A hybrid ergodic-splitting method for pseudo-monotone equilibrium problems

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Abstract In this paper, we consider the equilibrium problem for a pseudo-monotone function. For solving this problem, a hybrid ergodic-splitting method is proposed. We decompose the equilibrium function into two functions and then solve auxiliary problems for each decomposition function separately. Under pseudo-monotonicity of the equilibrium function and some additional assumptions, we prove that the sequence generated by algorithm converges to a solution of the problem.

Keywords Equilibrium problem · variational inequality problem · splitting method · ergodic method

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1 Introduction

Throughout this paper, let $C$ be a nonempty closed convex subset of $\mathbb{R}^m$ and denote by $S$ the set of solutions to the following equilibrium problem

$$\text{find } x \in C \text{ such that } f(x, y) \geq 0, \forall y \in C,$$

$EP(f, C)$

We suppose that $f(x, y) = f_1(x, y) + f_2(x, y)$ where $f_1, f_2 : C \times C \to \mathbb{R}$ are two given bifunction satisfy $f_i(x, x) = 0$ for all $x \in C, i = 1, 2$. As usual, we call a bifunction satisfying this property an equilibrium bifunction on $C$.

In applied sciences, there are many problems are reduced to finding solutions of equilibrium problems, which cover variational inequalities, fixed point problems, saddle point problems, complementarity problems as special cases (see [4–6,11,15,

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17, 20, 21). For example, if \( f(x, y) = \varphi(x) - \varphi(y) \) then Problem \( EP(f, C) \) collapses into the optimization problem

\[
\max \{ \varphi(x) : x \in C \}. \tag{1}
\]

If we have \( f(x, y) = \langle F(x), y - x \rangle + \varphi(y) - \varphi(x) \) where \( F : C \to \mathbb{R}^m \) and \( \varphi : C \to \mathbb{R} \), then \( EP(f, C) \) becomes the generalized variational inequality problem

\[
\text{find } x \in C \text{ such that } \langle F(x), y - x \rangle + \varphi(y) - \varphi(x) \geq 0 \quad \forall y \in C. \quad (GVIP)
\]

One more special case of Problem \( EP(f, C) \) is Nash-Cournot equilibrium problem. If we define \( f(x, y) = \varphi(x) - \varphi(x_i), \varphi(x, y) = \sum_{i=1}^{n} h_i(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n) \) where \( h_i : C \to \mathbb{R}, i = 1, \ldots, n \) are give functions, then \( EP(f, C) \) collapses into the Nash-Cournot equilibrium problem

\[
\text{find } x^* = (x_1^*, \ldots, x_n^*) \in C \text{ such that } \\
h_i(x_1^*, \ldots, x_{i-1}^*, y, x_{i+1}^*, \ldots, x_n^*) \leq h_i(x_1^*, \ldots, x_n^*) \quad \forall y \in C, \quad \forall i = 1, \ldots, n.
\]

In recent years, methods for solving equilibrium problems have been studied extensively. Let \( f(x, \cdot) \) is convex for each \( x \in C \). We define the mapping \( U \)

\[
U(x) = \text{argmin} \{ \lambda f(x, y) + G(y) - \langle \nabla G(x), y - x \rangle : y \in C \} \quad (AuP)
\]

where \( \lambda > 0, G : \mathbb{R}^m \to \mathbb{R} \) is a strongly convex and continuous differentiable function. Since objective function is strongly convex, the problem \( (AuP) \) admits a unique solution. Hence, the mapping \( U \) is well defined and single valued. It is well known that, \( x^* \in C \) is a solution to \( EP(f, C) \) if and only if \( x^* = U(x^*) \) [16]. This fact motivates the following algorithm

\[
\begin{cases}
    x^0 \in C \\
    x^{k+1} = U(x^k).
\end{cases} \tag{2}
\]

In [16], Mastroeni proved that the sequence \( \{x^k\} \) generated by algorithm (2) converges to the unique solution of \( EP(f, C) \) when \( f \) is strongly monotone and Lipschitz-type continuous.

