A Sample-gradient-based Algorithm for a Multiple-OR and PACU Surgery Scheduling Problem

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ABSTRACT
In this paper, we study a multiple-OR surgery scheduling problem constrained by shared PACU capacity within the block-booking framework. With surgery sequences predetermined in each OR, a Discrete Event Dynamic System (DEDS) is devised for the problem by applying the First-Come-First-Served (FCFS) rule. A DEDS-based stochastic optimization model is formulated in order to minimize the cost incurred from patient waiting time, OR idle time, OR blocking time, OR overtime and PACU overtime. A sample-gradient-based algorithm is thus proposed for the sample average approximation of our formulation. Numerical experiments suggest that the proposed method identifies near-optimal solutions and outperforms previous methods.

Keywords: Surgery scheduling, PACU, stochastic optimization, sample average approximation, sample-based gradients

1 Introduction
Facing an aging population and increasing expenditures, health care providers are under growing pressure to improve efficiency. As a sector accounting for more than 40% of a hospital’s revenue and expenses, operating rooms (ORs) have recently been the focus of research (Gupta & Denton, 2008; Gupta, 2007; Erdogan & Denton, 2010; May et al., 2011). OR efficiency criteria, such as utilization, overtime and punctuality have been found to be "well below the achievable target" in most hospitals (Denton et al., 2007). As an important area to reduce cost and improve overall satisfaction of both patients and service providers, OR scheduling has
received particularly intensive research attention.

Block-booking, a widely implemented OR scheduling system, is the framework within which we conduct our study. In the block-booking system, ORs are reserved for each surgeon or specialty within a specific time period or block. For example, a surgeon might be assigned blocks on Tuesdays and Thursdays from 7am to 3pm. After consulting with patients, surgeons can book surgeries into their allotted block time until the block is 'full'. The scheduled start time for each surgery is subsequently determined prior to the day of surgery with the help of OR managers (Erdogan & Denton, 2010). Patients are informed of their scheduled start times and expected to arrive accordingly. On the day of surgery, time overruns of surgeries and temporary unavailability of resources may occur due to the random durations of surgical activities. In response, OR staff make sequential decisions throughout the day to reschedule surgeries based on the information about ongoing activities. Various performance criteria are evaluated in this decision making process, such as patient waiting time, surgeon idle time and OR overtime (Cardoen et al., 2010).

Although the random nature of surgical care has been considered in OR scheduling, few have yet integrated downstream constraints in a surgical suite into their studies. The Post-Anesthesia Care Unit (PACU), a shared downstream resource for patients to recover from surgery, can be a bottleneck to overall performance of a surgical suite (Iser et al., 2008). Unavailability of PACU beds can result in OR blocking, in which case a patient waits for a PACU bed in the OR after his/her surgery. OR blocking may result in suboptimal patient care, delay of surgeries, overtime work for staff and surgery cancellation, which can be costly to both service providers and patients (Gordon et al., 1988; Jonnalagadda et al., 2005).

In this paper, we study a multiple-OR scheduling problem constrained by shared PACU capacity within the block-booking framework. This multistage stochastic problem is formulated as a DEDS-based stochastic optimization model by applying the First-Come-First-Served (FCFS) rule. The objective is to minimize the expected cost of patient waiting time, surgeon idle time, OR blocking time, OR overtime and PACU overtime. A sample-gradient-based algorithm is proposed to solve the sample average approximation (SAA) of our formulation. Based on Monte Carlo simulation results, our proposed method is compared with stochastic approximation, the mean-value method and other solution strategies. A series of numerical experiments are also conducted to provide more insights into this OR scheduling problem.

The remainder of this paper is organized as follows: A brief review of relevant literature
is presented in the next section. In Section 3, we define the problem under study and present a DEDS model. In Section 4, the sample-gradient-based algorithm is proposed. Numerical studies are shown in Section 5. Section 6 concludes the paper with some general remarks.

2 Literature Review

Once surgeons have booked surgeries into blocks in the block-booking system, the OR manager needs to schedule particular start times for the given surgeries. In this section, we focus on literature that investigates two topics: surgery scheduling in ORs without PACU constraints and surgery scheduling in ORs with PACU constraints, assuming that no decision on patient admission is involved. More general reviews on the OR management issues can be found in (Cayirli & Veral, 2003; Gupta, 2007; Gupta & Denton, 2008; Erdogan & Denton, 2010; Cardoen et al., 2010; May et al., 2011).

2.1 PACU not Considered

Given that surgeries have been assigned to ORs and no resources, for example PACUs, are shared among ORs, surgery scheduling in multiple ORs is reduceable to scheduling every OR independently. Various methods have been studied to address surgery scheduling in a single OR. Robinson and Chen (2003) study the problem of scheduling a fixed sequence of surgeries in an OR with the objective to minimize surgeon idle cost and patient waiting cost. They solve the sample average approximation of their stochastic linear programming model and propose a closed-form heuristic to set up surgery start times. Zhang and Xie (2015) study a multiple-OR surgery scheduling problem with a given surgery sequence within the open-scheduling framework. The problem is formulated as a discrete-event simulation-based model by assigning surgeries using the First-Come-First-Served rule. The objective is to minimize the weighted cost of surgeon waiting, OR idling and OR overtime. They prove that the sample cost function is unimodal, Lipschitz continuous and almost surely differentiable and the expected cost function is continuously differentiable. Stochastic approximation (SA) is proposed to solve the problem and numerical experiments demonstrate that SA converges to a unique global minimizer.

Another important stream of research is based on stochastic programming. Denton and Gupta (2003) propose a two-stage stochastic linear programming model with simple recourse to find the appointment time for a given sequence of surgeries with random durations in an OR. To minimize the expected cost of patient waiting, OR idling and overtime, this problem is solved
by a variant of standard L-shaped method with sequentially partitioning the duration space. A similar case is studied by Begen and Queyranne (2011), but with overage and underage cost associated with scheduled start times in the objective function. Given that all the durations follow a known discrete integer-value joint distribution, the existence of an integer optimal schedule is guaranteed. It is further proved that, if cost parameters satisfy $\alpha$-monotonicity, the objective function is L-convex and can be optimized in polynomial time.

Denton et al. (2007), and Mancilla and Storer (2012) both examine the surgery sequencing and scheduling problem in a single OR to minimize the expected cost of patient waiting, OR idling and overtime. With random surgery durations, Denton et al. (2007) present a two-stage mixed-integer linear programming model as the sample-average approximation (SAA) to the original stochastic problem. Simple heuristics are proposed to sequence the surgeries, among which Sequencing-by-Variance gives the best performance in the numerical study. Mancilla and Storer (2012) formally prove that this SAA problem with two scenarios is NP-complete when idle costs are equal but waiting costs are different for each job. A Benders’ decomposition-based heuristic is proposed to address the problem, which outperforms the Sequencing-by-Variance heuristic in numeral experiments.

Other studies of interest can be found in (Lamiri et al., 2008; Batun et al., 2011; Begen et al., 2012). It is worth noting that none of the papers mentioned above consider PACU constraints. Realizing that PACU can be a bottleneck to patient flow within many surgical suits, some researchers directly address the PACU issue in their OR scheduling studies.

### 2.2 PACU Considered

PACU capacity impacts surgery scheduling decisions while surgery schedules will in turn influence the performance of the PACU. Some simulation studies examine the impact of different surgery sequencing rules on PACU staffing or utilization (Marcon & Dexter, 2006; Iser et al., 2008). Marcon and Dexter (2006) test 7 sequencing rules and their influence on ORs and PACU. Longest-Duration-First, a rule that is commonly used in practice, is shown to be one of the worst performing rules.

With the PACU in consideration, ORs are linked by this shared resource and scheduling multiple ORs is more difficult than scheduling multiple independent ORs. To overcome the computational challenges, many studies assume deterministic service times when the PACU is considered. Hsu et al. (2003) formulate a two-stage job shop model for sequencing and scheduling surgeries in multiple ORs. In their study, all service times are assumed deterministic.
and no OR blocking is allowed (patients are sent to the PACU immediately after surgery). The objective is to minimize the number of PACU nurses and PACU makespan. A tabu search heuristic framework is developed based on solving two subproblems iteratively by a greedy heuristic. Their algorithm is shown to be effective in the experiments with four to six ORs and 12 to 26 surgeries.

Pham and Klinkert (2008) borrow the idea of Job Shop planning to schedule patients with deterministic surgical durations into multiple ORs. Surgery-to-OR assignment and scheduled start time for each surgery are to be optimized. OR blocking is allowed in their research and recovery is not started until a patient enters the PACU. A mixed-integer linear optimization model is constructed to minimize the makespan and meanwhile allot the limited medical resources to competing jobs. As reported by the authors, only small to medium-sized instances can be solved by CPLEX within a reasonable time. Augusto et al. (2010) also consider the problem of scheduling a group of patients with deterministic surgical service times into multiple ORs. In their study, recovery starts in ORs if patients are blocked due to unavailability of PACU beds. A deterministic flowshop is modeled to represent flows of patient between wards, ORs and the PACU. A mixed-integer linear programming model is established to minimize the costs associated with patients’ completion times and a Lagrangian relaxation-based heuristic is employed to solve the problem by relaxing all resource-related constraints. Their heuristic is reported to solve problems with 10-30 surgeries within 10 minutes.

Other relevant papers can be found in (Price et al., 2011; Min & Yih, 2010b). To our knowledge, only two papers are closely related to our study in that they consider both randomness of medical service times and PACU constraints. Lee and Yih (2012) investigate the problem of scheduling a pool of surgeries with lognormally-distributed durations into multiple ORs with limited downstream PACU capacity. OR blocking is allowed in their research and recovery starts when the patient enters the PACU. The surgery-to-OR assignment and scheduled start times of the surgeries in each OR are found to minimize the total cost associated with patient waiting, OR blocking, OR idling and OR overtime. A Genetic Algorithm (GA) is proposed to solve the problem. Solutions after 500 GA generations are shown to outperform, in terms of objective value, six sequencing rules used in hospitals such as Longest-Duration-First rule.

