Formulas for calculating the maximum and minimum ranks of products of generalized inverses of matrices

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Abstract. A matrix \(X\) is called an \(\{i, \ldots, j\}\)-inverse of \(A\), denoted by \(A^{(i, \ldots, j)}\), if it satisfies the \(i, \ldots, j\)th equations of the four matrix equations (i) \(AXA = A\), (ii) \(XAX = X\), (iii) \((AX)^* = AX\), (iv) \((XA)^* =XA\). The \(\{i, \ldots, j\}\)-inverse of \(A\) is not necessarily unique and their general expressions can be written as certain linear or quadratical matrix-valued functions that involve one or more variable matrices. Let \(A\) and \(B\) be two matrices such that the product \(AB\) is defined and let \(A^{(i, \ldots, j)}\) and \(B^{(i, \ldots, j)}\) be the \(\{i, \ldots, j\}\)-inverses of \(A\) and \(B\), respectively. Then the product \(B^{(i, \ldots, j)}A^{(i, \ldots, j)}\) will occur in the reverse-order law \((AB)^{(i, \ldots, j)} = B^{(i, \ldots, j)}A^{(i, \ldots, j)}\). The product \((B^{(i, \ldots, j)}A^{(i, \ldots, j)})^\dagger\) is usually not unique and is certain linear or nonlinear matrix-valued function with one or more independent variable matrices. In order to characterize the performance of \((B^{(i, \ldots, j)}A^{(i, \ldots, j)})^\dagger\), this paper considers maximum and minimum possible ranks of \((B^{(i, \ldots, j)}A^{(i, \ldots, j)})^\dagger\) for the eight frequently used \(\{i, \ldots, j\}\)-inverses of matrices. We shall establish 126 analytical formulas for calculating the global maximum and minimum values of the ranks of \((B^{(i, \ldots, j)}A^{(i, \ldots, j)})^\dagger\), and use the formulas to characterize some fundamental algebraic properties of these products. Especially, we establish necessary and sufficient conditions for rank\((B^{(i, \ldots, j)}A^{(i, \ldots, j)})^\dagger\) = rank\((AB)\) and \(B^{(i, \ldots, j)}A^{(i, \ldots, j)} = 0\) to hold, respectively.

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1 Introduction

Throughout this paper, \(\mathbb{C}^{m \times n}\) stands for the collection of all \(m \times n\) complex matrices and all \(m \times m\). The symbols \(r(A)\), \(\mathcal{R}(A)\) and \(\mathcal{N}(A)\) stand for the rank, range (column space) and kernel (null space) of a matrix \(A \in \mathbb{C}^{m \times n}\), respectively. \(I_m\) denotes the identity matrix of order \(m\); \([A, B]\) denotes a row block matrix consisting of \(A\) and \(B\).

The Moore–Penrose inverse of \(A \in \mathbb{C}^{m \times n}\), denoted by \(A^\dagger\), is the unique matrix \(X \in \mathbb{C}^{n \times m}\) satisfying the four Penrose equations

\[
\begin{align*}
(i) \quad AXA &= A, \\
(ii) \quad XAX &= X, \\
(iii) \quad (AX)^* &= AX, \\
(iv) \quad (XA)^* &=XA.
\end{align*}
\]

(1.1)

Further let \(E_A = I_m - AA^\dagger\) and \(F_A = I_n - A^\dagger A\) stand for two projectors induced by \(A\). Moreover, a matrix \(X\) is called an \(\{i, \ldots, j\}\)-inverse of \(A\), denoted by \(A^{(i, \ldots, j)}\), if it satisfies the \(i, \ldots, j\)th equations. The collection of all \(\{i, \ldots, j\}\)-inverses of \(A\) is denoted by \(\{A^{(i, \ldots, j)}\}\). The eight frequently used generalized inverses of \(A\) are

\[
A^\dagger, A^{(1,3,4)}, A^{(1,2,4)}, A^{(1,2,3)}, A^{(1,4)}, A^{(1,3)}, A^{(1,2)}, A^{(1)}; 
\]

(1.2)

see, e.g., [3, 4, 10]. Generalized inverses of matrices are common tools to deal with singular matrices, and now become fruitful and core parts of matrix theory and applications.

In matrix theory, a fundamental matrix operation is to find the inverse of a square matrix when it is nonsingular, or to find generalized inverses of the matrix when it is singular. It is a common fact that for a pair of nonsingular matrices \(A\) and \(B\), of the same size, the product \(AB\) is nonsingular as well, and the ordinary inverse of \(AB\) can be expressed as \((AB)^{-1} = B^{-1}A^{-1}\). This basic equality is called the reverse-order law of the inverse of the product of two nonsingular matrices in linear algebra. This law shows that if both \(A^{-1}\) and \(B^{-1}\) are given, we can use their product \(B^{-1}A^{-1}\) instead of \((AB)^{-1}\), so that this law can be used to simplify various matrix expressions that involve inverse operations of products of nonsingular matrices. For generalized inverses of the product of two singular matrices, the reverse-order laws can accordingly be written in the following forms

\[
(AB)^{(i, \ldots, j)} = B^{(i, \ldots, j)}A^{(i, \ldots, j)}
\]

(1.3)

for the multiple choices of \(\{i, \ldots, j\}\)-inverses of \(A\), \(B\) and \(AB\). The two well-known forms of (1.3) are \((AB)^\dagger = B^\dagger A^\dagger\) and \((AB)^{(1)} = B^{(1)}A^{(1)}\); which were recognized and studied in the literature since 1960s; see, e.g., [1, 2, 5, 6, 11, 12, 15, 16, 17, 19, 20]. Different to the situations for nonsingular matrices, reverse-order laws for generalized inverses of products of matrices have many forms due to the multiple choices

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of \{i, \ldots, j\}\)-inverses, while these matrix equalities do not necessarily hold. In these cases, people were naturally interested in establishing necessary and sufficient conditions for the reverse-order laws to hold since Penrose defined in 1950s the four matrix equations in (1.1). Now the work on generalized inverses of matrix products become one of the core issues in the theory of generalized inverses. In fact, the researches on reverse-order laws led to some essential developments of the theory of generalized inverses from the theoretical point of view. In particular, it greatly prompted establishments of many expansion formulas for calculating the ranks of matrices and their operations, and these rank formulas, as demonstrated below, now are widely used in matrix theory and applications.

Note from (1.3) that the products \(B^{(i, \ldots, j)}A^{(i, \ldots, j)}\) are building blocks for the constructions of the reverse-order laws. Hence it is necessary to study the constructions the products \(B^{(i, \ldots, j)}A^{(i, \ldots, j)}\) and their properties from the view-point of matrix-valued functions. In order to reveal more deep and fundamental connections between both sides in (1.3), we consider in this paper some fundamental algebraic properties of the products on the right-hand side of (1.3). Our main focus is on a particular class of optimization problems of establishing we establish in this paper a group of analytical formulas for calculating the products on the right-hand side of (1.3). Our main focus is on a particular class of optimization problems of establishing we establish in this paper a group of analytical formulas for calculating the products on the right-hand side of (1.3). Our main focus is on a particular class of optimization problems of establishing we establish in this paper a group of analytical formulas for calculating the products on the right-hand side of (1.3).

2 Preliminaries

We briefly review the mathematical foundations of our algebraic approaches to make the paper self-contained. The results in the following lemma are well known or follow from the definition of \(\{i, \ldots, j\}\)-inverse of matrix; see, e.g., [5, 6, 10].

Lemma 2.1. Let \(A \in \mathbb{C}^{n \times p}\). Then, the following results hold.

(a) The general expressions of the last seven generalized inverses of \(A\) in (1.2) can be written in the following parametric forms

\[
\begin{align*}
A^{(1,3,4)} &= A^\dagger + F_AV E_A, \\
A^{(1,2,4)} &= A^\dagger + A^\dagger A W E_A, \\
A^{(1,2,3)} &= A^\dagger + F_AV A^\dagger, \\
A^{(1,4)} &= A^\dagger + W E_A, \\
A^{(1,3)} &= A^\dagger + F_A V, \\
A^{(1,2)} &= (A^\dagger + F_A V) A (A^\dagger + W E_A), \\
A^{(1)} &= A^\dagger + F_A V + W E_A,
\end{align*}
\]

where the two matrices \(V, W \in \mathbb{C}^{n \times m}\) are arbitrary.

(b) The following matrix equalities hold

\[
\begin{align*}
AA^{(1,3,4)} &= AA^{(1,2,3)} = AA^{(1,3)} = AA^\dagger, \\
A^{(1,3,4)} A &= A^{(1,2,4)} A = A^{(1,4)} A = A^\dagger A, \\
AA^{(1,2,4)} &= AA^{(1,4)} = AA^{(1,2)} = AA^{(1)} = AA^\dagger + A W E_A, \\
A^{(1,2,3)} A &= A^{(1,3)} A = A^{(1,2)} A = A^{(1)} A = A^\dagger A + F_A V A,
\end{align*}
\]

where the two matrices \(V\) and \(W\) are arbitrary.

(c) The following matrix set inclusions hold

\[
\begin{align*}
A^\dagger &\in \{A^{(1,3,4)}\} \subseteq \{A^{(1,4)}\} \subseteq \{A^{(1)}\}, \\
A^\dagger &\in \{A^{(1,3,4)}\} \subseteq \{A^{(1,3)}\} \subseteq \{A^{(1)}\}, \\
A^\dagger &\in \{A^{(1,2,4)}\} \subseteq \{A^{(1,4)}\} \subseteq \{A^{(1)}\}, \\
A^\dagger &\in \{A^{(1,2,4)}\} \subseteq \{A^{(1,2)}\} \subseteq \{A^{(1)}\}, \\
A^\dagger &\in \{A^{(1,2,3)}\} \subseteq \{A^{(1,3)}\} \subseteq \{A^{(1)}\}, \\
A^\dagger &\in \{A^{(1,2,3)}\} \subseteq \{A^{(1,2)}\} \subseteq \{A^{(1)}\}.
\end{align*}
\]
Lemma 2.2 ([9]). Let $A \in \mathbb{C}^{m \times n}$, $B \in \mathbb{C}^{m \times k}$ and $C \in \mathbb{C}^{l \times n}$. Then

$$
\left\{ (A^{(1,3,4)})^* \right\} = \left\{ (A^*)^{(1,3,4)} \right\},
$$
(2.18)

$$
\left\{ (A^{(1,2,4)})^* \right\} = \left\{ (A^*)^{(1,2,3)} \right\},
$$
(2.19)

$$
\left\{ (A^{(1,2,3)})^* \right\} = \left\{ (A^*)^{(1,2,4)} \right\},
$$
(2.20)

$$
\left\{ (A^{(1,4)})^* \right\} = \left\{ (A^*)^{(1,3)} \right\},
$$
(2.21)

$$
\left\{ (A^{(1,3)})^* \right\} = \left\{ (A^*)^{(1,4)} \right\},
$$
(2.22)

$$
\left\{ (A^{(1,2)})^* \right\} = \left\{ (A^*)^{(1,2)} \right\},
$$
(2.23)

$$
\left\{ (A^{(1)})^* \right\} = \left\{ (A^*)^{(1)} \right\}.
$$
(2.24)

Lemma 2.3 ([9]). Let $A \in \mathbb{C}^{m \times n}$, $B \in \mathbb{C}^{m \times k}$, $C \in \mathbb{C}^{l \times n}$ and $D \in \mathbb{C}^{l \times k}$. Then, the following results hold.

(a) $r[A, B] = r(A) \Leftrightarrow \mathcal{R}(B) \subseteq \mathcal{R}(A) \Rightarrow AA^TB = B \Leftrightarrow EA_B = 0$.

(b) $r\begin{bmatrix} A \\ C \end{bmatrix} = r(A) \Leftrightarrow \mathcal{R}(C^*) \subseteq \mathcal{R}(A^*) \Leftrightarrow CA^TA = C \Leftrightarrow CF_A = 0$.

(c) $r[A, B] = r(A) + r(B) \Leftrightarrow \mathcal{R}(A) \cap \mathcal{R}(B) = \{0\} \Leftrightarrow \mathcal{R}[(E_AB)^*] = \mathcal{R}(B^*) \Leftrightarrow \mathcal{R}[(E_BA)^*] = \mathcal{R}(A^*)$.

(d) $r\begin{bmatrix} A \\ B \\ C \end{bmatrix} = r(A) + r(C) \Leftrightarrow \mathcal{R}(A) \cap \mathcal{R}(B) \cap \mathcal{R}(C) = \{0\} \Leftrightarrow \mathcal{R}(CF_A) = \mathcal{R}(C) \Leftrightarrow \mathcal{R}(AF_C) = \mathcal{R}(A)$.

Lemma 2.4 ([9]). Let $A \in \mathbb{C}^{m \times n}$ and $B \in \mathbb{C}^{n \times p}$. Then the following results hold.

(a) The rank of $AB$ satisfies the expansion formulas

$$
r(AB) = r(A) + r(B) - n + r[(I_n - BB^T)(I_n - A^TA)],
$$
(2.28)

$$
r(AB) = r(A) + r(B) - n + r[(I_n - BB^{(1)})(I_n - A^{(1)}A)]
$$
(2.29)

for all $A^{(1)}$ and $B^{(1)}$.

(b) The rank of $AB$ satisfies the inequalities

$$
r(AB) \leq \min\{r(A), r(B)\} \leq \min\{m, n, p\},
$$
(2.30)

$$
r(AB) \geq r(A) + r(B) - r[A^*, B] \geq \max\{0, r(A) + r(B) - n\}.
$$
(2.31)

(c) The following statements are equivalent:

(i) $r(AB) = r(A) + r(B) - n$.

(ii) $(I_n - BB^T)(I_n - A^TA) = 0$.

(iii) $(I_n - BB^{(1)})(I_n - A^{(1)}A) = 0$ for all $A^{(1)}$ and $B^{(1)}$.

(iv) $A^T \subseteq \mathcal{R}(B)$.

(v) $A^T(B^*) \subseteq \mathcal{R}(A^*)$.

(d) The following statements are equivalent:

(i) $AB = 0$.

(ii) $r(A) + r(B) = n - r[(I_n - BB^T)(I_n - A^TA)]$. 

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(iii) $\mathcal{R}(B) \subseteq \mathcal{N}(A)$.

**Lemma 2.5.** Let $A \in \mathbb{C}^{m \times n}$, $M \in \mathbb{C}^{p \times q}$ and $B \in \mathbb{C}^{p \times q}$. Then, the following rank equalities hold

\begin{align*}
    r(B^1 MA^1) &= r(B^* MA^*) = r(B^* MA^*), \quad (2.32) \\
    r(BB^1 MA^1 A) &= r(BB^1 MA^1) = r(BB^1 MA^* A) = r(BB^1 MA^*). \quad (2.33)
\end{align*}

**Lemma 2.6 ([14]).** Let $A \in \mathbb{C}^{m \times n}$, $B \in \mathbb{C}^{m \times k}$ and $C \in \mathbb{C}^{l \times n}$. Then, the following rank equalities hold

\begin{align*}
    \max_{A^{(1,3,4)}} r(A^{(1,3,4)}B) &= \min \{ n + r(A^* B) - r(A), \ r(B) \}, \quad (2.34) \\
    \min_{A^{(1,3,4)}} r(A^{(1,3,4)}B) &= r(A^* B), \quad (2.35) \\
    \max_{A^{(1,2,4)}} r(A^{(1,2,4)}B) &= \max_{A^{(1,2,4)}} r(AA^{(1,2,4)}B) = \min \{ n, \ r(A), \ r(B) \}, \quad (2.36) \\
    \min_{A^{(1,2,4)}} r(A^{(1,2,4)}B) &= \min_{A^{(1,2,4)}} r(AA^{(1,2,4)}B) = r(A) + r(B) - r[A, B], \quad (2.37) \\
    \max_{A^{(1,2,3)}} r(A^{(1,2,3)}B) &= \max_{A^{(1,2,3)}} r(AA^{(1,2,3)}B) = r(A^* B), \quad (2.38) \\
    \min_{A^{(1,2,3)}} r(A^{(1,2,3)}B) &= \min_{A^{(1,2,3)}} r(AA^{(1,2,3)}B) = r(A^* B), \quad (2.39) \\
    \max_{A^{(1,3)}} r(A^{(1,3)}B) &= \min \{ n + r(A^* B) - r(A), \ r(B) \}, \quad (2.40) \\
    \min_{A^{(1,3)}} r(A^{(1,3)}B) &= r(A^* B), \quad (2.41) \\
    \max_{A^{(1,2)}} r(A^{(1,2)}B) &= \max_{A^{(1,2)}} r(AA^{(1,2)}B) = \min \{ n, \ r(B) \}, \quad (2.42) \\
    \min_{A^{(1,2)}} r(A^{(1,2)}B) &= \min_{A^{(1,2)}} r(AA^{(1,2)}B) = r(A) + r(B) - r[A, B], \quad (2.43) \\
    \max_{A^{(1)}} r(A^{(1)}B) &= \min \{ n, \ r(B) \}, \quad (2.44) \\
    \min_{A^{(1)}} r(A^{(1)}B) &= r(A) + r(B) - r[A, B]. \quad (2.45)
\end{align*}

and

\begin{align*}
    \max_{A^{(1,3,4)}} r(CA^{(1,3,4)}B) &= \min \{ m + r(CA^*) - r(A), \ r(C) \}, \quad (2.46) \\
    \min_{A^{(1,3,4)}} r(CA^{(1,3,4)}B) &= r(CA^*), \quad (2.47) \\
    \max_{A^{(1,2,4)}} r(CA^{(1,2,4)}B) &= \max_{A^{(1,2,4)}} r(CA^{(1,2,4)}A) = r(CA^*), \quad (2.48) \\
    \min_{A^{(1,2,4)}} r(CA^{(1,2,4)}B) &= \min_{A^{(1,2,4)}} r(CA^{(1,2,4)}A) = r(CA^*), \quad (2.49) \\
    \max_{A^{(1,2,3)}} r(CA^{(1,2,3)}B) &= \max_{A^{(1,2,3)}} r(CA^{(1,2,3)}A) = \min \{ n, \ r(C) \}, \quad (2.50) \\
    \min_{A^{(1,2,3)}} r(CA^{(1,2,3)}B) &= \min_{A^{(1,2,3)}} r(CA^{(1,2,3)}A) = r(A) + r(C) - r[A^*, C^*], \quad (2.51) \\
    \max_{A^{(1,4)}} r(CA^{(1,4)}B) &= \min \{ m + r(CA^*) - r(A), \ r(C) \}, \quad (2.52) \\
    \min_{A^{(1,4)}} r(CA^{(1,4)}B) &= r(CA^*), \quad (2.53) \\
    \max_{A^{(1,3)}} r(CA^{(1,3)}B) &= \min \{ m, \ r(C) \}, \quad (2.54) \\
    \min_{A^{(1,3)}} r(CA^{(1,3)}B) &= r(A) + r(C) - r[A^*, C^*], \quad (2.55) \\
    \max_{A^{(1,2)}} r(CA^{(1,2)}B) &= \max_{A^{(1,2)}} r(CA^{(1,2)}A) = \min \{ n, \ r(C) \}, \quad (2.56) \\
    \min_{A^{(1,2)}} r(CA^{(1,2)}B) &= \min_{A^{(1,2)}} r(CA^{(1,2)}A) = r(A) + r(C) - r[A^*, C^*], \quad (2.57) \\
    \max_{A^{(1)}} r(CA^{(1)}B) &= \min \{ m, \ r(C) \}, \quad (2.58) \\
    \min_{A^{(1)}} r(CA^{(1)}B) &= r(A) + r(C) - r[A^*, C^*]. \quad (2.59)
\end{align*}
Lemma 2.7 ([18]). Let $q(X_1, X_2) = (A_1 + B_1X_1)D(A_2 + B_2X_2C_2)$ be a quadratic matrix-valued function of appropriate sizes. Then,

$$\max_{X_1, X_2} r[q(X_1, X_2)] = \min \left\{ r[A_1DA_2, A_1DB_2, B_1], r \left[ \begin{bmatrix} A_1DA_2 \\ C_1DA_2 \end{bmatrix} \right], r \left[ \begin{bmatrix} A_1DA_2 \\ C_2 \end{bmatrix} \right], r \left[ \begin{bmatrix} A_1DA_2 \\ C_1DA_2 \end{bmatrix} \right], r \left[ \begin{bmatrix} A_1DA_2 \\ A_1DB_2 \end{bmatrix} \right] \right\},$$

(2.62)

$$\min_{X_1, X_2} r[q(X_1, X_2)] = r \left[ \begin{bmatrix} A_1DA_2 \\ C_1DA_2 \end{bmatrix} \right] + r[A_1DA_2, A_1DB_2, B_1] + \max \{ s_1, s_2 \},$$

(2.63)

where

$$s_1 = r \left[ \begin{bmatrix} A_1DA_2 \\ C_2 \end{bmatrix} B_1 \right] - r \left[ \begin{bmatrix} A_1DA_2 \\ C_2 \end{bmatrix} B_1 \right] - r \left[ \begin{bmatrix} A_1DA_2 \\ C_1DA_2 \end{bmatrix} \right],$$

$$s_2 = r \left[ \begin{bmatrix} A_1DA_2 \\ C_1DA_2 \end{bmatrix} \right] - r \left[ \begin{bmatrix} A_1DA_2 \\ C_1DA_2 \end{bmatrix} \right] + r \left[ \begin{bmatrix} A_1DA_2 \\ C_1DA_2 \end{bmatrix} \right].$$

There do exist matrices $X_1$ and $X_2$ satisfying the two formulas in (2.62) and (2.63), while the constructions of the matrices were described in [18] and other references.