If \( f \) is monotone and Lipschitz-type continuous, the algorithm (2) may not converge. To deal with this situation, Muu in [19] proposed the following extra-gradient algorithm

\[
\begin{cases}
    x^0 \in C \\
    y^k = \text{argmin} \{ \lambda f(x^k, y) + G(y) - \langle \nabla G(x^k), y - x^k \rangle : y \in C \} \\
    x^{k+1} = \text{argmin} \{ \lambda f(x^k, y) + G(y) - \langle \nabla G(x^k), y - x^k \rangle : y \in C \}.
\end{cases} \tag{3}
\]

Under assumptions that \( f \) is pseudomonotone and Lipschitz-type continuous, the authors showed that the sequence \( \{x^k\} \) generated by algorithm converges to a solution of \( EP(f, C) \).

In general, solving the auxiliary problems \( (AuP) \) of objective function \( f \) may be computationally expensive due to complexity of this function. One alternative is to decompose the given bifunction into the sum of two bifunctions whose auxiliary
problems are easier to solve than the original one. Such method is known as splitting method. This is a very efficient method, since one can treat each part of the original bifunction independently. The splitting methods have been studied by Bello Cruz and Millán in [3], Konnov in [12] and Moudafi in [18]. Motivated by these works and that by Iiduka [9], we introduce a hybrid ergodic-splitting algorithm to solve approximately $EP(f, C)$. In our algorithm, the objective function $f$ will be decomposed into two functions: $f = f_1 + f_2$. At each iteration, we solve two auxiliary problems separately corresponding to $f_1$ and $f_2$. The obtained sequence, however, may not converge under pseudo-monotonicity of $f$. Hence, an ergodic procedure is added in order to obtain the convergence of the sequence generated by the proposed algorithm.

The rest of the paper is organized as follows. Section 2 recalls some concepts and results in equilibrium problems that will be used in the sequel. Section 3 presents the hybrid ergodic-splitting algorithm for solving problem $EP(f, C)$ and proof of convergence for this method as the main results of our paper.

2 Preliminaries

We recall some well known definitions and properties of the projection under the Euclidean norm which will be required in our following analysis.

**Definition 1** Let $C$ be a nonempty closed convex in $\mathbb{R}^m$. By $P_C$ we denote the projection operator on $C$ with the norm $\|\cdot\|$, that is

$$P_C(x) = \arg\min_{y \in C} \{\|y - x\| : y \in C\} \forall x \in C.$$  

Since $\|\cdot - x\|$ is a strongly convex function, therefore $P_C(x)$ is singleton and well defined for every $x$.  

**Proposition 1** (See [7]) Let $C$ be a nonempty closed convex in $\mathbb{R}^m$. Then

(a) $\langle x - P_C(x), y - P_C(x) \rangle \leq 0 \forall x \in \mathbb{R}^m, y \in C,$

(b) $\|P_C(x) - P_C(y)\| \leq \|x - y\| \forall x, y \in \mathbb{R}^m.$

**Definition 2** A bifunction $f : C \times C \to \mathbb{R}$ is said to be

(a) strongly monotone on $C$ with modulus $\gamma > 0$, if

$$f(x, y) + f(y, x) \leq -\gamma\|x - y\|^2 \forall x, y \in C,$$

(b) monotone on $C$, if

$$f(x, y) + f(y, x) \leq 0 \forall x, y \in C.$$

(c) pseudo-monotone on $C$, if

$$f(x, y) \geq 0 \Rightarrow f(y, x) \leq 0 \forall x, y \in C.$$

From the definitions, it is easy seen that $(a) \Rightarrow (b) \Rightarrow (c)$.

**Definition 3** Let $D$ is a subset in $\mathbb{R}^m$. A function $h : D \to \mathbb{R}$ is said to be $\tau$–Hölder continuous on $D$ if there exist constants $Q > 0$ and $\tau \in (0, 1]$ such that $\|h(x) - h(y)\| \leq Q\|x - y\|^\tau \forall x, y \in D$. If $\tau = 1$ then $h$ is Lipschitz continuous.
The next proposition is regarded to the properties of the solution set of $EP(f, C)$.

**Proposition 2** (See [28]) Suppose that $f : C \times C \to \mathbb{R}$ is continuous, pseudo-monotone, for each $x \in C$, $f(x, \cdot)$ is convex and the problem $EP(f, C)$ admits a solution. Then

(a) $S = \{ y \in C : f(x, y) \leq 0 \ \forall x \in C \}$,
(b) $S$ is closed and convex.