Subsequent research by Lee and Yih (2014) determines scheduled start times for surgeries in multiple ORs constrained by limited PACU capacity when the sequence of surgeries in each OR is given. Surgical service times are assumed to be triangular fuzzy numbers (TFNs) and recovery
is not started until a patient enters the PACU. They formulate a multi-objective problem and solve it in two stages: minimizing OR blocking time and total completion time of all medical processes in the first stage and then patient waiting and OR idle time in the second stage. The first-stage problem is modeled as a flexible job shop (FJSP) with a satisfaction degree objective function, an approximation of which is solved by a Genetic Algorithm. Five heuristics are presented in the second stage to construct feasible schedules, among which the Newsvendor-based heuristic gives the best performance. The proposed algorithm can achieve shorter patient waiting time in simulation studies compared with the GA method in their previous paper (Lee & Yih, 2012).

In contrast to most of the works cited above, we study a multiple-OR and PACU surgery scheduling problem. Given the sequence of surgeries in each OR, random service duration distributions and limited PACU capacity are integrated into our models. Most of the papers presumably follow the First-Come-First Served (FCFS) rule in their simulation studies. We apply FCFS to formulate a Discrete-Event-Dynamic-System (DEDS) for the problem. The objective is to minimize the expected cost of patient waiting time, OR blocking, surgeon idle time, and OR and PACU overtime. With ideas borrowed from Perturbation Analysis, a sample-gradient-based algorithm is proposed to solve the sample average approximation to our stochastic problem.

3 Problem Statement and Model Development

3.1 Problem Description

We study the problem of determining scheduled start times (SST) for elective surgeries in multiple ORs with shared PACU resources after surgeons have booked surgeries into ORs. In our study, it is assumed that each surgeon is dedicated to one OR over the whole time horizon under study and each will submit a sequence of surgeries to the OR manager. Such sequences are determined by surgeons based on their experience and personal preference, and cannot be altered. It is assumed that all surgeries have to be scheduled within regular work hours. For example surgeries can only be scheduled between 7am and 3pm.

We assume that the first surgery in each OR is scheduled at the beginning of the day (time 0) as in other studies including (Robinson & Chen, 2003; Marcon & Dexter, 2006; Iser et al., 2008). If the first surgery is allowed to start later than time 0 (late-start) and the late start is not penalized, the overall cost can be improved by 0.34% on average at the expense of 16.4% increase in solution time in numerical experiments over 100 test problems. We also observe that
allowing late-start brings in as much as 15% cost reduction in some test problems. Therefore one may choose to implement a late-start policy based on the particular problem under study and our algorithm can solve the late-start problem with minor modifications. In this paper, we assume the first surgery in each OR is scheduled at time 0.

To accommodate the random nature of medical service times, surgical durations and length of stay (LOS) in the PACU of patients are assumed to follow some known random distributions that are independent from the choice of SST. Surgical durations and LOS in the PACU of different patients are mutually independent. We further assume that these random distributions are truncated, which ensures that surgical durations and LOS in the PACU will be both upper and lower bounded. Truncated random distributions have been adopted in a number of papers to model durations of surgical procedures, such as (Marcon & Dexter, 2006; Denton et al., 2010; Mancilla & Storer, 2012; Begen & Queyranne, 2011; Begen et al., 2012). These truncated random distributions could be constructed based on historical data and patients’ conditions. Turnover times (pre-operative and post-operative tasks) are not explicitly formulated, but instead contained in the surgical durations in our formulation.

Informed of the scheduled start time prior to the day of surgery, patients and surgeons are assumed to be punctual at the allotted time on the day of surgery, and thus surgeries cannot be started before their SST. After surgery, patients are transferred to the PACU if a PACU bed is available, otherwise they are blocked in the OR and wait for a spot in the PACU. Patients are released from the PACU after spending LOS in the PACU. It is assumed that both transport from OR to PACU and release from the PACU take no time.

First-Come-First-Served (FCFS) is an important rule enforced in our model, which have three significant implications:

1. Surgeries in each OR are performed in the same sequence as their SST.
2. Surgeons start working from the beginning of the time horizon (time 0) and every surgery is started as soon as the OR, surgeon and patient are ready.
3. A patient will be sent to the PACU instantly after surgery when there is a PACU bed available. If more than one patient is waiting for a PACU bed, the one with the earliest surgery finish time will get the next PACU bed. If two or more surgeries are finished at the same time and compete for a PACU bed, we arbitrarily admit the one from the smallest-indexed OR into the PACU first.
The first implication eliminates the case of re-sequencing surgeries in an OR. The second and third FCFS implications enforce the non-anticipative rule, which requires responses not to depend on inputs at a later time. The second FCFS implication also implies that the first surgery in every OR is started at time 0, since surgeons start working at time 0 and patients are punctual. Furthermore, since all surgical durations and LOS in the PACU are bounded and the FCFS rule is strictly applied, it is not hard to see that all the activities can be completed in a large enough time period and hence the time horizon under study is bounded.

FCFS may not be mathematically optimal, but it has been widely implemented in different fields including OR scheduling because of its fairness. FCFS is explicitly applied in prior literature including (Zhang & Xie, 2015), while others are most likely to implement FCFS in their simulation study, such as: (Pham & Klinkert, 2008; Lee & Yih, 2012, 2014; Augusto et al., 2010).

The surgery schedule is evaluated by the expected cost of patient waiting time, surgeon idle time, OR blocking time, OR overtime and PACU overtime. Patient waiting time is the time a patient has to wait between his/her scheduled start time (SST) and actual start time (AST). Surgeon idle time is defined as the time a surgeon is waiting for the start of the next surgery after finishing one. OR blocking time is the time a patient is held in an OR after surgery before being sent to the PACU. OR overtime is calculated for every OR and reflects the total amount of time surgeons have worked past regular work hours. PACU overtime is the time that PACU work has exceeded regular hours.

3.2 Continuous-time Model

In this section, the scheduling problem under study is formulated as an optimization model based on a Discrete Event Dynamic System (DEDS). We first define the DEDS model of the scheduling process with a given set of scheduled start times (SST) and a scenario of random surgical durations and LOS in the PACU. Then Perturbation Analysis will be conducted to estimate the gradient of the sample cost function at a given set of SST, which is later used in a sample-gradient-based algorithm to solve the problem.

3.2.1 Discrete Event Dynamic System

First, we would like to define the notations in our DEDS. There are in total \(orr\) ORs and the corresponding OR index set is \(J = \{0, 1, \ldots, orr - 1\}\). \(SR_j\) patients are scheduled in OR \(j \in J\) and their indices are \(K_j = \{0, 1, \ldots, SR_j - 1\}\). We use \(y_{jk}\) to indicate patient or surgery \(k\) in OR \(j\), \(j \in J, k \in K_j\) and set \(Y\) contains all patients or surgeries in ORs. Scenarios are
indexed by \( \omega \) and the index set of all scenarios is \( \Omega \). Surgery durations and LOS in the PACU of patient \( y_{jk} \) in scenario \( \omega \) are denoted as \( d_{\omega jk} \) and \( p_{\omega jk} \), respectively.

The time horizon we study is \( HT \), within which all activities can be finished; and regular work time is \( MT \). The total number of PACU beds is \( pcap \). Parameters \( C_{PW}, C_{I}, C_{B}, C_{O} \) and \( C_{PO} \) represent the unit cost of patient waiting time, surgeon idle time, OR blocking time, OR overtime and PACU overtime, respectively. For simplicity, we assume \( C_{PW} \) and \( C_{B} \) are identical for different patients, and \( C_{I} \) and \( C_{O} \) are identical for different surgeons and ORs. It is also assumed that ORs and the PACU share the same regular work hours. Our model can, however, be modified to accommodate distinct costs for patients, surgeons and ORs and different work hours for ORs and the PACU.

We denote the set of scheduled start times of all patients as \( SST, SST = [SST_{00}, \ldots, SST_{jk}, \ldots] \) and \( SST_{jk} \) is the scheduled start time of patient \( y_{jk} \in Y \). The actual surgery start time and PACU admission time of patient \( y_{jk} \) in scenario \( \omega \in \Omega \) are represented by \( AST_{\omega jk} \) and \( APT_{\omega jk} \), respectively. We also define \( PF_{\omega} \) to indicate the time when the last patient leaves the PACU in scenario \( \omega \) and \( PF_{\omega} = \max_{j,k}(APT_{\omega jk} + p_{\omega jk}). \)

Patient waiting time \( PW_{\omega jk} \) and OR blocking time \( B_{\omega jk} \) of patient \( y_{jk} \) in scenario \( \omega \) can be calculated as: \( PW_{\omega jk} = AST_{\omega jk} - SST_{jk} \) and \( B_{\omega jk} = APT_{\omega jk} - AST_{\omega jk} - d_{\omega jk} \), where \( y_{jk} \in Y \) and \( \omega \in \Omega \). Surgeon idle time before \( y_{jk} \) in scenario \( \omega \) is defined as \( I_{\omega jk} = AST_{\omega jk} - SST_{(k-1)} - d_{\omega jk} \). OR overtime is penalized if the last patient in the OR leaves for the PACU after regular hours, and hence overtime of OR \( j \) in scenario \( \omega \) is \( O_{\omega j} = \max(APT_{\omega j(k-1)} - MT, 0). \) PACU overtime is incurred in scenario \( \omega \) if the last patient leaves after regular hours and thus \( PO_{\omega} = \max(PF_{\omega} - MT, 0). \) Please note that dependence on \( SST \) and \( \omega \) is omitted for simplicity.

Given a set of \( SST \) and a scenario \( \omega \), the day of surgery can be formulated as a Discrete Event Dynamic System (DEDS). An event in this system is defined to be a patient’s arrival at the OR, a patient’s admission into the PACU or release from the PACU. A state in our DEDS is described by sets of patients in different conditions: patients whose predecessors have not entered the PACU, patients who have not entered the PACU but whose predecessors in the OR have been admitted into the PACU, and patients in the PACU. Also every patient is associated with a time stamp in each state to reflect his/her SST, surgery finish time or the time when he/she is released from the PACU.

