An Inexact Proximal Method with Proximal Distances for Quasimonotone Equilibrium Problems

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Abstract

In this paper we propose an inexact proximal point method to solve equilibrium problem using proximal distances and the diagonal subdifferential. Under some natural assumptions on the problem and the quasimonotonicity condition on the bifunction, we prove that the sequence generated for the method converges to a solution point of the problem.

Keywords: Equilibrium problems; quasimonotonicity; proximal distance; proximal method.

1 Introduction

We are interested in studying the equilibrium problem (EP) in the Euclidean space: given a nonempty convex set $C \subset I\!R^n$ and $f : \bar{C} \times \bar{C} \to I\!R$, find $x \in \bar{C}$ such that

$$f(x, y) \geq 0, \quad \forall y \in \bar{C}. \tag{1.1}$$

Equilibrium problems generalize minimization and variational inequalities problems, see for example Blum and Oettli [10], as also the fixed point problems, nonlinear and linear complemetarity problems, vector minimization problems and Nash equilibria problems with noncooperative games. Due to this reason that (EP) is very attractive to many researchers, both in theory and applications, see Aoyama et al. [2], Nasri and Sosa [38], Castellani and Giuli [13], Blum and Oettli [10], Flores-Bázan [16], Oliveira et al. [26], Iusem and Nasri [21], Bianchi and Pini [8],

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Chadli et al. [12], Bianchi et al. [9], Iusem and Sosa [22] and Iusem et al. [23]. Equilibrium problems has been studied in different vectorial spaces, for example, in Banach spaces, see Burachik [34], Iusem and Narsi [21]; and in general vector spaces, see Balaj [4], Flores-Bazán et al. [18] and Bianchi et al. [6].

In previous works, the bifunction f in (1.1) was considered monotone and pseudomonotone, see Flores-Bazán [17], Chadli et al. [11], Konnov and Schaible [31], Bianchi and Schaible [7], Bianchi and Pini [8]. These problems also have been solved by different methods: splitting proximal methods, see Moudafi [37]; hybrid extragradient methods, see Anh [1]; extragradient methods, see Nguyen et al. [39]; double projection-type method, see Quoc and Muu [44] and proximal point algorithm (remember that the proximal method was developed by Rockafellar [45]), see Khatibzadeh et al. [30].

In Euclidean spaces some proximal methods to solve (EP) has been considered, for example Konnov [32] using the euclidean norm; Mashreghi and Nasri [36], Iusem and Sosa [21], Langenberg [35] with Bregman distances; Nguyen et al. [39] with $\varphi-$divergences distances; da Cruz Neto et al. [14] with second-order homogeneous distance. All them have been considered both the monotone and pseudomonotone case. However, the quasimonotone case was not considered, this is we are interested in develop an inexact proximal method for solving (EP) considering the quasimonotonicity of the bifunction f.

To achieve the objective, we propose the following iteration: given $x^{k-1} \in C$, find $x^k \in C$ such that:

$$g^k + \lambda_k \nabla_1 d(x^k, x^{k-1}) = e^k,$$

where d is a proximal distance, see Subsection 2.1 for a definition of this concept, $g^k \in \partial_2 f(x^k, x^k)$, see Section 2, e^k is an approximation error and λ_k is a positive parameter. To ensure the convergence of the proposed algorithm, we will consider some appropriate conditions for e^k, which will be introduced later in Section 4.

The proposed algorithm is motivated of our recently paper [42] where we introduce an inexact proximal method to solve variational inequalities problems, as also, from the paper of da Cruz Neto et al. [14] which solve (EP), where $f(x, .)$ is quasiconvex, using an exact proximal method with a second-order homogeneus distances.

The main contributions of this paper are the following: We propose an inexact proximal point algorithm to solve the (EP) when f in (1.1) quasimonotone, observe that this condition was not considered in previous works, as also we introduced in our algorithm the proximal distance of Auslender and Teboulle [3], for this reason we cover a great class of distances in the literature, for example, Bregman, $\varphi-$divergence and second order homogeneous distances.

This paper is organized as follows: Section 2 gives some basic results used throughout the paper. In the Section 3 we introduce the proposed method. In Section 4 we study the convergence of the sequence generated by the method, analyzing the quasimonotone case. In the Section 5 we study the convergence of the method in the pseudomonotone case.

