i,j}, X\rangle = 1$ implies that $\langle x_i, y_j\rangle = 1$. As before this implies that $x_i = y_j$. The proof is concluded by setting $k := x_i$.

We proceed with part *(b)*. It is easy to see that *(i)* implies *(ii)*. For the other direction we have from part *(a)* that there exists $k' \in \mathcal{X}$ such that $\sum_{a\in A} x_a^s = \sum_{b\in B} y_b^t = k',$ for all $s \in S$, $t \in T$, and $\langle k', k'\rangle = 1$. It suffices to show that $k = k'$. For this, notice that

$$\langle k - k', k - k'\rangle = 2 - 2\langle k, k'\rangle = 2 - 2\sum_{a\in A}\tilde{X}[0,(s,a)] = 0,$$

and the proof is concluded. □

A close inspection of the proof of Lemma 3.6 *(a)* reveals that in condition *(ii)* it suffices to only include one pair of indices $(i, j) \in S \times T$. We now state and prove our main result in this section.

Theorem 3.7. *Consider a Bell scenario with question sets S, T and answer sets A, B. Then,*

(i) $\mathcal{C} = \Pi\big(\mathcal{A} \cap \mathcal{CP}^N\big)$.

(ii) $\mathcal{Q} = \Pi\big(\mathcal{A} \cap \mathcal{CS}_+^N\big)$.

(iii) $\mathcal{NS} = \Pi\big(\mathcal{A} \cap \mathcal{NSO}^N\big)$.

(iv) $\mathcal{P} = \Pi\big(\mathcal{A} \cap \mathcal{N}^N\big)$.

Proof. Case *(i)* follows from Lemma 3.6 *(a)* and Theorem 3.5 and case *(ii)* from Lemma 3.6 *(a)* and Theorem 3.2. Lastly, *(iii)* and *(iv)* follow easily from the definitions of the corresponding cones. □

As exemplified by Theorem 3.7 the notion of $\mathcal{K}$-correlations has significant expressive power as it captures many correlation sets of physical significance. In our next result we continue the study of conic correlations and identify a sufficient condition in terms of the cone $\mathcal{K}$ so that the corresponding set of correlations $\mathrm{Corr}(\mathcal{K})$ satisfies the no-signaling conditions.

Theorem 3.8. *For any convex cone $\mathcal{K} \subseteq \mathcal{DNN}$ we have that* $\mathrm{Corr}(\mathcal{K}) \subseteq \mathcal{NS}$.

Proof. For any $p = (p(a,b|s,t)) \in \mathrm{Corr}(\mathcal{K})$ there exists $X \in \mathcal{DNN}^N$ such that $\langle J_{i,j}, X\rangle = 1$ for all $i,j \in S \cup T$ and $p(a,b|s,t) = X[(s,a),(t,b)]$ for all a,b,s,t. If $X = \mathrm{Gram}(\{x_a^s\}_{s\in S, a\in A}, \{y_b^t\}_{t\in T, b\in B})$ it follows from Lemma 3.6 (a) that there exists a vector k such that

$$\sum_{a\in A} x_a^s = \sum_{b\in B} y_b^t = k, \quad \text{for all } s \in S, t \in T, \quad \text{and} \quad \langle k,k\rangle = 1. \tag{26}$$

For $s \neq s' \in S$ and $t \in T$ we get from (26) that

$$\sum_{a\in A} X[(s,a),(t,b)] = \sum_{a\in A} \langle x_a^s, y_b^t\rangle = \langle k, y_b^t\rangle = \sum_{a\in A}\langle x_a^{s'}, y_b^t\rangle = \sum_{a\in A} X[(s',a),(t,b)], \tag{27}$$

and thus $\sum_{a\in A} p(a,b|s,t) = \sum_{a\in A} p(a,b|s',t)$ for all $t \in T$ and $s \neq s' \in S$. Symmetrically, we have that $\sum_{b\in B} p(a,b|s,t) = \sum_{b\in B} p(a,b|s,t')$ for all $s \in S$ and $t \neq t' \in T$ and thus $p \in \mathcal{NS}$. □

We now use Theorem 3.7 to derive a sufficient condition for showing that the set of quantum correlations is closed. It follows from the definition of the affine space $\mathcal{A}$ (recall (13)) that for every cone $\mathcal{K} \subseteq \mathcal{N}$ the set $\mathcal{A} \cap \mathcal{K}$ is bounded. Consequently, if the cone $\mathcal{K}$ is closed it follows that $\mathcal{A} \cap \mathcal{K}$ is a compact set and thus $\Pi(\mathcal{A} \cap \mathcal{K})$ is also compact. This gives the following corollary.

Proposition 3.9. *If the cone $\mathcal{CS}_+$ is closed then the set of quantum correlations is also closed.*

We conclude this section with a second conic formulation for the sets of $\mathcal{CP}, \mathcal{CS}_+$ and $\mathcal{DNN}$-correlations. We use these formulations in Section 3.4 where we compare $\mathrm{Corr}(\mathcal{DNN})$ with the first level of the NPA hierarchy and in Section 4.4 where we recover the conic programming formulations for certain quantum graph parameters that were recently obtained in the literature [LP14].

Lemma 3.10. *Let $p = (p(a,b|s,t)) \in \mathcal{P}$ and set $P := \sum_{a,b,s,t} p(a,b|s,t)\ e_s e_t^{\mathsf{T}} \otimes e_a e_b^{\mathsf{T}}$. For any cone $\mathcal{K} \in \{\mathcal{CP}, \mathcal{CS}_+, \mathcal{DNN}\}$ we have that $p = (p(a,b|s,t)) \in \mathrm{Corr}(\mathcal{K})$ if and only if there exists a matrix*

$$\tilde{X} = \begin{pmatrix} 1 & x^{\mathsf{T}} & y^{\mathsf{T}} \\ x & X & P \\ y & P^{\mathsf{T}} & Y \end{pmatrix} \in \mathcal{K}^{1+N}, \tag{28}$$

such that $\langle J_{i,j}, \tilde{X}\rangle = 1$, for all $i,j \in \{0\} \cup S \cup T$.

Proof. This follows easily from the definition of $\mathrm{Corr}(\mathcal{K})$ combined with Lemma 3.6 *(b)*. □

Remark 3.11. *Let $\mathcal{K} \in \{\mathcal{CP}, \mathcal{CS}_+, \mathcal{DNN}\}$. Given a correlation $p = (p(a,b|s,t)) \in \mathrm{Corr}(\mathcal{K})$ it follows from the proof of Lemma 3.10 that every feasible solution to* (28) *satisfies $\tilde{X}[0,(s,a)] = p_A(a|s)$, for all $a \in A, s \in S$ and $\tilde{X}[0,(t,b)] = p_B(b|t)$, for all $t \in T, b \in B$. We make use of this fact in Theorem 3.16.*

3.4 A spectrahedral outer approximation for quantum correlations

In this section we use the conic formulations from Theorem 3.7 to derive a new spectrahedral outer approximation for the set of quantum correlations. Furthermore, we show that our approximation is at least as strong as the first level of the NPA hierarchy.

In Theorem 3.7 we showed that $\mathcal{Q} = \mathrm{Corr}(\mathcal{CS}_+)$. As $\mathcal{CS}_+ \subseteq \mathcal{DNN}$ we immediately get a necessary and efficiently verifiable condition for membership in the set of quantum correlations.

Proposition 3.12. *For any Bell scenario we have that*

$$\mathcal{Q} \subseteq \mathrm{Corr}(\mathcal{DNN}) \subseteq \mathcal{NS}. \tag{29}$$

As already mentioned the set of quantum correlations is a non-polyhedral set whose structure is poorly understood. In [NPA08] Navascués, Pironio and Acín constructed a hierarchy of spectrahedral outer approximations to the set of quantum correlations. The precise mathematical derivation of the NPA hierarchy is involved and is beyond the scope of this paper. For the precise definition and its properties the reader is referred to [NPA08]. In this work we only consider the first level of the NPA hierarchy, denoted by $\mathrm{NPA}^{(1)}$, that we now introduce.