The states can be updated following the patient flow in ORs and the PACU. The first surgery in each OR is started at the beginning of the day and other surgeries are started in
the given sequence throughout the day. A surgery is finished when the patient has spent the
length of time in the OR equal to the surgical duration in the given scenario. After surgery,
a patient is sent to the PACU if a PACU bed is available; otherwise the patient is blocked in
the OR and waits for a released PACU bed. If more than one patient is competing for a PACU
bed, they are admitted in the same sequence as their surgery finish times (smallest-index rule
to break the tie). When a patient is admitted into the PACU, the surgery following him/her
(if any) will be started as soon as possible: if the following patient has arrived, his/her surgery
will be started immediately; otherwise the surgery is started at the time he/she arrives, i.e. at
his/her scheduled start time. Every patient spends LOS in the PACU in the given scenario
before he/she is discharged. The DEDS is terminated when all patients are discharged from the
PACU. The mathematical formulation of this DEDS can be found in Appendix A.

Throughout the DEDS, all the patient-related times are determined and hence the sample
cost function can be expressed as:

$$C(SST,\omega) = CPW \sum_{j \in J} PW_{jk}^\omega + CI \sum_{j \in J} I_{jk}^\omega + CB \sum_{j \in J} B_{jk}^\omega + CO \sum_{j \in J} O_j^\omega + CPOP^\omega$$

Our objective is to minimize the expected cost over set $\Theta$,

$$\min_{SST \in \Theta} J(SST) = \min_{SST \in \Theta} E_\omega [C(SST,\omega)] \quad (3.2.1)$$

where $\Theta$ is the set of all feasible SST,

$$\Theta = \left\{ SST \in \mathbb{R}^N, N = \sum_{j \in J} SR_j \left| 0 = SST_{j0} \leq SST_{jk} \leq \cdots \leq SST_{j(SR_j-1)} \leq MT, \forall j \in J \right. \right\}$$

3.2.2 Differentiability of the Sample Cost Function

In this section, the behavior of sample cost function $C(SST,\omega)$ is studied using ideas from
Perturbation Analysis (PA) (Ho & Cao, 1991; Glasserman, 1991; Fu & Hu, 1997; Fu, 2015).
By observing the system behavior for the given SST, we use PA to analyze the change in the
DEDS after a small perturbation is added to SST. We would like to investigate whether the
perturbation to a patient’s scheduled start time will bring changes to his/her actual surgery
start time and PACU admission time. It is also important to study whether such a perturbation
will propagate to other patients in the system. The sample gradients can be derived based on
PA. This analysis is not applicable when our DEDS has different event sequences before and
after the perturbation, or an overtime cost is triggered after the perturbation. Thus, we need to
find out the cases of event sequence change and overtime being triggered, and show that their
occurrence has null probability for a given SST before analyzing the sample gradients using Perturbation Analysis (PA).

First, note that all performance measures can be determined in terms of event times and the given SST, and that the sample cost function $C(SST, \omega)$ is a linear function of patient waiting time, OR idle time, OR blocking time, OR overtime and PACU overtime. Therefore the sample cost function is a linear function of event times and SST. It is not difficult to see that every event time is determined by a series of plus operations, min and max functions of terms $SST_{jk}, d^e_{jk}$ and $p^e_{jk}$. Consequently, the sample cost function can be written as a piecewise linear function of $SST_{jk}, d^e_{jk}$ and $p^e_{jk}$.

Due to this piecewise linearity, a small perturbation on SST will change $C(SST, \omega)$ linearly in a small enough neighborhood of any $SST \in \Theta$ for almost any scenario $\omega$ unless a 'turning point', a point of nondifferentiability or discontinuity, is hit. Those 'turning points' are encountered when perturbing a given SST changes the results of the min or max functions involved in the DEDS in some scenarios. It is observed that non-differentiability and discontinuity occur in the case of event sequence change or overtime cost being incurred. Similar observations have been made by Zhang and Xie in scheduling patients into multiple ORs without PACU constraints (Zhang & Xie, 2015). The cases corresponding to 'turning points' are written mathematically in Appendix B, a descriptive summary of which is listed as follows:

- $\Omega_1$: If $y_{ab}$ is the patient with the earliest surgery completion time among all patients who have not entered the PACU and $y_{lm}$ is the patient with the earliest PACU discharge among patients in the fully-occupied PACU, then in this case $AST_{ab} + d_{ab} = APT_{lm} + p_{lm}$. A perturbation on SST may result in $AST_{ab} + d_{ab} < APT_{lm} + p_{lm}$ and $AST_{ab} + d_{ab} > APT_{lm} + p_{lm}$, which correspond to two different sequences of states. Patient $y_{ab}$ is blocked until $y_{lm}$ is released from the PACU in the former case, while $y_{ab}$ is directly moved into the PACU after surgery in the latter case, which results in unequal left and right derivatives. Since the sequence of remaining events is recovered after $y_{ab}$ gets into the PACU and $y_{lm}$ is released, the current condition will not result in a discontinuity in the sample function. So $\Omega_1$ corresponds to a nondifferentiable but continuous case for a given SST.

- $\Omega_2$: This represents the condition where a patient is blocked waiting for a PACU bed and two patients $y_{ab}$ and $y_{lm}$ finish their recovery at the same time, i.e., $APT_{ab} + p_{ab} = APT_{lm} + p_{lm}$. A perturbation on SST may reduce $APT_{ab} + p_{ab}$, which reduces the blocking time for the blocked patients. OR blocking time, however, remains the same when
$APT_{ab} + p_{ab}$ is increased by perturbing $SST$. In spite of unequal one-sided derivatives, sequences of remaining events are the same in the perturbed and nominal sample paths after both patients are admitted into the PACU. So $\Omega_2$ describes a nondifferentiable but continuous case for a given $SST$.

- $\Omega_3$: This condition describes a case where $SST$ of a patient is the same as his/her predecessor’s PACU admission time. Increasing the patient’s $SST$ will not change his/her waiting time while reducing the $SST$ will increase his/her waiting time. So a perturbation on $SST$ may lead to unequal one-sided derivatives. Since the sequence of events recovered after the patient has his/her surgery started, $\Omega_3$ outlines a continuous but nondifferentiable case.

- $\Omega_4$: This describes a case in which two surgeries $y_{ab}$ and $y_{lm}$ are finished at the same time and compete for a PACU spot, i.e., $APT_{ab} + p_{ab} = APT_{lm} + p_{lm}$. A perturbation on $SST$ may change the sequence of patients into the PACU. For example, patient $y_{lm}$ is blocked and $y_{ab}$ gets into the PACU in the nominal path while $y_{ab}$ is blocked and $y_{lm}$ is admitted into the PACU after perturbation. The change in PACU admission sequence could result in a different sequence of the remaining events, because the blocked patient cannot be admitted until the next PACU release event. In such a case, an infinitesimal perturbation on $SST$ triggers finite differences in patient-related times and the sample cost. Therefore $\Omega_4$ represents a discontinuous case.

- $\Omega_5$: In this condition, a patient is transferred into the PACU at the end of regular work hours. Increasing his/her $SST$ could trigger OR overtime cost while reducing the $SST$ does not incur an overtime penalty, which results in unequal left and right derivatives. $\Omega_5$ does not create discontinuity in the sample cost function, since the sequence of the events is not changed.

- $\Omega_6$: This is a case in which the last patient leaves the PACU at the end of regular work hours. Similar to $\Omega_5$, this is a continuous but nondifferentiable case.

Although we only summarize the conditions in which two patients finish their surgeries or recovery simultaneously in $\Omega_2$ and $\Omega_4$, one can easily generalize them to cases with more than two patients involved.

As shown, local nondifferentiability occurs in the sample path function $C(SST, \omega)$ at a given $SST$ when $\omega \in \bigcup_{i=1,2,3,5,6} \Omega_i$ and discontinuity occurs when $\omega \in \Omega_4$. In all cases discussed,
there are requirements on random surgery durations $d_{jk}$ and LOS in the PACU $p_{jk}$ of different patients. For example in $\Omega_1$, nondifferentiability occurs at a given $SST$ when $AST_{jk} + d_{jk} = APT_{lm} + p_{lm}$ with some extra conditions. Since it is assumed that surgery durations and LOS in the PACU of different patients follow mutually independent random distributions and they are independent from the choice of $SST$, $\Omega_1$ occurs with null probability for any given $SST$. Similar arguments can be made for other cases to show their null probability of occurrence. Thus $\tilde{\Omega} = \{ \Omega_1 \cup \Omega_2 \cup \Omega_3 \cup \Omega_4 \cup \Omega_5 \cup \Omega_6 \}$ happens with probability zero. In $\omega \in \Omega \backslash \tilde{\Omega}$, there always exists a small enough neighborhood of a given $SST$ where the sequence of events is unchanged and every surgery or recovery finishes before or after regular work hours. Consequently, $C(SST, \omega)$ is continuous and differentiable at any $SST \in \Theta$ with probability one (w.p.1).

**Proposition 1.** $C(SST, \omega)$ is a.s. differentiable at any $SST \in \Theta$.

In the second part of this section, perturbation analysis (PA) is used to determine the sample gradient at a given $SST$. It is known that:

$$ \nabla_{SST} C(SST, \omega) = \left( \frac{\partial C}{\partial SST_{00}}, \ldots, \frac{\partial C}{\partial SST_{jk}}, \ldots, \frac{\partial C}{\partial SST_{(orr-1)(SRT_{(orr-1)-1})}} \right) $$

where $\frac{\partial C}{\partial SST_{jk}}$ calculates the sample partial derivative in sample $\omega$.

We will only study the performance of the perturbed system with $(SST + \Delta SST_{jk})$ and derive right derivatives because of the almost-sure differentiability of $C(SST, \omega)$ at any $SST$. The choice of $\Delta SST_{jk}$ is small enough so that the sample cost function maintains local linearity.

Besides partial derivatives of $C(SST, \omega)$ w.r.t. $SST$, two rates of change are necessary to express the sample gradient.

