Existence of Nash equilibrium for distributionally robust chance-constrained games

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Abstract We consider an $n$-player finite strategic game. The payoff vector of each player is a random vector whose distribution is not completely known. We assume that the distribution of the random payoff vector of each player belongs to a distributional uncertainty set. Using distributionally robust approach, we define a chance-constrained game with respect to the worst-case chance-constraint. We call such a game as distributionally robust chance-constrained game. We consider three different types of distributional uncertainty sets. For each case, we show that there always exists a mixed strategy Nash equilibrium for the corresponding distributionally robust chance-constrained game.

Keywords Distributionally robust chance-constrained games · Chance-constraints · Nash equilibrium · Semidefinite programming.

1 Introduction

The work on the existence of equilibrium in game theory started with the paper by John von Neumann [14]. He showed that there always exists a mixed strategy saddle point equilibrium for a two player zero sum matrix game. In 1950, John Nash [13] showed that there always exists a Nash equilibrium for a finite

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player strategic game where each player has finite number of actions. Both 
[13] [14] considered the games where the payoffs of the players are determinis-
tic. But, in many situations payoffs are random variables due to uncertainty 
caused by various external factors. The electricity markets are good examples 
that capture this situation [7] [12] [21] [22]. One way to handle these games is by 
taking the expected value of the random variables (if it exists) and solve the 
corresponding deterministic game [21] [22]. But, this criterion does not take 
a proper account of stochasticity because the observed sample payoffs can be 
large amounts with very small probabilities, the players may be satisfied to 
get the payoffs with certain probability. Such a situation can be handled using 
chance-constrained programming [3] [6] [17]. In this case, the payoff function 
of each player is defined using a chance-constraint due to which these games 
are called chance-constrained games. The applications of chance-constrained 
games can be found in electricity markets [7] [12]. To the best of our knowl-
edge only few theoretical results on two player zero sum chance-constrained 
games, with finite action sets for both the players, are available in the litera-
ure [1] [2] [4] [20]. Recently, Singh et al. [19] considered an \( n \)-player finite 
strategic game where a payoff vector of each player follows an elliptically sym-
metric distribution. They showed that there always exists a mixed strategy 
Nash equilibrium for the corresponding chance-constrained game.

Under many circumstances we do not have a complete knowledge of a dis-
tribution. The only information we have of a distribution is that it belongs 
to some distributional uncertainty set. In this paper, we consider an \( n \)-player 
strategic game where the distribution of the payoff vector of each player be-
ongs to a distributional uncertainty set. We consider three different types of 
distributional uncertainty sets depending on the information about the mean 
and the covariance matrix of the random payoff vector. We consider the distri-
butionally robust approach which is suitable for such cases. That is, we define 
the payoff of each player using his worst-case chance-constraint. We call such 
games distributionally robust chance-constrained games. We show that there 
always exists a mixed strategy Nash equilibrium for these games. In [10] [15], 
slightly related game models have been studied. Both [10] [15] considered the 
games where each player can neither evaluate his cost function exactly nor es-
timate his opponents’ strategies accurately. The cost functions and strategies 
of the players belong to certain Euclidean uncertainty sets.

The rest of the paper is organized as follows. Section 2 contains the defi-
nition of a distributionally robust chance-constrained game. In Section 3 we 
show the existence of a mixed strategy Nash equilibrium for the distribution-
ally robust chance-constrained game corresponding to three different cases of 
distributional uncertainties. We conclude our paper in Section 4.