Definition 3.13. *Let $p = (p(a,b|s,t)) \in \mathcal{NS}$ and set $P := \sum_{a,b,s,t} p(a,b|s,t)\; e_s e_t^\top \otimes e_a e_b^\top$, and p_A, p_B as defined in* (6)*. Then $p \in \mathrm{NPA}^{(1)}$ if and only if there exists a matrix*

$$\tilde{X} := \begin{pmatrix} 1 & p_A^\top & p_B^\top \\ p_A & X & P \\ p_B & P^\top & Y \end{pmatrix} \in \mathcal{S}_+^{1+N}, \tag{30}$$

satisfying

(i) $X[(s,a),(s,a')] = \delta_{a,a'}\; p_A(a|s)$, *for all* $s \in S, a, a' \in A$,

(ii) $Y[(t,b),(t,b')] = \delta_{b,b'}\; p_B(b|t)$, *for all* $t \in T, b, b' \in B$.

Remark 3.14. *Using Lemma 3.6 it is easy to verify that $\mathrm{NPA}^{(1)}$ can be expressed as the projection (onto the blocks that are indexed by $S \times T$) of the set of matrices in $\mathcal{S}_+^{1+N}$ satisfying the following constraints*

(i) $\langle J_{i,j}, \tilde{X} \rangle = 1$, *for all* $i, j \in \{0\} \cup S \cup T$,

(ii) $\tilde{X}[(s,a),(t,b)] \geq 0$, *for all* $s \in S$, $t \in T$, $a \in A$, $b \in B$,

(iii) $\tilde{X}[(s,a),(s,a')] = 0$, *for all* $s \in S$, $a \neq a' \in A$,

(iv) $\tilde{X}[(t,b),(t,b')] = 0$, *for all* $t \in T$, $b \neq b' \in B$.

We use this fact in Section 4.3.

Our last result in this section is that the set of $\mathcal{DNN}$-correlations is contained in $\mathrm{NPA}^{(1)}$. We start with a simple lemma that we use in the proof.

Lemma 3.15. *Consider two vectors $x, y \in \mathbb{R}_+^n$ with $\langle e, x \rangle = \langle e, y \rangle = 1$. If $\begin{pmatrix} 1 & x^\top \\ x & \mathrm{Diag}(y) \end{pmatrix} \in \mathcal{S}_+^{n+1}$ then we have that $x = y$.*

Proof. By Schur complements we have $\begin{pmatrix} 1 & x^\top \\ x & \mathrm{Diag}(y) \end{pmatrix} \in \mathcal{S}_+^{n+1}$ if and only if $\mathrm{Diag}(y) - xx^\top \in \mathcal{S}_+^n$. Notice that $\langle ee^\top, \mathrm{Diag}(y) - xx^\top \rangle = 0$ and as both matrices are positive semidefinite we get that $(\mathrm{Diag}(y) - xx^\top)e = 0$. Lastly, since $\langle e, x \rangle = 1$, it follows from the preceding equality that $x = y$. ☐

We are now ready to prove the last result in this section.

Theorem 3.16. *For any Bell scenario we have that*

$$\mathrm{Corr}(\mathcal{DNN}) \subseteq \mathrm{NPA}^{(1)}.$$

Proof. Consider a correlation $p \in \mathrm{Corr}(\mathcal{DNN})$. By Theorem 3.8 we have that $p \in \mathcal{NS}$ and thus the marginal probability distributions p_A and p_B are well-defined. By Remark 3.11 there exists

$$\tilde{X} := \begin{pmatrix} 1 & p_A^\mathsf{T} & p_B^\mathsf{T} \\ p_A & X & P \\ p_B & P^\mathsf{T} & Y \end{pmatrix} \in \mathcal{DNN}^{1+N}, \tag{31}$$

satisfying $\langle J_{i,j}, \tilde{X}\rangle = 1,$ for all $i, j \in \{0\} \cup S \cup T$. Fix $s \in S$ and $a \neq a' \in A$ and set $E^s_{a,a'} \in \mathcal{S}_+^{1+N}$ to be the matrix with entries $E^s_{a,a'}[(s,a),(s,a)] = 1$, $E^s_{a,a'}[(s,a'),(s,a')] = 1$, $E^s_{a,a'}[(s,a),(s,a')] = -1$, $E^s_{a,a'}[(s,a'),(s,a)] = -1$ and 0 otherwise. Furthermore, define

$$X' := \tilde{X} + \tilde{X}[(s,a),(s,a')]E^s_{a,a'}, \tag{32}$$

and notice that $X'[(s,a),(s,a')] = 0$. Moreover, since $\tilde{X} \in \mathcal{DNN}^{1+N}$ we have that $X' \in \mathcal{S}_+^{1+N}$ and since $\langle J_{i,j}, E^s_{a,a'}\rangle = 0$ it follows from (32) that $\langle J_{i,j}, X'\rangle = 1$ for all $i, j \in \{0\} \cup S \cup T$. Clearly, this argument can be repeated for all $s \in S, a \neq a' \in A$ and symmetrically for all $t \in T$ and $b \neq b' \in B$. In this way we construct a matrix

$$Z := \begin{pmatrix} 1 & p_A^\mathsf{T} & p_B^\mathsf{T} \\ p_A & Z_1 & P \\ p_B & P^\mathsf{T} & Z_2 \end{pmatrix} \in \mathcal{S}_+^{1+N}, \tag{33}$$

satisfying $\langle J_{i,j}, Z\rangle = 1$ for all $i, j \in \{0\} \cup S \cup T$ and

$$Z_1[(s,a)(s,a')] = 0 \text{ for } s \in S, a \neq a' \in A \quad \text{and} \quad Z_2[(t,b),(t,b')] = 0 \text{ for } t \in T, b \neq b' \in B.$$

It remains to show that $Z[(s,a),(s,a)] = p_A(a|s)$ for all $a \in A, s \in S$ and $Z[(t,b),(t,b)] = p_B(b|t)$ for all $t \in T$ and $b \in B$. To see this, fix $a \in A, s \in S$ and notice that the principal submatrix of Z indexed by $\{[0,0]\} \cup \{[0,(s,a)] : a \in A\}$ is given by $\begin{pmatrix} 1 & p_A(s)^\mathsf{T} \\ p_A(s) & \mathrm{Diag}(y) \end{pmatrix} \in \mathcal{S}_+^{1+|A|}$. Since $\langle y, e\rangle = 1$ it follows by Lemma 3.15 that $y = p_A(s)$. Since the same argument can be repeated for all other diagonal blocks of Z, the proof is concluded. ☐

We note that at present we do not know if the containment given in Theorem 3.16 is strict. One obvious difficulty in proving the converse inclusion is that for any matrix feasible for (30) we do not have control of the signs of the entries corresponding to off-diagonal blocks.

4 Conic programming formulations for game values

In this section we study the value of a nonlocal game when the players use strategies that generate classical, quantum, no-signaling or unrestricted correlations. By Theorem 3.7, the classical, quantum, no-signaling and unrestricted values can be formulated as linear conic programs over appropriate convex cones. Any conic program has an associated dual which we derive in our setting and investigate its properties. This allows us to identify a sufficient condition for showing that the $\mathcal{CS}_+$ cone is *not* closed. Furthermore, we identify a new SDP upper bound to the quantum value of an arbitrary nonlocal game which we show is at most the value of the SDP obtained when we optimize over the first level of the NPA hierarchy. Lastly, we show that the problem of deciding whether a nonlocal game admits a perfect $\mathcal{K}$-strategy is equivalent to deciding the feasibility of a linear conic program over $\mathcal{K}$. In particular, deciding whether a nonlocal game admits a perfect quantum strategy is equivalent to the feasibility of a linear conic program over the $\mathcal{CS}_+$ cone.