- $\frac{\Delta C}{\Delta AST_{jk}}$ demonstrates how much the sample cost will be influenced, if the actual start time (AST) of a patient $y_{jk}$ is postponed without considering his/her SST change.
- $\frac{\Delta C}{\Delta APT_{jk}}$ shows the effect of delaying PACU admission time (APT) of $y_{jk}$ on the sample cost without considering the adjustments on his/her SST and AST.

To calculate $\frac{\partial C}{\partial SST_{jk}}$, one needs to check whether a perturbation on SST of patient $y_{jk}$ will delay his/her actual AST. If $y_{jk}$ is started later than the SST ($AST_{jk} > SST_{jk}$), a change on SST will not affect his/her AST but only reduces his/her patient waiting time. If $y_{jk}$ is started punctually as scheduled ($AST_{jk} = SST_{jk}$), a perturbation on SST will impact his/her AST by
the same amount. Therefore $\frac{\partial C}{\partial SST_{jk}}$ can be written mathematically as:

$$\frac{\partial C}{\partial SST_{jk}} = \begin{cases} 
-C_{PW} & \text{if } AST_{jk} > SST_{jk} \\
\frac{\Delta C}{\Delta AST_{jk}} - C_{PW} & \text{if } AST_{jk} = SST_{jk}
\end{cases}$$

$\frac{\Delta C}{\Delta AST_{jk}}$ demonstrates how much the sample cost is influenced, if AST of $y_{jk}$ is delayed without his/her SST being changed. In addition to an increase in the patient waiting time, two other conditions need to be investigated:

1. Whether his/her PACU admission time (APT) will be affected. If $y_{jk}$ is currently blocked ($APT_{jk} > AST_{jk} + d_{jk}$), a small change on AST will change the OR blocking time, but not his/her APT. If he/she is not blocked, OR blocking time stays zero but APT is changed after the perturbation.

2. If $y_{jk}$ is the last patient in his/her OR ($k = SR_j - 1$), AST adjustment will change his/her surgery finish time and hence alter the surgeon idle time.

So $\frac{\Delta C}{\Delta AST_{jk}}$ can be written as the following 'if' conditions.

$$\frac{\Delta C}{\Delta AST_{jk}} = \begin{cases} 
C_{PW} - C_B & \text{if } AST_{jk} + d_{jk} < APT_{jk}, \ k < SR_j - 1 \\
C_{PW} - C_B + C_I & \text{if } AST_{jk} + d_{jk} < APT_{jk}, \ k = SR_j - 1 \\
C_{PW} - C_B + \frac{\Delta C}{\Delta APT_{jk}} & \text{if } AST_{jk} + d_{jk} = APT_{jk}, \ k < SR_j - 1 \\
C_{PW} - C_B + C_I + \frac{\Delta C}{\Delta APT_{jk}} & \text{if } AST_{jk} + d_{jk} = APT_{jk}, \ k = SR_j - 1
\end{cases}$$

Similarly, $\frac{\Delta C}{\Delta APT_{jk}}$ shows the effect of delaying the PACU admission time (APT) of $y_{jk}$ without adjusting his/her SST and AST. OR blocking time will be increased no matter whether $y_{jk}$ is blocked or not. Additionally, other conditions need to be examined:

- Whether $y_{jk}$ is the last patient discharged from the PACU ($APT_{jk} + p_{jk} = PF$).
  (a) If he/she is the last to be discharged from the PACU, one needs to determine whether a delay on his/her APT will incur the PACU overtime penalty ($PF > MT$).
  (b) If not, he/she might have a close follower $y_{mn}$ in the PACU queue who cannot enter the PACU until $y_{jk}$ is discharged, ($\exists m \in J, n \in K_m, s.t. APT_{mn} = APT_{jk} + p_{jk}$). In such cases, delaying APT of $y_{jk}$ will postpone APT of $y_{mn}$ and thus the perturbation is propagated.

- Whether $y_{jk}$ is the last patient in his/her OR ($k = SR_j - 1$).
(c) If he/she is the last patient in his/her OR and his/her surgery is finished after regular hours, OR overtime is extended by increasing his/her APT \( (APT_{jk} \geq MT) \).

(d) If not, he/she may have a close follower \( y_{jk} \) in his/her OR whose surgery cannot be started until \( y_{jk} \) is transferred into the PACU, \( (AST_{j(k+1)} = APT_{jk}) \). The AST of \( y_{jk} \) is changed if APT of \( y_{jk} \) is altered.

Different conditions in calculating \( \frac{\Delta C}{\Delta APT_{jk}} \) are defined as follows:

(a1) \( APT_{jk} + p_{jk} = PF, PF \geq MT \)  \quad (a2) \( APT_{jk} + p_{jk} = PF, PF < MT \)

(c1) \( k = SR_j - 1, APT_{jk} \geq MT \)  \quad (c2) \( k = SR_j - 1, APT_{jk} < MT \)

(d1) \( k < SR_j - 1, AST_{j(k+1)} = APT_{jk} \)  \quad (d2) \( k < SR_j - 1, AST_{j(k+1)} > APT_{jk} \)

(b1) \( APT_{jk} + p_{jk} < PF, \exists m \in J, n \in K, s.t. APT_{mn} = APT_{jk} + p_{jk} \)

(b2) \( APT_{jk} + p_{jk} < PF, \not\exists m \in J, n \in K, s.t. APT_{mn} = APT_{jk} + p_{jk} \)

With all conditions included, \( \frac{\Delta C}{\Delta APT_{jk}} \) is written as follows:

\[
\frac{\Delta C}{\Delta APT_{jk}} = \begin{cases} 
C_O + C_{PO} + C_B & \text{if (a1) + (c1)} \\
C_{PO} + C_B & \text{if (a1) + (c2)} \\
C_{PO} + \frac{\Delta C}{\Delta AST_{j(k+1)}} + C_B & \text{if (a1) + (d1)} \\
C_{PO} + C_B & \text{if (a1) + (d2)} \\
C_B & \text{if (a2) + (c2)} \\
\frac{\Delta C}{\Delta AST_{j(k+1)}} + C_B & \text{if (a2) + (d1)} \\
C_B & \text{if (a2) + (d2)} \\
C_O + \frac{\Delta C}{\Delta APT_{mn}} + C_B & \text{if (b1) + (c1)} \\
\frac{\Delta C}{\Delta APT_{mn}} + C_B & \text{if (b1) + (c2)} \\
\frac{\Delta C}{\Delta APT_{mn}} + C_B & \text{if (b1) + (d1)} \\
\frac{\Delta C}{\Delta AST_{j(k+1)}} + C_B & \text{if (b1) + (d2)} \\
C_O + C_B & \text{if (b2) + (c1)} \\
C_B & \text{if (b2) + (c2)} \\
\frac{\Delta C}{\Delta AST_{j(k+1)}} + C_B & \text{if (b2) + (d1)} \\
C_B & \text{if (b2) + (d2)}
\end{cases}
\]

From the derivation of the sample gradients, the local linear performance of the sample cost
function is further confirmed. Also, the sample gradients are consistent with our earlier findings that derivative changes when nondifferentiable and discontinuous cases occur with $\omega \in \tilde{\Omega}$ for a given SST.

4 SAA-Gradient Descent Algorithm

4.1 Sample Average Approximation

As two of the most widely used stochastic optimization methods, stochastic approximation (SA) and sample average approximation (SAA) are within our consideration. The SA method can be viewed as a stochastic version of the steepest descent algorithm (Chau et al., 2014). An estimator of the gradient of the expected cost function is derived based on a new set of scenarios generated in every iteration and used to find the descent direction (van Ryzin & Vulcano, 2008). It has been extensively studied and applied in various operations research fields. Two recent examples are van Ryzin and Vulcano’s study (2008) in network revenue management and Zhang and Xie’s research (2015) in surgery appointment scheduling.

SAA converts a stochastic problem into a deterministic counterpart by taking a finite number of samples. It has been shown to be most effective when the expected cost function is continuous (Kim et al., 2015). The theory of SAA is comprehensively covered in (Kleywegt et al., 2002; Ahmed et al., 2002) and it has been widely used in solving OR scheduling problems, including two articles we have discussed (Denton et al., 2007; Mancilla & Storer, 2012).

Typically with sample gradient information obtainable, one would show in SA that sample gradients are unbiased estimators of the gradients of $J(SST)$. Due to the discontinuity of the sample cost function $C(SST, \omega)$, however, it is not possible for us to prove the result. Kim et al. (2015), Fu (1997) and Ho (1991) all give sets of sufficient conditions under which the gradient of a discontinuous sample function is an unbiased estimator, but none of them can be effectively applied in our problem. All of their conditions need information about the probability of discontinuity occurring in $[SST, SST + \Delta]$, which would require us to write a complex convolution of multivariate distributions.

In contrast, sample gradients can be directly used in SAA to solve the deterministic problem. We are able to prove the continuity of our expected cost function in Proposition 2 in Appendix C, in which condition SAA is effective (Kim et al., 2015). In addition, SAA is shown to outperform the SA method in our numerical experiments in Section 5.2. Therefore, SAA is adopted to fully exploit the efficiency of perturbation analysis and the continuity of $J(SST)$. 
In SAA, \( n \) independent samples \( \omega_1, \omega_2, \ldots, \omega_n \) are generated from the distribution of \( \omega \) and let

\[
J_n(SST) = \frac{1}{n} \sum_{i=1}^{n} C(SST, \omega_i)
\]

Then deterministic optimization algorithms can be applied to solve the SAA formulation of the stochastic scheduling problem:

\[
\min_{SST \in \Theta} J_n(SST)
\]  

(4.1.1)

We are able to show the consistency of the SAA estimators of the optimal value and the optimal solutions in Appendix C. This ensures that, when the number of scenarios is sufficiently large, the minimizer or a solution with a near-infimum value of the SAA problem will converge to a minimizer of the stochastic problem w.p.1. In other words, if solutions with near-infimum value could be found in the SAA problem with a large enough number of scenarios, they are a.s. near-optimal solutions to the stochastic problem. In the next section, an SAA-gradient descent algorithm with random restarts (SAA-GDR) is proposed to find solutions with near-infimum value to our SAA formulation.