2 The model

We consider an \( n \)-player strategic game. Let \( I = \{1, 2, \cdots, n\} \) be a set of all 
the players. For each \( i \in I \), let \( A_i \) be a finite action set of player \( i \) and its
A vector $a = (a_1, a_2, \cdots, a_n)$ denotes an action profile of the game. Let $A = \times_{i=1}^{n} A_i$, where $\times$ denotes the Cartesian product, be the set of all action profiles of the game. Denote, $A_{-i} = \times_{j=1, j \neq i}^{n} A_i$, and $a_{-i} \in A_{-i}$ is a vector of actions $a_j$, $j \neq i$. The action set $A_i$ of player $i$ is also called the set of pure strategies of player $i$. A mixed strategy of a player is represented by a probability distribution over its action set. For each $i \in I$, let $X_i$ be the set of mixed strategies of player $i$, i.e., it is the set of all probability distributions over the action set $A_i$. A mixed strategy $\tau_i \in X_i$ is represented by $\tau_i = (\tau_i(a_i))_{a_i \in A_i}$, where $\tau_i(a_i) \geq 0$ is a probability with which the player $i$ chooses an action $a_i$, and $\sum_{a_i \in A_i} \tau_i(a_i) = 1$. Let $X = \times_{i=1}^{n} X_i$ be the set of all mixed strategy profiles of the game and its element is denoted by $\tau = (\tau_i)_{i \in I}$. Denote, $X_{-i} = \times_{j=1, j \neq i}^{n} X_i$, and $\tau_{-i} \in X_{-i}$ is a vector of mixed strategies of all the players excluding player $i$. We define $(\nu_i, \tau_{-i})$ to be a strategy profile where player $i$ uses the strategy $\nu_i$ and each player $j$, $j \neq i$, uses the strategy $\tau_j$. Let $r_i = (r_i(a))_{a \in A}$ be a payoff vector of player $i$, where $r_i(a)$ is a payoff that player $i$ receives at action profile $a$. For a given strategy profile $\tau \in X$, the payoff of player $i$, $i \in I$, is given by,

$$r_i(\tau) = \sum_{a \in A} \prod_{j=1}^{n} \tau_j(a_j)r_i(a).$$

Given a $\tau \in X$, let $\eta^\tau = (\eta^\tau(a))_{a \in A}$ be a vector, where $\eta^\tau(a) = \prod_{i=1}^{n} \tau_i(a_i)$. From [1], $r_i(\tau) = r^T \eta^\tau$, where $T$ is transposition. Let $|B|$ denotes the cardinality of a given set $B$. The case where $r_i \in \mathbb{R}^{|A_i|}$, $i \in I$, has been studied by Nash [13]. He showed that there always exists a mixed strategy Nash equilibrium.

We consider the games where the payoff vector of each player is a random vector. Let $\Omega$ denotes the sample space on which random variables are defined. Let $r_i : \Omega \rightarrow \mathbb{R}^{|A_i|}$, $i \in I$, be a random payoff vector of player $i$. Let $\mathcal{M}$ denotes the set of all probability measures over the set of all measurable subsets of $\mathbb{R}^{|A_i|}$. A probability distribution $F$ of $r_i$ is a member of set $\mathcal{M}$. To study such games, a satisfying payoff criterion has been introduced in the literature [1, 2, 4, 20]. In this payoff criterion, the payoff of a player is defined using a chance-constraint due to which these games are called chance-constrained games. We assume that each player uses satisfying payoff criterion. At strategy profile $\tau \in X$, the payoff of each player is the highest level of his payoff that he can attain with at least a specified level of confidence. The confidence level of each player is given a priori and it is not known to the other players. Let $\alpha_i \in [0, 1]$ be a confidence level of player $i$ and $\alpha = (\alpha_i)_{i \in I}$ denotes a vector of the confidence levels of all the players. For a given strategy profile $\tau \in X$, and a given confidence level vector $\alpha$, the payoff of player $i$, $i \in I$, is given by

$$u_i^\alpha(\tau) = \sup\{v | P(r_i(\tau) \geq v) \geq \alpha_i\}. \tag{2}$$

When the distribution of $r_i$ is known, e.g., if $r_i$ follows a multivariate normal distribution with the mean $\mu_i$ and the positive definite covariance matrix $\Sigma_i$, then