4.1 Primal formulations

Recall that the maximum probability of winning a game $\mathcal{G}$ using $\mathcal{K}$-correlations is given by

$$(\mathcal{P}_{\mathcal{K}}) \qquad \begin{aligned} \omega(\mathcal{K},\mathcal{G}) := \text{supremum} \quad & \sum_{s\in S}\sum_{t\in T}\pi(s,t)\sum_{a\in A}\sum_{b\in B}V(a,b|s,t)p(a,b|s,t) \\ \text{subject to} \quad & p = (p(a,b|s,t)) \in \mathrm{Corr}(\mathcal{K}). \end{aligned}$$

By Theorem 3.7 the sets of classical, quantum, no-signaling and unrestricted correlations can be expressed as the set of $\mathcal{K}$-correlations over some appropriate convex cone $\mathcal{K} \subseteq \mathcal{N}^N$.

Theorem 4.1. *For any nonlocal game $\mathcal{G}(\pi, V)$ we have:*

(i) The classical value $\omega_{\mathcal{C}}(\mathcal{G})$ equal to $\omega(\mathcal{CP},\mathcal{G})$.

(ii) The quantum value $\omega_{\mathcal{Q}}(\mathcal{G})$ equal to $\omega(\mathcal{CS}_+,\mathcal{G})$.

(iii) The no-signaling value $\omega_{\mathcal{NS}}(\mathcal{G})$ equal to $\omega(\mathcal{NSO},\mathcal{G})$.

(iv) The unrestricted value $\omega_{\mathcal{P}}(\mathcal{G})$ equal to $\omega(\mathcal{N},\mathcal{G})$.

Notice that ($\mathcal{P}_{\mathcal{K}}$) is a linear conic program over the convex cone $\mathcal{K}$. Our goal in this section is to apply the theory of linear conic optimization to ($\mathcal{P}_{\mathcal{K}}$) with the aim to better understand how the various values of a nonlocal game relate to each other and to study their properties.

4.2 Dual formulations

Any conic program has an associated dual, which in the case of ($\mathcal{P}_{\mathcal{K}}$) is given by:

$$(\mathcal{D}_{\mathcal{K}}) \qquad \begin{aligned} \xi(\mathcal{K},\mathcal{G}) := \text{infimum} \quad & \sum_{i,j\in S\cup T} v_{i,j} \\ \text{subject to} \quad & \sum_{i,j\in S\cup T} v_{i,j}J_{i,j} - \hat{C} \in \mathcal{K}^*. \end{aligned}$$

In this section, we analyze the primal-dual pair of conic programs ($\mathcal{P}_{\mathcal{K}}$) and ($\mathcal{D}_{\mathcal{K}}$) and explore to what extent duality theory can be used to gain information on the various game values.

By weak duality (cf. Theorem 2.1 *(i)*) the optimal value of the dual program upper bounds the optimal value of the primal, i.e., for any game $\mathcal{G}$ we have $\omega(\mathcal{K},\mathcal{G}) \le \xi(\mathcal{K},\mathcal{G})$. For this to hold with equality, it suffices to determine whether strong duality holds for the primal or the dual (cf. Theorem 2.1 *(ii)*). Notice that for any cone $\mathcal{K} \subseteq \mathcal{S}^N_+$ the primal program ($\mathcal{P}_{\mathcal{K}}$) fails to be strictly feasible. To see this, fix indices $i \in S$, $j \in T$, and define the (nonzero) positive semidefinite matrix

$$M := J_{i,i} - J_{i,j} - J_{j,i} + J_{j,j} \in \mathcal{S}^N_+.$$

Any matrix X feasible for ($\mathcal{P}_{\mathcal{K}}$) satisfies $\langle M, X\rangle = 0$ and thus $X \notin \mathrm{int}(\mathcal{K}) \subseteq \mathcal{S}^N_{++}$.

Also, notice that if the cone $\mathcal{K}$ is not closed then we cannot apply strong duality directly to the primal-dual pair. However, as we now show, under the additional assumption that $\mathcal{K}$ is a closed convex cone, strong duality holds for the primal-dual pair of conic programs ($\mathcal{P}_{\mathcal{K}}$) and ($\mathcal{D}_{\mathcal{K}}$).

Proposition 4.2. *Consider a game $\mathcal{G}$ and let $\mathcal{K} \subseteq \mathcal{N}$ be a closed convex cone such that ($\mathcal{P}_{\mathcal{K}}$) is primal feasible. Then we have that $\omega(\mathcal{K},\mathcal{G}) = \xi(\mathcal{K},\mathcal{G})$ and moreover there exists an optimal solution for ($\mathcal{P}_{\mathcal{K}}$).*

Proof. Since $(\mathcal{P}_\mathcal{K})$ is feasible, the dual value $\xi(\mathcal{K},\mathcal{G})$ is bounded from below by 0. It remains to show that the dual program is strictly feasible for the range of cones we consider. Notice that the program $(\mathcal{D}_\mathcal{K})$ is strictly feasible for $\mathcal{K} = \mathcal{N}$ since $\mathrm{int}(\mathcal{N}) = \{ X : X[i,j] > 0 \text{ for all } i,j \}$ and $\sum_{i,j \in S\cup T} v_{i,j} J_{i,j} - \hat{C} \in \mathrm{int}(\mathcal{N})$ by setting each $v_{i,j}$ to be a very large positive constant. Furthermore, for $\mathcal{K} \subseteq \mathcal{N}$ we have that $\mathcal{N} = \mathcal{N}^* \subseteq \mathcal{K}^*$ implying $\mathrm{int}(\mathcal{N}) \subseteq \mathrm{int}(\mathcal{K}^*)$. Thus $(\mathcal{D}_\mathcal{K})$ is strictly feasible for all cones $\mathcal{K} \subseteq \mathcal{N}$. The proof is concluded by Theorem 2.1 *(ii)*. □

As an immediate consequence we get:

Corollary 4.3. *For any nonlocal game $\mathcal{G}$, we have that $\omega(\mathrm{cl}(\mathcal{CS}_+),\mathcal{G}) = \xi(\mathcal{CS}_+,\mathcal{G})$.*

Proof. By Proposition 4.2 for $\mathcal{K} = \mathrm{cl}(\mathcal{CS}_+)$ we get that $\omega(\mathrm{cl}(\mathcal{CS}_+),\mathcal{G}) = \xi(\mathrm{cl}(\mathcal{CS}_+),\mathcal{G}))$. Lastly, as $\mathcal{CS}_+^* = (\mathrm{cl}(\mathcal{CS}_+))^*$ it follows from the definition of $\xi(\mathcal{K},\mathcal{G})$ that $\xi(\mathrm{cl}(\mathcal{CS}_+),\mathcal{G}) = \xi(\mathcal{CS}_+,\mathcal{G})$. □