4.2 SAA-Gradient Descent Algorithm with Random Restarts (SAA-GDR)

In SAA-GDR (Algorithm 1), a backtracking line search scheme (Nocedal & Wright, 2006) is implemented. Sample gradients are calculated by PA and used to determine the improving direction in each iteration. Step size is dynamically adapted based on initial step size \( \rho \) and step size updating factor \( \alpha \). A sufficient decrease percentage requirement is enforced in the line search, i.e., a step is taken only when the improvement in the objective value is no less than a threshold percentage, \( c \). This rule is shown in the computational experiments to be as good as the Armijo rule (Nocedal & Wright, 2006) in the quality of solutions but more efficient in terms of running time. The search is randomly restarted when a potential local minimum is identified to explore more broadly in the feasible region. The number of scenarios \( N \) and the number of random restarts \( K \) need to be specified in the algorithm, selection of which will be discussed in Section 5.1.

A move in the steepest descent direction may result in an infeasible solution, in which case \( \text{Proj}(SST) \) will project the infeasible SST into the feasible region \( \Theta \). With all the patients examined in the predetermined sequence in each OR, \( \text{Proj}(SST) \) will identify and adjust the scheduled start times that violate the predetermined sequence or exceed the regular work hours in Function \( \text{Proj} \).
\begin{align*}
&\text{for } j = 0 \text{ to } n - 1 \text{ do} \\
&\quad \text{for } k = 1 \text{ to } S_{Rj} - 1 \text{ do} \\
&\quad \quad \text{if } SST_{jk} < SST_{j(k-1)} \text{ then} \\
&\quad \quad \quad SST_{jk} = SST_{j(k-1)} \\
&\quad \quad \text{else if } SST_{jk} > MT \text{ then} \\
&\quad \quad \quad SST_{jk} = MT \\
&\quad \quad \text{else} \\
&\quad \quad \quad SST_{jk} = SST_{jk} \\
&\text{end} \\
&\text{end}
\end{align*}

\textbf{Function} Proj(SST)

It is not possible for us to theoretically prove that the SAA-GDR algorithm identifies optimal solutions in the stochastic problem, considering the discontinuity and nondifferentiability of the sample cost function. Instead, extensive computational results are presented in Section 5.1 to demonstrate the performance of the SAA-GDR algorithm and verify that it converges to a set of near-optimal points regardless of where the search starts.

5 \ Numerical Results

The DEDS and SAA-GDR algorithm are implemented in C++, compiled by gcc version 4.7.2 and tested on a compute node with dual eight-core 800MHz AMD Opteron™ processors 6128 with 32GB memory and 512KB cache per core. In our experiments, patients’ surgery durations and LOS in the PACU are modeled as independent truncated lognormal distributions. The lognormal distribution has been shown to be effective in surgical process modeling and used in papers including (Marcon & Dexter, 2006; Marcon et al., 2003; Min & Yih, 2010a; Zhang & Xie, 2015; Mancilla & Storer, 2012). Truncated random distributions, as discussed in Section 3.1, are also widely adopted in modeling medical procedures.

Test data in Lee and Yih(2014) is adopted in our experiments, which includes 15 ORs and two to six patients in a given sequence in each OR. Their data is originally given in the form of triangular fuzzy numbers and we convert them to truncated lognormal distributions in an approximation method described in their paper. Each patient has unique surgical duration and LOS in the PACU are modeled as independent truncated lognormal distributions. After conversion, the mean surgical durations range from 0.5 hours to over 5 hours and the average LOS in the PACU from 0.5 to 2.5 hours. In our experiments, the number of ORs are randomly generated and ORs are arbitrarily drawn from the pool of 15 ORs. Scenarios of surgical durations and LOS in the PACU are generated for patients using the corresponding means and standard deviations in these selected ORs. The
Data: Scenarios $\omega_1, \ldots, \omega_N$, percentage threshold $c$, initial step size $\rho$ and step size updating factor $\alpha$, step size threshold $\gamma_s$

Result: Best solution $\text{SST}^*$ and objective value $J_N^*$

\begin{verbatim}
begin
    $J_N^* \leftarrow \infty$; $\text{SST}^* \leftarrow 0$
    for $n=1$ to $K$ do
        $\gamma_0 = \rho$;
        Randomly choose $\text{SST}_0 \in \Theta$. Run DEDS with all $N$ scenarios and obtain
        
        $J_N(\text{SST}_0) = \frac{1}{N} \sum_{i=1}^{N} C(\text{SST}_0, \omega_i)$

        for $m=1$ to $M$ do
            Calculate the sample gradient at the current SST by Perturbation Analysis:
            
            $\nabla J_N(\text{SST}_{m-1}) = \frac{1}{N} \sum_{i=1}^{N} \nabla C(\text{SST}_{m-1}, \omega_i)$

            Update the SST by moving along the steepest descent direction.
            
            $\text{SST}_m = \text{Proj}(\text{SST}_{m-1} - \gamma_m \frac{\nabla J_N(\text{SST}_{m-1})}{\|\nabla J_N(\text{SST}_{m-1})\|})$

            where $\text{Proj}(\text{SST})$ projects SST into the feasible region $\Theta$;

            Run DEDS and obtain $J_N(\text{SST}_m)$;

            Enforce the sufficient decrease percentage requirement;
            
            if $J_N(\text{SST}_m) < J_N(\text{SST}_{m-1}) \ast (1 - c)$ then
                $\gamma_m = \alpha \gamma_{m-1}$;
            else
                $J_N(\text{SST}_m) = J_N(\text{SST}_{m-1})$, $\text{SST}_m = \text{SST}_{m-1}$, $\gamma_m = \frac{\gamma_{m-1}}{\alpha}$;
            end

            if $\gamma_m < \gamma_s$ then
                break;
            end
        end
        if $J_N(\text{SST}_M) < J_N^*$ then
            $J_N^* = J_N(\text{SST}_M)$ and $\text{SST}^* = \text{SST}_M$
        end
    end
end
\end{verbatim}

Algorithm 1: SAA-gradient descent algorithm with random restarts (SAA-GDR)

number of PACU beds is determined so that the ratios of PACU beds to ORs range from 0.6 to 0.75, similar to the the recommended ratio of 0.7 in (Lee & Yih, 2014). We conduct experiments with a wide range of cost parameters given the fact that hospitals vary in costs.

We also assume that ORs and the PACU have the same 8-hour regular work time ($MT$), though our method could be easily modified to accommodate different length of regular work time for ORs and the PACU as described in Section 3.2.1. Algorithmic parameters are selected in our pilot tests and used in all our numerical experiments: percentage threshold $c = 10^{-5}$,
initial step size $\rho = 1$ and step size updating factor $\alpha = 2$.

### 5.1 SAA-GDR Algorithm

Our first step is to conduct a set of experiments in a similar way as Linderoth et al. (2006), which demonstrates the quality of solutions obtained in our SAA-GDR algorithm. The key idea is to derive an estimate of the optimality gap based on (Mak et al., 1999):

$$E[J^*_N] \leq J^* \leq J(SST)$$

where $J^*_N$ is the infimum of the SAA problem with the number of scenario $N$, $J(SST)$ is the objective value of a feasible solution $SST$ and $J^*$ is the optimal value in the original stochastic problem. In Linderoth et al.’s (2006) experiment, the optimality gap $J(SST) - J^*$ is estimated by an over-estimator $J(SST) - E[J^*_N]$, the bias of which can be reduced by increasing the number of scenarios $N$. Therefore different numbers of scenarios are tested in the experiments. $J(SST)$ is estimated by the Monte-Carlo sampling method with batches of sufficiently large samples. An estimate of $E[J^*_N]$ can be derived statistically if we can estimate the infimum $J^*_N$ of the SAA problems. If $J_{NK}$ is the best objective value obtained with $K$ random restarts, the gap between $J^*_N$ and $J_{NK}$ can be narrowed by increasing $K$. Therefore, we will estimate $J^*_N$ using $J_{NK}$ with a sufficiently large number of random restarts $K$ and further use $E[J_{NK}]$ as an estimate of $E[J^*_N]$. The details of this experiment are given in Appendix D.

Random test instances with $N$ scenarios are generated and solved using SAA-GDR with $K$ random restarts. Estimates of the upper bound $J(SST)$ and the lower bound $E[J^*_N]$ are derived and part of the results are presented in Tables 1 to 3 with corresponding setup in Appendix E. In all test problems, the relative gap is very small with $N = 5000$ and $K = 5000$. When the number of scenarios $N$ is increased and the number of random restarts $K$ is unchanged, we observe an increase in the lower bound estimates, which is consistent with our expectation that $J^* - E[J^*_N]$ vanishes as $N$ increases. When $K$ is increased and $N$ is unchanged, a decrease in the lower bound estimates is observed, as we expect that $J_{NK}$ decreases to $J^*_N$ as $K$ increases. Additionally, we have a similar observation as noted by Linderoth et al. (2006) that the upper bound estimate is almost unaffected by the choice of $N$ and $K$, suggesting that the solutions obtained with $N = 50$ and $K = 50$ are of similar quality to those with $N = 5000$ and $K = 5000$. This property is particularly helpful when solutions are needed in a limited time frame.

Secondly, we study the convergence of the approximate solutions in the same way as Linderoth et al. did. The results are shown in Appendix F. Similar to Linderoth et al.’s conjecture,
it is likely that in our problem, there is a "feasible neighborhood of the solution set with which the objective value is not much different from the optimal value" \( J^* \) (Linderoth et al., 2006).

5.2 Comparisons with Other Algorithms

We would like to compare our SAA-GDR algorithm with four current algorithms in terms of running time and solution quality.