$$u_i^\alpha(\tau) = \sum_{a \in A} \prod_{j=1}^{n} \tau_j(a_j)\mu_i(a) + ||\Sigma^\frac{1}{2}\eta^\tau||_2 F^{-1}_n(1 - \alpha_i),$$

where $F^{-1}_n$ denotes the quantile function of the standard normal distribution.
where $Z_i$ is a standard normal distribution and $||·||_2$ is the Euclidean norm. Singh et al. \cite{Singh19} showed that there always exists a mixed strategy Nash equilibrium for the chance-constrained game when the payoff vector of each player follows an elliptically symmetric distribution. For more details about the elliptically symmetric distribution see \cite{9}.

In this paper, we consider the case where we do not have the complete knowledge of the distributions of the players’ payoff vectors. The only knowledge we have of a distribution of the payoff vector of each player is that it belongs to some uncertainty set. Such an uncertainty set is based on the partially available information about the distribution, it is called distributional uncertainty set. Let $\mathcal{D}_i$ denotes a distributional uncertainty set for player $i$, $i \in I$. We assume that the distributional uncertainty set of each player is known to all the players. We use distributionally robust approach which is more suitable for such situations. In this case, the payoff of a player is defined using his worst-case chance-constraint. That is, for a given strategy profile $\tau \in X$, and a given confidence level vector $\alpha$, the payoff of player $i$, $i \in I$, is given by

$$u_{\alpha i}(\tau) = \sup \left\{ v \mid \inf_{F \in \mathcal{D}_i} P_F(r_i(\tau) \geq v) \geq \alpha_i \right\}.$$  

(3)

For a given $\tau \in X$, $u_{\alpha i}(\tau) = -\infty$, if the worst case chance-constraint is not satisfied. We call such games distributionally robust chance-constrained games. For a given $\alpha \in [0, 1]^n$, the above distributionally robust chance-constrained game is a non-cooperative game with complete information. The payoffs of each player defined by (3) is known to all the other players. The set of best response strategies of player $i$, $i \in I$, against a given strategy profile $\tau_{-i}$ of the other players is given by

$$BR_{\alpha i}(\tau_{-i}) = \{ \bar{\tau}_i \in X_i \mid u_{\alpha i}(\bar{\tau}_i, \tau_{-i}) \geq u_{\alpha i}(\tau_i, \tau_{-i}), \forall \tau_i \in X_i \}.$$  

(4)

Next, we give the definition of Nash equilibrium.

**Definition 1 (Nash equilibrium)** A strategy profile $\tau^* \in X$ is said to be a Nash equilibrium for a given $\alpha \in [0, 1]^n$, if for all $i \in I$, the following inequality holds,

$$u_{\alpha i}(\tau^*_i, \tau^*_{-i}) \geq u_{\alpha i}(\tau_i, \tau^*_{-i}), \forall \tau_i \in X_i.$$  

That is, $\tau^*$ is a Nash equilibrium if and only if $\tau^*_i \in BR_{\alpha i}(\tau^*_{-i})$ for all $i \in I$.

### 3 Existence of Nash equilibrium

If for a given strategy profile the worst-case chance-constraint of a player does not hold, the payoff of the player is not finite and the player can increase his payoff by deviating to the strategies that give him a finite payoff. If the worst-case chance-constraint for a player does not hold at any strategy profile, all the strategies of the player would be part of a Nash equilibrium. So, we consider the
case when for each strategy profile $\tau$, the worst-case chance-constraint for each player holds. We consider three different types of distributional uncertainty sets. For each case we show the existence of a mixed strategy Nash equilibrium.