2 Preliminaries

Throughout this paper $I\!R^n$ is the Euclidean space endowed with the canonical inner product $\langle , \rangle$ and the norm of x given by $||x|| := \langle x, x\rangle^{1/2}$, and $bd(X)$, $\bar{C}$ denotes the boundary and closure of the subset $X \subset I\!R^n$ respectively.

Definition 2.1 *Let C be a nonempty convex set. A bifunction $f : \bar{C} \times \bar{C} \to I\!R$ is said to be* *(i) monotone on $\bar{C}$ if*

$$f(x,y) + f(y,x) \leq 0, \ \forall x, y \in \bar{C};$$

(ii) pseudomonotone on $\bar{C}$ if

$$f(x,y) \geq 0 \Rightarrow f(y,x) \leq 0, \ \forall x, y \in \bar{C};$$

(iii) quasimonotone on $\bar{C}$ if

$$f(x,y) > 0 \Rightarrow f(y,x) \leq 0, \ \forall x, y \in \bar{C}.$$

We can easily verify that $(i) \Rightarrow (ii) \Rightarrow (iii)$. On the other hand, in general the converse of $(ii) \Rightarrow (iii)$ is not true, for an example, consider $\bar{C} = [0,1] \times [0,1]$ and

$$f(x_1, x_2) = \left(\frac{-(\frac{x_1+\sqrt{x_1^2+4x_2}}{2})}{\frac{x_1+\sqrt{x_1^2+4x_2}}{2}+1} \ , \frac{-1}{\frac{x_1+\sqrt{x_1^2+4x_2}}{2}+1} \right),$$

see Example 3.1 in Hadjisavvas and Schaible [19].

Lemma 2.1 *Let $\{v_k\}$,$\{\gamma_k\}$, and $\{\beta_k\}$ be nonnegative sequences of real numbers satisfying $v_{k+1} \leq (1+\gamma_k)\, v_k + \beta_k$ and such that $\sum_{k=1}^{\infty} \beta_k < \infty$, $\sum_{k=1}^{\infty} \gamma_k < \infty$. Then, the sequence $\{v_k\}$ converges.*

Proof. See Lemma 2, pp. 44, in Polyak [43]. ■

Definition 2.2 *Let $f : \bar{C} \times \bar{C} \to I\!R$ be a bifunction. For each fixed $z \in \bar{C}$, the diagonal subdifferential of $f(z,.)$ at $x \in \bar{C}$, denoted by $\partial_2 f(z,x)$, is defined and denoted by*

$$\partial_2 f(z,x) = \{g \in I\!R^n : f(z,y) \geq f(z,x) + \langle g, y-x \rangle, \forall y \in \bar{C}\}.$$

Furthermore, if $f(x,x) = 0$, then

$$\partial_2 f(x,x) = \{g \in I\!R^n : f(x,y) \geq \langle g, y-x \rangle, \forall y \in \bar{C}\}.$$

Remark 2.1 *The above subdifferential has been used by Iusem [24], Vuong et al. [47] and Bello et al. [5]. Observe also that if $x \in C$ and $f(x,.)$ is convex in $\bar{C}$ and $f(x,x) = 0$, then $\partial_2 f(x,x) \neq \emptyset$.*

2.1 Proximal Distances

Now we present the definitions of proximal distance and induced proximal distance, introduced by Auslender and Teboulle [3]. This approach has been used in the works of Villacorta and Oliveira [46], Papa Quiroz and Oliveira [40], Papa Quiroz et al. [41] and Papa Quiroz et al. [42].

Definition 2.3 *A function $d : I\!R^n \times I\!R^n \to I\!R_+ \cup \{+\infty\}$ is called a proximal distance with respect to an open nonempty convex set C if for each $y \in C$ it satisfies the following properties:*

i. $d(\cdot, y)$ *is proper, lower semicontinuous, strictly convex and continuously differentiable on* C*;*

ii. $\text{dom}\,(d(\cdot, y)) \subset \bar{C}$ *and* $\text{dom}(\,\partial_1 d(\cdot, y)) = C$*, where* $\partial_1 d(\cdot, y)$ *denotes the classical subgradient map of the function* $d(\cdot, y)$ *with respect to the first variable;*

iii. $d(\cdot, y)$ *is coercive on* $I\!R^n$ *(i.e.,* $\lim_{||u||\to\infty} d(u, y) = +\infty$);

iv. $d(y, y) = 0$.