To prove our main results, we need the following lemmas

**Lemma 1** (See [27]) Assume that $\{a_n\}$ and $\{b_n\}$ are sequences of non-negative numbers such that $a_{n+1} \leq a_n + b_n \ \forall n \in \mathbb{N}$. If $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n \to \infty} a_n$ exists.

**Lemma 2** (See [26]) Let $\{a_n\}$ and $\{\lambda_n\}$ are sequences of non-negative numbers satisfying

$$
\lim_{n \to \infty} a_n = a \in \mathbb{R}, \quad \lim_{n \to \infty} \lambda_n = 0 \quad \text{and} \quad \sum_{n=1}^{\infty} \lambda_n = \infty.
$$

Then we have

$$
\lim_{n \to \infty} \frac{\sum_{k=1}^{\infty} \lambda_k a_n}{\sum_{k=1}^{\infty} \lambda_k} = a.
$$

3 The hybrid ergodic-splitting algorithm and its convergence.

We consider the problem $EP(f, C)$. Suppose that $f = f_1 + f_2$ where $f_1, f_2 : C \times C \to \mathbb{R}$ are equilibrium functions. In this article, we assume the following conditions.

**Assumption 1** Assume that the functions $f, f_1, f_2$ satisfy

(a) $f : C \times C \to \mathbb{R}$ is pseudo-monotone,
(b) for each $y \in C$, $f(\cdot, y)$ is concave,
(c) $f_1, f_2 : C \times C \to \mathbb{R}$ are $\tau$–Hölder continuous,
(d) for each $x \in C$, $f_i(x, \cdot)$, $i = 1, 2$ is convex,
(e) the solution set $S$ of problem $EP(f, C)$ is nonempty.

We now formally state our algorithm described in section 1.

**Algorithm 1** Choose the sequence $\{\lambda_n\} \subset (0, \infty)$ and a $(n \times n)$ symmetric and positive definite matrix $A$.

**Step 0.** Take $x^1 \in C$. Set $n = 1$.

**Step 1.** Given $x^n$, compute $y^n$, $z^n$ and $x^{n+1}$ as

$$
y^n = \arg\min \{ \lambda_n f_1(x^n, t) + \frac{1}{2} \langle A(t - x^n), t - x^n \rangle : t \in C \},
$$

$$
x^{n+1} = \arg\min \{ \lambda_n f_2(y^n, t) + \frac{1}{2} \langle A(t - y^n), t - y^n \rangle : t \in C \}
$$

and

$$
z^n = \frac{\sum_{k=1}^{n} \lambda_k x^k}{\sum_{k=1}^{n} \lambda_k}.
$$

**Step 2.** Update $n := n + 1$ and go to **Step 1**.
Define $\|x\|_A = \sqrt{(Ax,x)}$. In order to investigate the convergence of algorithm, we first prove the following lemma.

**Lemma 3** Let $C$ be a nonempty closed convex subset of $\mathbb{R}^m$ and all conditions in Assumption 1 hold. Then the sequences $\{x^n\}, \{y^n\}$ generated by Algorithm 1 satisfy the following properties.

(a) There exist $M > 0$ such that

$$\|x^n - y^n\|_A \leq M \lambda_n \frac{1}{\alpha}, \|x^{n+1} - y^n\|_A \leq M \lambda_n \frac{1}{\alpha} \forall n \geq 1.$$  

(b) There exist $L > 0$ such that

$$\|x^{n+1} - x\|_A^2 \leq \|x^n - x\|_A^2 + 2\lambda_n f(x^n, x) + L \lambda_n \frac{2}{\alpha} \forall x \in C.$$  