The first method is a time-index sample-average-approximation integer formulation (TIndex-SAA) for the problem under study. By defining the length of a basic time interval, the time horizon can be discretized into a finite number of time buckets. If surgery durations and LOS in the PACU are rounded to integer multiples of the time bucket, this problem can be represented as a two-stage time-index model, which can be approximated by TIndex-SAA. TIndex-SAA is difficult to solve and it takes CPLEX more than \( 10^5 \) seconds on average to solve a 5-scenario TIndex-SAA with 5-minute time bucket, 5 ORs of in total 19 patients and 3 PACU beds. Various algorithms for solving two-stage stochastic binary problems have also been extensively tested by us on the TIndex-SAA model, including Lagrangian relaxation, L-shaped method and their variants. None of them are able to handle the problem efficiently. In this comparison test, we choose to solve TIndex-SAA with 3 scenarios and 10-minute time buckets, which can be solved by CPLEX within 1 hour in most test instances.

The second algorithm included is the mean-value method (MV), which assumes that all random variables take their mean values. The MV method in our comparison test is a single-scenario TIndex-SAA formulation that takes mean values of surgery durations and LOS in the PACU with 10-minute time buckets.

The third method is scheduling the surgeries without considering PACU constraints (NoPACU). This method can be formulated as a stochastic linear programming problem and approximated by an SAA formulation. The model we use is similar to the one in (Robinson & Chen, 2003) but with overtime cost of ORs and PACU included. The number of scenarios can be selected using Linderoth et al. (2006)’s experiment. We choose 1000 scenarios based on our judgment in this comparison test.

The fourth algorithm used for comparison is stochastic approximation (SA) with random restarts. We understand that choices of parameters can impact the performance of SA, but tuning parameters for SA is beyond the scope of this paper. Instead, we implement SA similar to the one in Zhang and Xie (2015). 10 scenarios are generated in each iteration and the total number of iterations is \( K = 10^5 \). Step sizes \( \rho_k \) are updated by \( \rho_k = a/k \) where \( k \) is the current
iteration number. Different values of $a = 0.1, 0.5, 1, 5$, used by Zhang and Xie (2015), have been tested in our problem and solutions of the best quality are obtained with $a = 5$, which is thus adopted in our comparisons. Random restarts are included in this method for the sake of a fair comparison. A point is randomly picked in the feasible region $\Theta$ to start a new round of SA after the iteration number achieves $K = 10^5$. After 50 random restarts, the solution with the best objective value among 50 is used for comparison.

This comparison test is conducted on random test problems. Service times and mean durations are rounded to integer multiples of 10 minutes in TIndex-SAA and the MV method, respectively. NoPACU uses 1000 scenarios to find a schedule without PACU constraints. We choose $N = 1000$ and $K = 50$ in SAA-GDR based on our judgment.

The performance of different methods are compared in terms of solution time and solution quality. In random test problems, each algorithm is run 20 times on independent batches of scenarios and hence 20 sets of scheduled start times are obtained by each algorithm. Scheduled start times are evaluated by their average performance in DEDS simulations with 50 independent batches of 20000 scenarios. The mean objective value of 20 solutions, $J$ are used as the criteria to compare solution quality of different algorithms. Solution quality comparison is shown in Figure 1 and solution times are presented in Table 6 (details in Table 9 in Appendix G) with corresponding setup in Appendix E.

In our comparison tests, SAA-GDR outperforms other methods on average solution quality in problems of different sizes and various cost parameters. Meanwhile, solution time is within 50 seconds for a medium-sized problem with 6 ORs of in total 21 patients and 4 PACU beds.

5.3 Impact of PACU Constraints

We would like to show the impact of PACU constraints on the system relative to schedules that ignore PACU constraints. We obtain 20 schedules from SAA-GDR and 20 schedules from the NoPACU method in each of Test Problem 4 to 12 (setup in Appendix E) and evaluate their performance in DEDS simulations with 50 independent batches of 20000 scenarios. We calculate, for each schedule, the total OR blocking time, total patient waiting time, total surgeon idle time, total OR overtime in all ORs and total PACU overtime. We then report the mean differences between SAA-GDR and NoPACU solutions.

It is observed that after considering the PACU, total OR blocking times and total patient waiting times are decreased while total surgeon idle time, total OR overtime and total PACU overtime are increased in all test problems. Patients’ scheduled start times in SAA-GDR so-
olutions tend to have more idle time added between surgeries. A possible explanation for these observations is: PACU constraints will trigger OR blocking events that result in delays of surgeries. To reduce the chance of blocking, patients’ scheduled start times in SAA-GDR solutions tend to be more spread out, which results in an increased SST span after including the PACU constraints. Consequently, OR blocking time is decreased and surgeries are more likely to be started punctually, i.e., patient waiting time is reduced. Since more idle time added between scheduled start times after considering the PACU, surgeon idle time, OR and PACU overtime may be increased as well.

Scheduling with PACU constraints have two-fold benefits. First, it can reduce the overall cost as shown in Figure 1. Second, schedules considering the PACU maintain a better balance between patients and service providers. From the patients’ perspective, waiting times are significantly reduced and blocking after surgeries are also decreased. These changes are good for patients, since excessive patient waiting time results in patient dissatisfaction (Cayirli & Veral, 2003) and OR blocking is critical to patients’ health (Lee & Yih, 2014). From the service providers’ perspective, schedules considering the PACU can be beneficial as well. Though OR and PACU overtimes and surgeon idle times are increased, the amounts of increase are much smaller compared with the significant reductions in patient waiting times and OR blocking times. Moreover, according to (Hall, 2008), improved patient satisfaction can lead to better work efficiency and increased revenue for service providers. Therefore, surgery scheduling that considers PACU capacity can also help the service provider financially in the long term.

6 Conclusions

This paper addresses a surgery scheduling problem with multiple ORs and shared PACU capacity. The objective is to minimize the expected cost of patient waiting time, surgeon idle time, OR blocking time, OR overtime and PACU overtime. With the surgery sequence predetermined in each OR, this problem is formulated as a stochastic optimization model based on a Discrete Event Dynamic System (DEDS) with the First-Come-First-Served (FCFS) rule. With sample gradients derived by Perturbation Analysis, an SAA-gradient descent algorithm with random restarts (SAA-GDR) is proposed to solve the SAA of our stochastic optimization model. Numerical experiments are conducted to select sample size and the number of random restarts. It is demonstrated that SAA-GDR with 1000 scenarios and 50 random restarts could identify a near-optimal schedule for more than 20 patients within 1 minute. SAA-GDR is shown to outperform other methods, including the time-indexed model and stochastic approximation.
method, in terms of solution quality and running time. Lastly, we present the change in the schedule after including PACU constraints and discuss their benefits.

Although the First-Come-First-Served (FCFS) rule is applied in our study, problems with other priority-based rules, such as rules based on patients’ physical conditions, could also be solved by the SAA-GDR with modifications in the sample-gradient derivation. Also the proposed algorithm can solve a broader range of optimization problems in different fields of study. It can be directly applied to optimizing two-stage scheduling processes with downstream capacity constraints. With modifications, it can also address multi-stage planning and scheduling problems with capacity constraints. Typical examples include scheduling processes in health care and production systems. Our algorithm can be used in appointment scheduling processes in hospitals with upstream or/and downstream constraints such as preparatory rooms, Step-down Units, Intensive Care Units and wards. Production planning with multi-stage processing, such as flow shop scheduling, could also be solved by the proposed algorithm.
References


Fu, M. (2015). Stochastic gradient estimation. In M. C. Fu (Ed.), *Handbook of simulation*


Figure 1: Comparisons on average solution quality in Test Problem 4 to 12

<table>
<thead>
<tr>
<th>Sample size</th>
<th>Random restarts</th>
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<th>Upper bound estimates</th>
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<td>95% confidence interval</td>
<td>95% confidence interval</td>
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<td>10.107 ± 0.006</td>
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<tr>
<td>1000</td>
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<td>10.082 ± 0.050</td>
<td>10.112 ± 0.006</td>
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<td>10.106 ± 0.005</td>
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Table 1: Lower and Upper bound estimates for $J^*$ in Test Problem 1
<table>
<thead>
<tr>
<th>Sample size</th>
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<th>Lower bound estimates</th>
<th>Upper bound estimates</th>
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<tr>
<td></td>
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<td>95% confidence interval</td>
<td>95% confidence interval</td>
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Table 2: Lower and Upper bound estimates for $J^*$ in Test Problem 2

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<th>Maximal Solution Time (sec)</th>
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<td>Tindex</td>
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<td>MV</td>
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<td>NoPACU</td>
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<td>SA</td>
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<td>SAA-GDR</td>
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Table 4: Solution times in Test Problem 4 to 12
Appendices

A Discrete Event Dynamic System

Given a set of $SST$ and a scenario $\omega$, the day of surgery can be formulated as a Discrete Event Dynamic System (DEDS). An event in this system is defined to be a patient’s admission into the PACU or release from the PACU. $S$ denotes the state space of our system, $s_n$ is the $n^{th}$ state that our DEDS visits and $\tau_n$ is the time of the $n^{th}$ state transition. A state in our DEDS is described by sets of patients in different conditions and time stamps associated with every patient. More specifically, $s_n = (O_{jn}, \forall j \in J, U_n, R_n, c_n)$. $O_{jn}$ is the set of patients in OR $j \in J$ whose predecessors have not entered the PACU in the $n^{th}$ state. $U_n$ is the set of patients in all ORs who have not entered the PACU but whose predecessors in the OR have been admitted into the PACU in the $n^{th}$ state. $R_n$ is the set of patients in the PACU in the $n^{th}$ state. $c_n(y_{jk})$ is the time stamp of patient $y_{jk}$ in the $n^{th}$ state. The time stamp of a patient in $O_{jn}$ reflects his/her SST; the time stamp of a patient in $U_n$ tells his/her surgery finish time; and the time stamp of a patient in $R_n$ is the time when he/she is released from the PACU.