We denote by $D(C)$ *the family of functions satisfying the above definition.*

Property **i**. is needed to preserve convexity of $d(\cdot, y)$, property **ii** will force the iteration of the proximal method to stay in C, and the property **iii** is useful to guarantee the existence of the proximal iterations. For each $y \in C$, let $\nabla_1 d(\cdot, y)$ denote the gradient map of the function $d(\cdot, y)$ with respect to the first variable. Note that by definition $d(\cdot, \cdot) \geq 0$ and from **iv**. the global minimum of $d(\cdot, y)$ is obtained at y, which shows that $\nabla_1 d(y, y) = 0$.

Definition 2.4 *Given* $d \in D(C)$*, a function* $H : I\!R^n \times I\!R^n \to I\!R_+ \cup \{+\infty\}$ *is called the induced proximal distance to* d *if* H *is a finite-valued function on* $C \times C$ *and for each* $a, b \in C$ *we have:*

(Ii) $H(a, a) = 0$.

(Iii) $\langle c - b, \nabla_1 d(b, a)\rangle \leq H(c, a) - H(c, b), \quad \forall\, c \in C.$

Let us denoted by $(d, H) \in \mathcal{F}(C)$ to the proximal distance that satisfies the conditions of Definition 2.4.
We also denote $(d, H) \in \mathcal{F}(\bar{C})$ if there exists H such that:

(Iiii) H is finite valued on $\bar{C} \times C$ satisfying (**Ii**) and (**Iii**), for each $c \in \bar{C}$.

(Iiv) For each $c \in \bar{C}$, $H(c, \cdot)$ has level bounded sets on C.

Finally, denote $(d, H) \in \mathcal{F}_+(\bar{C})$ if

(Iv) $(d, H) \in \mathcal{F}(\bar{C})$.

(Ivi) $\forall\, y \in \bar{C}$ y $\forall\, \{y^k\} \subset C$ bounded with $\lim_{k\to+\infty} H(y, y^k) = 0$, then $\lim_{k\to+\infty} y^k = y$.

(Ivii) $\forall\, y \in \bar{C}$, y $\forall\, \{y^k\} \subset C$ such that $\lim_{k\to+\infty} y^k = y$, then $\lim_{k\to+\infty} H(y, y^k) = 0$.

The main result on proximal point method will be when $(d, H) \in \mathcal{F}_+(\bar{C})$. Several examples of proximal distances which satisfy the above definitions, for example Bregman distances, distances based on $\varphi-$ divergences and distances based on second order homogeneous proximal distances, were given by Auslender and Teboulle [3], Section 3.

Remark 2.2 *The conditions* (**Ivi**) *and* (**Ivii**) *will ensure the global convergence of the sequence generates by the proposed algorithm in this paper. As we will see in Proposition 2.2, the condition* (**Ivii**) *may be substituted by the following:*

(Iviii) $H(.,.)$ *is continuous in* $C \times C$ *and if* $\{y^k\} \subset C$ *such that* $\lim_{k\to+\infty} y^k = y \in bd(C)$ *and* $\bar{y} \neq y$ *is another point in* $bd(C)$ *then* $\lim_{k\to+\infty} H(\bar{y}, y^k) = +\infty$.

According Langenberg and Tichatschke [33], page 643, which is based on the papers of Kaplan and Tichatschke [28] and Kaplan and Tichatschke [29], the above condition for induced Bregman distances holds when nonlinear constraints are active at $y = \lim_{k\to+\infty} y^k$ *while the condition* (**Ivii**) *holds when only affine constraints are active at* y.