**Proof** (a) Using the well-known necessary and sufficient condition for optimality of convex program [24] we see that $y^k$ solves the convex program

$$\text{argmin}\{\lambda_n f_1(x^n, t) + \frac{1}{2} \langle A(t - x^n), t - x^n \rangle : t \in C\}$$

if and only if

$$0 \in \partial_2 \left\{ \lambda_n f_1(x^n, t) + \frac{1}{2} \langle A(t - x^n), t - x^n \rangle \right\} + N_C(y^n),$$

where $N_C(y^n)$ is the (outward) normal cone of $C$ at $y^n \in K$. By the Moreau-Rockafellar theorem, there exist $w \in \partial f(x^n,.)(y^n)$ and $v \in N_C(y^n) := \{z \in \mathbb{R}^m : \langle z, x - y^n \rangle \leq 0 \forall x \in C\}$ such that

$$0 = \lambda_n w + A(y^n - x^n) + v.$$  

Hence

$$v = A(x^n - y^n) - \lambda_n w.$$  

By the definition of $N_C(y^n)$, we have

$$\langle A(x^n - y^n) - \lambda_n w, x - y^n \rangle \leq 0 \forall x \in C,$$

or equivalently,

$$\langle A(x^n - y^n), x - y^n \rangle \leq \lambda_n \langle w, x - y^n \rangle \forall x \in C.$$  

From $w \in \partial f(x^n,.)(y^n)$, we obtain

$$\lambda_n (f_1(x^n, x) - f_1(x^n, y^n)) \geq \lambda_n \langle w, x - y^n \rangle$$

$$\geq \langle A(x^n - y^n), x - y^n \rangle \forall x \in C \quad (4)$$

In (4), taking $x = x^n \in C$, we get

$$0 \leq \|x^n - y^n\|_A^2 \leq -\lambda_n f_1(x^n, y^n) = \lambda_n |f_1(x^n, y^n)|. \quad (5)$$

On the other hand, from the $\tau-$Hölder continuity of $f_1$, there exists $Q > 0$ such that

$$|f_1(x^n, y^n)| = |f_1(x^n, y^n) - f_1(x^n, x^n)| \leq Q \|x^n - y^n\|^2_A. \quad (6)$$
Combining (5) and (6), we obtain \( \|x^n - y^n\|^{2-\tau} \leq \lambda_n Q \) or equivalently, \( \|x^n - y^n\| \leq (Q \lambda_n)^{-\frac{1}{2-\tau}} \). Analogously, from

\[
x^{n+1} = \text{argmin} \{ \lambda_n f_2(y^n, t) + \frac{1}{2} \langle A(t - y^n), t - y^n \rangle : t \in C \}
\]

we obtain

\[
\lambda_n (f_2(y^n, x) - f_2(y^n, x^{n+1})) \geq \langle A(y^n - x^{n+1}), x - x^{n+1} \rangle \quad \forall x \in C.
\] (7)

In (7), taking \( x = y^n \in C \) and using the \( \tau \)-Hölder continuity of \( f_2 \) we have \( \|x^{n+1} - y^n\| \leq (Q \lambda_n)^{-\frac{1}{2}} \).

(b) From (7), for each \( x \in C \), we have

\[
\|x^{n+1} - x\|^2_A = \|x^{n+1} - y^n\|^2_A + \|y^n - x\|^2_A + 2\langle A(x^{n+1} - y^n), y^n - x \rangle
\]

\[
= \|y^n - x\|^2_A - \|x^{n+1} - y^n\|^2_A + 2\langle A(x^{n+1} - y^n), x^{n+1} - x \rangle
\]

\[
\leq \|y^n - x\|^2_A - \|x^{n+1} - y^n\|^2_A + 2\lambda_n (f_2(y^n, x) - f_2(y^n, x^{n+1})).
\] (8)

Analogously, from (4), we have

\[
\|y^n - x\|^2_A \leq \|x^n - x\|^2_A - \|y^n - x^n\|^2_A + 2\lambda_n (f_1(x^n, x) - f_1(x^n, y^n)).
\] (9)

Adding (8), (9) and using the \( \tau \)-Hölder continuity of \( f_1, f_2 \) we obtain

\[
\|x^{n+1} - x\|^2_A \leq \|x^n - x\|^2_A + 2\lambda_n (f_1(x^n, x) + f_2(y^n, x) - f_1(x^n, y^n) - f_2(y^n, x^{n+1}))
\]

\[
\leq \|x^n - x\|^2_A + 2\lambda_n (f(x^n, x) + f_2(y^n, x) - f_2(y^n, x^n) + \langle f_1(x^n, y^n), y^n - x^n \rangle)
\]

\[
\leq \|x^n - x\|^2_A + 2\lambda_n (f(x^n, x) + Q\|x^n - y^n\|^2_A + Q\|x^n - y^n\|^2_A + Q\|y^n - x^n\|^2_A)
\]

\[
\leq \|x^n - x\|^2_A + 2\lambda_n f(x^n, x) + L\lambda_n^{-\frac{1}{2-\tau}}
\] (10)

where \( L = 6Q^{-\frac{1}{2}} - \frac{2}{\tau} \).