Lemma 2.8. Let $S$ and $T$ be two matrix sets consisting of matrices of the same size, and $P$ and $Q$ be two matrices of appropriate sizes. Then, the following results hold.

(a) The following implication holds

$$S \supseteq T \Rightarrow PSQ \supseteq PTQ.$$  

(2.64)

(b) The following rank inequalities hold

$$\max_{X \in S} r(X) \geq \max_{Y \in PSQ} r(Y), \quad \min_{X \in S} r(X) \geq \min_{Y \in PSQ} r(Y).$$  

(2.65)

(c) The following implications hold

$$S \supseteq T \Rightarrow \max_{X \in S} r(X) \geq \max_{Y \in T} r(Y) \text{ and } \min_{X \in S} r(X) \leq \min_{Y \in T} r(Y).$$  

(2.66)

Proof. Result (a) is obvious from the fact that the matrix equality $S = T$ implies the matrix equality $PSQ = PTQ$. The two rank inequalities (2.65) follow from the well-known rank inequality $r(S) \geq r(PSQ)$. The implications in (2.66) are obvious.

Lemma 2.9. Let $A \in \mathbb{C}^{m \times n}$. Then, the following rank inequalities hold.

$$r(A) = r(A^T) \leq \max_{A \in \mathbb{C}^{1,3,4}} r(A^{1,3,4}) \leq \max_{A \in \mathbb{C}^{1,4}} r(A^{1,4}) \leq \max_{A \in \mathbb{C}^{1,3}} r(A^{1,3}) \leq \min_{A \in \mathbb{C}^{1,4}} r(A^{1,4}) = \min_{A \in \mathbb{C}^{1,3}} r(A^{1,3}),$$

(2.67)

$$r(A) = r(A^T) \leq \max_{A \in \mathbb{C}^{1,3,4}} r(A^{1,3,4}) \leq \max_{A \in \mathbb{C}^{1,3}} r(A^{1,3}) \leq \min_{A \in \mathbb{C}^{1,4}} r(A^{1,4}) = \min_{A \in \mathbb{C}^{1,3}} r(A^{1,3}),$$

(2.68)

$$r(A) = r(A^T) \leq \max_{A \in \mathbb{C}^{1,2,4}} r(A^{1,2,4}) \leq \max_{A \in \mathbb{C}^{1,4}} r(A^{1,4}) \leq \min_{A \in \mathbb{C}^{1,2}} r(A^{1,2}) = \min_{A \in \mathbb{C}^{1,3}} r(A^{1,3}),$$

(2.69)

$$r(A) = r(A^T) \leq \max_{A \in \mathbb{C}^{1,2,4}} r(A^{1,2,4}) \leq \max_{A \in \mathbb{C}^{1,3}} r(A^{1,3}) \leq \min_{A \in \mathbb{C}^{1,2}} r(A^{1,2}) = \min_{A \in \mathbb{C}^{1,3}} r(A^{1,3}),$$

(2.70)

$$r(A) = r(A^T) \leq \max_{A \in \mathbb{C}^{1,2,3}} r(A^{1,2,3}) \leq \max_{A \in \mathbb{C}^{1,3}} r(A^{1,3}) \leq \min_{A \in \mathbb{C}^{1,2}} r(A^{1,2}) = \min_{A \in \mathbb{C}^{1,3}} r(A^{1,3}),$$

(2.71)

$$r(A) = r(A^T) \leq \max_{A \in \mathbb{C}^{1,2,3}} r(A^{1,2,3}) \leq \max_{A \in \mathbb{C}^{1,3}} r(A^{1,3}) \leq \min_{A \in \mathbb{C}^{1,2}} r(A^{1,2}) = \min_{A \in \mathbb{C}^{1,3}} r(A^{1,3}),$$

(2.72)

Proof. It follows from (2.12)–(2.17).

3 Main results

Let $A \in \mathbb{C}^{m \times n}$ and $B \in \mathbb{C}^{n \times p}$. In order to characterize performances of the product $B^{(i,\ldots,j)}A^{(i,\ldots,j)}$, we first give their parametric forms of the products

$$B^{(i,\ldots,j)}A^{(i,\ldots,j)}$$

(3.1)
for the eight commonly used generalized inverses $A^{(i,...,j)}$ and $B^{(i,...,j)}$, respectively. From (2.1)–(2.7), the 63 parametric expressions of (3.1) corresponding to $A^{(i,...,j)}$ and $B^{(i,...,j)}$ can be written as

\[
\begin{align*}
B^1A^{(1,3,4)} &= B^1A^1 + B^1F_AVWE_A, \\
B^1A^{(1,2,4)} &= B^1A^1 + B^1A^2AWE_A, \\
B^1A^{(1,2,3)} &= B^1A^1 + B^1F_AWAA^1, \\
B^1A^{(1,4)} &= B^1A^1 + B^1WE_A, \\
B^1A^{(3)} &= B^1A^1 + B^1F_AVWE_A, \\
B^1A^{(1,2)} &= (B^1A^1 + B^1F_AW_1)A(A^1 + W_2E_A), \\
B^1A^{(1)} &= B^1A^1 + B^1F_AW_1 + B^1W_2E_A, \\
B^{(1,3,4)}A^1 &= B^1A^1 + F_BVBE_BA^1, \\
B^{(1,3,4)}A^{(1,3,4)} &= (B^1 + B^1B^1BV_B)(A^1 + F_AW_EA), \\
B^{(1,3,4)}A^{(1,2,4)} &= (B^1 + B^1BV_B)(A^1 + A^1AWE_A), \\
B^{(1,3,4)}A^{(1,2,3)} &= (B^1 + B^1BV_B)(A^1 + F_AWAA^1), \\
B^{(1,3,4)}A^{(1,4)} &= (B^1 + B^1BV_B)(A^1 + WEA), \\
B^{(1,2,4)}A^{(1,3)} &= (B^1 + B^1BV_B)(A^1 + FAW), \\
B^{(1,2,4)}A^{(1,2)} &= (B^1 + B^1BV_B)(A^1 + FAW_1)A(A^1 + W_2E_A), \\
B^{(1,2,4)}A^{(1)} &= (B^1 + B^1BV_B)(A^1 + FAW_1 + W_2E_A), \\
B^{(1,2,3)}A^1 &= B^1A^1 + B^1FBVB^1A^1, \\
B^{(1,2,3)}A^{(1,3,4)} &= (B^1 + B^1BV_B)(A^1 + F_AW_EA), \\
B^{(1,2,3)}A^{(1,2,4)} &= (B^1 + B^1BV_B)(A^1 + A^1AWE_A), \\
B^{(1,2,3)}A^{(1,2,3)} &= (B^1 + B^1BV_B)(A^1 + F_AWAA^1), \\
B^{(1,2,3)}A^{(1,4)} &= (B^1 + B^1BV_B)(A^1 + WEA), \\
B^{(1,2,3)}A^{(1,3)} &= (B^1 + B^1BV_B)(A^1 + FAW), \\
B^{(1,2,3)}A^{(1,2)} &= (B^1 + B^1BV_B)(A^1 + FAW_1)A(A^1 + W_2E_A), \\
B^{(1,2,3)}A^{(1)} &= (B^1 + B^1BV_B)(A^1 + FAW_1 + W_2E_A), \\
B^{(1,4)}A^1 &= B^1A^1 + VEB_A^1, \\
B^{(1,4)}A^{(1,3,4)} &= (B^1 + VEB)(A^1 + FAW_EA), \\
B^{(1,4)}A^{(1,2,4)} &= (B^1 + VEB)(A^1 + A^1AWE_A), \\
B^{(1,4)}A^{(1,2,3)} &= (B^1 + VEB)(A^1 + FAWAA^1), \\
B^{(1,4)}A^{(1,4)} &= (B^1 + VEB)(A^1 + WEA), \\
B^{(1,4)}A^{(1,3)} &= (B^1 + VEB)(A^1 + FAW), \\
B^{(1,4)}A^{(1,2)} &= (B^1 + VEB)(A^1 + FAW_1)A(A^1 + W_2E_A), \\
B^{(1,4)}A^{(1)} &= (B^1 + VEB)(A^1 + FAW_1 + W_2E_A), 
\end{align*}
\]
\[ B^{(1,3)}A^\dagger = B^\dagger A^\dagger + F_BVA^\dagger, \]  
\[ B^{(1,3)}A^{(1,3,4)} = (B^\dagger + F_BV)(A^\dagger + F_AWE_A), \]  
\[ B^{(1,3)}A^{(1,2,4)} = (B^\dagger + F_BV)(A^\dagger + A^\dagger AW_E A), \]  
\[ B^{(1,3)}A^{(1,2,3)} = (B^\dagger + F_BV)(A^\dagger + F_AWA^\dagger), \]  
\[ B^{(1,3)}A^{(1,4)} = (B^\dagger + F_BV)(A^\dagger + W_E A), \]  
\[ B^{(1,3)}A^{(1,3)} = (B^\dagger + F_BV)(A^\dagger + F_AW), \]  
\[ B^{(1,3)}A^{(1,2)} = (B^\dagger + F_BV)(A^\dagger + F_AW_1)A(A^\dagger + W_2E_A), \]  
\[ B^{(1,3)}A^{(1)} = (B^\dagger + F_BV)(A^\dagger + F_AW_1 + W_2E_A), \]  
\[ B^{(1,2)}A^\dagger = B^\dagger A^\dagger + F_BV_1 A^\dagger + V_2EB_A^\dagger, \]  
\[ B^{(1,3)}A^{(1,3,4)} = (B^\dagger + F_BV_1 + V_2EB)(A^\dagger + F_AWE_A), \]  
\[ B^{(1,3)}A^{(1,2,4)} = (B^\dagger + F_BV_1 + V_2EB)(A^\dagger + A^\dagger AW_E A), \]  
\[ B^{(1,3)}A^{(1,2,3)} = (B^\dagger + F_BV_1 + V_2EB)(A^\dagger + F_AWA^\dagger), \]  
\[ B^{(1,3)}A^{(1,4)} = (B^\dagger + F_BV_1 + V_2EB)(A^\dagger + W_E A), \]  
\[ B^{(1,3)}A^{(1,3)} = (B^\dagger + F_BV_1 + V_2EB)(A^\dagger + F_AW), \]  
\[ B^{(1,3)}A^{(1,2)} = (B^\dagger + F_BV_1 + V_2EB)(A^\dagger + F_AW_1)A(A^\dagger + W_2E_A), \]  
\[ B^{(1,2)}A^{(1)} = (B^\dagger + F_BV_1)B(B^\dagger + V_2EB)(A^\dagger + F_AW_1 + W_2E_A), \]  
\[ B^{(1,2)}A = B^\dagger A^\dagger + F_BV_1 A^\dagger + V_2EB_A^\dagger, \]  
\[ B^{(1,3)}A^{(1,3,4)} = (B^\dagger + F_BV_1 + V_2EB)(A^\dagger + F_AWE_A), \]  
\[ B^{(1,3)}A^{(1,2,4)} = (B^\dagger + F_BV_1 + V_2EB)(A^\dagger + A^\dagger AW_E A), \]  
\[ B^{(1,3)}A^{(1,2,3)} = (B^\dagger + F_BV_1 + V_2EB)(A^\dagger + F_AWA^\dagger), \]  
\[ B^{(1,3)}A^{(1,4)} = (B^\dagger + F_BV_1 + V_2EB)(A^\dagger + W_E A), \]  
\[ B^{(1,3)}A^{(1,3)} = (B^\dagger + F_BV_1 + V_2EB)(A^\dagger + F_AW), \]  
\[ B^{(1,3)}A^{(1,2)} = (B^\dagger + F_BV_1 + V_2EB)(A^\dagger + F_AW_1)A(A^\dagger + W_2E_A), \]  
\[ B^{(1,2)}A^{(1)} = (B^\dagger + F_BV_1 + V_2EB)(A^\dagger + F_AW_1 + W_2E_A), \]  
\[ B^{(1,2)}A = B^\dagger A^\dagger + F_BV_1 A^\dagger + V_2EB_A^\dagger, \]  
where \( V, V_1, V_2, W, W_1, W_2 \) are arbitrary matrices of appropriate sizes. Eqs. (3.2)–(3.64) show that the matrix products in (3.1) are in fact a group of linear or nonlinear matrix-valued functions with one or more independent variables. It is obvious that the ranks of the linear or nonlinear matrix-valued functions in (3.2)–(3.64) may vary with respect to the choices of the variable matrices in them.

Recall that the rank of a matrix is the conceptual foundation in elementary linear algebra, and is the most significant finite integer in reflecting intrinsic properties of the matrix. But it took a long time in the development of matrix theory to establish thousands of influential and effective matrix rank formulas and to use the formulas, as demonstrated below, in the intuitive and rigorous investigations of matrix-valued functions. Because the rank of a matrix is nonnegative finite integer, the ranks of matrix-valued functions are always bounded no matter what the variable matrices in them are taken. This fact motivates us to establish analytical formulas for calculating the global maximum and minimum ranks of (3.2)–(3.64) when \( V, V_1, V_2, W, W_1, W_2 \) run over the corresponding matrix spaces. Concerning the upper and lower bounds of the ranks of (3.1), we have the following results.

**Lemma 3.1.** Let \( A \in \mathbb{C}^{m \times n} \) and \( B \in \mathbb{C}^{n \times p} \). Then, the following inequalities

\[ r(B^{(i,...,j)}A^{(i,...,j)}) \leq \min \left\{ r(A^{(i,...,j)}), \ r(B^{(i,...,j)}) \right\} \leq \min \{ m, \ n, \ p \}, \]  
\[ r(B^{(i,...,j)}A^{(i,...,j)}) \geq \max \left\{ 0, \ r(A^{(i,...,j)}) + r(B^{(i,...,j)}) - n \right\} \geq \max \{ 0, \ r(A) + r(B) - n \} \]  

hold, where \( A^{(i,...,j)} \) and \( B^{(i,...,j)} \) are the eight commonly used generalized inverses of \( A \) and \( B \), respectively.

**Proof.** Follows from (2.30), (2.31) and Lemma 2.9. \( \square \)

The upper and lower bounds of the ranks of (3.65) and (3.66), as shown below, are attainable for certain choices of \( A^{(i,...,j)} \) and \( B^{(i,...,j)} \). Concerning the global maximum and minimum ranks of (3.1), we have the following results.
Theorem 3.2. Let $A \in \mathbb{C}^{m \times n}$ and $B \in \mathbb{C}^{n \times p}$ be given. Then, the following 126 formulas for calculating the global maximum and minimum ranks of (3.2)–(3.64) hold