Recall that the $\mathcal{CS}_+$ cone is not known to be closed [LP14, BLP15]. It follows from Corollary 4.3 that a sufficient condition for showing that the cone $\mathcal{CS}_+$ is *not* closed is to identify a game $\mathcal{G}$ for which $\omega(\mathcal{CS}_+,\mathcal{G}) < \xi(\mathcal{CS}_+,\mathcal{G})$. We conclude this section with a table that summarizes the known relationships between the various primal and dual values:

$$\begin{array}{ccccccccccc}
\omega(\mathcal{CP},\mathcal{G}) & \leq & \omega(\mathcal{CS}_+,\mathcal{G}) & \leq & \omega(\mathrm{cl}(\mathcal{CS}_+),\mathcal{G}) & \leq & \omega(\mathcal{DNN},\mathcal{G}) & \leq & \omega(\mathcal{NSO},\mathcal{G}) & \leq & \omega(\mathcal{N},\mathcal{G}) \\
\parallel & & \wedge\!\!\mid & & \parallel & & \parallel & & \parallel & & \parallel \\
\xi(\mathcal{CP},\mathcal{G}) & \leq & \xi(\mathcal{CS}_+,\mathcal{G}) & = & \xi(\mathrm{cl}(\mathcal{CS}_+),\mathcal{G}) & \leq & \xi(\mathcal{DNN},\mathcal{G}) & \leq & \xi(\mathcal{NSO},\mathcal{G}) & \leq & \xi(\mathcal{N},\mathcal{G})
\end{array}$$

4.3 The Feige-Lovász SDP relaxation

Notice that the tractability of the conic program $(\mathcal{P}_\mathcal{K})$ depends on the underlying cone $\mathcal{K}$. In this section we focus on the case $\mathcal{K} = \mathcal{DNN}$ for which $(\mathcal{P}_\mathcal{K})$ becomes an instance of a semidefinite program. Specifically, using the definition of $\mathrm{Corr}(\mathcal{DNN})$ we have:

$$(\mathcal{P}_{\mathcal{DNN}}) \qquad \begin{array}{rll}
\omega(\mathcal{DNN},\mathcal{G}) := \text{ maximum} & \langle \hat{C}, X\rangle & \\
\text{subject to} & \langle J_{i,j}, X\rangle = 1, & \text{for all } i,j \in S \cup T, \\
& X \in \mathcal{DNN}^N. &
\end{array}$$

Notice that we have defined $(\mathcal{P}_{\mathcal{DNN}})$ as a maximization since its feasible region is a compact set.

For two convex cones satisfying $\mathrm{Corr}(\mathcal{K}_1) \subseteq \mathrm{Corr}(\mathcal{K}_2)$ we have that $\omega(\mathcal{K}_1,\mathcal{G}) \leq \omega(\mathcal{K}_2,\mathcal{G})$. By Proposition 3.12 it follows that $\omega(\mathcal{DNN},\mathcal{G})$ is a semidefinite programming upper bound to the quantum value of an arbitrary nonlocal game, that never exceeds the no-signaling value.

Proposition 4.4. *For any nonlocal game $\mathcal{G}$ we have that*

$$\omega_\mathcal{Q}(\mathcal{G}) \leq \omega(\mathcal{DNN},\mathcal{G}) \leq \omega_{\mathcal{NS}}(\mathcal{G}).$$

As it turns out, the semidefinite program $(\mathcal{P}_{\mathcal{DNN}})$ was already studied by Feige and Lovász as an upper bound to the *classical value* of an arbitrary nonlocal game (cf. Equations (5)-(9) in [FL92]). On the other hand, Proposition 4.4 yields a much stronger result, namely that $\omega(\mathcal{DNN},\mathcal{G})$ is in fact an upper bound to the quantum value, so in particular it also upper bounds $\omega_\mathcal{C}(\mathcal{G})$. To the best of our knowledge, prior to this work, the only known result relating $(\mathcal{P}_{\mathcal{DNN}})$ with the quantum value is that they are equal for the special class of XOR games [Upa11, Theorem 22].

We conclude this section by comparing $\omega(\mathcal{DNN}, \mathcal{G})$ with the maximum probability of winning the game $\mathcal{G}$ when the players use strategies that generate correlations in the first level of the NPA hierarchy, denoted by $\mathrm{SDP}^{(1)}(\mathcal{G})$. As an immediate consequence of Theorem 3.16 we have that:

Proposition 4.5. *For any game $\mathcal{G}$ we have that $\omega(\mathcal{DNN}, \mathcal{G}) \leq \mathrm{SDP}^{(1)}(\mathcal{G})$.*

At present, we have not been able to identify a game for which this inequality is strict. Lastly, by Remark 3.14 it is easy to see that $\mathrm{SDP}^{(1)}$ is equal to:

$$
\begin{aligned}
\text{maximize} \quad & \langle \hat{C}, X \rangle \\
\text{subject to} \quad & \langle J_{i,j}, X \rangle = 1, \ \text{ for all } i, j \in S \cup T, \\
& X[(s,a),(t,b)] \geq 0, \ \text{ for all } \ s \in S, t \in T, a \in A, b \in B, \\
& X[(s,a),(s,a')] = 0, \ \text{ for all } s \in S, a \neq a' \in A, \\
& X[(t,b),(t,b')] = 0, \ \text{ for all } t \in T, b \neq b' \in B, \\
& X \in \mathcal{S}^N_+ .
\end{aligned} \tag{34}
$$

The SDP given in (34) is the "canonical" SDP relaxation for $\omega_{\mathcal{Q}}(\mathcal{G})$ that is usually considered in the quantum information literature. For example, based on (34) Kempe, Regev and Toner devised an algorithm for approximating the quantum value of unique games [KRT10].

4.4 Perfect strategies

A strategy for a nonlocal game $\mathcal{G}$ is called *perfect* if it wins the game $\mathcal{G}$ with certainty (i.e., with probability 1). In this section we consider the problem of deciding the existence of a perfect strategy (in some fixed correlation set) when we are given as input the description of a nonlocal game. Using our conic formulations for the various game values we show that this problem is equivalent to deciding the feasibility of a linear conic program over some appropriate convex cone.

Recall that for a convex cone $\mathcal{K} \subseteq \mathcal{N}$ we say the game $\mathcal{G}(\pi, V)$ admits a *perfect* $\mathcal{K}$-strategy if there exists a correlation $p = (p(a,b|s,t)) \in \mathrm{Corr}(\mathcal{K})$ such that

$$\sum_{s \in S, t \in T} \pi(s,t) \sum_{a \in A, b \in B} V(a,b|s,t) p(a,b|s,t) = 1.$$

We now show that deciding the existence of a perfect $\mathcal{K}$-strategy for an arbitrary nonlocal game can be cast as a conic program over the cone $\mathcal{K}$.

Lemma 4.6. *Let $\mathcal{G}(\pi, V)$ be a game with questions sets S, T and answer sets A, B and let $\mathcal{K} \subseteq \mathcal{N}^N$. The game $\mathcal{G}$ admits a perfect $\mathcal{K}$-strategy if and only if there exists $p = (p(a,b|s,t)) \in \mathrm{Corr}(\mathcal{K})$ satisfying:*

$$p(a,b|s,t) = 0 \ \text{ when } \pi(s,t) > 0 \text{ and } V(a,b|s,t) = 0. \tag{35}$$

Proof. For any $p = (p(a,b|s,t)) \in \mathrm{Corr}(\mathcal{K})$ we have that

$$\sum_{s \in S, t \in T} \pi(s,t) \sum_{a \in A, b \in B} V(a,b|s,t) p(a,b|s,t) \leq \sum_{s \in S, t \in T} \pi(s,t) \sum_{a \in A, b \in B} p(a,b|s,t) = 1. \tag{36}$$

Therefore $\mathcal{G}$ admits a perfect $\mathcal{K}$-strategy if and only if there exists $p = (p(a,b|s,t)) \in \mathrm{Corr}(\mathcal{K})$ for which (36) holds throughout with equality. This is equivalent to

$$\pi(s,t)(V(a,b|s,t) - 1) p(a,b|s,t) = 0, \ \text{ for all } a \in A, b \in B, s \in S, t \in T,$$

which is in turn equivalent to $p(a,b|s,t) = 0$ when $\pi(s,t) > 0$ and $V(a,b|s,t) = 0$. $\square$

By Lemma 4.6 and the definition of $\mathrm{Corr}(\mathcal{K})$ it follows that a nonlocal game $\mathcal{G}(\pi, V)$ admits a perfect $\mathcal{K}$-strategy if and only if the following conic program is feasible:

$$(\mathcal{F}_{\mathcal{K}}) \qquad \begin{aligned} \langle J_{i,j}, X\rangle = 1, &\quad \text{for all } i,j \in S \cup T, \\ X[(s,a),(t,b)] = 0, &\quad \text{for all } a,b,s,t \text{ such that } \pi(s,t) > 0 \text{ and } V(a,b|s,t) = 0, \\ X \in \mathcal{K}. & \end{aligned}$$

In particular, using the conic formulations for the sets of classical, quantum, no-signaling and unrestricted correlations from Theorem 3.7 we have the following theorem.