The initial state $s_0$ is defined according to the FCFS rule. The first surgery in each OR is scheduled and started at time 0, that is $SST_j0 = AST_j0 = 0, \forall j \in J$. Therefore at time $\tau_0 = 0$, the system visits state $s_0$, where $s_0 = (O_{j0}, \forall j \in J, U_0, R_0, c_0)$.

\[
O_{j0} = \{y_{jk}, \forall k \in K_j: k > 0\} \\
U_0 = \{y_{j0}, \forall j \in J\} \\
c_0(y_{jk}) = \begin{cases} SST_{jk}, & \forall y_{jk} \in O_{jn} \\ d_{jk}, & \forall y_{jk} \in U_n \end{cases}
\] (A.0.1)

Functions $\Phi$ and $\Gamma$ update states and transition times of this system by $s_{n+1} = \Phi(s_n)$ and $\tau_{n+1} = \Gamma(\tau_n, s_n)$. To write functions $\Phi$ and $\Gamma$, two functions need to be defined first:

\[
t_1(s_n) = \min \{c_n(y_{jk}) : y_{jk} \in U_n\} \quad t_2(s_n) = \min \{c_n(y_{jk}) : y_{jk} \in R_n\}
\]

where min() breaks tie by choosing the patient from the smallest-indexed OR, as described in the FCFS rule. $t_1(s_n)$ returns the earliest time when a patient in any OR has his/her surgery finished. Such a patient may have his/her surgery finished just at $t_1(s_n)$ or may have been blocked for a certain time. $t_2(s_n)$ returns the earliest time when a patient in the PACU is ready for release.

By comparing $t_1(s_n)$ and $t_2(s_n)$, the next event and the next state of the system can be
determined. Therefore the state-updating function $\Phi$ and transition-time-updating function $\Gamma$ can be written in the following conditions:

1. When $t_2(s_n) \leq t_1(s_n)$, the next event is a patient’s departure from the PACU. This patient is eliminated from the set of patients in the PACU $R_n$ and the time of the next event is the same as the time stamp of him/her. The same results also work in the case when $t_2(s_n) > t_1(s_n)$, but no PACU admission is allowed because of a fully-occupied PACU.

$$\text{if } t_2(s_n) \leq t_1(s_n) \text{ or } \{ t_2(s_n) > t_1(s_n) \text{ and } |R_n| = \text{pcap} \}$$

$$\tau_{n+1} = t_2(s_n) = \Gamma(\tau_n, s_n) \quad R_{n+1} = R_n \setminus \{ r_{\text{rq}} \}$$

$$y_{\text{rq}} = \arg\min_{y_{\text{jk}}} \{ c_n(y_{\text{jk}}) : y_{\text{jk}} \in R_n \}$$

$$O_{j(n+1)} = O_{jn}, \forall j \in J$$

$$U_{n+1} = U_n$$

$$s_{n+1} = \Phi(s_n) = (O_{j(n+1)}, \forall j \in J, U_{n+1}, R_{n+1}, c_{n+1}) \quad (A.0.2)$$

where $|R_n|$ is the cardinality of set $R_n$. If $O_{j(n+1)} = U_{n+1} = R_{n+1} = \emptyset, \forall j \in J$, then the system is terminated, and $PF = \tau_{n+1}$.

2. When $t_2(s_n) > t_1(s_n)$ and $|R_n| < \text{pcap}$, the next event is transferring a patient into the PACU. The newly-admitted patient will be removed from set $U_n$ and added to set $R_n$. If this patient has a follower in his/her OR, i.e. $|O_{rn}| > 0$, this follower will be moved from $O_{rn}$ into $U_n$. One should note that the time stamp of a patient in set $U_n$ can be smaller than $\tau_n$ if he/she is blocked in the OR. Therefore the time of the next event should be determined by the maximum between $t_1(s_n)$ and $\tau_n$.

$$\text{if } t_2(s_n) > t_1(s_n), |R_n| < \text{pcap} \text{ and } |O_{rn}| > 0$$

$$y_{\text{rq}} = \arg\min_{y_{\text{jk}}} \{ c_n(y_{\text{jk}}) : y_{\text{jk}} \in U_n \} \quad O_{r(n+1)} = O_{rn} \setminus \{ r_{\text{rq}} \}$$

$$\tau_{n+1} = \max(\tau_n, t_1(s_n)) = \Gamma(\tau_n, s_n) \quad U_{n+1} = U_n - y_{\text{rq}} + y_{r(q+1)}$$

$$O_{j(n+1)} = O_{jn}, \forall j \in J, j \neq r$$

$$R_{n+1} = R_n \cup \{ y_{\text{rq}} \}$$

$$c_{n+1}(y_{\text{jk}}) = \begin{cases} \tau_{n+1} + p_{\text{rq}}, & y_{\text{jk}} = y_{\text{rq}} \\ \max(SST_{r(q+1)}, \tau_{n+1}) + d_{r(q+1)}, & y_{\text{jk}} = y_{r(q+1)} \\ c_n(y_{\text{jk}}), & \text{otherwise} \end{cases}$$

$$s_{n+1} = (O_{j(n+1)}, \forall j \in J, U_{n+1}, R_{n+1}, c_{n+1}) \quad (A.0.3)$$
Under this condition, it can also be determined that

\[ APT_{rq} = \tau_{n+1} \] \[ ASTR_{r(q+1)} = \max(SST_{r(q+1)}, \tau_{n+1}) \]

3. When \( t_2(s_n) > t_1(s_n) \) and \( |R_n| < pcap \), a patient is moved into the PACU, but he/she is the last patient scheduled in his/her OR.

\[
\text{if } t_2(s_n) > t_1(s_n), |R_n| < pcap \text{ and } |O_{rn}| = 0
\]

\[
y_{rq} = \arg\min_{y_{jk}} \{c_n(y_{jk}): y_{jk} \in U_n\}
\]

\[
\tau_{n+1} = \max(\tau_n, t_1(s_n)) = \Gamma(\tau_n, s_n)
\]

\[
c_{n+1}(y_{jk}) = \begin{cases} 
\tau_{n+1} + p_{rq}, & y_{jk} = y_{rq} \\
\tau_n, & \text{otherwise} 
\end{cases}
\]

\[
O_{j(n+1)} = O_{jn}, \forall j \in J 
\]

\[
R_{n+1} = R_n \cup y_{rq} 
\]

\[
U_{n+1} = U_n - y_{rq} 
\]

\[
s_{n+1} = (O_{j(n+1)}, \forall j \in J, U_{n+1}, R_{n+1}, c_{n+1}) \quad (A.0.4)
\]

Under this condition, one can determine that \( APT_{rq} = \tau_{n+1} \).

Combining all conditions, the DEDS can be written as follows:

**Input**: \( SST_{jk}, d_{jk}^{\omega} \) and \( p_{jk}^{\omega} \) for all \( y_{jk} \in Y \) in scenario \( \omega \)

**Output**: \( ASTR_{jk}^{\omega}, APT_{jk}^{\omega} \) and \( PF^{\omega} \) for \( y_{jk} \in Y \) in scenario \( \omega \)

**begin**

\[
\text{Initialize } s_0 = (O_{j0}, \forall j \in J, U_0, R_0, c_0) \text{ based on equations (A.0.1)};
\]

\[
\tau_0 \leftarrow 0, n \leftarrow 1;
\]

**while** \((O_{jn} = U_n = R_n = \phi, \forall j \in J) \ false \) **do**

\[
\text{Update using Equations (A.0.2) to (A.0.4), } s_n = \Phi(s_{n-1}), \tau_n = \Gamma(\tau_{n-1}, s_{n-1});
\]

\[
n \leftarrow n + 1;
\]

**end**

**end**

**Procedure** Discrete Event Dynamic System

## B Nondifferentiability and Discontinuity in the Sample Cost Function

- \( \Omega_1 = \{\omega \in \Omega : t_1(s_n) = t_2(s_n), |R_n| = pcap\} \).

Based on the way \( AST \) and \( APT \) are determined in DEDS, one can also write this condition as \( t_1(s_n) = c_n(y_{ab}) = ASTR_{ab} + d_{ab} = t_2(s_n) = c_n(y_{lm}) = APT_{lm} + p_{lm} \) where \( y_{ab} \in R_n, y_{lm} \in U_n \). In this condition, the earliest surgery completion time in \( U_n \) is the same as
the earliest patient release time in $R_n$ when the PACU is full.

- $\Omega_2 = \{ \omega \in \Omega : t_2(s_n) = c_n(y_{ab}) = c_n(y_{lm}) > t_1(s_n), |R_n| = pcap, y_{ab}, y_{lm} \in R_n \}.$

Written as $c_n(y_{ab}) = APT_{ab} + p_{ab}$ and $c_n(y_{lm}) = APT_{lm} + p_{lm},$ it represents the condition where a patient is blocked waiting for a PACU bed and two patients finish their recovery at the same time.

- $\Omega_3 = \{ \omega \in \Omega : c_{n+1}(y_{r(q+1)}) - d_{r(q+1)} = \tau_{n+1} = SST_{r(q+1)}, t_2(s_n) > t_1(s_n), |R_n| < pcap, |O_{rq}| > 0, y_{rq} \in U_n \}$

Written in patient-associated time as $\tau_{n+1} = APT_{rq} = SST_{r(q+1)},$ the current condition describes the case where SST of a patient is the same as his/her predecessor’s PACU admission time. Please note that although patients’ arrivals are not explicitly modeled as events in our DEDS, this case is included when event sequence change is discussed.

- $\Omega_4 = \{ \omega \in \Omega : t_1(s_n) = c_n(y_{ab}) = c_n(y_{lm}) < t_2(s_n), |R_n| \geq pcap - 1, y_{ab}, y_{lm} \in U_n \}.$

This describes the case in which two surgeries are finished at the same time and compete for a PACU spot, if we write $c_n(y_{ab}) = AST_{ab} + d_{ab}$ and $c_n(y_{lm}) = AST_{lm} + d_{lm}.$

- $\Omega_5 = \{ \omega \in \Omega : \tau_{n+1} = MT, t_2(s_n) > t_1(s_n), |R_n| < pcap, |O_{rn}| = 0 \}$

Written as $\tau_{n+1} = APT_{rq}, y_{rq} \in U_n,$ it can be seen that a patient is transferred into the PACU at the end of regular work hours in this condition.