Now, we state and prove the main convergence theorem for the proposed algorithm.

**Theorem 1** Let \( C \) be a nonempty closed convex subset of \( \mathbb{R}^m \) and all conditions in Assumption 1 hold. Suppose that the sequence \( \{\lambda_n\} \) satisfies \( \sum_{n=1}^\infty \lambda_n = \infty \), \( \sum_{n=1}^\infty \lambda_n^{-\frac{1}{2-\tau}} < \infty \). Then the sequence \( \{x^n\} \) generated by Algorithm 1 converges to a solution of EP(\( f, C \)).

**Proof** The proof of this theorem is divided into several steps.

**Step 1.** Claim that the sequences \( \{x^n\}, \{\xi^n\} \) are bounded.

**Proof of Step 1.** In (10), take \( x = x^n \in S \subseteq C \). Since \( f \) is pseudo-monotone, we have \( f(x^n, x^*) \leq 0 \ \forall n \geq 1 \). Hence, from (10) we obtain

\[
\|x^{n+1} - x^n\|^2_A \leq \|x^n - x^n\|^2_A + L\lambda_n^{-\frac{1}{2}}
\]

From Lemma 1, it follows that there exists the limit \( \lim_{n \to \infty} \|x^n - x^*\|^2_A \). Hence, \( \{x^n\} \) is bounded, i.e., there exists \( K > 0 \) such that \( \|x^n\| \leq K \ \forall n \geq 1 \). By the definition
of $z^n$, it follows that $\|z^n\| \leq \frac{\sum_{k=1}^{m} \lambda_k \|x^k\|}{\sum_{k=1}^{m} \lambda_k} \leq K$. So $\{z^n\}$ is bounded and there exists a subsequence $\{z^{n_i}\} \subset \{z^n\}$ such that $\lim_{i \to \infty} z^{n_i} = \bar{z} \in C$.

**Step 2.** Claim that $\bar{z} \in S$.

**Proof of Step 2.** Applying Lemma 1, we have

$$\|x^{n+1} - x\|^2_A - \|x^n - x\|^2_A \leq 2\lambda_n f(x^n, x) + L\lambda_n^2 \quad \forall x \in C.$$  

Hence, using the concavity of $f(., y)$, we have

$$\frac{\|x^{n_i+1} - x\|^2_A - \|x^{n_i} - x\|^2_A}{\sum_{k=1}^{m} \lambda_k} \leq 2 \frac{\sum_{k=1}^{m} \lambda_k f(x^n, x)}{\sum_{k=1}^{m} \lambda_k} + L \frac{\sum_{k=1}^{m} \lambda_k^{\frac{1}{2}}}{\sum_{k=1}^{m} \lambda_k}$$

$$\leq 2f\left(\frac{\sum_{k=1}^{m} \lambda_k x^n}{\sum_{k=1}^{m} \lambda_k}, x\right) + L \frac{\sum_{k=1}^{m} \lambda_k^{\frac{1}{2}}}{\sum_{k=1}^{m} \lambda_k}$$

$$= 2f(z^n, x) + L \frac{\sum_{k=1}^{m} \lambda_k^{\frac{1}{2}}}{\sum_{k=1}^{m} \lambda_k}.$$  

Taking the limit as $i \to \infty$, using assumptions $\sum_{k=1}^{m} \lambda_k = \infty$, $\sum_{k=1}^{m} \lambda_k^{\frac{1}{2}} < \infty$ we find that $f(\bar{z}, x) \geq 0 \quad \forall x \in C$ which ensures that $\bar{z} \in S$.