$$\begin{align*}
\max_{A^{(1,3,4)}} r(B^t A^{(1,3,4)}) &= \min \{ r(B), \ m + r(AB) - r(A) \}, \\
\min_{A^{(1,3,4)}} r(B^t A^{(1,3,4)}) &= r(AB), \\
\max_{A^{(1,2,4)}} r(B^t A^{(1,2,4)}) &= r(AB), \\
\min_{A^{(1,2,4)}} r(B^t A^{(1,2,4)}) &= r(AB), \\
\max_{A^{(1,3,3)}} r(B^t A^{(1,2,3)}) &= \min \{ r(A), \ r(B) \}, \\
\min_{A^{(1,2,3)}} r(B^t A^{(1,2,3)}) &= r(A) + r(B) - r[A^*, B], \\
\max_{A^{(1,4)}} r(B^t A^{(1,4)}) &= \min \{ r(B), \ m + r(AB) - r(A) \}, \\
\min_{A^{(1,4)}} r(B^t A^{(1,4)}) &= r(AB), \\
\max_{A^{(1,3,3)}} r(B^t A^{(1,3,3)}) &= \min \{ m, \ r(B) \}, \\
\min_{A^{(1,3,3)}} r(B^t A^{(1,3,3)}) &= r(A) + r(B) - r[A^*, B], \\
\max_{A^{(1,2,2)}} r(B^t A^{(1,2,2)}) &= \min \{ r(A), \ r(B) \}, \\
\min_{A^{(1,2,2)}} r(B^t A^{(1,2,2)}) &= r(A) + r(B) - r[A^*, B], \\
\min_{B^{(1,3,4)}} r(B^{(1,3,4)} A^{(1,3,4)}) &= \min \{ r(A), \ p + r(AB) - r(B) \}, \\
\min_{B^{(1,3,4)}} r(B^{(1,3,4)} A^{(1,3,4)}) &= r(AB), \\
\max_{B^{(1,3,4)}, A^{(1,3,4)}} r(B^{(1,3,4)} A^{(1,3,4)}) &= \min \{ m, \ n, \ p, \ m + p + r(AB) - r(A) - r(B) \}, \\
\min_{B^{(1,3,4)}, A^{(1,3,4)}} r(B^{(1,3,4)} A^{(1,3,4)}) &= r(AB), \\
\max_{B^{(1,3,4)}, A^{(1,2,4)}} r(B^{(1,3,4)} A^{(1,2,4)}) &= \min \{ r(A), \ p + r(AB) - r(B) \}, \\
\min_{B^{(1,3,4)}, A^{(1,2,4)}} r(B^{(1,3,4)} A^{(1,2,4)}) &= r(AB), \\
\max_{B^{(1,3,3)}, A^{(1,2,3)}} r(B^{(1,3,4)} A^{(1,2,3)}) &= \min \{ p, \ r(A) \}, \\
\min_{B^{(1,3,3)}, A^{(1,2,3)}} r(B^{(1,3,4)} A^{(1,2,3)}) &= r(A) + r(B) - r[A^*, B], \\
\max_{B^{(1,3,4)}, A^{(1,4)}} r(B^{(1,3,4)} A^{(1,4)}) &= \min \{ m, \ n, \ p, \ m + p + r(AB) - r(A) - r(B) \}, \\
\min_{B^{(1,3,4)}, A^{(1,4)}} r(B^{(1,3,4)} A^{(1,4)}) &= r(AB), \\
\max_{B^{(1,3,4)}, A^{(1,3,3)}} r(B^{(1,3,4)} A^{(1,3,3)}) &= \min \{ m, \ n, \ p \}, \\
\min_{B^{(1,3,4)}, A^{(1,3,3)}} r(B^{(1,3,4)} A^{(1,3,3)}) &= r(A) + r(B) - r[A^*, B], \\
\max_{B^{(1,3,4)}, A^{(1,2,2)}} r(B^{(1,3,4)} A^{(1,2,2)}) &= \min \{ p, \ r(A) \}, \\
\min_{B^{(1,3,4)}, A^{(1,2,2)}} r(B^{(1,3,4)} A^{(1,2,2)}) &= r(A) + r(B) - r[A^*, B], \\
\max_{B^{(1,3,4)}, A^{(1)}} r(B^{(1,3,4)} A^{(1)}) &= \min \{ m, \ n, \ p \}, \\
\min_{B^{(1,3,4)}, A^{(1)}} r(B^{(1,3,4)} A^{(1)}) &= r(A) + r(B) - r[A^*, B], \\
\max_{B^{(1,2,4)}} r(B^{(1,2,4)} A^{(1)}) &= \min \{ r(A), \ r(B) \}, \\
\min_{B^{(1,2,4)}} r(B^{(1,2,4)} A^{(1)}) &= r(A) + r(B) - r[A^*, B],
\end{align*}$$
\begin{align}
\max_{B^{(1,2,4)}, A^{(1,3,4)}} r(B^{(1,2,4)} A^{(1,3,4)}) &= \min \{ m, \ r(B) \}, \\
\min_{B^{(1,2,4)}, A^{(1,3,4)}} r(B^{(1,2,4)} A^{(1,3,4)}) &= r(A) + r(B) - r[A^*, B], \\
\max_{B^{(1,2,4)}, A^{(1,2,4)}} r(B^{(1,2,4)} A^{(1,2,4)}) &= \min \{ r(A), \ r(B) \}, \\
\min_{B^{(1,2,4)}, A^{(1,2,4)}} r(B^{(1,2,4)} A^{(1,2,4)}) &= r(A) + r(B) - r[A^*, B], \\
\max_{B^{(1,2,4)}, A^{(1,2,3)}} r(B^{(1,2,4)} A^{(1,2,3)}) &= \min \{ r(A), \ r(B) \}, \\
\min_{B^{(1,2,4)}, A^{(1,2,3)}} r(B^{(1,2,4)} A^{(1,2,3)}) &= \max \{ 0, \ r(A) + r(B) - n \}, \\
\max_{B^{(1,2,4)}, A^{(1,4)}} r(B^{(1,2,4)} A^{(1,4)}) &= \min \{ m, \ r(B) \}, \\
\min_{B^{(1,2,4)}, A^{(1,4)}} r(B^{(1,2,4)} A^{(1,4)}) &= r(A) + r(B) - r[A^*, B], \\
\max_{B^{(1,2,4)}, A^{(1,3)}} r(B^{(1,2,4)} A^{(1,3)}) &= \min \{ m, \ r(B) \}, \\
\min_{B^{(1,2,4)}, A^{(1,3)}} r(B^{(1,2,4)} A^{(1,3)}) &= \max \{ 0, \ r(A) + r(B) - n \}, \\
\max_{B^{(1,2,4)}, A^{(1,2)}} r(B^{(1,2,4)} A^{(1,2)}) &= \min \{ r(A), \ r(B) \}, \\
\min_{B^{(1,2,4)}, A^{(1,2)}} r(B^{(1,2,4)} A^{(1,2)}) &= \max \{ 0, \ r(A) + r(B) - n \}, \\
\max_{B^{(1,2,4)}, A^{(1)}} r(B^{(1,2,4)} A^{(1)}) &= \min \{ m, \ r(B) \}, \\
\min_{B^{(1,2,4)}, A^{(1)}} r(B^{(1,2,4)} A^{(1)}) &= \max \{ 0, \ r(A) + r(B) - n \}, \\
\max_{B^{(1,2,3)}, A^{(1,3,4)}} r(B^{(1,2,3)} A^{(1,3,4)}) &= \min \{ r(B), \ m + r(AB) - r(A) \}, \\
\min_{B^{(1,2,3)}, A^{(1,3,4)}} r(B^{(1,2,3)} A^{(1,3,4)}) &= r(AB), \\
\max_{B^{(1,2,3)}, A^{(1,2,4)}} r(B^{(1,2,3)} A^{(1,2,4)}) &= r(AB), \\
\min_{B^{(1,2,3)}, A^{(1,2,4)}} r(B^{(1,2,3)} A^{(1,2,4)}) &= r(AB), \\
\max_{B^{(1,2,3)}, A^{(1,2,3)}} r(B^{(1,2,3)} A^{(1,2,3)}) &= \min \{ r(A), \ r(B) \}, \\
\min_{B^{(1,2,3)}, A^{(1,2,3)}} r(B^{(1,2,3)} A^{(1,2,3)}) &= r(A) + r(B) - r[A^*, B], \\
\max_{B^{(1,2,3)}, A^{(1,4)}} r(B^{(1,2,3)} A^{(1,4)}) &= \min \{ m, \ r(B) \}, \\
\min_{B^{(1,2,3)}, A^{(1,4)}} r(B^{(1,2,3)} A^{(1,4)}) &= r(AB), \\
\max_{B^{(1,2,3)}, A^{(1,3)}} r(B^{(1,2,3)} A^{(1,3)}) &= \min \{ m, \ r(B) \}, \\
\min_{B^{(1,2,3)}, A^{(1,3)}} r(B^{(1,2,3)} A^{(1,3)}) &= r(A) + r(B) - r[A^*, B], \\
\max_{B^{(1,2,3)}, A^{(1,2)}} r(B^{(1,2,3)} A^{(1,2)}) &= \min \{ r(A), \ r(B) \}, \\
\min_{B^{(1,2,3)}, A^{(1,2)}} r(B^{(1,2,3)} A^{(1,2)}) &= r(A) + r(B) - r[A^*, B], \\
\max_{B^{(1,2,3)}, A^{(1)}} r(B^{(1,2,3)} A^{(1)}) &= \min \{ m, \ r(B) \}, \\
\min_{B^{(1,2,3)}, A^{(1)}} r(B^{(1,2,3)} A^{(1)}) &= r(A) + r(B) - r[A^*, B], \\
\max_{B^{(1,1,4)}, A^{(1)}} r(B^{(1,1,4)} A^{(1)}) &= \min \{ p, \ r(A) \}, \\
\min_{B^{(1,1,4)}, A^{(1)}} r(B^{(1,1,4)} A^{(1)}) &= r(A) + r(B) - r[A^*, B].
\end{align}
\[
\begin{align*}
\max_{B^{(1,3), A^{(1,4)}}} r(B^{(1,4)} A^{(1,3,4)}) &= \min \{ m, n, p \}, \\
\min_{B^{(1,3), A^{(1,4)}}} r(B^{(1,4)} A^{(1,3,4)}) &= r(A) + r(B) - r[A^*, B], \\
\max_{B^{(1,3), A^{(1,2,4)}}} r(B^{(1,4)} A^{(1,2,4)}) &= \min \{ p, r(A) \}, \\
\min_{B^{(1,3), A^{(1,2,4)}}} r(B^{(1,4)} A^{(1,2,4)}) &= r(A) + r(B) - r[A^*, B], \\
\max_{B^{(1,3), A^{(1,2,3)}}} r(B^{(1,4)} A^{(1,2,3)}) &= \min \{ p, r(A) \}, \\
\min_{B^{(1,3), A^{(1,2,3)}}} r(B^{(1,4)} A^{(1,2,3)}) &= \max \{ 0, r(A) + r(B) - n \}, \\
\max_{B^{(1,3), A^{(1,1)}}} r(B^{(1,4)} A^{(1,1)}) &= \min \{ m, n, p \}, \\
\min_{B^{(1,3), A^{(1,1)}}} r(B^{(1,4)} A^{(1,1)}) &= \max \{ 0, r(A) + r(B) - n \}, \\
\max_{B^{(1,2), A^{(1,1)}}} r(B^{(1,4)} A^{(1,2)}) &= \min \{ m, n, p \}, \\
\min_{B^{(1,2), A^{(1,1)}}} r(B^{(1,4)} A^{(1,2)}) &= \max \{ 0, r(A) + r(B) - n \}, \\
\max_{B^{(1,3), A^{(1,3,4)}}} r(B^{(1,3)} A^{(1,3,4)}) &= \min \{ m, n, p, m + p + r(AB) - r(A) - r(B) \}, \\
\min_{B^{(1,3), A^{(1,3,4)}}} r(B^{(1,3)} A^{(1,3,4)}) &= r(AB), \\
\max_{B^{(1,3), A^{(1,2,4)}}} r(B^{(1,3)} A^{(1,2,4)}) &= \min \{ r(A), p + r(AB) - r(B) \}, \\
\min_{B^{(1,3), A^{(1,2,4)}}} r(B^{(1,3)} A^{(1,2,4)}) &= r(AB), \\
\max_{B^{(1,3), A^{(1,2,3)}}} r(B^{(1,3)} A^{(1,2,3)}) &= \min \{ p, r(A) \}, \\
\min_{B^{(1,3), A^{(1,2,3)}}} r(B^{(1,3)} A^{(1,2,3)}) &= r(A) + r(B) - r[A^*, B], \\
\max_{B^{(1,3), A^{(1,1)}}} r(B^{(1,3)} A^{(1,1)}) &= \min \{ m, n, p, m + p + r(AB) - r(A) - r(B) \}, \\
\min_{B^{(1,3), A^{(1,1)}}} r(B^{(1,3)} A^{(1,1)}) &= r(AB), \\
\max_{B^{(1,3), A^{(1,3)}}} r(B^{(1,3)} A^{(1,3)}) &= \min \{ m, n, p \}, \\
\min_{B^{(1,3), A^{(1,3)}}} r(B^{(1,3)} A^{(1,3)}) &= r(A) + r(B) - r[A^*, B], \\
\max_{B^{(1,3), A^{(1,2,4)}}} r(B^{(1,3)} A^{(1,2,4)}) &= \min \{ p, r(A) \}, \\
\min_{B^{(1,3), A^{(1,2,4)}}} r(B^{(1,3)} A^{(1,2,4)}) &= r(A) + r(B) - r[A^*, B], \\
\max_{B^{(1,3), A^{(1,2)}}} r(B^{(1,3)} A^{(1,2)}) &= \min \{ m, n, p \}, \\
\min_{B^{(1,3), A^{(1,2)}}} r(B^{(1,3)} A^{(1,2)}) &= r(A) + r(B) - r[A^*, B], \\
\max_{B^{(1,3), A^{(1,1)}}} r(B^{(1,3)} A^{(1,1)}) &= \min \{ r(A), r(B) \}, \\
\min_{B^{(1,3), A^{(1,1)}}} r(B^{(1,3)} A^{(1,1)}) &= r(A) + r(B) - r[A^*, B],
\end{align*}
\]
\[
\begin{align*}
\max_{B^{(1,2)}A^{(1,3,4)}} r(B^{(1,2)}A^{(1,3,4)}) &= \min \{ m, \; r(B) \}, \\
\min_{B^{(1,2)}A^{(1,3,4)}} r(B^{(1,2)}A^{(1,3,4)}) &= r(A) + r(B) - r[A^*, B], \\
\max_{B^{(1,2)}A^{(1,4,2)}} r(B^{(1,2)}A^{(1,4,2)}) &= \min \{ r(A), \; r(B) \}, \\
\min_{B^{(1,2)}A^{(1,4,2)}} r(B^{(1,2)}A^{(1,4,2)}) &= r(A) + r(B) - r[A^*, B], \\
\max_{B^{(1,2)}A^{(1,2,3)}} r(B^{(1,2)}A^{(1,2,3)}) &= \min \{ r(A), \; r(B) \}, \\
\min_{B^{(1,2)}A^{(1,2,3)}} r(B^{(1,2)}A^{(1,2,3)}) &= \max \{ 0, \; r(A) + r(B) - n \}, \\
\max_{B^{(1,2)}A^{(1,4)}} r(B^{(1,2)}A^{(1,4)}) &= \min \{ m, \; r(B) \}, \\
\min_{B^{(1,2)}A^{(1,4)}} r(B^{(1,2)}A^{(1,4)}) &= r(A) + r(B) - r[A^*, B], \\
\max_{B^{(1,2)}A^{(1,3,3)}} r(B^{(1,2)}A^{(1,3,3)}) &= \min \{ m, \; r(B) \}, \\
\min_{B^{(1,2)}A^{(1,3,3)}} r(B^{(1,2)}A^{(1,3,3)}) &= \max \{ 0, \; r(A) + r(B) - n \}, \\
\max_{B^{(1,2)}A^{(1,2,2)}} r(B^{(1,2)}A^{(1,2,2)}) &= \min \{ r(A), \; r(B) \}, \\
\min_{B^{(1,2)}A^{(1,2,2)}} r(B^{(1,2)}A^{(1,2,2)}) &= \max \{ 0, \; r(A) + r(B) - n \}, \\
\max_{B^{(1,2)}A^{(1,4)}} r(B^{(1,2)}A^{(1,4)}) &= \min \{ m, \; r(B) \}, \\
\min_{B^{(1,2)}A^{(1,4)}} r(B^{(1,2)}A^{(1,4)}) &= r(A) + r(B) - r[A^*, B], \\
\max_{B^{(1,2)}A^{(1,2,2)}} r(B^{(1,2)}A^{(1,2,2)}) &= \min \{ r(A), \; r(B) \}, \\
\min_{B^{(1,2)}A^{(1,2,2)}} r(B^{(1,2)}A^{(1,2,2)}) &= \max \{ 0, \; r(A) + r(B) - n \}, \\
\max_{B^{(1,2)}A^{(1,4)}} r(B^{(1,2)}A^{(1,4)}) &= \min \{ m, \; n, \; p \}, \\
\min_{B^{(1,2)}A^{(1,4)}} r(B^{(1,2)}A^{(1,4)}) &= \max \{ 0, \; r(A) + r(B) - n \}, \\
\max_{B^{(1,2)}A^{(1,4)}} r(B^{(1,2)}A^{(1,4)}) &= \min \{ m, \; n, \; p \}, \\
\min_{B^{(1,2)}A^{(1,4)}} r(B^{(1,2)}A^{(1,4)}) &= r(A) + r(B) - r[A^*, B], \\
\max_{B^{(1,2)}A^{(1,4)}} r(B^{(1,2)}A^{(1,4)}) &= \min \{ r(A), \; r(B) \}, \\
\min_{B^{(1,2)}A^{(1,4)}} r(B^{(1,2)}A^{(1,4)}) &= \max \{ 0, \; r(A) + r(B) - n \}, \\
\max_{B^{(1,2)}A^{(1,4)}} r(B^{(1,2)}A^{(1,4)}) &= \min \{ m, \; n, \; p \}, \\
\min_{B^{(1,2)}A^{(1,4)}} r(B^{(1,2)}A^{(1,4)}) &= \max \{ 0, \; r(A) + r(B) - n \}. 
\end{align*}
\]

Proof. Note from (2.32) that \(r(B^tA^{(i,...,j)}) = r(B^*A^{(i,...,j)})\) holds for all \(A^{(i,...,j)}\). Applying (2.48)–(2.61)
to it yields

\[
\begin{align*}
\max_{A^{(1,3,4)}} \quad & r(B^* A^{(1,3,4)}) = \min \{ r(B^*), \ m + r(B^* A^*) - r(A) \} = \min \{ r(B), \ m + r(AB) - r(A) \}, \\
\min_{A^{(1,3,4)}} \quad & r(B^* A^{(1,3,4)}) = r(B^* A^*) = r(AB), \\
\max_{A^{(1,2,3)}} \quad & r(B^* A^{(1,2,3)}) = r(B^* A^*) = r(AB), \\
\min_{A^{(1,2,3)}} \quad & r(B^* A^{(1,2,3)}) = \min \{ r(A), \ r(B^*) \} = \min \{ r(A), \ r(B) \}, \\
\min_{A^{(1,2,3)}} \quad & r(B^* A^{(1,2,3)}) = r(A) + r(B^*) - r[A^*, (B^*)^*] = r(A) + r(B) - r[A^*, B], \\
\max_{A^{(1,3)}} \quad & r(B^* A^{(1,3)}) = \min \{ r(B^*), \ m + r(B^* A^*) - r(A) \} = \min \{ r(B), \ m + r(AB) - r(A) \}, \\
\min_{A^{(1,3)}} \quad & r(B^* A^{(1,3)}) = r(B^* A^*) = r(AB), \\
\max_{A^{(1,3)}} \quad & r(B^* A^{(1,3)}) = \min \{ m, \ r(B^*) \} = \min \{ m, \ r(B) \}, \\
\min_{A^{(1,3)}} \quad & r(B^* A^{(1,3)}) = r(A) + r(B^*) - r[A^*, (B^*)^*] = r(A) + r(B) - r[A^*, B], \\
\max_{A^{(1,2)}} \quad & r(B^* A^{(1,2)}) = \min \{ r(A), \ r(B^*) \} = \min \{ r(A), \ r(B) \}, \\
\min_{A^{(1,2)}} \quad & r(B^* A^{(1,2)}) = r(A) + r(B^*) - r[A^*, (B^*)^*] = r(A) + r(B) - r[A^*, B], \\
\max_{A^{(1)}} \quad & r(B^* A^{(1)}) = \min \{ m, \ r(B^*) \} = \min \{ m, \ r(B) \}, \\
\min_{A^{(1)}} \quad & r(B^* A^{(1)}) = r(A) + r(B^*) - r[A^*, (B^*)^*] = r(A) + r(B) - r[A^*, B],
\end{align*}
\]

thus establishing (3.67)–(3.80).

Note from (2.32) that \( r(B^{(i,\ldots,j)} A^*) = r(B^{(i,\ldots,j)}) A^* \) holds for all \( B^{(i,\ldots,j)} \). Hence, (3.81), (3.82), (3.97), (3.98), (3.113), (3.114), (3.129), (3.130), (3.145), (3.146), (3.161), (3.162), (3.177) and (3.178) follow from (2.34)–(2.47).

Applying (2.62) and (2.63) to (3.14), we obtain

\[
\begin{align*}
\max_{B^{(1,3,4)}, A^{(1,3)}} \min_{V, W} r(B^{(1,3,4)} A^{(1,3)}) &= \max_{V, W} r[(B^\dagger + F_B V E_B)(A^\dagger + F_A W)] \\
&= \min \left\{ r[B^\dagger A^\dagger, B^\dagger F_A, F_B], r\left[ \begin{bmatrix} B^\dagger A^\dagger \\ E_B A^\dagger \end{bmatrix} \right], r\left[ \begin{bmatrix} B^\dagger A^\dagger \\ F_B I_m \end{bmatrix} \right], r\left[ \begin{bmatrix} B^\dagger A^\dagger \\ E_B F_A \end{bmatrix} \right] \right\}, \\
&= \min \left\{ r[B^\dagger A^\dagger], r[B^\dagger F_A], r[B^\dagger F_B] \right\} + \max\{ s_1, s_2 \}, \\
&= r\left[ \begin{bmatrix} B^\dagger A^\dagger \\ E_B A^\dagger \end{bmatrix} \right] + r[B^\dagger A^\dagger, B^\dagger F_A, F_B] + \max\{ s_1, s_2 \}.
\end{align*}
\]

where

\[
\begin{align*}
s_1 &= r\left[ \begin{bmatrix} B^\dagger A^\dagger \\ F_B \\ I_m \end{bmatrix} \right] - r\left[ \begin{bmatrix} B^\dagger A^\dagger \\ F_B \\ 0 \end{bmatrix} \right] - r\left[ \begin{bmatrix} B^\dagger A^\dagger \\ I_m \end{bmatrix} \right], \\
s_2 &= r\left[ \begin{bmatrix} B^\dagger A^\dagger \\ E_B F_A \end{bmatrix} \right] - r\left[ \begin{bmatrix} B^\dagger A^\dagger \\ E_B A^\dagger \end{bmatrix} \right] - r\left[ \begin{bmatrix} B^\dagger A^\dagger \\ E_B F_A \end{bmatrix} \right] - r\left[ \begin{bmatrix} B^\dagger A^\dagger \\ F_B \\ E_B A^\dagger \end{bmatrix} \right].
\end{align*}
\]

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Simplifying the formulas by (2.25)–(2.27) gives

\[ r[B^\dagger A^\dagger, B^\dagger F_A, F_B] = r[B^\dagger A^\dagger, B^\dagger, F_B] = r[B^\dagger A^\dagger, B^\dagger, I_p] = p, \quad r \left[ \frac{B^\dagger A^\dagger}{I_m} \right] = m, \]  

(3.195)

\[ r \left[ \frac{B^\dagger A^\dagger}{I_m} \begin{bmatrix} F_B & 0 \\ 0 & 0 \end{bmatrix} \right] = m + r(F_B) = m + p - r(B), \]  

(3.196)

\[ r \left[ \frac{B^\dagger A^\dagger}{E_B A^\dagger} \begin{bmatrix} B^\dagger F_A & 0 \\ 0 & 0 \end{bmatrix} \right] = r \left[ \begin{bmatrix} B^\dagger A^\dagger & B^\dagger \\ A^\dagger & 0 \end{bmatrix} \right] - r(A) - r(B) \]  

(3.197)

and

\[ r \left[ \frac{B^\dagger A^\dagger}{F_B} \begin{bmatrix} B^\dagger F_A & 0 \\ 0 & 0 \end{bmatrix} \right] = m + r(F_B) = m + p - r(B), \]  

(3.198)

\[ r \left[ \frac{B^\dagger A^\dagger}{E_B A^\dagger} \begin{bmatrix} B^\dagger F_A & 0 \\ 0 & 0 \end{bmatrix} \right] = r \left[ \begin{bmatrix} B^\dagger A^\dagger & B^\dagger F_A \\ E_B A^\dagger & E_B F_A \end{bmatrix} \right] + r(F_B) = n + p - r(B), \]  

(3.199)

\[ r \left[ \frac{B^\dagger A^\dagger}{E_B F_A} \begin{bmatrix} B^\dagger F_A & 0 \\ 0 & 0 \end{bmatrix} \right] = m + r \left[ \begin{bmatrix} B^\dagger F_A \\ E_B F_A \end{bmatrix} \right] = m + r(F_A) = m + n - r(A). \]  

(3.200)

Substituting (3.195)–(3.201) into (3.193) and (3.194) and simplifying leads to (3.91) and (3.92).

Also from (2.18), (2.21) and symmetry of patterns, we obtain from (3.91) and (3.92) the following two

\[ \max_{B(1,4), A(1,3,4)} r(B(1,4)A(1,3,4)) = \max_{B(1,4), A(1,3,4)} r[(B(1,4)A(1,3,4))^*] \]  

(3.201)

\[ \min_{B(1,4), A(1,3,4)} r(B(1,4)A(1,3,4)) = \min_{B(1,4), A(1,3,4)} r[(B(1,4)A(1,3,4))^*] \]  

(3.202)

establishing (3.131) and (3.132).