### 3.1 Distribution with known moments

We start with the case where the mean vector $\mu_i$ and the covariance matrix $\Sigma_i$ of the distribution of the payoff vector $r_i$ of the player $i$, $i \in I$, are known exactly, and otherwise the distribution is arbitrary. We assume that the mean $\mu_i$ and the covariance matrix $\Sigma_i$ for each $i \in I$ is known to all the players. For a given matrix $B$, $B \succeq 0$ (resp. $B \succ 0$) means $B$ is a positive semidefinite (resp. definite) matrix. We assume that $\Sigma_i \succ 0$. Let $\mathcal{D}_i(\mu_i, \Sigma_i)$ be the set of all such probability distributions. This type of distributional uncertainty set is considered in [8]. For each $i \in I$, (3) can be equivalently written as,

$$u_i^{\alpha_i}(\tau) = -\inf \left\{ u \mid \sup_{F \in \mathcal{D}_i(\mu_i, \Sigma_i)} P_F(u \leq -r_i(\tau)) \leq 1 - \alpha_i \right\}. \quad (5)$$

The problem $\inf \{u | \sup_{F \in \mathcal{D}_i(\mu_i, \Sigma_i)} P_F(u \leq -r_i(\tau)) \leq 1 - \alpha_i \}$ is the same as the worst-case value-at-risk with known moments problem considered in [8].

It follows from [3] that, for each $i \in I$,

$$u_i^{\alpha_i}(\tau_i, \tau_{-i}) = \mu_i^T \eta^r - \sqrt{\frac{\alpha_i}{1 - \alpha_i}} ||\Sigma_i^{\frac{1}{2}} \eta^r||_2, \forall \tau \in X. \quad (6)$$

**Lemma 1** For each $i \in I$, $u_i^{\alpha_i}(\cdot, \tau_{-i})$ given by (6) is a concave function of $\tau_i$ for all $\alpha_i \in [0, 1)$.

**Proof** Fix $i \in I$, $\alpha_i \in [0, 1)$, and $\tau_{-i} \in X_{-i}$. Let $\tau_i^1$, $\tau_i^2 \in X_i$. Take $\lambda \in [0, 1]$. Then, for a strategy profile $(\lambda \tau_i^1 + (1 - \lambda)\tau_i^2, \tau_{-i})$ we have $\eta(\lambda \tau_i^1 + (1 - \lambda)\tau_i^2, \tau_{-i})(a) = (\lambda \tau_i^1(a_i) + (1 - \lambda)\tau_i^2(a_i)) \prod_{j \in I, j \neq i} \tau_j(a_j)$ for each $a \in A$. That is, $\eta(\lambda \tau_i^1 + (1 - \lambda)\tau_i^2, \tau_{-i}) = \lambda \eta(\tau_i^1, \tau_{-i}) + (1 - \lambda)\eta(\tau_i^2, \tau_{-i})$. From (6), we have

$$u_i^{\alpha_i}(\lambda \tau_i^1 + (1 - \lambda)\tau_i^2, \tau_{-i}) = \mu_i^T (\lambda \eta(\tau_i^1, \tau_{-i}) + (1 - \lambda)\eta(\tau_i^2, \tau_{-i}))$$

$$= \sqrt{\frac{\alpha_i}{1 - \alpha_i}} ||\Sigma_i^{\frac{1}{2}} (\lambda \eta(\tau_i^1, \tau_{-i}) + (1 - \lambda)\eta(\tau_i^2, \tau_{-i}))||_2$$

$$\geq \lambda u_i^{\alpha_i}(\tau_i^1, \tau_{-i}) + (1 - \lambda)u_i^{\alpha_i}(\tau_i^2, \tau_{-i}).$$

That is, $u_i^{\alpha_i}(\cdot, \tau_{-i})$ is a concave function. \hfill \Box

**Theorem 1** Consider an $n$-player finite strategic game, where, the payoff vector $r_i = (r_i(a))_{a \in A}$ of player $i$, $i \in I$, is a random vector. If for each $i \in I$, the mean vector $\mu_i$, and the covariance matrix $\Sigma_i$ which is positive definite, of the distribution of $r_i$ are known exactly, and otherwise the distribution is arbitrary, there always exists a mixed strategy Nash equilibrium of the corresponding distributionally robust chance-constrained game for all $\alpha \in [0, 1)^n$. 