Since $S$ is closed, convex and nonempty set, hence for each $x^n$, there exists an unique point $u^n$ such that

$$u^n = \arg\min\{\|y - x^n\|_A : y \in S\}.$$  

To complete the proof of theorem, it is sufficent to prove that $u^n \to \bar{z}$.

**Step 3.** Claim that $u^n$ converges.

**Proof of Step 3.** Since $u^n \in S$, it follows that $f(x^n, u^n) \leq 0 \quad \forall k, n \geq 1$. From (10), we get

$$\|x^{n+p} - u^n\|^2_A \leq \|x^n - u^n\|^2_A + L \sum_{k=n}^{\infty} \lambda_k^{\frac{1}{2}}.$$  

(11)

Since $u^{n+k} = \arg\min\{\|y - x^{n+k}\|_A : y \in S\}$, we have

$$\|x^{n+p} - u^{n+p}\|^2_A \leq \|x^{n+p} - \frac{1}{2}(u^n + u^{n+p})\|^2_A.$$  

(12)

Combining (11) and (12) we have

$$\|u^{n+p} - u^n\|^2_A = \|(u^{n+p} - x^{n+p}) + (x^{n+p} - u^n)\|^2_A$$

$$= 2\|u^{n+p} - x^{n+p}\|^2_A + 2\|x^{n+p} - u^n\|^2_A - 4\|x^{n+p} - \frac{1}{2}(u^n + u^{n+p})\|^2_A$$

$$\leq 2\|x^{n+p} - u^n\|^2_A - 2\|u^{n+p} - x^{n+p}\|^2_A$$

$$\leq 2\|x^n - u^n\|^2_A - 2\|u^{n+p} - x^{n+p}\|^2_A + 2L \sum_{k=n}^{\infty} \lambda_k^{\frac{1}{2}}.$$  

(13)
It implies that
\[
\|u^{n+p} - x^{n+p}\|^2_A \leq \|x^n - u^n\|^2_A + L \sum_{k=n}^{\infty} \lambda_k^{2} \forall n, p \geq 1.
\]

Hence,
\[
\limsup_{n \to \infty} \|u^n - x^n\|^2_A \leq \|u^n - x^n\|^2_A + L \sum_{k=n}^{\infty} \lambda_k^{2} \forall n \geq 1.
\]

Since \( \lim_{n \to \infty} \sum_{k=n}^{\infty} \lambda_k^{2} = 0 \), we deduce that the limit \( \lim_{n \to \infty} \|x^n - u^n\|^2_A \) exists. Combining this and (13), it follows that \( \lim_{n \to \infty} \|u^{n+p} - u^n\|^2_A = 0 \) \( \forall p \geq 1 \) which means that \( \{u^n\} \) is a Cauchy sequence. Then, there exists the limit \( \lim_{n \to \infty} u^n = \bar{z} \).

**Step 4.** Claim that \( \bar{z} = \bar{z} \).

*Proof of Step 4.* Since \( S \) is convex, for each \( y \in S \) and \( \mu \in (0,1) \), we have \( y_n^\mu = \mu y + (1 - \mu) u^n \in S \). Hence
\[
\|x^n - u^n\|^2_A \leq \|x^n - y_n^\mu\|^2_A
\]
\[
= \|\mu y + (1 - \mu) u^n - x^n\|^2_A
\]
\[
= \|\mu (y - u^n) + (1 - \mu) (u^n - x^n)\|^2_A
\]
\[
= \mu^2 \|y - u^n\|^2_A + 2\mu \langle A(y - u^n), u^n - x^n \rangle + \|u^n - x^n\|^2_A.
\]

It implies that
\[
\mu \|y - u^n\|^2_A + 2 \langle A(y - u^n), u^n - x^n \rangle \geq 0.
\]