Theorem 4.7. *For any nonlocal game $\mathcal{G}$ we have that:*

(i) $\mathcal{G}$ admits a perfect classical strategy if and only if $(\mathcal{F}_{\mathcal{CP}})$ is feasible.

(ii) $\mathcal{G}$ admits a perfect quantum strategy if and only if $(\mathcal{F}_{\mathcal{CS}_+})$ is feasible.

(iii) $\mathcal{G}$ admits a perfect no-signaling strategy if and only if $(\mathcal{F}_{\mathcal{NSO}})$ is feasible.

(iv) $\mathcal{G}$ admits a perfect unrestricted strategy if and only if $(\mathcal{F}_{\mathcal{N}})$ is feasible.

As already mentioned in the introduction, no algorithm is currently known for deciding the existence of a perfect quantum strategy to a nonlocal game. In Theorem 4.7 *(ii)* we showed that this problem is equivalent to deciding the feasibility of a conic program over $\mathcal{CS}_+$. Nevertheless, this does not render the problem decidable as no algorithms are currently known for deciding the feasibility of a $\mathcal{CS}_+$ program.

We conclude this section with an equivalent form of Theorem 4.7 which is used in Section 5.3.

Proposition 4.8. *Consider a nonlocal game $\mathcal{G}(\pi, V)$ and let $\mathcal{K} \in \{\mathcal{CP}, \mathcal{CS}_+, \mathcal{DNN}\}$. The game $\mathcal{G}$ admits a perfect $\mathcal{K}$-strategy if and only if there exists a matrix $\tilde{X} \in \mathcal{K}^{1+N}$ satisfying:*

$$\langle J_{i,j}, \tilde{X}\rangle = 1, \ \textit{for all } i,j \in \{0\} \cup S \cup T, \ \textit{and}$$

$$\tilde{X}[(s,a),(t,b)] = 0, \ \ \textit{for all } a,b,s,t \textit{ such that } \pi(s,t) > 0 \textit{ and } V(a,b|s,t) = 0.$$

Proof. The claim follows easily using Lemma 3.10. □

5 Synchronous correlations and game values

Throughout this section we focus on Bell scenarios where $A = B$ and $S = T$. Given such a scenario we study *synchronous* correlations, i.e., correlations with the property that the players respond with the same answer whenever they receive the same question. In Section 5.1 we specialize Theorem 3.7 to the class of synchronous correlations and show that our conic characterizations assume a particularly simple form. Using these simplified characterizations, in Section 5.2 we study the maximum probability of wining a nonlocal game when the players use strategies that generate synchronous correlations. In Section 5.3 we derive conic programming formulations for deciding the existence of perfect strategies for synchronous nonlocal games (cf. Definition 1.9). This enables us to recover in a uniform manner the conic programming formulations for deciding the existence of a classical and quantum graph homomorphisms [Rob14] and also the conic programming formulations for the quantum chromatic and the quantum independence number [LP14].

5.1 Synchronous correlations

Recall that a correlation $p = (p(a, a'|s, s')) \in \mathcal{P}$ is called *synchronous* if it satisfies:

$$p(a, a'|s, s) = 0, \text{ for all } s \in S \text{ and } a \neq a' \in A, \tag{37}$$

i.e., the players respond with the same answer whenever they receive the same question. Our first result in this section is a geometric lemma that is essential in obtaining simplified conic characterizations for quantum and classical synchronous correlations.

Lemma 5.1. *Let $\mathcal{X}$ be a Euclidean space and consider two families of psd matrices $\{X_i : i \in [n]\} \subseteq \mathcal{H}_+(\mathcal{X})$ and $\{Y_i : i \in [n]\} \subseteq \mathcal{H}_+(\mathcal{X})$ satisfying:*

(i) $\sum_{i=1}^n X_i = \sum_{i=1}^n Y_i$, *and*

(ii) $\langle X_i, Y_j \rangle = 0$, *for all $i \neq j \in [n]$.*

Then we have that $X_i = Y_i$ for all $i \in [n]$.

Proof. Fix an index $i \in [n]$ and let λ be the largest eigenvalue of X_i with corresponding (normalized) eigenvector v and let μ be the largest eigenvalue of Y_i. By condition *(ii)*, we know that $Y_j v = 0$ for all $j \neq i \in [n]$. Using this, we have

$$\lambda = v^* X_i v \leq v^* \left(\sum_{j=1}^n X_j \right) v = v^* \left(\sum_{j=1}^n Y_j \right) v = v^* Y_i v \leq \mu, \tag{38}$$

proving that $\mu \geq \lambda$. By symmetry, we also have $\lambda \geq \mu$ proving that $\lambda = \mu$. This shows that (38) holds throughout with equality and thus v is also an eigenvector of Y_i corresponding to eigenvalue $\lambda = \mu$ as well. Lastly, define $X_i' := X_i - \lambda v v^*, Y_i' := Y_i - \lambda v v^*$ and for $j \neq i$ set $X_j' := X_j$ and $Y_j' := Y_j$. Notice that the matrices $\{X_i' : i \in [n]\}$ and $\{Y_i' : i \in [n]\}$ satisfy conditions *(i)* and *(ii)* and the proof is concluded by an inductive argument. ☐

Based on Lemma 5.1 we now derive a second result that we use in our main theorem below and also in Section 5.3 where we study perfect strategies for synchronous games.

Lemma 5.2. *Consider a family of vectors $\{x_a^s\}_{s \in S, a \in A}$ in some Euclidean space $\mathcal{X}$.*

(a) For $X := \mathrm{Gram}(\{x_a^s\}_{s \in S, a \in A})$ the following are equivalent:

  (i) There exists $k \in \mathcal{X}$ satisfying $\sum_{a \in A} x_a^s = k$ for all $s \in S$ and $\langle k, k \rangle = 1$.

  (ii) $\langle J_{s,s'}, X \rangle = 1$, for all $s, s' \in S$.

(b) Set $\tilde{X} := \mathrm{Gram}(k, \{x_a^s\}_{s \in S, a \in A})$ where $k \in \mathcal{X}$ with $\langle k, k \rangle = 1$. The following are equivalent:

  (i) $\sum_{a \in A} x_a^s = k$, for all $s \in S$.

  (ii) $\langle J_{s,s}, \tilde{X} \rangle = \langle J_{0,s}, \tilde{X} \rangle = 1$, for all $s \in \{0\} \cup S$.

Proof. We only consider case *(b)* and show that *(ii)* implies *(i)*. Notice that

$$\big\langle k - \sum_{a \in A} x_a^s, k - \sum_{a \in A} x_a^s \big\rangle = \langle k, k \rangle - 2 \sum_{a \in A} \big\langle k, x_a^s \big\rangle + \big\langle \sum_{a \in A} x_a^s, \sum_{a \in A} x_a^s \big\rangle. \tag{39}$$

By assumption we have that $\langle k, k \rangle = 1$, $\langle J_{0,s}, \tilde{X} \rangle = \sum_{a \in A} \langle k, x_a^s \rangle = 1$ and $\big\langle \sum_{a \in A} x_a^s, \sum_{a \in A} x_a^s \big\rangle = \langle J_{s,s}, \tilde{X} \rangle = 1$. Substituting everything in (39) the proof is concluded. ☐

We now arrive at the main result in this section.