Proof Let $\mathcal{P}(X)$ be a power set of $X$. Define a set valued map $G : X \to \mathcal{P}(X)$ such that $G(\tau) = \bigcap_{i=1}^{n} BR_{\mu_i}^{i}(\tau_{-i})$. A strategy profile $\tau \in X$ is said to be a fixed point of the set valued map $G$ if $\tau \in G(\tau)$. It is easy to see that a fixed point of $G$ is a Nash equilibrium. In order to show that $G$ has a fixed point, we show that $G$ satisfies all the following conditions of Kakutani fixed point theorem [11]:

1. $X$ is a non-empty, convex, and compact subset of a finite dimensional Euclidean space.
2. $G(\tau)$ is non-empty and convex for all $\tau \in X$.
3. $G(\cdot)$ has closed graph: If $(\tau_n, \bar{\tau}_n) \to (\tau, \bar{\tau})$ with $\bar{\tau}_n \in G(\tau_n)$ for all $n$ then $\bar{\tau} \in G(\tau)$.

Condition 1 holds from the definition of $X$. Fix $\alpha \in [0,1)^n$. For each $i \in I$, $BR_{\mu_i}^{i}(\tau_{-i})$ is non-empty because $u_i^{\alpha_i}(\cdot, \tau_{-i})$ defined by (6) is a continuous function of $\tau_i$ and $X_i$ is a compact set. For each $i \in I$, $BR_{\mu_i}^{i}(\tau_{-i})$ is a convex set because $u_i^{\alpha_i}(\cdot, \tau_{-i})$ is a concave function of $\tau_i$ from Lemma 1. This shows that the condition 2 also holds. The payoff function $u_i^{\alpha_i}(\cdot)$, $i \in I$, defined by (6) is a continuous function of $\tau$. The closed graph condition of $G(\cdot)$ can be proved using the continuity of functions $u_i^{\alpha_i}(\cdot)$, $i \in I$. For detailed proof of closed graph condition see Theorem 3.1 of [19]. That is, the set valued map $G(\cdot)$ satisfies all the conditions of Kakutani fixed point theorem. Hence, there exists a strategy profile $\tau^*$ which is a fixed point of $G(\cdot)$. Such $\tau^*$ is a Nash equilibrium of the game. The $\alpha \in [0,1)^n$ is arbitrary, so, there always exists a mixed strategy Nash equilibrium for all $\alpha \in [0,1)^n$. □

3.2 Distribution with polytopic uncertainty

We now consider the case when for each $i \in I$, the mean vector $\mu_i$ and the covariance matrix $\Sigma_i$ of the distribution of $r_i$ are only known to belong to some polytopes described by their vertices. That is, $\mu_i \in U_{\mu_i}$ and $\Sigma_i \in U_{\Sigma_i}$, where

$$U_{\mu_i} := \text{Co}\{\mu_i^1, \mu_i^2, \cdots, \mu_i^l\}, \quad U_{\Sigma_i} := \text{Co}\{\Sigma_i^1, \Sigma_i^2, \cdots, \Sigma_i^l\}.$$ 

$\text{Co}$ stands for convex hull. The vertices $\{\mu_i^k\}_{k=1}^l$ and $\{\Sigma_i^k\}_{k=1}^l$ are given and known to all the players. We assume that $\Sigma_i^k > 0$ for all $k = 1, 2, \cdots, l$. Again, let $\mathcal{D}_i(\mu_i, \Sigma_i)$ denotes the set of all probability distributions that have the mean $\mu_i \in U_{\mu_i}$ and the covariance matrix $\Sigma_i \in U_{\Sigma_i}$. Such polytopic uncertainty is considered in [8]. From [8], for each $i \in I$ and $\tau \in X$, we have,

$$u_i^{\alpha_i}(\tau_i, \tau_{-i}) = \min_{1 \leq k \leq l} (\mu_i^k)^T \eta - \max_{1 \leq k \leq l} \sqrt{\frac{\alpha_i}{1 - \alpha_i}} \|((\Sigma_i^k)^{1/2}) \eta^*\|_2. \quad (7)$$

Lemma 2 For each $i \in I$, $u_i^{\alpha_i}(\cdot, \tau_{-i})$ given by (7) is a concave function of $\tau_i$ for all $\alpha_i \in [0,1)$. 