Note from the definitions of \{1, 2\}, \{1, 2, 3\} and \{1, 2, 4\}-inverses of matrices that

\[ r(B(1,2)A(1,2,3)) = r(B(1,2)A(1,2,3)) = r(B(1,2)A(1,2)) = r(B(1,2)A(1,2)) = r(B(1,2)A(1,2)) \]  

always hold for all \( A(1,2), A(1,2,3), B(1,2) \), and \( B(1,2,4) \). Also see from (3.55) that

\[ \begin{bmatrix} B^\dagger & + & B^\dagger B^\dagger + B E_B \end{bmatrix}(A^\dagger A + F_A W A). \]  

(3.203)

Applying (2.62) and (2.63) to (3.203), we obtain

\[ \max_{V, W} r[(B^\dagger + B V E_B)(A^\dagger A + F_A W A)] \]  

(3.204)

\[ \min_{V, W} r[(B^\dagger A^\dagger, B B^\dagger A^\dagger, B)] + \max_{s_1, s_2} \]  

(3.205)
where

\[
\begin{align*}
s_1 &= r \begin{bmatrix} BB^t A^t A & B \\ A & 0 \end{bmatrix} - r \begin{bmatrix} BB^t A^t A & B & BB^t F_A \\ A & 0 & 0 \end{bmatrix} - r \begin{bmatrix} BB^t A^t A & B \\ A & 0 \end{bmatrix}, \\
s_2 &= r \begin{bmatrix} BB^t A^t A & BB^t F_A \\ E_B A^t A & E_B F_A \end{bmatrix} - r \begin{bmatrix} BB^t A^t A & BB^t F_A & B \\ E_B A^t A & E_B F_A & 0 \end{bmatrix} - r \begin{bmatrix} BB^t A^t A & BB^t F_A \\ E_B A^t A & E_B F_A \end{bmatrix}.
\end{align*}
\]

Simplifying the formulas by (2.25)–(2.27) gives

\[
\begin{align*}
r[BB^t A^t A, BB^t F_A, B] &= r(B), \quad r \begin{bmatrix} BB^t A^t A \\ E_B^t A^t A \end{bmatrix} = r(A), \quad (3.206) \\
r \begin{bmatrix} BB^t A^t A & B \\ A & 0 \end{bmatrix} &= r \begin{bmatrix} BB^t A^t A & B \\ A & 0 \end{bmatrix} = r(A) + r(B), \quad (3.207) \\
r \begin{bmatrix} BB^t A^t A & BB^t F_A \\ E_B A^t A & E_B F_A \end{bmatrix} &= r \begin{bmatrix} BB^t A^t A & BB^t F_A \\ E_B A^t A & E_B F_A \end{bmatrix} = n, \quad (3.208) \\
r \begin{bmatrix} BB^t A^t A & BB^t F_A \\ E_B A^t A & E_B F_A \end{bmatrix} &= r(B) + r \begin{bmatrix} E_B A^t A, E_B F_A \end{bmatrix} = r(B) + r(E_B) = n, \quad (3.209) \\
r \begin{bmatrix} BB^t A^t A & BB^t F_A \\ E_B A^t A & E_B F_A \end{bmatrix} &= r(A) + r \begin{bmatrix} BB^t F_A \\ E_B F_A \end{bmatrix} = r(A) + r(F_A) = n. \quad (3.210)
\end{align*}
\]

Substituting (3.206)–(3.210) into (3.204) and (3.205) and combining (3.202) and (3.203) leads to (3.103), (3.104), (3.109), (3.167), (3.173) and (3.174), respectively.

Note from (2.12)–(2.17) that

\[
\begin{align*}
\{B^{(1,3,4)} A^{(1,3)}\} &\subseteq \{B^{(1,3,4)} A^{(1)}\} \subseteq \{B^{(1)} A^{(1)}\}, \\
\{B^{(1,3,4)} A^{(1,3)}\} &\subseteq \{B^{(1,4)} A^{(1,3)}\} \subseteq \{B^{(1)} A^{(1,3)}\}, \\
\{B^{(1,3,4)} A^{(1,3)}\} &\subseteq \{B^{(1,3)} A^{(1,3)}\} \subseteq \{B^{(1,3)} A^{(1)}\}, \\
\{B^{(1,4)} A^{(1,3,4)}\} &\subseteq \{B^{(1)} A^{(1,3,4)}\}, \\
\{B^{(1,4)} A^{(1,4)}\} &\subseteq \{B^{(1)} A^{(1,4)}\} \subseteq \{B^{(1)} A^{(1)}\}, \\
\{B^{(1,4)} A^{(1,3,4)}\} &\subseteq \{B^{(1,4)} A^{(1)}\} \subseteq \{B^{(1,4)} A^{(1)}\},
\end{align*}
\]

and

\[
\begin{align*}
\{B^{(1,2,4)} A^{(1,2,3)}\} &\subseteq \{B^{(1,2,4)} A^{(1,3)}\} \subseteq \{B^{(1,2,4)} A^{(1)}\} \subseteq \{B^{(1,4)} A^{(1)}\} \subseteq \{B^{(1)} A^{(1)}\}, \\
\{B^{(1,2,4)} A^{(1,2,3)}\} &\subseteq \{B^{(1,2,4)} A^{(1,2)}\} \subseteq \{B^{(1,2,4)} A^{(1)}\}, \\
\{B^{(1,2,4)} A^{(1,2,3)}\} &\subseteq \{B^{(1,2,4)} A^{(1,3)}\} \subseteq \{B^{(1,2)} A^{(1)}\}, \\
\{B^{(1,2,4)} A^{(1,2,3)}\} &\subseteq \{B^{(1,2,4)} A^{(1,2)}\} \subseteq \{B^{(1)} A^{(1,2)}\}, \\
\{B^{(1,2,4)} A^{(1,2)}\} &\subseteq \{B^{(1,2,4)} A^{(1,2,3)}\} \subseteq \{B^{(1)} A^{(1,2,3)}\} \subseteq \{B^{(1)} A^{(1,3)}\}.
\end{align*}
\]

Hence, we obtain from Lemma 2.8 that

\[
\begin{align*}
\max_{B^{(1,3,4)}, A^{(1,3)}} r(B^{(1,3,4)} A^{(1,3)}) &\leq \max_{B^{(1,3,4)}, A^{(1)}} r(B^{(1,3,4)} A^{(1)}) \leq \max_{B^{(1)}, A^{(1)}} r(B^{(1)} A^{(1)}), \quad (3.211) \\
\max_{B^{(1,3,4)}, A^{(1,3)}} r(B^{(1,3,4)} A^{(1,3)}) &\leq \max_{B^{(1,4)}, A^{(1,3)}} r(B^{(1,4)} A^{(1,3)}) \leq \max_{B^{(1)}, A^{(1,3)}} r(B^{(1)} A^{(1,3)}), \quad (3.212) \\
\max_{B^{(1,3,4)}, A^{(1,3)}} r(B^{(1,3,4)} A^{(1,3)}) &\leq \max_{B^{(1,3)}, A^{(1,3)}} r(B^{(1,3)} A^{(1,3)}) \leq \max_{B^{(1)}, A^{(1,3)}} r(B^{(1)} A^{(1,3)}), \quad (3.213) \\
\max_{B^{(1,4)}, A^{(1,3,4)}} r(B^{(1,4)} A^{(1,3,4)}) &\leq \max_{B^{(1)}, A^{(1,3,4)}} r(B^{(1)} A^{(1,3,4)}), \quad (3.214) \\
\max_{B^{(1,4)}, A^{(1,3,4)}} r(B^{(1,4)} A^{(1,3,4)}) &\leq \max_{B^{(1,4)}, A^{(1,4)}} r(B^{(1,4)} A^{(1,4)}) \leq \max_{B^{(1)}, A^{(1,4)}} r(B^{(1)} A^{(1,4)}), \quad (3.215) \\
\max_{B^{(1,4)}, A^{(1,3,4)}} r(B^{(1,4)} A^{(1,3,4)}) &\leq \max_{B^{(1,4)}, A^{(1)}} r(B^{(1,4)} A^{(1)}), \quad (3.216)
\end{align*}
\]
and
\[
\min_{B^{(1,2,4)}, A^{(1,2,3)}} r(B^{(1,2,4)} A^{(1,2,3)}) \geq \min_{B^{(1,2,4)}, A^{(1,3)}} r(B^{(1,2,4)} A^{(1,3)}) \geq \min_{B^{(1,2,4)}, A^{(1)}} r(B^{(1,2,4)} A^{(1)})
\]
\[
\geq \min_{B^{(1,4)}, A^{(1)}} r(B^{(1,4)} A^{(1)}) \geq \min_{B^{(1), A^{(1)}}} r(B^{(1)} A^{(1)}),
\] (3.217)
\[
\min_{B^{(1,2,4)}, A^{(1,2,3)}} r(B^{(1,2,4)} A^{(1,2,3)}) \geq \min_{B^{(1,2,4)}, A^{(1,2)}} r(B^{(1,2,4)} A^{(1,2)}) \geq \min_{B^{(1,4)}, A^{(1,2)}} r(B^{(1,4)} A^{(1,2)}),
\] (3.218)
\[
\min_{B^{(1,2,4)}, A^{(1,2,3)}} r(B^{(1,2,4)} A^{(1,2,3)}) \geq \min_{B^{(1,4)}, A^{(1,2,3)}} r(B^{(1,4)} A^{(1,2,3)}) \geq \min_{B^{(1,4)}, A^{(1,3)}} r(B^{(1,4)} A^{(1,3)}),
\] (3.219)
\[
\min_{B^{(1,2,4)}, A^{(1,2,3)}} r(B^{(1,2,4)} A^{(1,2,3)}) \geq \min_{B^{(1,2), A^{(1,2,3)}}} r(B^{(1,2)} A^{(1,2,3)}) \geq \min_{B^{(1,2), A^{(1,3)}}} r(B^{(1,2)} A^{(1,3)}),
\] (3.220)
\[
\min_{B^{(1,2,4)}, A^{(1,2,3)}} r(B^{(1,2,4)} A^{(1,2,3)}) \geq \min_{B^{(1,2,2)}, A^{(1,2,3)}} r(B^{(1,2,2)} A^{(1,2,3)}) \geq \min_{B^{(1,1), A^{(1,2,3)}}} r(B^{(1,2)} A^{(1,3)}),
\] (3.221)
\[
\min_{B^{(1,2,4)}, A^{(1,2,3)}} r(B^{(1,2,4)} A^{(1,2,3)}) \geq \min_{B^{(1,1), A^{(1,2,3)}}} r(B^{(1,1)} A^{(1,2,3)}) \geq \min_{B^{(1,1), A^{(1,3)}}} r(B^{(1)} A^{(1)}),
\] (3.222)

Combining (3.211)–(3.216) with (3.91), (3.131) and (3.65), we obtain the following results
\[
\max_{B^{(1,3,4), A^{(1,3)}}} r(B^{(1,3,4)} A^{(1,3)}) = \max_{B^{(1,4), A^{(1,3)}}} r(B^{(1,4)} A^{(1,3)}) = \max_{B^{(1,4), A^{(1)}}} r(B^{(1,4)} A^{(1)})
\]
\[
= \max_{B^{(1,3,4), A^{(1)}}} r(B^{(1,3,4)} A^{(1)}) = \max_{B^{(1,3), A^{(1)}}} r(B^{(1,3)} A^{(1)}) = \max_{B^{(1,1), A^{(1)}}} r(B^{(1,1)} A^{(1)})
\]
\[
= \max_{B^{(1,1), A^{(1)}}} r(B^{(1)} A^{(1)}) \geq \max_{B^{(1,1), A^{(1)}}} r(B^{(1)} A^{(1)}) = \max_{B^{(1,1), A^{(1)}}} r(B^{(1)} A^{(1)}) = \max_{\{m, n, p\}} \min\{m, n, p\},
\]

establishing (3.95), (3.137), (3.139), (3.143), (3.155), (3.159), (3.179), (3.185), (3.187), (3.191), respectively.

Combining (3.217)–(3.222) with (3.104) and (3.66), we obtain the following results
\[
\min_{B^{(1,2,4), A^{(1,3,4)}}} r(B^{(1,2,4)} A^{(1,2,3)}) = \min_{B^{(1,2,4), A^{(1,3)}}} r(B^{(1,2,4)} A^{(1,3)}) = \min_{B^{(1,2,4), A^{(1)}}} r(B^{(1,2,4)} A^{(1,2)})
\]
\[
= \min_{B^{(1,4), A^{(1)}}} r(B^{(1,4)} A^{(1)}) = \min_{B^{(1,4), A^{(1,3)}}} r(B^{(1,4)} A^{(1,3)}) = \min_{B^{(1,4), A^{(1,3)}}} r(B^{(1,4)} A^{(1,3)})
\]
\[
= \min_{B^{(1,2), A^{(1)}}} r(B^{(1,2)} A^{(1)}) = \min_{B^{(1,2), A^{(1,3)}}} r(B^{(1,2)} A^{(1,3)}) = \min_{B^{(1,1), A^{(1,3)}}} r(B^{(1,2)} A^{(1,2)})
\]
\[
= \min_{B^{(1,1), A^{(1,3)}}} r(B^{(1,1)} A^{(1,2,3)}) \geq \min_{B^{(1,1), A^{(1,3)}}} r(B^{(1,1)} A^{(1,3)}) \geq \min_{B^{(1,1), A^{(1,2,3)}}} r(B^{(1,1)} A^{(1,2)})
\]
\[
= \min_{B^{(1,1), A^{(1,3)}}} r(B^{(1)} A^{(1)}) \geq \max\{0, r(A) + r(B) - n\},
\]

establishing (3.108), (3.110), (3.112), (3.136), (3.140), (3.142), (3.144), (3.168), (3.172), (3.174), (3.176), (3.184), (3.188), (3.190), (3.192), respectively.

Applying (2.62) and (2.63) to (3.10) yields
\[
\max_{B^{(1,3,4), A^{(1,3,4)}}} r(B^{(1,3,4)} A^{(1,3,4)}) = \max_{V, W} \left[ r \left( V^\dagger + F_B V E_B \right) (A^\dagger + F_A W E_A) \right]
\]
\[
= \min \left\{ r \left[ B^\dagger A^\dagger \right], r \left[ B^\dagger A^\dagger \right], r \left[ B^\dagger A^\dagger \right], r \left[ B^\dagger A^\dagger \right] \right\},
\] (3.223)
\[
\min_{B^{(1,3,4), A^{(1,3,4)}}} r(B^{(1,3,4)} A^{(1,3,4)}) = \min_{V, W} \left[ r \left( V^\dagger + F_B V E_B \right) (A^\dagger + F_A W E_A) \right]
\]
\[
= r \begin{bmatrix} B^\dagger A^\dagger & E_B A^\dagger & E_B A^\dagger & E_B A^\dagger \end{bmatrix} + r \left[ B^\dagger A^\dagger, B^\dagger A^\dagger, F_B + \max\{ s_1, s_2 \},
\] (3.224)
where

\[ s_1 = r \left[ \begin{array}{ccc} B^\dagger A^\dagger & F_B & 0 \\ E_A & 0 & 0 \\ \end{array} \right] - r \left[ \begin{array}{ccc} B^\dagger A^\dagger & F_B & B^\dagger F_A \\ E_A & 0 & 0 \\ \end{array} \right] - r \left[ \begin{array}{ccc} B^\dagger A^\dagger & F_B \\ E_A & 0 \\ \end{array} \right], \]

\[ s_2 = r \left[ \begin{array}{ccc} B^\dagger A^\dagger & B^\dagger F_A \\ E_B A^\dagger & E_B F_A \\ \end{array} \right] - r \left[ \begin{array}{ccc} B^\dagger A^\dagger & B^\dagger F_A & F_B \\ E_B A^\dagger & E_B F_A & 0 \\ \end{array} \right] - r \left[ \begin{array}{ccc} B^\dagger A^\dagger & B^\dagger F_A \\ E_B A^\dagger & E_B F_A & 0 \\ \end{array} \right]. \]

Simplifying the formulas by (2.25)–(2.27) gives

\[ r[B^\dagger A^\dagger, B^\dagger F_A, F_B] = r[B^\dagger A^\dagger, B^\dagger F_A] + r(F_B) = r(B) + p - r(B) = p, \] (3.225)

\[ r\left[ \begin{array}{ccc} B^\dagger A^\dagger & F_B \\ E_A & 0 \\ \end{array} \right] = r\left[ \begin{array}{ccc} B^\dagger A^\dagger \\ E_A \\ \end{array} \right] + r(E_A) = r(A) + m - r(A) = m, \] (3.226)

\[ r\left[ \begin{array}{ccc} B^\dagger A^\dagger & F_B \\ E_A & 0 \\ \end{array} \right] = r(B^\dagger A^\dagger) + r(F_B) = m + r(AB) - r(A) - r(B), \] (3.227)

\[ r\left[ \begin{array}{ccc} B^\dagger A^\dagger & F_B \\ E_A & 0 \\ \end{array} \right] = r\left[ \begin{array}{ccc} B^\dagger A^\dagger \bigg| B^\dagger F_A \\ E_B A^\dagger & E_B F_A \\ \end{array} \bigg] \right] + r\left( E_A \bigg| F_B \right) = m + r(A), \] (3.228)

\[ r\left[ \begin{array}{ccc} B^\dagger A^\dagger & F_B \\ E_A & 0 \\ \end{array} \right] = r\left[ \begin{array}{ccc} B^\dagger A^\dagger \bigg| B^\dagger F_A \\ E_B A^\dagger & E_B F_A \\ \end{array} \bigg] \right] + r\left( E_A \bigg| F_B \right) = m + r(B), \] (3.229)

\[ r\left[ \begin{array}{ccc} B^\dagger A^\dagger & B^\dagger F_A \\ E_B A^\dagger & E_B F_A \\ \end{array} \right] = r\left[ \begin{array}{ccc} B^\dagger A^\dagger \\ E_B A^\dagger \\ \end{array} \right] + r( E_A ) + r(F_B) = n + p - r(B), \] (3.230)

\[ r\left[ \begin{array}{ccc} B^\dagger A^\dagger & B^\dagger F_A \\ E_B A^\dagger & E_B F_A \\ \end{array} \right] = r\left[ \begin{array}{ccc} B^\dagger A^\dagger \\ E_B A^\dagger \\ \end{array} \right] + r( E_A ) = m + n - r(A). \] (3.231)

Substituting (3.226)–(3.31) into (3.233) and (3.24) and simplifying leads to (3.83) and (3.84), respectively.

Note from (2.22) that

\[ r(B^{(1,3,4)} A^{(1,2,4)}) = r(B^{(1,3,4)} A^{(1,2,4)}) = r(B^{(1,3,4)} A^\dagger) \] (3.232)

hold for all $B^{(1,3,4)}$ and $A^{(1,2,4)}$. Hence, (3.85) and (3.86) follow from (3.81) and (3.82). Eqs. (3.115) and (3.116) can be established by pattern symmetry.

Note from (2.3) and (2.6) that

\[ r(B^{(1,3,4)} A^{(1,2,3)}) = r(B^{(1,3,4)} A^{(1,2,3)}) = r(B^{(1,3,4)} A^{(1,2)}) \] (3.233)

hold for all $B^{(1,3,4)}$, $A^{(1,2,3)}$ and $A^{(1,2)}$, where by (3.12) and (3.15),

\[ B^{(1,3,4)} A^{(1,2,3)} = B^{(1,3,4)} A^{(1,2)} = ( B^\dagger + F_B V E_B )( A^\dagger + F_A W A ). \] (3.234)

Applying (2.62) and (2.63) to the right-hand side of (3.234) yields

\[ \max_{V,W} \min_{B,B^\dagger} r(B^\dagger + F_B V E_B ) ( A^\dagger + F_A W A ) \]

\[ = \min_{V,W} \max_{B,B^\dagger} r(B^\dagger, B^\dagger F_A, F_B), \]

\[ \min_{V,W} r(B^\dagger, B^\dagger F_A, F_B) \}

\[ = r( E_B A^\dagger ) + r( B^\dagger A^\dagger F_A, F_B ) + \max\{ s_1, s_2 \} \],

\[ \text{where} \]

\[ s_1 = r \left[ \begin{array}{ccc} B^\dagger A^\dagger & F_B \\ A & 0 \\ \end{array} \right] - r \left[ \begin{array}{ccc} B^\dagger A^\dagger & F_B & B^\dagger F_A \\ A & 0 & 0 \\ \end{array} \right] - r \left[ \begin{array}{ccc} B^\dagger A^\dagger & F_B \\ A & 0 \\ \end{array} \right], \]

\[ s_2 = r \left[ \begin{array}{ccc} B^\dagger A^\dagger & B^\dagger F_A \\ E_B A^\dagger & E_B F_A \\ \end{array} \right] - r \left[ \begin{array}{ccc} B^\dagger A^\dagger & B^\dagger F_A & F_B \\ E_B A^\dagger & E_B F_A & 0 \\ \end{array} \right] - r \left[ \begin{array}{ccc} B^\dagger A^\dagger & B^\dagger F_A \\ E_B A^\dagger & E_B F_A & 0 \\ \end{array} \right]. \]
Simplifying the formulas by (2.25)–(2.27) gives

\[ r[B^\dagger A^\dagger, B^\dagger F_A, F_B] = r[B^\dagger A^\dagger, B^\dagger F_A] + r(F_B) = r(B) + p + r(B) = p, \quad (3.237) \]

\[ r \begin{bmatrix} B^\dagger A^\dagger & B^\dagger F_A \\ E_B A^\dagger & E_B F_A \end{bmatrix} = r(A), \quad r \begin{bmatrix} B^\dagger A^\dagger & F_B \\ A & 0 \end{bmatrix} = r(A) + r(F_B) = p + r(A) - r(B), \quad (3.238) \]

\[ r \begin{bmatrix} B^\dagger A^\dagger & B^\dagger F_A \\ F_B & B^\dagger F_A \end{bmatrix} = r \begin{bmatrix} B^\dagger A^\dagger & B^\dagger F_A \\ E_B A^\dagger & E_B F_A \end{bmatrix} = n, \quad (3.239) \]

\[ r \begin{bmatrix} B^\dagger A^\dagger & F_B \\ A & 0 \end{bmatrix} = r(A) + r(F_B) + r(B^\dagger F_A) = p + r[A^*, B] - r(B), \quad (3.240) \]

Substituting (3.237)–(3.243) into (3.25) and (3.26) and combining them with (3.233) and (3.234) lead to (3.87), (3.88), (3.93) and (3.94), respectively. Eqs. (3.99), (3.100), (3.163) and (3.164) can be established by pattern symmetry.