Let \( \mu \to 0^+ \), we have \( \langle A(y - u^n), u^n - x^n \rangle \geq 0 \) \( \forall y \in S \). Since \( \bar{z} \in S \), we have \( \langle A(\bar{z} - u^n), u^n - x^n \rangle \leq 0 \). We have
\[
\langle A(\bar{z} - \bar{z}), x^n - u^n \rangle = \langle A(\bar{z} - u^n), x^n - u^n \rangle + \langle A(u^n - \bar{z}), x^n - u^n \rangle
\]
\[
\leq \langle A(u^n - \bar{z}), x^n - u^n \rangle
\]
\[
\leq \|u^n - \bar{z}\| \cdot \|A(x^n - u^n)\|
\]
\[
\leq P \|u^n - \bar{z}\|,
\]

where \( P = \sup \{ \|A(x^n - u^n)\| : n \geq 1 \} < \infty \). On summing we find
\[
\langle A(\bar{z} - \bar{z}), \sum_{k=1}^{n} \lambda_k u_k - \sum_{k=1}^{n} \lambda_k u_k \rangle \leq P \sum_{k=1}^{n} \lambda_k \|u_k - \bar{z}\|.
\]

Hence
\[
\langle A(\bar{z} - \bar{z}), z^n - \sum_{k=1}^{n} \lambda_k u_k \rangle \leq P \sum_{k=1}^{n} \lambda_k \|u_k - \bar{z}\| / \sum_{k=1}^{n} \lambda_k.
\]

Since \( u^n \to \bar{z} \), \( \sum_{k=1}^{\infty} \lambda_k = \infty \), applying Lemma 1 we have \( \lim_{n \to \infty} \sum_{k=1}^{n} \lambda_k u_k / \sum_{k=1}^{n} \lambda_k = \bar{z} \) and
\[
\lim_{n \to \infty} \frac{\sum_{k=1}^{n} \lambda_k \|u_k - \bar{z}\| / \sum_{k=1}^{n} \lambda_k}{\sum_{k=1}^{n} \lambda_k} = 0. \text{ Thus, } \langle A(\bar{z} - \bar{z}), \bar{z} - \bar{z} \rangle \leq 0, \text{ which implies that } \bar{z} = \bar{z}. \quad \square
Remark 1 Since $\tau \in (0, 1]$, then $\frac{2}{2-\tau} \in (1, 2]$. There exists a positive sequence $\{\lambda_n\}$ satisfying

$$\sum_{n=1}^{\infty} \lambda_n = \infty, \sum_{n=1}^{\infty} \lambda_n^{\frac{2}{2-\tau}} < \infty,$$

for example $\lambda_n = \frac{1}{n^\alpha}$ with $\alpha \in (\frac{2}{2-\tau}, 1]$.

If $f_2 = 0$ in Theorem 1, we have the following corollary.

**Corollary 1** Let $C$ be a nonempty, closed, convex subset in $\mathbb{R}^m$, $f : C \times C \to \mathbb{R}$ be an equilibrium pseudo-monotone and $\tau$-Hölder continuous bifunction satisfying: for each $y \in C$, $f(\cdot, y)$ is concave, for each $x \in C$, $f(x, \cdot)$ is convex. Suppose that the solution set of problem $EP(f, C)$ is nonempty. Let $\{\lambda_n\}$ be a positive sequence satisfying $\sum_{n=1}^{\infty} \lambda_n = \infty, \sum_{n=1}^{\infty} \lambda_n^{\frac{2}{2-\tau}} < \infty$ and $A$ be a symmetric and positive definite matrix. Then the sequence $\{\tilde{x}^n\}$ generated by algorithm

$$\begin{align*}
\text{Choose } x^1 & \in C. \\
\text{Choose } \tilde{x}^n, \xi^n, \zeta^n, \tilde{x}^n, \tilde{z}^n & \text{ as} \\
\tilde{x}^{n+1} & = \arg\min_{t \in C} \{\tilde{\lambda}_n f(x^n, t) + \frac{1}{2}]A(t - x^n), t - x^n] : t \in C\}, \\
\tilde{z}^n & = \frac{\sum_{k=1}^{n} \lambda_k \xi^k}{\sum_{k=1}^{n} \lambda_k},
\end{align*}$$

converges to a solution of $EP(f, C)$.

4 Conclusion

A new iterative algorithm based on hybrid method is proposed in this paper to solve the equilibrium problems. In this algorithm, the objective function is decomposed into two functions, so that we can treat each part of the given function separately. This method may reduce the computationally cost in practical situations, when the objective function has a complex form. Using an ergodic procedure, we prove that the proposed algorithm converges under the pseudo-monotonicity and Hölder continuity of the functions.

References