Proof Fix $\tau_i \in X_i$ and $i \in I$. For each $k, k = 1, 2, \cdots, l$, $(\mu_k^i)^T \eta^r$ is a linear function of $\tau_i$, hence it is a concave function of $\tau_i$. The minimum of a set of concave functions is a concave function, so the first term of (7) is a concave function of $\tau_i$. As shown in Lemma 1, for each $k = 1, 2, \cdots, l$, $(\mu_k^i)^T (\Sigma_k^i)^{1/2} \eta^r$ is a convex function of $\tau_i$. The maximum of a set of convex functions is a convex function, and the negative of a convex function is a concave function. That is, the second term of (7) is also a concave function of $\tau_i$. So, $u_{\alpha_i}^i(\cdot, \tau_i)$ is a concave function of $\tau_i$. \hfill \qed

Theorem 2 Consider an $n$-player finite strategic game, where, the payoff vector $r_i = (r_i(a))_{a \in A}$ of the player $i$, $i \in I$, is a random vector. If for each $i \in I$, the mean-covariance pair $(\mu_i, \Sigma_i)$ of the distribution of $r_i$ are such that $\mu_i \in U_{\mu_i}$ and $\Sigma_i \in U_{\Sigma_i}$, and otherwise the distribution is arbitrary, there always exists a mixed strategy Nash equilibrium of the corresponding distributionally robust chance-constrained game for all $\alpha \in [0, 1)^n$.

Proof From Lemma 2, $u_{\alpha_i}^i(\cdot, \tau_i)$, $i \in I$, is a concave function. So, $BR_{\alpha_i}^i(\tau_i)$, $i \in I$, is a convex set. From (7), $u_{\alpha_i}^i(\cdot)$, $i \in I$ is also a continuous function of $\tau$. Using these properties the proof follows from the similar arguments used in Theorem 1. \hfill \qed

3.3 Distribution with known mean and an upper bound on covariance matrix

We consider the case where the distributional uncertainty set for player $i$, $i \in I$, accounts for the information about the mean vector $\mu_i$ and an upper bound $\Sigma_i > 0$ on the covariance matrix of the random payoff vector $r_i$. We assume that, for each $i \in I$, the mean $\mu_i$ and the upper bound $\Sigma_i$ on the covariance matrix are known to all the players. The distributional uncertainty set for player $i$, $i \in I$, is given by

$$
D_i(\mu_i, \Sigma_i) = \left\{ F \in M \left| \begin{array}{c} \mathbb{E}_F[r_i] = \mu_i \\ \mathbb{E}_F[(r_i - \mu_i)(r_i - \mu_i)^T] \preceq \Sigma_i \end{array} \right\} \right. 
$$

Such an uncertainty set is considered by Cheng et al. \cite{5}. For given two matrices $B$ and $C$, $B \succeq C$ means $B - C \succeq 0$.

To show the existence of a mixed strategy Nash equilibrium, first we get a closed form expression for the payoff function of each player defined by (3). For this we need to further simplify the worst-case chance-constraint used in the definition of the payoff function. For each $i \in I$, the worst-case chance-constraint from (3) can be written as

$$
\sup_{F \in D_i(\mu_i, \Sigma_i)} \mathbb{E}_F[\mathbb{I}_{\{r_i^T \eta^r \leq 1\}}] \leq 1 - \alpha_i, \quad (9)
$$
where \( I(\cdot) \) is an indicator function that gives value one if the condition is true and zero otherwise. For each \( i \in I \), we begin with the optimization problem