Definition 2.5 *Let* $(d, H) \in \mathcal{F}(\bar{C})$. *We say that the sequence* $\{z^l\} \subset C$ *is* H*-quasi-Fejér convergent to a set* $U \subset \bar{C}$ *if for each* $u \in U$ *there exists a sequence* $\{\epsilon_l\}$, *with* $\epsilon_l \geq 0$ *and* $\sum_{l=1}^{+\infty} \epsilon_l < +\infty$ *such that*

$$H(u, z^l) \leq H(u, z^{l-1}) + \epsilon_l.$$

Proposition 2.1 *Let* $(d, H) \in \mathcal{F}_+(\bar{C})$ *and* $\{z^l\} \subset C$ *be a sequence* H*-quasi-Fejér convergent to* $U \subset \bar{C}$ *then* $\{z^l\}$ *is bounded. If, furthermore, there exists a cluster point* $\bar{z}$ *of* $\{z^l\}$, *belongs to* U, *then the whole sequence* $\{z^l\}$ *converges to* $\bar{z}$.

Proof. See Proposition 2.2 in Papa Quiroz et al. [42]. ■

The following proposition weaken the above result and it will be important to stabilize the global convergence of the proposed algorithm, when we substitute the condition (**Iviii**) instead of (**Ivii**) in Definition 2.4.

Proposition 2.2 *Let* $(d, H) \in \mathcal{F}_+(\bar{C})$ *satisfy the condition* (**Iviii**) *instead of* (**Ivii**) *and* $\{z^l\} \subset C$ *be a sequence* H*-quasi-Fejér convergent to* $U \subset \bar{C}$ *then* $\{z^l\}$ *is bounded. If, furthermore, any cluster point of* $\{z^l\}$, *belongs to* U, *then the whole sequence* $\{z^l\}$ *converges.*

Proof. See Proposition 2.3 in Papa Quiroz et al. [42]. ■

3 Inexact Proximal Method for Equilibrium Problem

Let C be a nonempty open convex set and $f : \bar{C} \times \bar{C} \to I\!R$ an equilibrium bifunction, i.e., satisfying $f(x, x) = 0$ for every $x \in \bar{C}$. The equilibrium problem, $EP(f, \bar{C})$ in short, consists in find a point $\bar{x} \in \bar{C}$ such that

$$EP(f, \bar{C}) \quad f(\bar{x}, y) \geq 0, \forall y \in \bar{C}. \tag{3.2}$$

The solution set of the $EP(f, \bar{C})$, is denoted by $S(f, \bar{C})$. Next, We give the following first and natural assumptions.

Assumption H1. $f(., y) : \bar{C} \to I\!R$ is upper semicontinuous for all $y \in \bar{C}$.

Assumption H2. $f(x, .) : \bar{C} \to I\!R$ is convex and $f(x, x) = 0$ for all $x \in \bar{C}$

Now, we propose an extension of the proximal point method using a proximal distance to solve the problem (3.2).

Inexact Algorithm

Initialization: Let $\{\lambda_k\}$ be a sequence of positive parameters and a starting point:

$$x^0 \in C. \tag{3.3}$$

Main Steps: For $k = 1, 2, \ldots$, and $x^{k-1} \in C$, find $x^k \in C$ and $g^k \in \partial_2 f(x^k, x^k)$ such that

$$g^k + \lambda_k \nabla_1 d(x^k, x^{k-1}) = e^k, \tag{3.4}$$

where d is a proximal distance such that $(d, H) \in \mathcal{F}_+(\bar{C})$ and e^k is an approximation error which satisfies some conditions to be specific later.

Stop Criterion: If $x^k = x^{k-1}$ or $e^k \in \partial_2 f(x^k, x^k)$, then finish. Otherwise, to do $k - 1 \leftarrow k$ and return to Main Steps.

We assume also the following assumptions:

Assumption H3. f is quasimonotone.

Assumption H4. For each $k \in I\!N$, there exists $x^k \in C$.

4 Convergence Results

In this section, we are interested in analyse the iterations when $x^k \neq x^{k-1}$ for each $k = 1, 2, ...$ In fact, if $x^k = x^{k-1}$, for some k, then $\nabla_1 d(x^k, x^{k-1}) = 0$ and thus we obtain $g^k = e^k \in \partial_2 f(x^k, x^k)$ and therefore the algorithm finishes.
Now we define the following particular solution set of $S(f, \bar{C})$.

$$S^*(f, \bar{C}) = \{x \in \bar{C} : x \in S(f, \bar{C}) \ \ such\ \ that\ \ \forall w \in C \ \ then\ \ f(x, w) > 0\}. \tag{4.5}$$

The definition of the above set is a variant of the Bianchi and Schaible definition, see pp.41 of [7] and it will be very important to obtain the convergence of he proposed algorithm.