Applying (2.62) and (2.63) to the right-hand side of (3.13) yields

\[ \max_{B^{(1,3,4)}, A^{(1,4)}} r(B^{(1,3,4)}A^{(1,4)}) = \max_{V, W} r[(B^\dagger + F_B V E_B)(A^\dagger + W E_A)] \]

\[ = \min \left\{ r[B^\dagger A^\dagger, B^\dagger, F_B], r \begin{bmatrix} B^\dagger A^\dagger \\ E_B A^\dagger \end{bmatrix}, r \begin{bmatrix} B^\dagger A^\dagger & F_B \\ E_A & 0 \end{bmatrix}, r \begin{bmatrix} B^\dagger A^\dagger & B^\dagger \\ E_B A^\dagger & E_B \end{bmatrix} \right\}, \quad (3.244) \]

\[ \min_{B^{(1,3,4)}, A^{(1,4)}} r(B^{(1,3,4)}A^{(1,4)}) = \min_{V, W} r[(B^\dagger + F_B V E_B)(A^\dagger + W E_A)] \]

\[ = r \begin{bmatrix} B^\dagger A^\dagger \\ E_A \end{bmatrix} + r[B^\dagger A^\dagger, B^\dagger, F_B] + \max\{ s_1, s_2 \}, \quad (3.245) \]

where

\[ s_1 = r \begin{bmatrix} B^\dagger A^\dagger & F_B \\ E_A & 0 \end{bmatrix} - r \begin{bmatrix} B^\dagger A^\dagger & F_B \\ E_A & 0 \end{bmatrix} - r \begin{bmatrix} B^\dagger A^\dagger & F_B \\ E_A & 0 \end{bmatrix}, \]

\[ s_2 = r \begin{bmatrix} B^\dagger A^\dagger & B^\dagger \\ E_B A^\dagger & E_B \end{bmatrix} - r \begin{bmatrix} B^\dagger A^\dagger & B^\dagger \\ E_B A^\dagger & E_B \end{bmatrix} - r \begin{bmatrix} B^\dagger A^\dagger & B^\dagger \\ E_A & 0 \end{bmatrix}. \]

Simplifying the formulas by (2.25)–(2.27) gives

\[ r[B^\dagger A^\dagger, B^\dagger, F_B] = r[B^\dagger, F_B] = p, \quad r \begin{bmatrix} B^\dagger A^\dagger \\ E_B A^\dagger \end{bmatrix} = m, \quad (3.246) \]

\[ r \begin{bmatrix} B^\dagger A^\dagger & F_B \\ E_A & 0 \end{bmatrix} = m + p + r(AB) - r(A) - r(B), \quad r \begin{bmatrix} B^\dagger A^\dagger & B^\dagger \\ E_B A^\dagger & E_B \end{bmatrix} = r \begin{bmatrix} B^\dagger \\ E_B \end{bmatrix} = n, \quad (3.247) \]

\[ r \begin{bmatrix} B^\dagger A^\dagger & F_B \\ E_A & 0 \end{bmatrix} = m + p - r(A), \quad r \begin{bmatrix} B^\dagger A^\dagger & F_B \\ E_B A^\dagger & 0 \end{bmatrix} = m + p - r(B), \quad (3.248) \]

\[ r \begin{bmatrix} B^\dagger A^\dagger & B^\dagger \\ E_B A^\dagger & E_B \end{bmatrix} = r \begin{bmatrix} B^\dagger \\ E_B \end{bmatrix} + r(F_B) = n + p - r(B), \quad (3.249) \]

\[ r \begin{bmatrix} B^\dagger A^\dagger & B^\dagger \\ E_A & 0 \end{bmatrix} = r \begin{bmatrix} B^\dagger \\ E_B \end{bmatrix} + r(E_A) = m + n - r(A). \quad (3.250) \]
Substituting (3.246)–(3.250) into (3.244) and (3.245) leads to (3.89) and (3.90), respectively. Eqs. (3.147) and (3.148) can be established by pattern symmetry.

Applying (2.34) and (2.35) gives

\[
\min_{B^{(1,3,4)}} r(B^{(1,3,4)} A^{(1)}) = r(B^* A^{(1)}), \quad \min_{A^{(1)}} r(B^* A^{(1)}) = r(A) + r(B) - r[A^*, B].
\]  

Combining the two equalities in (3.251) yields

\[
\min_{B^{(1,3,4)}, A^{(1)}} r(B^{(1,3,4)} A^{(1)}) = r(A) + r(B) - r[A^*, B],
\]
as required for (3.96). Eq. (3.180) can be established by pattern symmetry.

Note from (2.2) and (2.6) that

\[
r(B^{(1,2,4)} A^{(1,4)}) = r(B^{(1,2,4)} A^{(1,4)}) = r(B^{(1,2,4)} A^{(1,4)}) = r(B^{(1,2,4)} A^{(1,4)})
\]  

hold for all \(B^{(1,2,4)}\) and \(A^{(1,4)}\). Applying (2.36) and (2.37) to (3.252) and simplifying, we obtain

\[
\max_{B^{(1,2,4)}, A^{(1,4)}} r(B^{(1,2,4)} A^{(1,4)}) = \max_{B^{(1,2,4)}} r(B^{(1,2,4)} A^*) = \min\{ r(A), r(B) \},
\]

\[
\min_{B^{(1,2,4)}, A^{(1,4)}} r(B^{(1,2,4)} A^{(1,4)}) = \min_{B^{(1,2,4)}} r(B^{(1,2,4)} A^*) = r(A) + r(B) - r[A^*, B],
\]
as required for (3.101) and (3.102). Eqs. (3.119) and (3.120) can be established by pattern symmetry.

Note from (2.2) and (2.6) that

\[
r(B^{(1,2,4)} A^{(1,4)}) = r(BB^{(1,2,4)} A^{(1,4)}) = r(B^{(1,2,4)} A^{(1,4)}) = r(B^{(1,2,4)} A^{(1,4)})
\]  

hold for all \(B^{(1,2,4)}\) and \(B^{(1,2,4,2)}\), where by (3.53),

\[
BB^{(1,2)} A^{(1,4)} = (BB^* + BV E_B)(A^* + WE_A).
\]  

Applying (2.62) and (2.63) to the right-hand side of (3.254) and simplifying, we obtain

\[
\max_{V, W} r[(BB^* + BV E_B)(A^* + WE_A)]
\]  

\[
= \min \left\{ r[BB^* A^*, BB^*, B], r \left[ \begin{bmatrix} BB^* A^* \\ E_B A^* \end{bmatrix} \right], \ r \left[ \begin{bmatrix} BB^* A^* \\ E_A \end{bmatrix} \right], \ r \left[ \begin{bmatrix} BB^* A^* \\ E_B A^* \\ E_B \end{bmatrix} \right] \right\},
\]  

\[
\min_{V, W} r[(BB^* + BV E_B)(A^* + WE_A)]
\]  

\[
= r \left[ \begin{bmatrix} BB^* A^* \\ E_B A^* \\ E_A \end{bmatrix} \right] + r[BB^* A^*, BB^*, B] + \max\{ s_1, s_2 \},
\]  

where

\[
s_1 = r \left[ \begin{bmatrix} BB^* A^* \\ E_A \end{bmatrix} \right] - r \left[ \begin{bmatrix} BB^* A^* \\ E_A \\ 0 \end{bmatrix} \right] - r \left[ \begin{bmatrix} BB^* A^* \\ E_B A^* \\ 0 \end{bmatrix} \right],
\]

\[
s_2 = r \left[ \begin{bmatrix} BB^* A^* \\ E_B A^* \\ E_B \end{bmatrix} \right] - r \left[ \begin{bmatrix} BB^* A^* \\ E_B A^* \\ 0 \end{bmatrix} \right] - r \left[ \begin{bmatrix} BB^* A^* \\ E_B A^* \\ E_A \end{bmatrix} \right].
\]

Simplifying the formulas by (2.25)–(2.27) gives

\[
r[BB^* A^*, BB^*, B] = r(B), \quad r \left[ \begin{bmatrix} BB^* A^* \\ E_B A^* \end{bmatrix} \right] = m,
\]  

\[
r \left[ \begin{bmatrix} BB^* A^* \\ E_A \end{bmatrix} \right] = m - r(A) + r(B), \quad r \left[ \begin{bmatrix} BB^* A^* \\ E_B A^* \end{bmatrix} \right] = n,
\]

\[
r \left[ \begin{bmatrix} BB^* A^* \\ E_A \end{bmatrix} \right] = r(B) + r(E_A) = m - r(A) + r(B),
\]

\[
r \left[ \begin{bmatrix} BB^* A^* \\ E_B A^* \end{bmatrix} \right] = r(E_A) + r(B) + r(E_B) = m + r[A^*, B] - r(A),
\]

\[
r \left[ \begin{bmatrix} BB^* A^* \\ E_B A^* \end{bmatrix} \right] = r(B) + r(E_B) = n,
\]

\[
r \left[ \begin{bmatrix} BB^* A^* \\ E_B A^* \end{bmatrix} \right] = r(E_A) + r \left[ \begin{bmatrix} BB^* \end{bmatrix} \right] = m + n - r(A).
\]
Substituting (3.257)–(3.262) into (3.255) and (3.256) and combining them with (3.253) and (3.254) lead to (3.105), (3.106), (3.169) and (3.170), respectively. Eqs. (3.151), (3.152), (3.157) and (3.158) can be established by pattern symmetry.

Note from (2.2) and (2.6) that
\[ r(B^{(1,2,4)}A^{(1,3)}) = r(BB^{(1,2,4)}A^{(1,3)}) = r(BB^{(1,2)}A^{(1,3)}) = r(BB^{(1,2)}A^{(1,3)}) \] (3.263)
hold for all \( B^{(1,2,4)}, B^{(1,2)} \) and \( A^{(1,3)} \), where by (3.54),
\[ BB^{(1,2)}A^{(1,3)} = (BB^\dagger + BV_E B)(A^\dagger + F_A W). \] (3.264)

Applying (2.62) to the right-hand side of (3.264) and simplifying, we obtain
\[
\max_{V, W} r[(BB^\dagger + BV_E B)(A^\dagger + F_A W)] \\
= \min \left\{ r[BB^\dagger A^\dagger, BB^\dagger F_A, B], r \left[ \begin{array}{cc} BB^\dagger A^\dagger & B \\ \frac{1}{m^2} & 0 \end{array} \right], r \left[ \begin{array}{cc} BB^\dagger A^\dagger & BB^\dagger F_A \\ E_B A^\dagger & E_B F_A \end{array} \right] \right\} \\
= \min \{ r(B), m, m + r(B), n \} = \min \{ m, r(B) \}. \] (3.265)

Combining (3.263), (3.264) and (3.265) yields (3.107) and (3.171). Eqs. (3.135) and (3.141) can be established by pattern symmetry.

Note from (2.2) and (2.6) that
\[ r(B^{(1,2,4)}A^{(1)}) = r(BB^{(1,2,4)}A^{(1)}) = r(BB^{(1,2)}A^{(1)}) = r(BB^{(1,2)}A^{(1)}) \] (3.266)
hold for all \( B^{(1,2,4)}, B^{(1,2)} \) and \( A^{(1)} \). Applying (2.44) to \( B^{(1,2)}A^{(1)} \) and simplifying, we obtain
\[
\max_{B^{(1,2,4)}} r(B^{(1,2,4)}A^{(1)}) = \max_{B^{(1,2)}} r(B^{(1,2)}A^{(1)}) = \min \{ r(A^{(1)}), r(B) \}, \max_{A^{(1)}} r(A^{(1)}) = \min \{ m, n \}. \] (3.267)

Combining the two equalities in (3.267) yields
\[
\max_{B^{(1,2,3)}, A^{(1)}} r(B^{(1,2,4)}A^{(1)}) = \max_{B^{(1,2)}, A^{(1)}} r(B^{(1,2)}A^{(1)}) = \min \{ m, r(B) \},
\]
as required for (3.111) and (3.175). Eqs. (3.183) and (3.189) can be established by pattern symmetry.

Note from (2.1) and (2.3), (2.32) and (2.33) that
\[
\begin{align*}
& r(B^{(1,2,3)}A^{(1,2,4)}) = r(BB^{(1,2,3)}A^{(1,2,4)}A) = r(BB^\dagger A^\dagger A) = r(AB), \\
& r(B^{(1,2,3)}A^{(1,2,3)}) = r(BB^{(1,2,3)}A^{(1,2,3)}) = r(BB^\dagger A^{(1,2,3)}) = r(B^\dagger A^{(1,2,3)}), \\
& r(B^{(1,2,3)}A^{(1,4)}) = r(BB^{(1,2,3)}A^{(1,4)}) = r(BB^\dagger A^{(1,4)}) = r(B^\dagger A^{(1,4)}), \\
& r(B^{(1,2,3)}A^{(1,3)}) = r(BB^{(1,2,3)}A^{(1,3)}) = r(BB^\dagger A^{(1,3)}) = r(B^\dagger A^{(1,3)}), \\
& r(B^{(1,2,3)}A^{(1,2)}) = r(BB^{(1,2,3)}A^{(1,2)}) = r(BB^\dagger A^{(1,2)}) = r(B^\dagger A^{(1,2)}), \\
& r(B^{(1,2,3)}A^{(1)}) = r(BB^{(1,2,3)}A^{(1)}) = r(BB^\dagger A^{(1)}) = r(B^\dagger A^{(1)}).
\end{align*}
\]

So that (3.117)–(3.128) follow from (3.69)–(3.80). Eqs. (3.133), (3.134), (3.149), (3.150), (3.165), (3.166), (3.181), (3.182) can be established by pattern symmetry.

Applying (2.63) to (3.37) and simplifying, we obtain
\[
\begin{align*}
\min_{B^{(1,4)}A^{(1,4)}} r(B^{(1,4)}A^{(1,4)}) &= \min_{V, W} r[(B^\dagger + VE_B)(A^\dagger + WE_A)] \\
&= r \left[ \begin{array}{cc} B^\dagger A^\dagger & I_p \\ E_A & 0 \end{array} \right] + r[BB^\dagger A^\dagger, B^\dagger, I_p] + \max \{ s_1, s_2 \},
\end{align*}\] (3.268)
Simplifying the formulas by (2.25)–(2.27) gives
\[ r[B^\dagger A^\dagger, B^\dagger, I_p] = p, \quad r \begin{bmatrix} B^\dagger A^\dagger \\ E_A \end{bmatrix} = m, \quad r \begin{bmatrix} B^\dagger A^\dagger \\ E_A \end{bmatrix} = m + p - r(A), \] (3.269)
\[ r \begin{bmatrix} B^\dagger A^\dagger \\ E_B A^\dagger \\ E_B \end{bmatrix} = n, \quad r \begin{bmatrix} B^\dagger A^\dagger \\ E_A \\ 0 \end{bmatrix} = p + r(E_A) = m + p - r(A), \] (3.270)
\[ r \begin{bmatrix} B^\dagger A^\dagger \\ E_A \\ 0 \end{bmatrix} = p + r(E_A) + r(E_B A^\dagger) = m + p + r[A^*, B] - r(A) - r(B), \] (3.271)
\[ r \begin{bmatrix} B^\dagger A^\dagger \\ E_B A^\dagger \\ E_B \end{bmatrix} = p + r(E_B) = n + p - r(B), \quad r \begin{bmatrix} B^\dagger A^\dagger \\ E_A \\ 0 \end{bmatrix} = m + n - r(A). \] (3.272)

Substituting (3.269)–(3.272) into (3.268) lead to (3.138). Eq. (3.156) can be established by pattern symmetry.

Applying (2.62) and (2.63) to (3.45) and simplifying, we obtain
\[
\max_{B^{(1,3)}, A^{(1,4)}} r(B^{(1,3)} A^{(1,4)}) = \max_{V,W} \{ r[(B^\dagger + F_B V)(A^\dagger + W E_A)] \}
\]
\[
= \min \left\{ r[B^\dagger A^\dagger, B^\dagger, F_B], r \begin{bmatrix} B^\dagger A^\dagger \\ A^\dagger \\ E_A \end{bmatrix}, r \begin{bmatrix} B^\dagger A^\dagger \\ E_A \\ 0 \end{bmatrix}, r \begin{bmatrix} B^\dagger A^\dagger \\ I_n \end{bmatrix} \right\},
\] (3.273)
\[
\min_{B^{(1,3)}, A^{(1,4)}} r(B^{(1,3)} A^{(1,4)}) = \min_{V,W} \{ r[(B^\dagger + F_B V)(A^\dagger + W E_A)] \}
\]
\[
= r \begin{bmatrix} B^\dagger A^\dagger \\ A^\dagger \\ E_A \end{bmatrix} + r[B^\dagger A^\dagger, B^\dagger, F_B] + \max \{ s_1, s_2 \},
\] (3.274)
where
\[
s_1 = r \begin{bmatrix} B^\dagger A^\dagger \\ A^\dagger \\ E_A \end{bmatrix} - r \begin{bmatrix} B^\dagger A^\dagger \\ E_A \\ 0 \end{bmatrix} - r \begin{bmatrix} B^\dagger A^\dagger \\ A^\dagger \\ 0 \end{bmatrix},
\]
\[
s_2 = r \begin{bmatrix} B^\dagger A^\dagger \\ A^\dagger \\ I_n \end{bmatrix} - r \begin{bmatrix} B^\dagger A^\dagger \\ E_A \\ 0 \end{bmatrix} - r \begin{bmatrix} B^\dagger A^\dagger \\ I_n \end{bmatrix}.
\]

Simplifying the formulas by (2.25)–(2.27) gives
\[ r[B^\dagger A^\dagger, B^\dagger, F_B] = p, \quad r \begin{bmatrix} B^\dagger A^\dagger \\ A^\dagger \\ E_A \end{bmatrix} = m, \quad r \begin{bmatrix} B^\dagger A^\dagger \\ E_A \\ 0 \end{bmatrix} = m + p + r(AB) - r(A) - r(B), \] (3.275)
\[ r \begin{bmatrix} B^\dagger A^\dagger \\ A^\dagger \\ I_n \end{bmatrix} = n, \quad r \begin{bmatrix} B^\dagger A^\dagger \\ E_A \\ 0 \end{bmatrix} = m + p - r(A), \quad r \begin{bmatrix} B^\dagger A^\dagger \\ E_A \\ 0 \end{bmatrix} = m + p - r(B), \] (3.276)
\[ r \begin{bmatrix} B^\dagger A^\dagger \\ A^\dagger \\ I_n \end{bmatrix} = n + p - r(B), \quad r \begin{bmatrix} B^\dagger A^\dagger \\ A^\dagger \\ I_n \end{bmatrix} = m + n - r(A). \] (3.277)

Substituting (3.275)–(3.277) into (3.273) and (3.274) leads to (3.153) and (3.154).