Theorem 5.3. *Consider a Bell scenario with question set S and answer set A. Furthermore, consider a* synchronous *correlation $p = (p(a,a'|s,s')) \in \mathcal{P}$ and set $P := \sum_{a,a',s,s'} p(a,a'|s,s')\ e_s e_{s'}^{\mathsf{T}} \otimes e_a e_{a'}^{\mathsf{T}}$. For every cone $\mathcal{K} \in \{\mathcal{CP}, \mathcal{CS}_+\}$ the following are equivalent:*

(i) $p \in \mathrm{Corr}(\mathcal{K})$.

(ii) $P \in \mathcal{K}^{|S\times A|}$.

(iii) There exists $\tilde{X} = \begin{pmatrix} 1 & x^{\mathsf{T}} \\ x & P \end{pmatrix} \in \mathcal{K}^{1+|S\times A|}$ *satisfying* $\langle J_{0,s}, \tilde{X}\rangle = 1,$ *for all* $s \in \{0\} \cup S$.

Proof. We only consider $\mathcal{K} = \mathcal{CS}_+$ the case $\mathcal{K} = \mathcal{CP}$ being similar.

(i) $\Longrightarrow$ *(ii)*: Since $p = (p(a,a'|s,s')) \in \mathrm{Corr}(\mathcal{CS}_+)$, by Theorem 3.2 there exist psd matrices $\{X_a^s\}_{s\in S, a\in A}$, $\{Y_a^s\}_{s\in S a\in A}$, $K \in \mathcal{S}_+^d$ (for some $d \geq 1$) such that $p(a,a'|s,s') = \langle X_a^s, Y_{a'}^{s'}\rangle$ for all a,a',s,s', $\sum_{a\in A} X_a^s = \sum_{a\in A} Y_a^s = K$ for all $s \in S$ and $\langle K, K\rangle = 1$. Since p is synchronous it follows from Lemma 5.1 that $X_a^s = Y_a^s$ for all $s \in S$ and $a \in A$. This implies that $P \in \mathcal{CS}_+^{|S\times A|}$.

(ii) $\Longrightarrow$ *(iii)*: Since $P \in \mathcal{CS}_+^{|S\times A|}$ there exist matrices $\{X_a^s\}_{s\in S,a\in A} \in \mathcal{S}_+^d$ (for some $d \geq 1$) such that $p(a,a'|s,s') = \langle X_a^s, X_{a'}^{s'}\rangle$ for all a,a',s,s'. By Lemma 5.2 *(a)* there exists $K \in \mathcal{S}_+^d$ such that $\langle K, K\rangle = 1$ and $\sum_{a\in A} X_a^s = K$ for all $s \in S$. The proof is concluded by noticing that the matrix $\mathrm{Gram}(K, \{X_a^s\}_{s\in S,a\in A})$ is feasible for *(iii)*.

(iii) $\Longrightarrow$ *(i)*: Let $\tilde{X} \in \mathcal{CS}_+^{1+|S\times A|}$ be feasible for *(iii)* and say $\tilde{X} = \mathrm{Gram}(K, \{X_a^s\}_{s\in S,a\in A})$ where $K, \{X_a^s\}_{s\in S,a\in A} \in \mathcal{S}_+^d$. By Lemma 5.2 *(b)* we have $\sum_{a\in A} X_a^s = K$ for all $s \in S$. Lastly, Theorem 3.2 implies that $p \in \mathrm{Corr}(\mathcal{CS}_+)$. □

As an immediate consequence of Theorem 5.3 we arrive at the following conic characterization of the sets of synchronous quantum and synchronous classical correlations:

Corollary 5.4. *Consider a Bell scenario with question set S and answer set A and let $\mathcal{K} \in \{\mathcal{CP}, \mathcal{CS}_+\}$. The set of synchronous $\mathcal{K}$-correlations is given by*

$$\{X \in \mathcal{K}^{|S\times A|} : \langle J_{s,s'}, X\rangle = 1, \text{ for } s,s' \in S \text{ and } X[(s,a),(s,a')] = 0, \text{ for all } a \neq a' \in A, s \in S\},$$

where we identify a correlation vector $p = (p(a,a'|s,s'))$ with a square matrix of size $|S \times A|$ (recall (19)*).*

5.2 Synchronous value

In this section we consider nonlocal games where the players share the same question set S and the same answer set A. We start with a definition which is central for this section.

Definition 5.5. *For any convex cone $\mathcal{K} \subseteq \mathcal{N}^N$, the $\mathcal{K}$-*synchronous value *of a nonlocal game $\mathcal{G}$, denoted $\omega_{syn}(\mathcal{K}, \mathcal{G})$, is defined as the maximum probability of winning the game when the players are only allowed to use strategies that generate synchronous $\mathcal{K}$-correlations.*

As we now show by Corollary 5.4 we get a conic programming formulation for the classical and quantum synchronous value of a nonlocal game with matrix variables of size $|S \times A|$.

Proposition 5.6. *Consider a game $\mathcal{G}$ with question set S and answer set A. For a cone $\mathcal{K} \subseteq \mathcal{DNN}^{|S\times A|}$ set*

$$\begin{aligned} \nu(\mathcal{K},\mathcal{G}) := \text{supremum} \quad & \frac{1}{2}\langle C + C^{\mathsf{T}}, X\rangle \\ \text{subject to} \quad & \langle J_{s,s'}, X\rangle = 1, \text{ for all } s, s' \in S, \\ & X[(s,a),(s,a')] = 0, \text{ for } s \in S, a \neq a' \in A, \\ & X \in \mathcal{K}^{|S\times A|}. \end{aligned}$$

Then $\omega_{syn}(\mathcal{CP},\mathcal{G}) = \nu(\mathcal{CP},\mathcal{G})$ and $\omega_{syn}(\mathcal{CS}_+,\mathcal{G}) = \nu(\mathcal{CS}_+,\mathcal{G})$.

There are many examples of games for which the optimal classical strategy generates a synchronous correlation but all known optimal quantum strategies do not (e.g. the CHSH game). This raises the following question: What is the optimal value for such games when one restricts to synchronous quantum strategies? Perhaps the interesting thing to see is if the power of quantum strategies comes from the fact that the optimal quantum strategies do not need to be synchronous. The following corollary of Proposition 5.6 gives a partial answer to the above question. It is a consequence of the fact that $\mathcal{CP}^n = \mathcal{CS}_+^n = \mathcal{DNN}^n$ for any $n \leq 4$ [LP14].

Corollary 5.7. *For any nonlocal game $\mathcal{G}$ with with identical binary questions sets (i.e., $S = T$ and $|S| = 2$) and identical binary answers sets (i.e., $A = B$ and $|A| = 2$), we have that the synchronous classical and synchronous quantum values coincide and are expressible as a semidefinite program, i.e.,*

$$\omega_{syn}(\mathcal{CP},\mathcal{G}) = \omega_{syn}(\mathcal{CS}_+,\mathcal{G}) = \nu(\mathcal{DNN},\mathcal{G}). \tag{40}$$

As an example, consider the CHSH game for which there exists an optimal classical strategy which is synchronous (Alice and Bob just output 0) with success probability $3/4$. Then, it follows from Corollary 5.7 that the synchronous quantum value is also $3/4$. That is, quantum strategies cannot be synchronous to win CHSH with greater probability than classical strategies. For games with large question or answer sets, Corollary 5.7 does not help. However, Proposition 5.6 implies that $\nu(\mathcal{DNN},\mathcal{G})$ is a tractable upper bound on the synchronous quantum value of $\mathcal{G}$.