\[
\begin{aligned}
&\sup_{F \in \mathcal{M}} \int_{\mathbb{R}^{|A|}} I(\bar{r}_i^T \eta^T \leq v) \, dF(\bar{r}_i) \\
\text{s.t.}
&\int_{\mathbb{R}^{|A|}} \bar{r}_i \, dF(\bar{r}_i) = \mu_i \\
&\int_{\mathbb{R}^{|A|}} (\bar{r}_i - \mu_i)(\bar{r}_i - \mu_i)^T \, dF(\bar{r}_i) \preceq \Sigma_i.
\end{aligned}
\tag{10}
\]

Let \( B \cdot C = \sum_{i,j} B_{ij} C_{ij} \) represents the Frobenius product between two given matrices \( B \) and \( C \) of the same dimensions. The dual of the above problem is

\[
\begin{aligned}
&\min_{\bar{t}_i, \bar{q}_i, \bar{Q}_i} \bar{t}_i + 2\bar{q}_i^T \mu_i + (\Sigma_i + \mu_i \mu_i^T) \cdot \bar{Q}_i \\
\text{s.t.}
&-1 \{\bar{r}_i^T \eta^T \leq v\} + \bar{t}_i + 2\bar{q}_i^T \bar{q}_i + \bar{r}_i^T \bar{Q}_i \bar{r}_i \geq 0, \quad \forall \bar{r}_i \in \mathbb{R}^{|A|} \\
&\bar{Q}_i \succeq 0.
\end{aligned}
\tag{11}
\]

For details about above duality formulation see [18]. The strong duality follows from [18] because Dirac measure \( \delta_{\mu_i} \) lies in the relative interior of the set \( \mathcal{D}_i(\mu_i, \Sigma_i) \). Hence, constraint [10] can be reformulated as

\[
\begin{aligned}
&\bar{t}_i + 2\bar{q}_i^T \mu_i + (\Sigma_i + \mu_i \mu_i^T) \cdot \bar{Q}_i \leq 1 - \alpha_i \\
&\bar{Q}_i \succeq 0 \\
&-1 + \bar{t}_i + 2\bar{q}_i^T \bar{q}_i + \bar{r}_i^T \bar{Q}_i \bar{r}_i \geq 0, \quad \forall \bar{r}_i \in \mathbb{R}^{|A|} \text{ such that } \bar{r}_i^T \eta^T - v \leq 0.
\end{aligned}
\tag{12}
\]

Given \( \tau \in X \) and \( v \in \mathbb{R} \) there always exists \( \bar{r}_0 \in \mathbb{R}^{|A|} \) such that \( \bar{r}_0^T \eta^T - v < 0 \), i.e., Slater condition holds. This is possible because \( \eta^T \) is a probability distribution over the set \( A \) of all the action profiles and hence it cannot be a zero vector. From Theorem 2.1 of [16], the last constraint from [12] is equivalent to:

\[
-1 + \bar{t}_i + 2\bar{q}_i^T \bar{q}_i + \bar{r}_i^T \bar{Q}_i \bar{r}_i + 2\lambda_i (\bar{r}_i^T \eta^T - v) \geq 0, \quad \forall \bar{r}_i \in \mathbb{R}^{|A|} \text{ such that } \bar{r}_i^T \eta^T - v \leq 0.
\]

So, the new set of constraints equivalent to [12] is

\[
\begin{aligned}
&M_i \cdot T_i \leq 1 - \alpha_i, \\
&M_i \succeq 0, \\
&M_i + \begin{bmatrix} 0_{|A| \times |A|} & \lambda_i \eta^T \\
\lambda_i (\eta^T)^T & -1 - 2\lambda_i v \end{bmatrix} \succeq 0, \\
&\lambda_i \geq 0.
\end{aligned}
\tag{13}
\]
where, $\mathbf{0}_{|A| \times |A|}$ is the $|A| \times |A|$ zero matrix, $M_i = \begin{bmatrix} Q_i \bar{q}_i \\ q_i^T \bar{t}_i \end{bmatrix}$ and

$$\Gamma_i = \begin{bmatrix} \Sigma_i + \mu_i \mu_i^T & \bar{\mu}_i \\ \bar{\mu}_i^T & 1 \end{bmatrix}.$$ From (3), we have

$$u_i^{\alpha_i}(\tau) = \sup_{v, M_i, \lambda_i} v$$

s.t.