Applying (2.43) and (2.61) to \(B^{(1,3)} A^{(1)}\) gives
\[
\min_{B^{(1,3)}, A^{(1)}} r(B^{(1,3)} A^{(1)}) = r(B^* A^{(1)}) = r(A) + r(B) - r[A^*, B]. \] (3.278)

Combining the two equalities in (3.278) yields
\[
\min_{B^{(1,3)}, A^{(1)}} r(B^{(1,3)} A^{(1)}) = r(A) + r(B) - r[A^*, B],
\]
as required for (3.160). Eq. (3.186) can be established by pattern symmetry.
Theorem 3.2 shows that there do exist analytical formulas for calculating the global maximum and minimum ranks of (3.2)–(3.64) even it is hard to believe this work can completely be finished at the very beginning. In fact, the present author made sufficient preparations for the matrix tricks and tools used in the above proofs since 1980s, while a systematic theory on matrix rank formulas and generalized inverses of matrices were well developed.

Many rank equalities and inequalities for the ranks of the products $B^{(i,\ldots,j)}A^{(i,\ldots,j)}$ can be established from the previous theorem. In particular, we can directly obtain the difference of the maximum and minimum ranks of (3.2)–(3.64) even it is hard to believe this work can completely be finished at the very beginning.

**Corollary 3.3.** Let $A \in \mathbb{C}^{m \times n}$ and $B \in \mathbb{C}^{n \times p}$ be given with $m \neq 0$, $n \neq 0$ and $p \neq 0$, and let $M = [A^*, B]$. Then, the following results hold.

1. The spread of the rank of $B^\dagger A^{(1,3,4)}$ is
   \[
   \max_{A^{(1,3,4)}} r(B^\dagger A^{(1,3,4)}) - \min_{A^{(1,3,4)}} r(B^\dagger A^{(1,3,4)}) = \min \{ r(B) - r(AB), \ m - r(A) \}. \tag{3.279}
   \]
   Hence, the following two statements are equivalent:
   (i) $r(B^\dagger A^{(1,3,4)}) = r(AB)$ holds for all $A^{(1,3,4)}$.
   (ii) $r(AB) = r(B)$ or $r(A) = m$.

2. The spread of the rank of $B^\dagger A^{(1,2,4)}$ is
   \[
   \max_{A^{(1,2,4)}} r(B^\dagger A^{(1,2,4)}) - \min_{A^{(1,2,4)}} r(B^\dagger A^{(1,2,4)}) = \min \{ r(M) - r(A), \ r(M) - r(B) \}. \tag{3.280}
   \]
   Hence, the following two statements are equivalent:
   (i) $r(B^\dagger A^{(1,2,4)}) = r(AB)$ holds for all $A^{(1,2,4)}$.
   (ii) $\mathcal{R}(A^*) \subseteq \mathcal{R}(B)$ or $\mathcal{R}(A^*) \supseteq \mathcal{R}(B)$.

3. The spread of the rank of $B^\dagger A^{(1,4)}$ is
   \[
   \max_{A^{(1,4)}} r(B^\dagger A^{(1,4)}) - \min_{A^{(1,4)}} r(B^\dagger A^{(1,4)}) = \min \{ r(B) - r(AB), \ m - r(A) \}. \tag{3.281}
   \]
   Hence, the following two statements are equivalent:
   (i) $r(B^\dagger A^{(1,4)}) = r(AB)$ holds for all $A^{(1,4)}$.
   (ii) $r(AB) = r(B)$ or $r(A) = m$.

4. The spread of the rank of $B^\dagger A^{(1,3)}$ is
   \[
   \max_{A^{(1,3)}} r(B^\dagger A^{(1,3)}) - \min_{A^{(1,3)}} r(B^\dagger A^{(1,3)}) = \min \{ m - r(A) - r(B) + r(M), \ r(M) - r(A) \}. \tag{3.282}
   \]
   Hence, the following two statements are equivalent:
   (i) $r(B^\dagger A^{(1,3)}) = r(AB)$ holds for all $A^{(1,3)}$.
   (ii) $\mathcal{R}(A^*) \supseteq \mathcal{R}(B)$ or $r(M) = r(A) + r(B) - m$.

5. The spread of the rank of $B^\dagger A^{(1,2)}$ is
   \[
   \max_{A^{(1,2)}} r(B^\dagger A^{(1,2)}) - \min_{A^{(1,2)}} r(B^\dagger A^{(1,2)}) = \min \{ r(M) - r(A), \ r(M) - r(B) \}. \tag{3.283}
   \]
   Hence, the following two statements are equivalent:
   (i) $r(B^\dagger A^{(1,2)}) = r(AB)$ holds for all $A^{(1,2)}$.
   (ii) $\mathcal{R}(A^*) \subseteq \mathcal{R}(B)$ or $\mathcal{R}(A^*) \supseteq \mathcal{R}(B)$.

6. The spread of the rank of $B^\dagger A^{(1)}$ is
   \[
   \max_{A^{(1)}} r(B^\dagger A^{(1)}) - \min_{A^{(1)}} r(B^\dagger A^{(1)}) = \min \{ m - r(A) - r(B) + r(M), \ r(M) - r(A) \}. \tag{3.284}
   \]
   Hence, the following two statements are equivalent:
The spread of the rank of $B^{(1,3,4)}A^{(1)}$ is
\[
\max_{B^{(1,3,4)}, A^{(1)}} r(B^{(1,3,4)} A^{(1)}) - \min_{B^{(1,3,4)}, A^{(1)}} r(B^{(1,3,4)} A^{(1)}) = \min\{ r(A) - r(AB), \ p - r(B) \}. \tag{3.285}
\]

Hence, the following two statements are equivalent:

(i) $r(B^{(1,3,4)} A^{(1)}) = r(AB)$ holds for all $B^{(1,3,4)}$.
(ii) $r(AB) = r(A) \text{ or } r(B) = p$.

The spread of the rank of $B^{(1,3,4)}A^{(1,3,4)}$ is
\[
\max_{B^{(1,3,4)}, A^{(1,3,4)}} r(B^{(1,3,4)} A^{(1,3,4)}) - \min_{B^{(1,3,4)}, A^{(1,3,4)}} r(B^{(1,3,4)} A^{(1,3,4)}) = \min\{ m - r(AB), \ n - r(AB), \ p - r(AB), \ m + p - r(A) - r(B) \}. \tag{3.286}
\]

Hence, the following two statements are equivalent:

(i) $r(B^{(1,3,4)} A^{(1,3,4)}) = r(AB)$ holds for all $B^{(1,3,4)}$ and $A^{(1,3,4)}$.
(ii) $r(AB) = m \text{ or } r(AB) = n \text{ or } r(AB) = p \text{ or } r(A) + r(B) = m + p$.

The spread of the rank of $B^{(1,3,4)}A^{(1,2,4)}$ is
\[
\max_{B^{(1,3,4)}, A^{(1,2,4)}} r(B^{(1,3,4)} A^{(1,2,4)}) - \min_{B^{(1,3,4)}, A^{(1,2,4)}} r(B^{(1,3,4)} A^{(1,2,4)}) = \min\{ r(A) - r(AB), \ p - r(B) \}. \tag{3.287}
\]

Hence, the following two statements are equivalent:

(i) $r(B^{(1,3,4)} A^{(1,2,4)}) = r(AB)$ holds for all $B^{(1,3,4)}$ and $A^{(1,2,4)}$.
(ii) $r(AB) = r(A) \text{ or } r(B) = p$.

The spread of the rank of $B^{(1,3,4)}A^{(1,2,3)}$ is
\[
\max_{B^{(1,3,4)}, A^{(1,2,3)}} r(B^{(1,3,4)} A^{(1,2,3)}) - \min_{B^{(1,3,4)}, A^{(1,2,3)}} r(B^{(1,3,4)} A^{(1,2,3)}) = \min\{ p - r(A) - r(B) + r(M), \ r(M) - r(B) \}. \tag{3.288}
\]

Hence, the following two statements are equivalent:

(i) $r(B^{(1,3,4)} A^{(1,2,3)}) = r(AB)$ holds for all $B^{(1,3,4)}$ and $A^{(1,2,3)}$.
(ii) $r(M) = r(A) + r(B) - p \text{ or } \mathcal{R}(A^*) \subseteq \mathcal{R}(B)$.

The spread of the rank of $B^{(1,3,4)}A^{(1,4)}$ is
\[
\max_{B^{(1,3,4)}, A^{(1,4)}} r(B^{(1,3,4)} A^{(1,4)}) - \min_{B^{(1,3,4)}, A^{(1,4)}} r(B^{(1,3,4)} A^{(1,4)}) = \min\{ m - r(AB), \ n - r(AB), \ p - r(AB), \ m + p - r(A) - r(B) \}. \tag{3.289}
\]

Hence, the following two statements are equivalent:

(i) $r(B^{(1,3,4)} A^{(1,4)}) = r(AB)$ holds for all $B^{(1,3,4)}$ and $A^{(1,4)}$.
(ii) $r(AB) = m \text{ or } r(AB) = n \text{ or } r(AB) = p \text{ or } r(A) + r(B) = m + p$.

The spread of the rank of $B^{(1,3,4)}A^{(1,3)}$ is
\[
\max_{B^{(1,3,4)}, A^{(1,3)}} r(B^{(1,3,4)} A^{(1,3)}) - \min_{B^{(1,3,4)}, A^{(1,3)}} r(B^{(1,3,4)} A^{(1,3)}) = \min\{ m - r(A) - r(B) + r(M), \ n - r(A) - r(B) + r(M), \ p - r(A) - r(B) + r(M) \}. \tag{3.290}
\]

Hence, the following two statements are equivalent:

(i) $r(B^{(1,3,4)} A^{(1,3)}) = r(AB)$ holds for all $B^{(1,3,4)}$ and $A^{(1,3)}$.
(ii) $r(A) + r(B) - r(M) = \min\{ m, \ n, \ p \}$.
The spread of the rank of $B^{(1,3,4)} A^{(1,2)}$ is

$$
\max_{B^{(1,3,4)}, A^{(1,2)}} r(B^{(1,3,4)} A^{(1,2)}) - \min_{B^{(1,3,4)}, A^{(1,2)}} r(B^{(1,3,4)} A^{(1,2)})
= \min \{ p - r(A) - r(B) + r(M), \ r(M) - r(B) \}. \tag{3.291}
$$

Hence, the following two statements are equivalent:

(i) $r(B^{(1,3,4)} A^{(1,2)}) = r(AB)$ holds for all $B^{(1,3,4)}$ and $A^{(1,2)}$.

(ii) $r(M) = r(A) + r(B) - p$ or $\mathcal{R}(A^*) \subseteq \mathcal{R}(B)$.

The spread of the rank of $B^{(1,3,4)} A^{(1)}$ is

$$
\max_{B^{(1,3,4)}, A^{(1)}} r(B^{(1,3,4)} A^{(1)}) - \min_{B^{(1,3,4)}, A^{(1)}} r(B^{(1,3,4)} A^{(1)})
= \min \{ m - r(A) - r(B) + r(M), \ n - r(A) - r(B) + r(M), \ p - r(A) - r(B) + r(M) \}. \tag{3.292}
$$

Hence, the following two statements are equivalent:

(i) $r(B^{(1,3,4)} A^{(1)}) = r(AB)$ holds for all $B^{(1,3,4)}$ and $A^{(1)}$.

(ii) $r(A) + r(B) - r(M) = \min \{ m, n, p \}$.

The spread of the rank of $B^{(1,2,4)} A^1$ is

$$
\max_{B^{(1,2,4)}} r(B^{(1,2,4)} A^1) - \min_{B^{(1,2,4)}} r(B^{(1,2,4)} A^1)
= \min \{ r(M) - r(A), \ r(M) - r(B) \}. \tag{3.293}
$$

Hence, the following two statements are equivalent:

(i) $r(B^{(1,2,4)} A^1) = r(AB)$ holds for all $B^{(1,2,4)}$ and $A^1$.

(ii) $\mathcal{R}(A^*) \subseteq \mathcal{R}(B)$ or $\mathcal{R}(A^*) \supseteq \mathcal{R}(B)$.

The spread of the rank of $B^{(1,2,4)} A^{(1,3,4)}$ is

$$
\max_{B^{(1,2,4)}, A^{(1,3,4)}} r(B^{(1,2,4)} A^{(1,3,4)}) - \min_{B^{(1,2,4)}, A^{(1,3,4)}} r(B^{(1,2,4)} A^{(1,3,4)})
= \min \{ m - r(A) - r(B) + r(M), \ r(M) - r(A) \}. \tag{3.294}
$$

Hence, the following two statements are equivalent:

(i) $r(B^{(1,2,4)} A^{(1,3,4)}) = r(AB)$ holds for all $B^{(1,2,4)}$ and $A^{(1,3,4)}$.

(ii) $r(M) = r(A) + r(B) - m$ or $\mathcal{R}(A^*) \supseteq \mathcal{R}(B)$.

The spread of the rank of $B^{(1,2,4)} A^{(1,2,4)}$ is

$$
\max_{B^{(1,2,4)}, A^{(1,2,4)}} r(B^{(1,2,4)} A^{(1,2,4)}) - \min_{B^{(1,2,4)}, A^{(1,2,4)}} r(B^{(1,2,4)} A^{(1,2,4)})
= \min \{ r(M) - r(A), \ r(M) - r(B) \}. \tag{3.295}
$$

Hence, the following two statements are equivalent:

(i) $r(B^{(1,2,4)} A^{(1,2,4)}) = r(AB)$ holds for all $B^{(1,2,4)}$ and $A^{(1,2,4)}$.

(ii) $\mathcal{R}(A^*) \subseteq \mathcal{R}(B)$ or $\mathcal{R}(A^*) \supseteq \mathcal{R}(B)$.

The spread of the rank of $B^{(1,2,4)} A^{(1,2,3)}$ is

$$
\max_{B^{(1,2,4)}, A^{(1,2,3)}} r(B^{(1,2,4)} A^{(1,2,3)}) - \min_{B^{(1,2,4)}, A^{(1,2,3)}} r(B^{(1,2,4)} A^{(1,2,3)})
= \min \{ r(A), \ r(B), \ n - r(A), \ n - r(B) \}. \tag{3.296}
$$

Hence, the following two statements are equivalent:

(i) $r(B^{(1,2,4)} A^{(1,2,3)}) = r(AB)$ for all $B^{(1,2,4)}$ and $A^{(1,2,3)}$.

(ii) $A = 0$ or $B = 0$ or $r(A) = n$ or $r(B) = n$.

The spread of the rank of $B^{(1,2,4)} A^{(1,4)}$ is

$$
\max_{B^{(1,2,4)}, A^{(1,4)}} r(B^{(1,2,4)} A^{(1,4)}) - \min_{B^{(1,2,4)}, A^{(1,4)}} r(B^{(1,2,4)} A^{(1,4)})
= \min \{ m - r(A) - r(B) + r(M), \ r(M) - r(A) \}. \tag{3.297}
$$

Hence, the following two statements are equivalent:
(i) $r(B^{(1,2,4)} A^{(1,4)}) = r(AB)$ holds for all $B^{(1,2,4)}$ and $A^{(1,4)}$.
(ii) $r(M) = r(A) + r(B) - m$ or $\mathcal{R}(A^*) \supseteq \mathcal{R}(B)$.

(21) The spread of the rank of $B^{(1,2,4)} A^{(1,3)}$ is

$$\max_{B^{(1,2,4)}, A^{(1,3)}} r(B^{(1,2,4)} A^{(1,3)}) - \min_{B^{(1,2,4)}, A^{(1,3)}} r(B^{(1,2,4)} A^{(1,3)})$$

$$= \min\{ m, \ n - r(A), \ r(B), \ m + n - r(A) - r(B) \}.$$  \hspace{1cm} (3.298)

Hence, the following two statements are equivalent:

(i) $r(B^{(1,2,4)} A^{(1,3)}) = r(AB)$ for all $B^{(1,2,4)}$ and $A^{(1,3)}$.
(ii) $r(A) = n$, or $B = 0$, or $r(A) = m$ and $r(B) = n$.

(22) The spread of the rank of $B^{(1,2,4)} A^{(1,2)}$ is

$$\max_{B^{(1,2,4)}, A^{(1,2)}} r(B^{(1,2,4)} A^{(1,2)}) - \min_{B^{(1,2,4)}, A^{(1,2)}} r(B^{(1,2,4)} A^{(1,2)})$$

$$= \min\{ r(A), \ r(B), \ n - r(A), \ n - r(B) \}.$$  \hspace{1cm} (3.299)

Hence, the following two statements are equivalent:

(i) $r(B^{(1,2,4)} A^{(1,2)}) = r(AB)$ holds for all $B^{(1,2,4)}$ and $A^{(1,2)}$
(ii) $A = 0$ or $B = 0$ or $r(A) = n$ or $r(B) = n$.

(23) The spread of the rank of $B^{(1,2,4)} A^{(1)}$ is

$$\max_{B^{(1,2,4)}, A^{(1)}} r(B^{(1,2,4)} A^{(1)}) - \min_{B^{(1,2,4)}, A^{(1)}} r(B^{(1,2,4)} A^{(1)})$$

$$= \min\{ m, \ n - r(A), \ r(B), \ m + n - r(A) - r(B) \}.$$  \hspace{1cm} (3.300)

Hence, the following two statements are equivalent:

(i) $r(B^{(1,2,4)} A^{(1)}) = r(AB)$ holds for all $B^{(1,2,4)}$ and $A^{(1)}$.
(ii) $r(A) = n$, or $B = 0$, or $r(A) = m$ and $r(B) = n$.

(24) $r(B^{(1,2,3)} A^{(1)}) = r(AB)$ holds for all $B^{(1,2,3)}$ and $A^{(1)}$.

(25) The spread of the rank of $B^{(1,2,3)} A^{(1,3,4)}$ is

$$\max_{B^{(1,2,3)}, A^{(1,3,4)}} r(B^{(1,2,3)} A^{(1,3,4)}) - \min_{B^{(1,2,3)}, A^{(1,3,4)}} r(B^{(1,2,3)} A^{(1,3,4)})$$

$$= \min\{ r(B) - r(AB), \ m - r(A) \}.$$  \hspace{1cm} (3.301)

Hence, the following two statements are equivalent:

(i) $r(B^{(1,2,3)} A^{(1,3,4)}) = r(AB)$ holds for all $B^{(1,2,3)}$ and $A^{(1,3,4)}$.
(ii) $r(AB) = r(B)$ or $r(A) = m$.

(26) $r(B^{(1,2,3)} A^{(1,2,4)}) = r(AB)$ holds for all $B^{(1,2,3)}$ and $A^{(1,2,4)}$.

(27) The spread of the rank of $B^{(1,2,3)} A^{(1,2,3)}$ is

$$\max_{B^{(1,2,3)}, A^{(1,2,3)}} r(B^{(1,2,3)} A^{(1,2,3)}) - \min_{B^{(1,2,3)}, A^{(1,2,3)}} r(B^{(1,2,3)} A^{(1,2,3)})$$

$$= \min\{ r(M) - r(A), \ r(M) - r(B) \}.$$  \hspace{1cm} (3.302)

Hence, the following two statements are equivalent:

(i) $r(B^{(1,2,3)} A^{(1,2,3)}) = r(AB)$ holds for all $B^{(1,2,3)}$ and $A^{(1,2,3)}$.
(ii) $\mathcal{R}(A^*) \subseteq \mathcal{R}(B)$ or $\mathcal{R}(A^*) \supseteq \mathcal{R}(B)$.

(28) The spread of the rank of $B^{(1,2,3)} A^{(1,4)}$ is

$$\max_{B^{(1,2,3)}, A^{(1,4)}} r(B^{(1,2,3)} A^{(1,4)}) - \min_{B^{(1,2,3)}, A^{(1,4)}} r(B^{(1,2,3)} A^{(1,4)})$$

$$= \min\{ r(B) - r(AB), \ m - r(A) \}.$$  \hspace{1cm} (3.303)

Hence, the following two statements are equivalent:
(i) \( r(B^{(1,2,3)} A^{(1,4)}) = r(AB) \) for all \( B^{(1,2,3)} \) and \( A^{(1,4)} \).
(ii) \( r(AB) = r(B) \) or \( r(A) = m \).

(29) The spread of the rank of \( B^{(1,2,3)} A^{(1,3)} \) is

\[
\max_{B^{(1,2,3)}, A^{(1,3)}} r(B^{(1,2,3)} A^{(1,3)}) - \min_{B^{(1,2,3)}, A^{(1,3)}} r(B^{(1,2,3)} A^{(1,3)})
= \min\{ m - r(A) - r(B) + r(M), \ r(M) - r(A) \}. \quad (3.304)
\]

Hence, the following two statements are equivalent:

(i) \( r(B^{(1,2,3)} A^{(1,3)}) = r(AB) \) holds for all \( B^{(1,2,3)} \) and \( A^{(1,3)} \).
(ii) \( r(M) = r(A) + r(B) - m \) or \( \mathcal{R}(A^*) \supseteq \mathcal{R}(B) \).