5.3 Perfect strategies for synchronous games

Recall that a nonlocal game $\mathcal{G}(\pi, V)$ is called *synchronous* if both players share the same question set S and the same answer set A, and moreover:

(i) $V(a,a'|s,s) = 0$, for all $s \in S$, $a \neq a' \in A$ and

(ii) $\pi(s,s) > 0$, for all $s \in S$.

As a consequence of Theorem 4.7, deciding the existence of a perfect $\mathcal{K}$-strategy is equivalent to the feasibility of a linear conic program with matrix variables of size $|(S\times A)\cup(T\times B)|$. As an immediate consequence of (35) it follows that any perfect strategy for a synchronous nonlocal game generates a synchronous correlation. This allows us to use the conic characterization of synchronous classical and quantum correlations from Theorem 5.3 to derive a conic program with matrix variables of size $|S\times A|$ whose feasibility is equivalent to the existence of a perfect strategy.

Theorem 5.8. *Let $\mathcal{G}(\pi, V)$ be a synchronous game and $\mathcal{K} \in \{\mathcal{CP},\mathcal{CS}_+\}$. The following are equivalent:*

(i) $\mathcal{G}$ admits a perfect $\mathcal{K}$-strategy.

(ii) There exists a matrix $X \in \mathcal{K}^{|S\times A|}$ satisfying:

- $\langle J_{s,s'}, X\rangle = 1,$ *for all* $s, s' \in S,$
- $X[(s,a),(s,a')] = 0,$ *for all* $s \in S, a \neq a' \in A,$
- $X[(s,a),(s',a')] = 0,$ *for all* $a, a' \in A, s, s' \in S,$ *such that* $\pi(s,s') > 0$ *and* $V(a,a'|s,s') = 0.$

(iii) There exists a matrix $\tilde{X} \in \mathcal{K}^{1+|S\times A|}$ *satisfying:*

- $\langle J_{s,s}, \tilde{X}\rangle = \langle J_{0,s}, \tilde{X}\rangle = 1,$ *for all* $s \in \{0\} \cup S,$
- $\tilde{X}[(s,a),(s,a')] = 0,$ *for all* $s \in S, a \neq a' \in A,$
- $\tilde{X}[(s,a),(s',a')] = 0,$ *for all* $a, a' \in A, s, s' \in S,$ *such that* $\pi(s,s') > 0$ *and* $V(a,a'|s,s') = 0.$

The proof is omitted as it is an easy consequence of Theorem 5.3. In the next two sections we specialize Theorem 5.8 to graph coloring and more generally, graph homomorphism games and derive conic formulations for the existence of perfect strategies for these classes of games.

5.3.1 Graph Homomorphisms

Given two undirected graphs H and G, a *graph homomorphism* from H to G, denoted $H \to G$, is an adjacency preserving map from the vertex set of H to the vertex set of G, i.e., a function $f : V(H) \to V(G)$ with the property that $f(h) \sim_G f(h')$ whenever $h \sim_H h'$. In this section we focus on the (H,G)-*homomorphism game* where Alice and Bob are trying to convince a referee that there exists a graph homomorphism from H to G. To verify their claim the referee sends each player a vertex of H. Each player responds with a vertex of G. The answers of the players are supposed to model a homomorphism $f : H \to G$, i.e., the answer to a question $h \in V(H)$ should be $f(h) \in V(G)$.

Formally, in the (H,G)-homomorphism game the players share the same question set $S := V(H)$ and the same answer set $A := V(G)$. The distribution over the question set is the uniform distribution on $\{(h,h) : h \in V(H)\} \cup \{(h,h') : h \sim_H h'\}$. Lastly, the verification predicate is given by

$$V(g,g'|h,h') = \begin{cases} 0, & \text{if } h = h' \text{ and } g \neq g,' \\ 0, & \text{if } h \sim_H h' \text{ and } [g \not\sim_G g \text{ or } g = g'], \\ 1, & \text{otherwise.} \end{cases}$$

Notice that the existence of a homomorphism $H \to G$ can be understood via the (H,G)-homomorphism game. Specifically, there exists a graph homomorphism from H to G if and only if the (H,G)-homomorphism game admits a perfect classical strategy. This is easy to see using the fact that in the classical setting we may assume without loss of generality that both players are using deterministic strategies. This motivates the following definition.

Definition 5.9. *For two graphs G and H, we say there exists a* quantum graph homomorphism *from H to G, denoted $H \xrightarrow{q} G$, if the (H,G)-homomorphism game admits a perfect quantum strategy.*

The notion of quantum graph homomorphisms was introduced and studied recently in [MR14]. In view of our conic characterizations for the sets of quantum and classical correlations we arrive at a natural conic generalization of the notion of graph homomorphism.

Definition 5.10. *For a convex cone $\mathcal{K} \subseteq \mathcal{N}$ we say that there exists a* $\mathcal{K}$-homomorphism *from H to G if and only if the (H,G)-homomorphism game admits a perfect $\mathcal{K}$-strategy.*

It follows from Lemma 4.6 that the existence of a $\mathcal{K}$-homomorphism from H to G is equivalent to the feasibility of a linear conic program over $\mathcal{K}$. The similar notion of *strong $\mathcal{K}$-homomorphism* was introduced recently in [Rob14]. Since the (H, G)-homomorphism game is synchronous we can use the conic formulations from Theorem 5.8 to show that the two notions of conic homomorphisms coincide for $\mathcal{K} \in \{\mathcal{CP}, \mathcal{CS}_+\}$. We note that strong $\mathcal{K}$-homomorphisms are only defined for a certain class of convex cones called *frabjous*. Working over frabjous cones ensures that the $\mathcal{K}$-homomorphism relation is reflexive and transitive, mimicking classical graph homomorphisms.

As an immediate consequence of Theorem 5.8 *(ii)* it follows that deciding the existence of a classical (resp. quantum) graph homomorphism can be formulated as a feasibility conic program over the cone of completely positive (resp. completely positive semidefinite) matrices.

Corollary 5.11. *Consider two undirected graphs H and G and let $\mathcal{K} \in \{\mathcal{CP}, \mathcal{CS}_+\}$. The (H, G)-homomorphism game admits a perfect $\mathcal{K}$-strategy if and only if there exists $X \in \mathcal{K}^{|V(H)\times V(G)|}$ such that*

$$\sum_{g\in V(G)} \sum_{g'\in V(G)} X[(h,g),(h',g')] = 1, \text{ for all } h, h' \in V(H), \text{ and}$$
$$X[(h,g),(h',g')] = 0, \text{ when } (h = h' \text{ and } g \neq g') \text{ or } (h \sim_H h' \text{ and } g \not\sim_G g').$$

We note that the case $\mathcal{K} = \mathcal{CP}$ corresponds to the first part of Theorem 4.1 in [Rob14] and the case $\mathcal{K} = \mathcal{CS}_+$ corresponds to the first part of Theorem 4.3 in [Rob14].

5.3.2 Chromatic and independence number

A *k-coloring* for an (undirected) graph G corresponds to an assignment of one out of k possible colors to its vertices so that adjacent vertices receive different colors. The *chromatic number* of a graph G, denoted $\chi(G)$, is equal to the smallest integer $k \geq 1$ for which G admits a k-coloring. Notice that G admits a k-coloring if and only if there exist a homomorphism from G into K_k, i.e., the complete graph on k vertices. Thus, $\chi(G)$ may be equivalently defined as the smallest integer $k \geq 1$ for which the (G, K_k)-homomorphism game admits a perfect classical strategy.