$$M_i \cdot \Gamma_i \leq 1 - \alpha_i,$$

$$M_i \succeq 0,$$

$$M_i + \begin{bmatrix} 0 & \lambda_i \eta^T \\ \lambda_i (\eta^T)^T - 1 - 2 \lambda_i v \end{bmatrix} \succeq 0,$$

$$\lambda_i \geq 0.$$...

The above problem is equivalent to

$$u_i^{\alpha_i}(\tau) = - \inf_{v, M_i, \lambda_i} v$$

s.t.

$$M_i \cdot \Gamma_i \leq 1 - \alpha_i,$$

$$M_i \succeq 0,$$

$$M_i + \begin{bmatrix} 0 & \lambda_i \eta^T \\ \lambda_i (\eta^T)^T - 1 + 2 \lambda_i v \end{bmatrix} \succeq 0,$$

$$\lambda_i \geq 0.$$...

From [8] it follows that $\lambda_i$-components of the optimal solutions of (15) are uniformly bounded from below by a positive number. So, we can divide by $\lambda_i$ in the matrix inequalities above and replace $\frac{1}{\lambda_i}$ by $\lambda_i$ and $\frac{M_i}{\lambda_i}$ by $M_i$. Now, we have the following semidefinite programming problem,

$$u_i^{\alpha_i}(\tau) = - \inf_{v, M_i} v$$

s.t.

$$M_i \cdot \Gamma_i \leq \lambda_i (1 - \alpha_i),$$

$$M_i \succeq 0,$$

$$M_i + \begin{bmatrix} 0 & \eta^T \\ (\eta^T)^T - \lambda_i + 2v \end{bmatrix} \succeq 0,$$

$$\lambda_i \geq 0.$$...

From Theorem 1 of [8] it follows that, for each $i \in I$,

$$u_i^{\alpha_i}(\tau, \tau_{-i}) = \mu_i^T \eta^T - \sqrt{\frac{\alpha_i}{1 - \alpha_i}} \| \Sigma_i^{\frac{1}{2}} \eta^T \|_2, \ \forall \ \tau \in X.$$
Theorem 3 Consider an $n$-player finite strategic game, where, the payoff vector $r_i = (r_i(a))_{a \in A}$ of the player $i, i \in I$, is a random vector whose distribution $F$ is not completely known. If $F$ belongs to an uncertainty set $\mathcal{D}_i(\mu_i, \Sigma_i)$ defined by (5), where $\Sigma_i$ is a positive definite matrix, there always exists a mixed strategy Nash equilibrium of the corresponding distributionally robust chance-constrained game for all $\alpha \in [0,1]^n$.

Proof From Lemma 1, $u^{\alpha}_i(\cdot, \tau_{-i}), i \in I$, given by (17) is a concave function. So, $BR^{\alpha}_i(\tau_{-i}), i \in I$, is a convex set. From (17), $u^{\alpha}_i(\cdot), i \in I$, is also a continuous function of $\tau$. Using these properties the proof follows from the similar arguments used in Theorem 1. $\square$

4 Conclusions

In this paper, we consider an $n$-player finite strategic game where the payoff vector of each player is a random vector. The distribution of each player’s payoff vector is not completely known. It belongs to a certain distributional uncertainty set. We use chance-constrained programming to study these games. We define a distributionally robust chance-constrained game using the worst-case chance-constraint. We consider three different types of distributional uncertainty sets. For each case we show that there always exists a mixed strategy Nash equilibrium.

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