We will use the following assumption.

Assumption H5. $S^*(f, \bar{C}) \neq \emptyset$.

Proposition 4.1 *Under the assumptions* **H2**, **H3**, **H4**, **H5** *and* $(d, H) \in \mathcal{F}(\bar{C})$, *we have*

$$H(x^*, x^k) \leq H(x^*, x^{k-1}) + \frac{1}{\lambda_k}\left\langle e^k, x^k - x^* \right\rangle, \tag{4.6}$$

for all $x^* \in S^*(f, \bar{C})$.

Proof. Given $x^* \in S^*(f, \bar{C})$ then $f(x^*, w) > 0, \forall w \in C$ and as $x^k \in C$, we obtain $f(x^*, x^k) > 0$. As f is quasimonotone, by **H3**, we have $f(x^k, x^*) \leq 0$. Due that $g^k \in \partial_2 f(x^k, x^k)$, we get from **H2** and from Definition 2.2

$$\left\langle g^k, x^* - x^k \right\rangle \leq f(x^k, x^*) \leq 0. \tag{4.7}$$

Replacing (3.4) in (4.7) gives

$$\left\langle e^k - \lambda_k \nabla_1 d(x^k, x^{k-1}), x^* - x^k \right\rangle \leq 0, \tag{4.8}$$

which implies

$$\frac{1}{\lambda_k}\left\langle e^k, x^* - x^k \right\rangle \leq \left\langle x^* - x^k, \nabla_1 d(x^k, x^{k-1}) \right\rangle.$$

Now using the property **(Iii)** of Definition 3.2 then

$$\frac{1}{\lambda_k}\left\langle e^k, x^* - x^k \right\rangle \leq \left\langle x^* - x^k, \nabla_1 d(x^k, x^{k-1}) \right\rangle \leq H(x^*, x^{k-1}) - H(x^*, x^k),$$

thus we obtain the result. ■

Proposition 4.2 *Suppose that the assumptions* **H2**, **H3**, **H4**, **H5** *and* $(d, H) \in \mathcal{F}(\bar{C})$, *are satisfied. If the following additional conditions hold:*

$$\sum_{k=1}^{+\infty} \frac{\left\|e^k\right\|}{\lambda_k} < +\infty \tag{4.9}$$

$$\sum_{k=1}^{+\infty} \frac{\left|\langle e^k, x^k \rangle\right|}{\lambda_k} < +\infty \tag{4.10}$$

then

a). $\{x^k\}$ *is* H*-quasi-Fejér convergent to the set* $S^*(f, \bar{C})$.

b). $\{H(\bar{x}, x^k)\}$ *converges for all* $x^* \in S^*(f, \bar{C})$.

c). $\{x^k\}$ *is bounded.*

Proof.
a). Using the Cauchy-Schwarz inequality in (4.6) have for all $x^* \in \bar{C}$ that:

$$H(x^*, x^k) \leq H(x^*, x^{k-1}) + \frac{1}{\lambda_k} \left(\|e^k\| \|x^*\| + |\langle e^k, x^k \rangle| \right), \tag{4.11}$$

let $\epsilon^k = \frac{1}{\lambda_k} \left(\|e^k\| \|x^*\| + |\langle e^k, x^k \rangle| \right)$, then $H\left(x^*, x^k\right) \leq H\left(x^*, x^{k-1}\right) + \epsilon^k$, and from (4.9) and (4.10) we have $\sum_{k=1}^{+\infty} \epsilon^k < \infty$.

b). It is immediately from **a)** and Lemma 2.1.

c). It is also immediately from **a)** and Proposition 2.1. ■

Conditions (4.9) and (4.10) has been introduced by Eckstein [15] and was used by several researches, for example see Iusem and Nasri [20], Xu et al. [48], Kaplan and Tichatschke [27]. It is possible to get rid the assumption (4.10) in Proposition 4.2, to obtain that $\{H(x, x^k)\}$ is convergent and $\{x^k\}$ is bounded, for a class of induced proximal distances which includes Bregman distances given by the standard entropy kernel, all strongly convex Bregman functions, $\phi-$divergence distances and second order homogeneous distances, see Kaplan and Tichatsche, [25]. We prove this fact in the following proposition.