The *independence number* of an undirected graph G, denoted $\alpha(G)$, is equal to the largest number of pairwise nonadjacent vertices of G. Notice that G contains k pairwise nonadjacent vertices if and only if there exists a homomorphism from K_k into $\overline{G}$, where $\overline{G}$ denotes the complement of the graph G. As a result, $\alpha(G)$ can be equivalently defined as the largest integer $k \geq 1$ for which the $(K_k, \overline{G})$-homomorphism game admits a perfect classical strategy.

As discussed above, the chromatic and the independence number can be reformulated in the framework of nonlocal games and this motivates the following definitions.

Definition 5.12. *The* quantum chromatic number *of an undirected graph G, denoted $\chi_q(G)$, is equal to the smallest integer $k \geq 1$ for which the (G, K_k)-homomorphism game admits a perfect quantum strategy. Analogously, the* quantum independence number *of an undirected graph G, denoted $\alpha_q(G)$, is equal to the largest integer $k \geq 1$ for which the $(K_k, \overline{G})$-homomorphism game admits a perfect quantum strategy.*

The quantum chromatic number of a graph was introduced and studied in [CMN$^+$07] and the quantum independence number in [MR15]. It has been recently shown that deciding whether the quantum chromatic number of a graph is at most 3 is NP-hard [Ji13].

Using Theorem 5.8 *(iii)* we immediately get conic programming formulations for the quantum chromatic number and the quantum independence number of a graph. We note that these formulations were also identified in Proposition 4.10 and Proposition 4.1 in [LP14], respectively.

Corollary 5.13. *The quantum chromatic number of an undirected graph G is equal to the smallest integer $k \geq 1$ for which there exists a matrix $X \in \mathcal{CS}_+^{|V(G)|k+1}$ satisfying:*

$$\begin{aligned}
& X[0,0] = 1, \\
(41) \qquad & \sum_{i,i' \in [k]} X[(g,i),(g,i')] = \sum_{i \in [k]} X[0,(g,i)] = 1, \ \textit{for all } g \in V(G), \\
& X[(g,i),(g',i')] = 0, \ \textit{when } (g = g' \textit{ and } i \neq i') \textit{ or } (g \sim_G g' \textit{ and } i = i').
\end{aligned}$$

The quantum independence number of an undirected graph G is equal to the largest integer $k \geq 1$ for which there exists a matrix $X \in \mathcal{CS}_+^{k|V(G)|+1}$ satisfying:

$$\begin{aligned}
& X[0,0] = 1, \\
(42) \qquad & \sum_{g,g' \in V(G)} X[(i,g),(i,g')] = \sum_{g \in V(G)} X[0,(i,g)] = 1, \ \textit{for all } i \in [k], \\
& X[(i,g),(i',g')] = 0, \ \textit{when } (i = i' \textit{ and } g \neq g') \textit{ or } (i \neq i' \textit{ and } [g = g' \textit{ or } g \sim_G g']).
\end{aligned}$$

5.3.3 Binary constraint satisfaction problems

In view of Sections 5.3.1 and 5.3.2 it is natural to ask what kind of combinatorial problems admit linear conic formulations of a similar form. In this section we show that all examples considered in this work can be cast in the common framework of binary constraint satisfaction problems.

An instance of a *constraint satisfaction problem* (CSP) is specified by a triple $(\mathcal{V}, \mathcal{D}, \mathcal{C})$ where the elements of $\mathcal{V} = \{x_1, .., x_n\}$ are called the *variables* of the CSP, the elements of $\mathcal{D} = \{D_1, \ldots, D_n\}$ are the *domains* of the corresponding variables and the elements of $\mathcal{C} = \{C_1, \ldots, C_m\}$ are called the *constraints* of the CSP. Each constraint C_i involves a subset of variables $\{x_{i_1}, \ldots, x_{i_{t_i}}\} \subseteq \mathcal{V}$ and is defined as some t_i-ary relation on $D_{i_1} \times \cdots \times D_{i_{t_i}}$. The number of variables t_i is called the *arity* of the constraint C_i. We say that a CSP is *satisfiable* if there exists an assignment of values to each variable from its corresponding domain so that every constraint is satisfied. A CSP that only involves constraints of arity 2 is called a *binary* CSP.

Deciding the existence of a homomorphism from a graph H to a graph G can be formulated as an instance of a binary CSP. Specifically, we have one variable for each vertex of H and the domain of each variable is the vertex set of G. Lastly, for every edge $e = (h, h') \in E(H)$ we have a constraint C_e of arity 2 involving the variables h and h'; the constraint is given by $C_e = \{E(G)\}$.

To any binary constraint satisfaction problem $\mathcal{P} := (\mathcal{V}, \mathcal{D}, \mathcal{C})$ we may associate a two-player nonlocal game, denoted $\mathcal{G}(\mathcal{P})$, having the property that the CSP is satisfiable if and only if the game admits a perfect classical strategy. The game is defined as follows: The referee selects uniformly at random a pair of variables $(x_i, x_j) \in \mathcal{V} \times \mathcal{V}$ and sends x_i to Alice and x_j to Bob. For the players to win they need to respond to the referee with an element of D_i and D_j, respectively. Furthermore, if there exists some constraint C_k that involves the variables x_i and x_j then the answers of the players must satisfy the constraint. Lastly, if the players receive the same variables as questions they have to provide identical answers ensuring that the game is synchronous.

Definition 5.14. *We say that the binary constraint satisfaction problem $\mathcal{P}$ is* quantumly satisfiable *if the nonlocal game $\mathcal{G}(\mathcal{P})$ admits a perfect quantum strategy.*

Notice that the notion of quantum satisfiability of binary CSP's generalizes the concept of quantum graph homomorphisms. Indeed, it is immediate from the definitions that there exists a quantum graph homomorphism from H to G if and only if the nonlocal game corresponding to the homomorphism CSP admits a perfect quantum strategy.

The majority of the literature concerning CSP's usually focuses on binary CSP's. The reason for this is that any non-binary CSP $\mathcal{P}$ can be converted to a binary CSP $\mathcal{P}'$ such that $\mathcal{P}$ is satisfiable if and only if $\mathcal{P}'$ is satisfiable. The transformation is straightforward: For each constraint C_i of $\mathcal{P}$ we introduce one variable c_i in $\mathcal{P}'$. The domain of the variable c_i is given by all assignments that satisfy the constraint C_i in $\mathcal{P}$. Lastly, for every two constraints C_i, C_j of $\mathcal{P}$ that share a variable x_k we add a binary constraint between the variables c_i, c_j in $\mathcal{P}'$. This constraint excludes those satisfying assignments for C_i and C_j where the common variable x_k receives different values.

The discussion above allows us to generalize the notion of quantum satisfiability from binary CSP's to arbitrary ones. Combining this fact with Theorem 5.8 we get the following corollary.

Corollary 5.15. *Deciding whether an arbitrary constraint satisfaction problem is satisfiable (resp. quantumly satisfiable) is equivalent to deciding the feasibility of a linear conic program over $\mathcal{CP}$ (resp. $\mathcal{CS}_+$).*

Acknowledgements. The authors thank S. Burgdorf, M. Laurent, L. Mančinska, T. Piovesan, D. E. Roberson, S. Upadhyay, T. Vidick and Z. Wei for useful discussions. AV is supported in part by the Singapore National Research Foundation under NRF RF Award No. NRF-NRFF2013-13. JS is supported in part by NSERC Canada. Research at the Centre for Quantum Technologies at the National University of Singapore is partially funded by the Singapore Ministry of Education and the National Research Foundation, also through the Tier 3 Grant "Random numbers from quantum processes," (MOE2012-T3-1-009).

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