- $\Omega_6 = \{ \omega \in \Omega : \tau_{n+1} = MT, O_{j(n+1)} = U_{n+1} = R_{n+1} = \psi, \forall j \in J \}$

Since the last event in the system is a patient’s release from the PACU, one can write $\tau_n = APT_{rq} + p_{rq} = MT, y_{rq} \in R_n.$

## C Consistency of the SAA estimators

In the following part, we would like to show the consistency of the SAA estimators of the optimal value and the optimal solutions. First, we will show the expected cost function $J(SST) = E_\omega[C(SST, \omega)]$ is continuous in the feasible region of $SST.$

**Proposition 2.** The expected cost function $J(SST) = E_\omega[C(SST, \omega)]$ is continuous in the feasible region $\Theta,$ where

$$\Theta = \left\{ SST \in \mathbb{R}^N, N = \sum_{j \in J} SR_j \left| 0 = SST_{j0} \leq SST_{jk} \leq \cdots \leq SST_{j(SR_j - 1)} \leq MT, \forall j \in J \right. \right\}$$

**Proof.** The proof is similar to those for Proposition 1 in (Kim et al., 2015) and Proposition 5 in (Kim & Henderson, 2008).

It has been shown that $C(SST, \omega)$ is a.s. continuous at a given $SST,$ that is, $C(SST +
\( \Delta, \omega \rightarrow C(SST, \omega) \) a.s. as \( \Delta \rightarrow 0 \). Since all activities can be finished within a large enough period \( HT \), \( C(SST, \omega) \) is bounded by a large number \( M \) for any \( SST \in \Theta \) and any sample \( \omega \in \Omega \), where

\[
M = \max \{ C_{PW}, C_B, C_I, CO, C_{PO} \} \ast (2 |Y| + 2 \ast orr + pcap) \ast HT
\]

and \( |Y| \), \( orr \) and \( pcap \) are the number of patients, ORs and PACU beds, respectively. Because of this boundedness, the Dominated Convergence Theorem (DCT) (c.f. (Kim & Henderson, 2008; Kim et al., 2015)) can be applied to the family of random variables \( \{ C(SST + \Delta, \omega) - C(SST, \omega) \} \).

For any \( SST, SST + \Delta \in \Theta \), we have

\[
\lim_{\Delta \rightarrow 0} [J(SST + \Delta) - J(SST)] = \lim_{\Delta \rightarrow 0} \{ E_{\omega} [C(SST + \Delta, \omega) - C(SST, \omega)] \}
= E_{\omega} \left[ \lim_{\Delta \rightarrow 0} \{ C(SST + \Delta, \omega) - C(SST, \omega) \} \right] = 0
\]

where the interchange of limit and expectation is justified by the DCT.

Since the result above can be applied to any \( SST \in \Theta \), \( J(SST) \) is continuous in \( \Theta \).

Based on the continuity of \( J(SST) \), we are able to follow the proof procedure of Shapiro et. al.’s Theorem 7.48 in (Shapiro et al., 2009) to show the objective function of the SAA formulation a.s. uniformly converges to the expected cost function.

**Proposition 3.** \( \{ J_n(SST) \} \rightarrow J(SST) \) uniformly on \( \Theta \), a.s. as \( n \rightarrow \infty \)

Since \( \Theta \) is bounded and the function \( J(SST) \) is continuous, the minimum can be attained in the feasible region. Let \( J^* \) be the optimal objective value of the stochastic problem (3.2.1) and \( \pi^* \) denote the set of corresponding optimal solutions. In the SAA problem (4.1.1), the objective function \( J_n(SST) \) may be discontinuous and might not attain the minimum in \( \Theta \). Instead, we define \( J^*_n = \inf \{ J_n(SST) \} \). Due to the boundedness of \( \Theta \), one could always find a sequence \( \{ SST_k \} \) in \( \Theta \) that \( \lim_{k \rightarrow \infty} J_n(SST_k) = J^*_n \). Since \( \Theta \) is compact, \( \lim_{k \rightarrow \infty} SST_k = SST_0 \).

Then we borrow the methods in proving Theorem 9 in (Kim et al., 2015) and Theorem 5.3 in (Shapiro et al., 2009) to show the consistency of the SAA estimators of the optimal value and the optimal solutions based on the previous two propositions.

**Proposition 4.** \( J^*_n \rightarrow J^* \) and \( d(SST_0, \pi^*) \rightarrow 0 \) a.s. as \( n \rightarrow \infty \). where \( d(x, B) = \inf_{y \in B} |x - y| \).
D Lower and Upper Bounds Derivation in Section 5.1

We first generate $M = 20$ independent batches of scenarios, $\omega_1^j, \omega_2^j, \ldots, \omega_N^j, j = 1, 2, \ldots, M$, each of size $N$ and solve $M = 20$ SAA problems

$$\min_{SST \in \Theta} \left\{ J_N^j(SST) = \frac{1}{N} \sum_{i=1}^{N} C(SST, \omega_i^j) \right\}, j = 1, 2, \ldots, M \quad (D.0.1)$$

by SAA-GDR with $K$ random restarts. Let $J_{NK}^j$ be the best objective value by solving (D.0.1) with $K$ random restarts and $sst_{NK}^j$ be the corresponding solution in each of the $M = 20$ SAA problems. According to (Linderoth et al., 2006), if $J_{NK}^j$ is optimal to (D.0.1), a $95\%$ confidence interval for $LB_N$, a lower bound on the optimal value $J^*$ of the stochastic problem (3.2.1) can be found by

$$\left[ L_{NK} - \frac{t_{\alpha/2, M-1}s_{lb}^{NK}}{\sqrt{M}}, L_{NK} + \frac{t_{\alpha/2, M-1}s_{lb}^{NK}}{\sqrt{M}} \right]$$

where

$$L_{NK} = \frac{1}{M} \sum_{j=1}^{M} J_{NK}^j \quad \text{and} \quad s_{lb}^{NK} = \sqrt{\frac{1}{M - 1} \sum_{j=1}^{M} (J_{NK}^j - L_{NK})^2}$$

Upper bound estimates on the optimal value $J^*$ are also obtained as described in (Linderoth et al., 2006). For each solution $sst_{NK}^j$ in the lower bound derivation, one could calculate

$$U_N(sst_{NK}^j) = \frac{1}{T} \sum_{i=1}^{T} J_N^i(sst_{NK}^j) = \frac{1}{T} \sum_{i=1}^{T} \sum_{l=1}^{N} C(sst_{NK}^j, \omega_l^j)$$

by sampling $T = 50$ independent batches of $N = 20000$ scenarios. If we select the $sst_{NK}^j$ with the smallest $U_N(sst_{NK}^j)$ value and generate a new and independent set of $T = 50$ batches of $N = 20000$ scenarios, a new estimate of $U_N(sst_{NK}^j)$ is obtained. Then a $95\%$ confidence interval for upper bound estimate $UB_N$ is

$$\left[ U_N(sst_{NK}^j) - \frac{t_{\alpha/2, T-1}s_{ub}^{NK}}{\sqrt{T}}, U_N(sst_{NK}^j) + \frac{t_{\alpha/2, T-1}s_{ub}^{NK}}{\sqrt{T}} \right]$$

where

$$s_{ub}^{NK} = \sqrt{\frac{1}{T - 1} \sum_{i=1}^{T} (J_N^i(sst_{NK}^j) - U_N(sst_{NK}^j))^2}$$

E Setup of the Experiments in Section 5.1 and 5.2

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<th>No. of ORs</th>
<th>Total Patients</th>
<th>No. of PACU beds</th>
<th>$C_{PW}$</th>
<th>$C_I$</th>
<th>$C_B$</th>
<th>$C_O$</th>
<th>$C_{PO}$</th>
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<td>2</td>
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<td>1.2</td>
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<td>1.2</td>
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Table 5: Setup of the Experiments

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Table 6: Distance Matrix of 5 solutions in Test 1 with \( N = K = 5000 \)

\[
\begin{array}{cccccc}
0.000 & 0.147 & 0.098 & 0.051 & 0.085 \\
0.147 & 0.000 & 0.108 & 0.124 & 0.161 \\
0.098 & 0.108 & 0.000 & 0.076 & 0.092 \\
0.051 & 0.124 & 0.076 & 0.000 & 0.065 \\
0.085 & 0.161 & 0.092 & 0.065 & 0.000 \\
\end{array}
\]

Table 7: Distance Matrix of 5 solutions in Test 2 with \( N = K = 5000 \)

\[
\begin{array}{cccccc}
0.000 & 2.056 & 1.203 & 2.714 & 2.164 \\
2.056 & 0.000 & 1.339 & 2.079 & 0.527 \\
1.203 & 1.339 & 0.000 & 2.355 & 1.342 \\
2.714 & 2.079 & 2.355 & 0.000 & 1.855 \\
2.164 & 0.527 & 1.342 & 1.855 & 0.000 \\
\end{array}
\]

Table 8: Distance Matrix of 5 solutions in Test 3 with \( N = K = 5000 \)

\[
\begin{array}{cccccc}
0.000 & 0.661 & 0.893 & 0.878 & 1.022 \\
0.661 & 0.000 & 0.484 & 0.552 & 0.679 \\
0.893 & 0.484 & 0.000 & 0.410 & 0.562 \\
0.878 & 0.552 & 0.410 & 0.000 & 0.787 \\
1.022 & 0.679 & 0.562 & 0.787 & 0.000 \\
\end{array}
\]

F Convergence Test

We generate 5 batches of \( N = 5000 \) scenarios and solve the problem by SAA-GDR with \( K = 5000 \) for Test 1 to 3. Five sets of scheduled start times are obtained and their pairwise Euclidean distances are calculated and presented in Tables 6 to 8. In all tests, the pairwise distances are not sufficiently small to declare the convergence of solutions. To rule out the possibility of insufficient number of scenarios and random restarts, we conduct additional tests by taking \( N = 20000 \) and \( K = 20000 \), and observe similar pairwise distances in all test instances. Similar to Linderoth et al.'s conjecture, it is likely that in our problem, there is a "feasible neighborhood of the solution set with which the objective value is not much different from the optimal value" \( J^* \) (Linderoth et al., 2006).
### G Comparison of Results

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<tr>
<th>Method</th>
<th>Test</th>
<th>$\mathcal{I}$</th>
<th>Time (s)</th>
<th>Test</th>
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*Table 9: Comparison with Other Methods*