(30) The spread of the rank of \( B^{(1,2,3)} A^{(1,2)} \) is

\[
\max_{B^{(1,2,3)}, A^{(1,2)}} r(B^{(1,2,3)} A^{(1,2)}) - \min_{B^{(1,2,3)}, A^{(1,2)}} r(B^{(1,2,3)} A^{(1,2)})
= \min\{ r(M) - r(A), \ r(M) - r(B) \}. \quad (3.305)
\]

Hence, the following two statements are equivalent:

(i) \( r(B^{(1,2,3)} A^{(1,2)}) = r(AB) \) holds for all \( B^{(1,2,3)} \) and \( A^{(1,2)} \).
(ii) \( \mathcal{R}(A^*) \subseteq \mathcal{R}(B) \) or \( \mathcal{R}(A^*) \supseteq \mathcal{R}(B) \).

(31) The spread of the rank of \( B^{(1,2,3)} A^{(1)} \) is

\[
\max_{B^{(1,2,3)}, A^{(1)}} r(B^{(1,2,3)} A^{(1)}) - \min_{B^{(1,2,3)}, A^{(1)}} r(B^{(1,2,3)} A^{(1)})
= \min\{ m - r(A) - r(B) + r(M), \ r(M) - r(A) \}. \quad (3.306)
\]

Hence, the following two statements are equivalent:

(i) \( r(B^{(1,2,3)} A^{(1)}) = r(AB) \) for all \( B^{(1,2,3)} \) and \( A^{(1)} \).
(ii) \( r(M) = r(A) + r(B) - m \) or \( \mathcal{R}(A^*) \supseteq \mathcal{R}(B) \).

(32) The spread of the rank of \( B^{(1,4)} A^\dagger \) is

\[
\max_{B^{(1,4)}} r(B^{(1,4)} A^\dagger) - \min_{B^{(1,4)}} r(B^{(1,4)} A^\dagger) = \min\{ p - r(A) - r(B) + r(M), \ r(M) - r(B) \}. \quad (3.307)
\]

Hence, the following two statements are equivalent:

(i) \( r(B^{(1,4)} A^\dagger) = r(AB) \) holds for all \( B^{(1,4)} \)
(ii) \( r(M) = r(A) + r(B) - p \) or \( \mathcal{R}(A^*) \subseteq \mathcal{R}(B) \).

(33) The spread of the rank of \( B^{(1,4)} A^{(1,3,4)} \) is

\[
\max_{B^{(1,4)}, A^{(1,3,4)}} r(B^{(1,4)} A^{(1,3,4)}) - \min_{B^{(1,4)}, A^{(1,3,4)}} r(B^{(1,4)} A^{(1,3,4)})
= \min\{ m - r(A) - r(B) + r(M), \ n - r(A) - r(B) + r(M), \ p - r(A) - r(B) + r(M) \}. \quad (3.308)
\]

Hence, the following two statements are equivalent:

(i) \( r(B^{(1,4)} A^{(1,3,4)}) = r(AB) \) holds for all \( B^{(1,4)} \) and \( A^{(1,3,4)} \).
(ii) \( r(A) + r(B) - r(M) = \min\{ m, n, p \} \).

(34) The spread of the rank of \( B^{(1,4)} A^{(1,2,4)} \) is

\[
\max_{B^{(1,4)}, A^{(1,2,4)}} r(B^{(1,4)} A^{(1,2,4)}) - \min_{B^{(1,4)}, A^{(1,2,4)}} r(B^{(1,4)} A^{(1,2,4)})
= \min\{ p - r(A) - r(B) + r(M), \ r(M) - r(B) \}. \quad (3.309)
\]

Hence, the following two statements are equivalent:

(i) \( r(B^{(1,4)} A^{(1,2,4)}) = r(AB) \) holds for all \( B^{(1,4)} \) and \( A^{(1,2,4)} \).
(ii) \( r(M) = r(A) + r(B) - p \) or \( \mathcal{R}(A^*) \subseteq \mathcal{R}(B) \).
(35) The spread of the rank of $B^{(1,4)}A^{(1,2,3)}$ is

$$\max_{B^{(1,4)}, A^{(1,2,3)}} r(B^{(1,4)}A^{(1,2,3)}) - \min_{B^{(1,4)}, A^{(1,2,3)}} r(B^{(1,4)}A^{(1,2,3)}) = \min\{ p, \ r(A), \ n - r(B), \ n + p - r(A) - r(B) \}. \quad (3.310)$$

Hence, the following two statements are equivalent:

(i) $r(B^{(1,4)}A^{(1,2,3)}) = r(AB)$ holds for all $B^{(1,4)}$ and $A^{(1,2,3)}$.

(ii) $A = 0$ or $r(B) = n$ or $r(A) + r(B) = n + p$

(36) The spread of the rank of $B^{(1,4)}A^{(1,4)}$ is

$$\max_{B^{(1,4)}, A^{(1,4)}} r(B^{(1,4)}A^{(1,4)}) - \min_{B^{(1,4)}, A^{(1,4)}} r(B^{(1,4)}A^{(1,4)}) = \min\{ m - r(A) - r(B) + r(M), \ n - r(A) - r(B) + r(M), \ p - r(A) - r(B) + r(M) \}. \quad (3.311)$$

Hence, the following two statements are equivalent:

(i) $r(B^{(1,4)}A^{(1,4)}) = r(AB)$ holds for all $B^{(1,4)}$ and $A^{(1,4)}$.

(ii) $r(A) + r(B) - r(M) = \min\{ m, n, p \}$.

(37) The spread of the rank of $B^{(1,4)}A^{(1,3)}$ is

$$\max_{B^{(1,4)}, A^{(1,3)}} r(B^{(1,4)}A^{(1,3)}) - \min_{B^{(1,4)}, A^{(1,3)}} r(B^{(1,4)}A^{(1,3)}) = \min\{ m, n, p, m + n - r(A) - r(B), 2n - r(A) - r(B), n + p - r(A) - r(B) \}. \quad (3.312)$$

Hence, the following two statements are equivalent:

(i) $r(B^{(1,4)}A^{(1,3)}) = r(AB)$ holds for all $B^{(1,4)}$ and $A^{(1,3)}$.

(ii) $r(A) + r(B) = \min\{ m + n, 2n, n + p \}$.

(38) The spread of the rank of $B^{(1,4)}A^{(1,2)}$ is

$$\max_{B^{(1,4)}, A^{(1,2)}} r(B^{(1,4)}A^{(1,2)}) - \min_{B^{(1,4)}, A^{(1,2)}} r(B^{(1,4)}A^{(1,2)}) = \min\{ p, \ r(A), \ n - r(B), \ n + p - r(A) - r(B) \}. \quad (3.313)$$

Hence, the following two statements are equivalent:

(i) $r(B^{(1,4)}A^{(1,2)}) = r(AB)$ holds for all $B^{(1,4)}$ and $A^{(1,2)}$.

(ii) $A = 0$, or $r(B) = n$, or $r(A) + r(B) = n + p$.

(39) The spread of the rank of $B^{(1,4)}A^{(1)}$ is

$$\max_{B^{(1,4)}, A^{(1)}} r(B^{(1,4)}A^{(1)}) - \min_{B^{(1,4)}, A^{(1)}} r(B^{(1,4)}A^{(1)}) = \min\{ m, n, p, m + n - r(A) - r(B), 2n - r(A) - r(B), n + p - r(A) - r(B) \}. \quad (3.314)$$

Hence, the following two statements are equivalent:

(i) $r(B^{(1,4)}A^{(1)}) = r(AB)$ for all $B^{(1,4)}$ and $A^{(1)}$.

(ii) $r(A) + r(B) = \min\{ m + n, 2n, n + p \}$.

(40) The spread of the rank of $B^{(1,3)}A^\dagger$ is

$$\max_{B^{(1,3)}} r(B^{(1,3)}A^\dagger) - \min_{B^{(1,3)}} r(B^{(1,3)}A^\dagger) = \min\{ r(A) - r(AB), \ p - r(B) \}. \quad (3.315)$$

Hence, the following two statements are equivalent:

(i) $r(B^{(1,3)}A^\dagger) = r(AB)$ holds for all $B^{(1,3)}$.

(ii) $r(AB) = r(A)$ or $r(B) = p$.

(41) The spread of the rank of $B^{(1,3)}A^{(1,3,4)}$ is

$$\max_{B^{(1,3)}, A^{(1,3,4)}} r(B^{(1,3)}A^{(1,3,4)}) - \min_{B^{(1,3)}, A^{(1,3,4)}} r(B^{(1,3)}A^{(1,3,4)}) = \min\{ m - r(AB), \ n - r(AB), \ p - r(AB), \ m + p - r(A) - r(B) \}. \quad (3.316)$$

Hence, the following two statements are equivalent:
\( (i) \) \( r(B^{(1,3)} A^{(1,3,4)}) = r(AB) \) holds for all \( B^{(1,3)} \) and \( A^{(1,3,4)} \).

\( (ii) \) \( r(AB) = m \) or \( r(AB) = n \) or \( r(AB) = p \) or \( r(A) + r(B) = m + p \).

\( (42) \) The spread of the rank of \( B^{(1,3)} A^{(1,2,4)} \) is

\[
\max_{B^{(1,3)}, A^{(1,2,4)}} r(B^{(1,3)} A^{(1,2,4)}) - \min_{B^{(1,3)}, A^{(1,2,4)}} r(B^{(1,3)} A^{(1,2,4)})
= \min\{ r(A) - r(AB), \ p - r(B) \}. \tag{3.317}
\]

Hence, the following two statements are equivalent:

\( (i) \) \( r(B^{(1,3)} A^{(1,2,4)}) = r(AB) \) holds for all \( B^{(1,3)} \) and \( A^{(1,2,4)} \).

\( (ii) \) \( r(AB) = r(A) \) or \( r(B) = p \).

\( (43) \) The spread of the rank of \( B^{(1,3)} A^{(1,2,3)} \) is

\[
\max_{B^{(1,3)}, A^{(1,2,3)}} r(B^{(1,3)} A^{(1,2,3)}) - \min_{B^{(1,3)}, A^{(1,2,3)}} r(B^{(1,3)} A^{(1,2,3)})
= \min\{ p - r(A) - r(B) + r(M), \ r(M) - r(B) \}. \tag{3.318}
\]

Hence, the following two statements are equivalent:

\( (i) \) \( r(B^{(1,3)} A^{(1,2,3)}) = r(AB) \) holds for all \( B^{(1,3)} \) and \( A^{(1,2,3)} \).

\( (ii) \) \( r(M) = r(A) + r(B) - p \) or \( \mathcal{R}(A^*) \subseteq \mathcal{R}(B) \).

\( (44) \) The spread of the rank of \( B^{(1,3)} A^{(1,4)} \) is

\[
\max_{B^{(1,3)}, A^{(1,4)}} r(B^{(1,3)} A^{(1,4)}) - \min_{B^{(1,3)}, A^{(1,4)}} r(B^{(1,3)} A^{(1,4)})
= \min\{ m - r(AB), \ n - r(AB), \ p - r(AB), \ m + p - r(A) - r(B) \}. \tag{3.319}
\]

Hence, the following two statements are equivalent:

\( (i) \) \( r(B^{(1,3)} A^{(1,4)}) = r(AB) \) holds for all \( B^{(1,3)} \) and \( A^{(1,4)} \).

\( (ii) \) \( r(AB) = m \) or \( r(AB) = n \) or \( r(AB) = p \) or \( r(A) + r(B) = m + p \).

\( (45) \) The spread of the rank of \( B^{(1,3)} A^{(1,3)} \) is

\[
\max_{B^{(1,3)}, A^{(1,3)}} r(B^{(1,3)} A^{(1,3)}) - \min_{B^{(1,3)}, A^{(1,3)}} r(B^{(1,3)} A^{(1,3)})
= \min\{ m - r(A) - r(B) + r(M), \ n - r(A) - r(B) + r(M), \ p - r(A) - r(B) + r(M) \}. \tag{3.320}
\]

Hence, the following two statements are equivalent:

\( (i) \) \( r(B^{(1,3)} A^{(1,3)}) = r(AB) \) holds for all \( B^{(1,3)} \) and \( A^{(1,3)} \).

\( (ii) \) \( r(A) + r(B) - r(M) = \min\{ m, n, p \} \).

\( (46) \) The spread of the rank of \( B^{(1,3)} A^{(1,2)} \) is

\[
\max_{B^{(1,3)}, A^{(1,2)}} r(B^{(1,3)} A^{(1,2)}) - \min_{B^{(1,3)}, A^{(1,2)}} r(B^{(1,3)} A^{(1,2)})
= \min\{ p - r(A) - r(B) + r(M), \ r(M) - r(B) \}. \tag{3.321}
\]

Hence, the following two statements are equivalent:

\( (i) \) \( r(B^{(1,3)} A^{(1,2)}) = r(AB) \) holds for all \( B^{(1,3)} \) and \( A^{(1,2)} \).

\( (ii) \) \( r(M) = r(A) + r(B) - p \) or \( \mathcal{R}(A^*) \subseteq \mathcal{R}(B) \).

\( (47) \) The spread of the rank of \( B^{(1,3)} A^{(1)} \) is

\[
\max_{B^{(1,3)}, A^{(1)}} r(B^{(1,3)} A^{(1)}) - \min_{B^{(1,3)}, A^{(1)}} r(B^{(1,3)} A^{(1)})
= \min\{ m - r(A) - r(B) + r(M), \ n - r(A) - r(B) + r(M), \ p - r(A) - r(B) + r(M) \}. \tag{3.322}
\]

Hence, the following two statements are equivalent:

\( (i) \) \( r(B^{(1,3)} A^{(1)}) = r(AB) \) holds for all \( B^{(1,3)} \) and \( A^{(1)} \).

\( (ii) \) \( r(A) + r(B) - r(M) = \min\{ m, n, p \} \).

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The spread of the rank of \(B^{(1,2)}A^+\) is
\[
\max_{B^{(1,2)}} r(B^{(1,2)}A^+) - \min_{B^{(1,2)}} r(B^{(1,2)}A^+) = \min\{ r(M) - r(A), \ r(M) - r(B) \}. \tag{3.323}
\]

Hence, the following two statements are equivalent:
(i) \(r(B^{(1,2)}A^+) = r(AB)\) holds for all \(B^{(1,2)}\) and \(A^+\).
(ii) \(\mathcal{R}(A^+) \subseteq \mathcal{R}(B)\) or \(\mathcal{R}(A^+) \supseteq \mathcal{R}(B)\).

The spread of the rank of \(B^{(1,2)}A^{(1,3,4)}\) is
\[
\max_{B^{(1,2)}, A^{(1,3,4)}} r(B^{(1,2)}A^{(1,3,4)}) - \min_{B^{(1,2)}, A^{(1,3,4)}} r(B^{(1,2)}A^{(1,3,4)}) = \min\{ m - r(A) - r(B) + r(M), \ r(M) - r(A) \}. \tag{3.324}
\]

Hence, the following two statements are equivalent:
(i) \(r(B^{(1,2)}A^{(1,3,4)}) = r(AB)\) holds for all \(B^{(1,2)}\) and \(A^{(1,3,4)}\).
(ii) \(r(M) = r(A) + r(B) - m\) or \(\mathcal{R}(A^+) \supseteq \mathcal{R}(B)\).

The spread of the rank of \(B^{(1,2)}A^{(1,2,4)}\) is
\[
\max_{B^{(1,2)}, A^{(1,2,4)}} r(B^{(1,2)}A^{(1,2,4)}) - \min_{B^{(1,2)}, A^{(1,2,4)}} r(B^{(1,2)}A^{(1,2,4)}) = \min\{ r(M) - r(A), \ r(M) - r(B) \}. \tag{3.325}
\]

Hence, the following two statements are equivalent:
(i) \(r(B^{(1,2)}A^{(1,2,4)}) = r(AB)\) holds for all \(B^{(1,2)}\) and \(A^{(1,2,4)}\).
(ii) \(\mathcal{R}(A^+) \subseteq \mathcal{R}(B)\) or \(\mathcal{R}(A^+) \supseteq \mathcal{R}(B)\).

The spread of the rank of \(B^{(1,2)}A^{(1,2,3)}\) is
\[
\max_{B^{(1,2)}, A^{(1,2,3)}} r(B^{(1,2)}A^{(1,2,3)}) - \min_{B^{(1,2)}, A^{(1,2,3)}} r(B^{(1,2)}A^{(1,2,3)}) = \min\{ r(A), \ r(B), \ n - r(A), \ n - r(B) \}. \tag{3.326}
\]

Hence, the following two statements are equivalent:
(i) \(r(B^{(1,2)}A^{(1,2,3)}) = r(AB)\) holds for all \(B^{(1,2)}\) and \(A^{(1,2,3)}\).
(ii) \(A = 0\) or \(B = 0\) or \(r(A) = n\) or \(r(B) = n\).

The spread of the rank of \(B^{(1,2)}A^{(1,4)}\) is
\[
\max_{B^{(1,2)}, A^{(1,4)}} r(B^{(1,2)}A^{(1,4)}) - \min_{B^{(1,2)}, A^{(1,4)}} r(B^{(1,2)}A^{(1,4)}) = \min\{ m - r(A) - r(B) + r(M), \ r(M) - r(A) \}. \tag{3.327}
\]

Hence, the following two statements are equivalent:
(i) \(r(B^{(1,2)}A^{(1,4)}) = r(AB)\) holds for all \(B^{(1,2)}\) and \(A^{(1,4)}\).
(ii) \(r(M) = r(A) + r(B) - m\) or \(\mathcal{R}(A^+) \supseteq \mathcal{R}(B)\).

The spread of the rank of \(B^{(1,2)}A^{(1,3)}\) is
\[
\max_{B^{(1,2)}, A^{(1,3)}} r(B^{(1,2)}A^{(1,3)}) - \min_{B^{(1,2)}, A^{(1,3)}} r(B^{(1,2)}A^{(1,3)}) = \min\{ m, \ r(B), \ m + n - r(A) - r(B), \ n - r(A) \}. \tag{3.328}
\]

Hence, the following two statements are equivalent:
(i) \(r(B^{(1,2)}A^{(1,3)}) = r(AB)\) holds for all \(B^{(1,2)}\) and \(A^{(1,3)}\).
(ii) \(r(A) = n\) or \(B = 0\) or \(r(A) + r(B) = m + n\).

The spread of the rank of \(B^{(1,2)}A^{(1,2)}\) is
\[
\max_{B^{(1,2)}, A^{(1,2)}} r(B^{(1,2)}A^{(1,2)}) - \min_{B^{(1,2)}, A^{(1,2)}} r(B^{(1,2)}A^{(1,2)}) = \min\{ r(A), \ r(B), \ n - r(A), \ n - r(B) \}. \tag{3.329}
\]

Hence, the following two statements are equivalent:
(i) \( r(B^{(1,2)} A^{(1,2)}) = r(AB) \) holds for all \( B^{(1,2)} \) and \( A^{(1,2)} \).
(ii) \( A = 0 \) or \( B = 0 \) or \( r(A) = n \) or \( r(B) = n \).

(55) The spread of the rank of \( B^{(1,2)} A^{(1)} \) is

\[
\max_{B^{(1,2)}, A^{(1)}} r(B^{(1,2)} A^{(1)}) - \min_{B^{(1,2)}, A^{(1)}} r(B^{(1,2)} A^{(1)})
= \min\{ m, \ n - r(A), \ r(B), \ m + n - r(A) - r(B) \}. \tag{3.330}
\]

Hence, the following two statements are equivalent:
(i) \( r(B^{(1,2)} A^{(1)}) = r(AB) \) holds for all \( B^{(1,2)} \) and \( A^{(1)} \),
(ii) \( r(A) = n \) or \( B = 0 \) or \( r(A) + r(B) = m + n \).

(56) The spread of the rank of \( B^{(1)} A^\dagger \) is

\[
\max_{B^{(1)}} r(B^{(1)} A^\dagger) - \min_{B^{(1)}} r(B^{(1)} A^\dagger) = \min\{ p - r(A) - r(B) + r(M), \ r(M) - r(B) \}. \tag{3.331}
\]

Hence, the following two statements are equivalent:
(i) \( r(B^{(1)} A^\dagger) = r(AB) \) holds for all \( B^{(1)} \) and \( A^\dagger \),
(ii) \( r(M) = r(A) + r(B) - p \) or \( \mathcal{R}(A^\dagger) \subseteq \mathcal{R}(B) \).

(57) The spread of the rank of \( B^{(1)} A^{(1,3,4)} \) is

\[
\max_{B^{(1)}, A^{(1,3,4)}} r(B^{(1)} A^{(1,3,4)}) - \min_{B^{(1)}, A^{(1,3,4)}} r(B^{(1)} A^{(1,3,4)})
= \min\{ m - r(A) - r(B) + r(M), \ n - r(A) - r(B) + r(M), \ p - r(A) - r(B) + r(M) \}. \tag{3.332}
\]

Hence, the following two statements are equivalent:
(i) \( r(B^{(1)} A^{(1,3,4)}) = r(AB) \) for all \( B^{(1)} \) and \( A^{(1,3,4)} \),
(ii) \( r(A) + r(B) - r(M) = \min\{ m, n, p \} \).

(58) The spread of the rank of \( B^{(1)} A^{(1,2,4)} \) is

\[
\max_{B^{(1)}, A^{(1,2,4)}} r(B^{(1)} A^{(1,2,4)}) - \min_{B^{(1)}, A^{(1,2,4)}} r(B^{(1)} A^{(1,2,4)})
= \min\{ p - r(A) - r(B) + r(M), \ r(M) - r(B) \}. \tag{3.333}
\]

Hence, the following two statements are equivalent:
(i) \( r(B^{(1)} A^{(1,2,4)}) = r(AB) \) holds for all \( B^{(1)} \) and \( A^{(1,2,4)} \),
(ii) \( r(M) = r(A) + r(B) - p \) or \( \mathcal{R}(A^\dagger) \subseteq \mathcal{R}(B) \).

(59) The spread of the rank of \( B^{(1)} A^{(1,2,3)} \) is

\[
\max_{B^{(1)}, A^{(1,2,3)}} r(B^{(1)} A^{(1,2,3)}) - \min_{B^{(1)}, A^{(1,2,3)}} r(B^{(1)} A^{(1,2,3)})
= \min\{ p, \ r(A), \ n - r(B), \ n + p - r(A) - r(B) \}. \tag{3.334}
\]

Hence, the following two statements are equivalent:
(i) \( r(B^{(1)} A^{(1,2,3)}) = r(AB) \) for all \( B^{(1)} \) and \( A^{(1,2,3)} \),
(ii) \( A = 0 \) or \( r(B) = n \) or \( r(A) + r(B) = n + p \).

(60) The spread of the rank of \( B^{(1)} A^{(1,4)} \) is

\[
\max_{B^{(1)}, A^{(1,4)}} r(B^{(1)} A^{(1,4)}) - \min_{B^{(1)}, A^{(1,4)}} r(B^{(1)} A^{(1,4)})
= \min\{ m - r(A) - r(B) + r(M), \ n - r(A) - r(B) + r(M), \ p - r(A) - r(B) + r(M) \}. \tag{3.335}
\]

Hence, the following two statements are equivalent:
(i) \( r(B^{(1)} A^{(1,4)}) = r(AB) \) for all \( B^{(1)} \) and \( A^{(1,4)} \),
(ii) \( r(A) + r(B) - r(M) = \min\{ m, n, p \} \).
Corollary 3.4. The spread of the rank of $B^{(1)} A^{(1,3)}$ is
\[
\max_{B^{(1)}, A^{(1,3)}} r(B^{(1)} A^{(1,3)}) - \min_{B^{(1)}, A^{(1,3)}} r(B^{(1)} A^{(1,3)}) = \min\{ m, n, p, m + n - r(A) - r(B), 2n - r(A) - r(B), n + p - r(A) - r(B) \}. \tag{3.336}
\]

Hence, the following two statements are equivalent:
(i) $r(B^{(1)} A^{(1,3)}) = r(AB)$ holds for all $B^{(1)}$ and $A^{(1,3)}$.
(ii) $r(A) + r(B) = \min\{ m + n, 2n, n + p \}$.

The spread of the rank of $B^{(1)} A^{(1,2)}$ is
\[
\max_{B^{(1)}, A^{(1,2)}} r(B^{(1)} A^{(1,2)}) - \min_{B^{(1)}, A^{(1,2)}} r(B^{(1)} A^{(1,2)}) = \min\{ p, r(A), n - r(B), n + p - r(A) - r(B) \}. \tag{3.337}
\]

Hence, the following two statements are equivalent:
(i) $r(B^{(1)} A^{(1,2)}) = r(AB)$ holds for all $B^{(1)}$ and $A^{(1,2)}$.
(ii) $A = 0$ or $r(B) = n$ or $r(A) + r(B) = n + p$.

The spread of the rank of $B^{(1)} A^{(1)}$ is
\[
\max_{B^{(1)}, A^{(1)}} r(B^{(1)} A^{(1)}) - \min_{B^{(1)}, A^{(1)}} r(B^{(1)} A^{(1)}) = \min\{ m, n, p, m + n - r(A) - r(B), 2n - r(A) - r(B), n + p - r(A) - r(B) \}. \tag{3.338}
\]

Hence, the following two statements are equivalent:
(i) $r(B^{(1)} A^{(1)}) = r(AB)$ holds for all $B^{(1)}$ and $A^{(1)}$.
(ii) $r(A) + r(B) = \min\{ m + n, 2n, n + p \}$.

It can be seen from the above corollary that the right-hand sides of many rank formulas are the same.

Corollary 3.4. Let $A \in \mathbb{C}^{m \times n}$ and $B \in \mathbb{C}^{n \times p}$ be given. Then, the following results hold.

(a) The following statements are equivalent:
(1) $r(B^t A^{(1,2,3)}) \geq r(AB)$ holds for all $A^{(1,2,3)}$.
(2) $r(B^t A^{(1,3)}) \geq r(AB)$ holds for all $A^{(1,3)}$.
(3) $r(B^t A^{(1,2)}) \geq r(AB)$ holds for all $A^{(1,2)}$.
(4) $r(B^t A^{(1)}) \geq r(AB)$ holds for all $A^{(1)}$.
(5) $r(B^{(1,3,4)} A^{(1,2,3)}) \geq r(AB)$ holds for all $B^{(1,3,4)}$ and $A^{(1,2,3)}$.
(6) $r(B^{(1,3,4)} A^{(1,3)}) \geq r(AB)$ holds for all $B^{(1,3,4)}$ and $A^{(1,3)}$.
(7) $r(B^{(1,3,4)} A^{(1,2)}) \geq r(AB)$ holds for all $B^{(1,3,4)}$ and $A^{(1,2)}$.
(8) $r(B^{(1,3,4)} A^{(1)}) \geq r(AB)$ holds for all $B^{(1,3,4)}$ and $A^{(1)}$.
(9) $r(B^{(1,2,4)} A^{(1)}) \geq r(AB)$ holds for all $B^{(1,2,4)}$.
(10) $r(B^{(1,2,4)} A^{(1,3,4)}) \geq r(AB)$ holds for all $B^{(1,2,4)}$ and $A^{(1,3,4)}$.
(11) $r(B^{(1,2,4)} A^{(1,2,4)}) \geq r(AB)$ holds for all $B^{(1,2,4)}$ and $A^{(1,2,4)}$.
(12) $r(B^{(1,2,4)} A^{(1,4)}) \geq r(AB)$ holds for all $B^{(1,2,4)}$ and $A^{(1,4)}$.
(13) $r(B^{(1,2,3)} A^{(1,2,3)}) \geq r(AB)$ holds for all $B^{(1,2,3)}$ and $A^{(1,2,3)}$.
(14) $r(B^{(1,2,3)} A^{(1,3)}) \geq r(AB)$ holds for all $B^{(1,2,3)}$ and $A^{(1,3)}$.
(15) $r(B^{(1,2,3)} A^{(1,2)}) \geq r(AB)$ holds for all $B^{(1,2,3)}$ and $A^{(1,2)}$.
(16) $r(B^{(1,2,3)} A^{(1)}) \geq r(AB)$ holds for all $B^{(1,2,3)}$ and $A^{(1)}$.
(17) $r(B^{(1,1,4)} A^{(1)}) \geq r(AB)$ holds for all $B^{(1,1,4)}$.
(18) $r(B^{(1,1,4)} A^{(1,3,4)}) \geq r(AB)$ holds for all $B^{(1,1,4)}$ and $A^{(1,3,4)}$.
(19) $r(B^{(1,1,4)} A^{(1,2,4)}) \geq r(AB)$ holds for all $B^{(1,1,4)}$ and $A^{(1,2,4)}$.
(20) $r(B^{(1,1,4)} A^{(1,4)}) \geq r(AB)$ holds for all $B^{(1,1,4)}$ and $A^{(1,4)}$.
(21) $r(B^{(1,1,3)} A^{(1,2,3)}) \geq r(AB)$ holds for all $B^{(1,1,3)}$ and $A^{(1,2,3)}$. 

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Corollary 3.5. It follows from setting the minimum ranks of the corresponding $A$, $B$, and $AB$.

(b) The following statements are equivalent:

1. $r(B(1,2,4)A(1,2,3)) \geq r(AB)$ holds for all $B(1,2,4)$ and $A(1,2,3)$.
2. $r(B(1,2,4)A(1,3)) \geq r(AB)$ holds for all $B(1,2,4)$ and $A(1,3)$.
3. $r(B(1,2,4)A(1,2)) \geq r(AB)$ holds for all $B(1,2,4)$ and $A(1,2)$.
4. $r(B(1,2,4)A(1)) \geq r(AB)$ holds for all $B(1,2,4)$ and $A(1)$.
5. $r(B(1,4)A(1,2,3)) \geq r(AB)$ holds for all $B(1,4)$ and $A(1,2,3)$.
6. $r(B(1,4)A(1,3)) \geq r(AB)$ holds for all $B(1,4)$ and $A(1,3)$.
7. $r(B(1,4)A(1,2)) \geq r(AB)$ holds for all $B(1,4)$ and $A(1,2)$.
8. $r(B(1,4)A(1)) \geq r(AB)$ holds for all $B(1,4)$ and $A(1)$.
9. $r(B(1,2,2)A(1,2,3)) \geq r(AB)$ holds for all $B(1,2,2)$ and $A(1,2,3)$.
10. $r(B(1,2)A(1,3)) \geq r(AB)$ holds for all $B(1,2)$ and $A(1,3)$.
11. $r(B(1,2)A(1,2)) \geq r(AB)$ holds for all $B(1,2)$ and $A(1,2)$.
12. $r(B(1,2)A(1)) \geq r(AB)$ holds for all $B(1,2)$ and $A(1)$.
13. $r(B(1)A(1,2,3)) \geq r(AB)$ holds for all $B(1)$ and $A(1,2,3)$.
14. $r(B(1)A(1,3)) \geq r(AB)$ holds for all $B(1)$ and $A(1,3)$.
15. $r(B(1)A(1,2)) \geq r(AB)$ holds for all $B(1)$ and $A(1,2)$.
16. $r(B(1)A(1)) \geq r(AB)$ holds for all $B(1)$ and $A(1)$.
17. $AB = 0$ or $r(AB) = r(A) + r(B) - n$.

Proof. It follows from setting the minimum ranks of the corresponding $B^{(1,...,j)}A^{(1,...,j)}$ equal to $r(AB)$. □

Corollary 3.5. Let $A \in \mathbb{C}^{m \times n}$ and $B \in \mathbb{C}^{n \times p}$ be given. Then, the following results hold.

(a) The following statements are equivalent:

1. There exists an $A(1,3,4)$ such that $B^\dagger A(1,3,4) = 0$.
2. $B^\dagger A(1,2,4) = 0$ holds for some/any $A(1,2,4)$.
3. There exists an $A(1,4)$ such that $B^\dagger A(1,4) = 0$.
4. There exists a $B(1,3,4)$ such that $B(1,3,4)A^\dagger = 0$.
5. There exist $B(1,3,4)$ and $A(1,3,4)$ such that $B(1,3,4)A(1,3,4) = 0$.
6. There exist $B(1,3,4)$ and $A(1,2,4)$ such that $B(1,3,4)A(1,2,4) = 0$.
7. There exist $B(1,3,4)$ and $A(1,4)$ such that $B(1,3,4)A(1,4) = 0$.
8. $B(1,2,3)A^\dagger = 0$ holds for some/any $B(1,2,3)$.
9. There exist $B(1,2,3)$ and $A(1,3,4)$ such that $B(1,2,3)A(1,3,4) = 0$.
10. $B(1,2,3)A(1,2,4) = 0$ holds for some/any $B(1,2,3)$ and $A(1,2,4)$.
11. There exist $B(1,2,3)$ and $A(1,4)$ such that $B(1,2,3)A(1,4) = 0$.
12. There exists a $B(1,3)$ such that $B(1,3)A^\dagger = 0$.
13. There exist $B(1,3)$ and $A(1,3,4)$ such that $B(1,3)A(1,3,4) = 0$.
(b) The following statements are equivalent:

1. There exists an $A^{(1,2,3)}$ such that $B^{\dagger} A^{(1,2,3)} = 0$.
2. There exists an $A^{(1,3)}$ such that $B^{\dagger} A^{(1,3)} = 0$.
3. There exists an $A^{(1,2)}$ such that $B^{\dagger} A^{(1,2)} = 0$.
4. There exists an $A^{(1)}$ such that $B^{\dagger} A^{(1)} = 0$.
5. There exist $B^{(1,3,4)}$ and $A^{(1,2,3)}$ such that $B^{(1,3,4)} A^{(1,2,3)} = 0$.
6. There exist $B^{(1,3,4)}$ and $A^{(1,3)}$ such that $B^{(1,3,4)} A^{(1,3)} = 0$.
7. There exist $B^{(1,3,4)}$ and $A^{(1,2)}$ such that $B^{(1,3,4)} A^{(1,2)} = 0$.
8. There exist $B^{(1,3,4)}$ and $A^{(1)}$ such that $B^{(1,3,4)} A^{(1)} = 0$.
9. There exists a $B^{(1,2,4)}$ such that $B^{(1,2,4)} A^{\dagger} = 0$.
10. There exist $B^{(1,2,4)}$ and $A^{(1,3,4)}$ such that $B^{(1,2,4)} A^{(1,3,4)} = 0$.
11. There exist $B^{(1,2,4)}$ and $A^{(1,2,4)}$ such that $B^{(1,2,4)} A^{(1,2,4)} = 0$.
12. There exist $B^{(1,2,4)}$ and $A^{(1,4)}$ such that $B^{(1,2,4)} A^{(1,4)} = 0$.
13. There exist $B^{(1,2,3)}$ and $A^{(1,2,3)}$ such that $B^{(1,2,3)} A^{(1,2,3)} = 0$.
14. There exist $B^{(1,2,3)}$ and $A^{(1,3)}$ such that $B^{(1,2,3)} A^{(1,3)} = 0$.
15. There exist $B^{(1,2,3)}$ and $A^{(1,2)}$ such that $B^{(1,2,3)} A^{(1,2)} = 0$.
16. There exist $B^{(1,2,3)}$ and $A^{(1)}$ such that $B^{(1,2,3)} A^{(1)} = 0$.
17. There exists a $B^{(1,4)}$ such that $B^{(1,4)} A^{\dagger} = 0$.
18. There exist $B^{(1,4)}$ and $A^{(1,3,4)}$ such that $B^{(1,4)} A^{(1,3,4)} = 0$.
19. There exist $B^{(1,4)}$ and $A^{(1,2,4)}$ such that $B^{(1,4)} A^{(1,2,4)} = 0$.
20. There exist $B^{(1,4)}$ and $A^{(1,4)}$ such that $B^{(1,4)} A^{(1,4)} = 0$.
21. There exist $B^{(1,3)}$ and $A^{(1,2,3)}$ such that $B^{(1,3)} A^{(1,2,3)} = 0$.
22. There exist $B^{(1,3)}$ and $A^{(1,3)}$ such that $B^{(1,3)} A^{(1,3)} = 0$.
23. There exist $B^{(1,3)}$ and $A^{(1,2)}$ such that $B^{(1,3)} A^{(1,2)} = 0$.
24. There exist $B^{(1,3)}$ and $A^{(1)}$ such that $B^{(1,3)} A^{(1)} = 0$.
25. There exists a $B^{(1,2)}$ such that $B^{(1,2)} A^{\dagger} = 0$.
26. There exist $B^{(1,2)}$ and $A^{(1,3,4)}$ such that $B^{(1,2)} A^{(1,3,4)} = 0$.
27. There exist $B^{(1,2)}$ and $A^{(1,2,4)}$ such that $B^{(1,2)} A^{(1,2,4)} = 0$.
28. There exist $B^{(1,2)}$ and $A^{(1,4)}$ such that $B^{(1,2)} A^{(1,4)} = 0$.
29. There exists a $B^{(1)}$ such that $B^{(1)} A^{\dagger} = 0$.
30. There exist $B^{(1)}$ and $A^{(1,3,4)}$ such that $B^{(1)} A^{(1,3,4)} = 0$.
31. There exist $B^{(1)}$ and $A^{(1,2,4)}$ such that $B^{(1)} A^{(1,2,4)} = 0$.
32. There exist $B^{(1)}$ and $A^{(1,4)}$ such that $B^{(1)} A^{(1,4)} = 0$.
33. $\mathcal{R}(A^*) \cap \mathcal{R}(B) = \{0\}$.

(c) The following statements are equivalent:

1. There exist $B^{(1,2,4)}$ and $A^{(1,2,3)}$ such that $B^{(1,2,4)} A^{(1,2,3)} = 0$.
2. There exist $B^{(1,2,4)}$ and $A^{(1,3)}$ such that $B^{(1,2,4)} A^{(1,3)} = 0$.
3. There exist $B^{(1,2,4)}$ and $A^{(1,2)}$ such that $B^{(1,2,4)} A^{(1,2)} = 0$.
4. There exist $B^{(1,2,4)}$ and $A^{(1)}$ such that $B^{(1,2,4)} A^{(1)} = 0$.
5. There exist $B^{(1,4)}$ and $A^{(1,2,3)}$ such that $B^{(1,4)} A^{(1,2,3)} = 0$.
6. There exist $B^{(1,4)}$ and $A^{(1,3)}$ such that $B^{(1,4)} A^{(1,3)} = 0$.
7. There exist $B^{(1,4)}$ and $A^{(1,2)}$ such that $B^{(1,4)} A^{(1,2)} = 0$.
8. There exist $B^{(1,4)}$ and $A^{(1)}$ such that $B^{(1,4)} A^{(1)} = 0$. 
There exist $B^{(1,2)}$ and $A^{(1,2,3)}$ such that $B^{(1,2)} A^{(1,2,3)} = 0$.

There exist $B^{(1,2)}$ and $A^{(1,3)}$ such that $B^{(1,2)} A^{(1,3)} = 0$.

There exist $B^{(1,2)}$ and $A^{(1,2)}$ such that $B^{(1,2)} A^{(1,2)} = 0$.

There exist $B^{(1,2)}$ and $A^{(1)}$ such that $B^{(1,2)} A^{(1)} = 0$.

There exist $B^{(1)}$ and $A^{(1,2,3)}$ such that $B^{(1)} A^{(1,2,3)} = 0$.

There exist $B^{(1)}$ and $A^{(1,3)}$ such that $B^{(1)} A^{(1,3)} = 0$.

There exist $B^{(1)}$ and $A^{(1,2)}$ such that $B^{(1)} A^{(1,2)} = 0$.

There exist $B^{(1)}$ and $A^{(1)}$ such that $B^{(1)} A^{(1)} = 0$.

$r(A) + r(B) \leq n$.

Proof. It follows from setting the minimum ranks of the corresponding $B^{(i,\ldots,j)} A^{(i,\ldots,j)}$ equal to zero.

We studied a group of fundamental problems on the products of generalized inverses of two matrices and their rank formulas by using pure algebraic operations of the two given matrices. The rank formulas are clearly in analytical forms, so that the results in this paper can be utilized as standard tools in solving many problems on the products of generalized inverses and properties. In particular, it is obvious that (1.3) holds only if

$$
\max_{\langle AB \rangle^{(i,\ldots,j)}} r[(AB)^{(i,\ldots,j)}] = \max_{B^{(i,\ldots,j)}} r(B^{(i,\ldots,j)} A^{(i,\ldots,j)}),
$$

$$
\min_{\langle AB \rangle^{(i,\ldots,j)}} r[(AB)^{(i,\ldots,j)}] = \min_{B^{(i,\ldots,j)}} r(B^{(i,\ldots,j)} A^{(i,\ldots,j)}).
$$

Hence, the results in the previous theorem and corollaries can be used to establish necessary and sufficient conditions for (1.3) to hold. In addition, pre-multiplying $B$ and $AB$ and/or post-multiplying $A$ and $AB$ to (1.3) yields

$$
B(AB)^{(i,\ldots,j)} A = BB^{(i,\ldots,j)} A^{(i,\ldots,j)},
$$

$$
AB(AB)^{(i,\ldots,j)} A = ABB^{(i,\ldots,j)} A^{(i,\ldots,j)}, \quad (AB)^{(i,\ldots,j)} AB = B^{(i,\ldots,j)} A^{(i,\ldots,j)} AB.
$$

It is obvious that the products $BB^{(i,\ldots,j)} A^{(i,\ldots,j)} A$, $AB^{(i,\ldots,j)} A^{(i,\ldots,j)}$, and $B^{(i,\ldots,j)} A^{(i,\ldots,j)} AB$ are linear or nonlinear matrix-valued functions, while formulas for calculating the maximum and minimum ranks of these matrix products can be established by a similar approach.

References