Proposition 4.3 *Suppose that assumptions* **H2**, **H3**, **H4**, **H5** *and* $(d, H) \in \mathcal{F}(\bar{C})$, *are satisfied. If only the condition* (4.9) *is satisfied and suppose that the induced proximal distance* $H(.,.)$ *satisfies the following additional property:*

(Iix) *For each* $x \in \bar{C}$ *there exist* $\alpha(x) > 0$ *and* $c(x) > 0$ *such that:*

$$H(x, v) + c(x) \geq \alpha(x)||x - v||, \forall v \in C;$$

then

a. *For all* $x^* \in S^*(f, \bar{C})$, *we have*

$$H(x^*, x^k) \leq \left(1 + 2\frac{\left\|e^k\right\|}{\lambda_k \alpha(x^*)}\right) H(x^*, x^{k-1}) + 2\frac{\left\|e^k\right\| c(x^*)}{\lambda_k \alpha(x^*)},$$

for k *sufficiently large and therefore* $\{H(x^*, x^k)\}$ *converges.*

b. $\{x^k\}$ *is bounded.*

Proof. Let $x^* \in S^*(f, \bar{C})$, from (4.6) we have

$$H(x^*, x^k) \leq H(x^*, x^{k-1}) + \frac{1}{\lambda_k} \left\| e^k \right\| \left\| x^k - x^* \right\|. \tag{4.12}$$

Taking $x = x^*$ and $v = x^k$ in (**Iix**) and using in (4.12) we obtain

$$\left(1 - \frac{\left\| e^k \right\|}{\lambda_k \alpha(x^*)}\right) H(x^*, x^k) \leq H(x^*, x^{k-1}) + \frac{c(x^*) \left\| e^k \right\|}{\lambda_k \alpha(x^*)}. \tag{4.13}$$

From (4.9), there exists $k_0 \equiv k_0(x^*)$ such that $\frac{\|e^k\|}{\lambda_k \alpha(x^*)} \leq \frac{1}{2}$, for all $k \geq k_0$. Then

$$1 \leq \left(1 - \frac{\left\| e^k \right\|}{\lambda_k \alpha(x^*)}\right)^{-1} \leq 1 + 2\frac{\left\| e^k \right\|}{\lambda_k \alpha(x^*)} \leq 2, \ \ \forall k \geq k_0. \tag{4.14}$$

From (4.13) and (4.14) we have

$$H(x^*, x^k) \leq \left(1 + 2\frac{\left\| e^k \right\|}{\lambda_k \alpha(x^*)}\right) H(x^*, x^{k-1}) + 2\frac{\left\| e^k \right\| c(x^*)}{\lambda_k \alpha(x^*)}, \forall k \geq k_0.$$

Thus, from Lemma 2.1 $\left\{H(x^*, x^k)\right\}$ is convergent and from Definition 2.4, (**Iiv**), $\left\{x^k\right\}$ is bounded. ■

Denote $Acc(x^k)$ as the set of all accumulation points of $\{x^k\}$

$$Acc(x^k) = \{z \in \bar{C} : \textit{there exists a subsequence } \{x^{k_l}\} \textit{ of } \{x^k\} : x^{k_l} \to z\}.$$

Theorem 4.1 *Suppose that assumptions* **H1**, **H2**, **H3**, **H4** *and* **H5** *are satisfied. If* $(d, H) \in \mathcal{F}_+(\bar{C})$, $0 < \lambda_k < \bar{\lambda}$, *for some* $\bar{\lambda} > 0$, *and one of the following condition is satisfied:*

i) The conditions (4.9)-(4.10) *are satisfied;*

ii) (d, H) *satisfies* **(Iix)** *and only the condition* (4.9) *is satisfied;*

then,

(a) $\{x^k\}$ *converges weakly to an element of* $S(f, \bar{C})$*, that is,* $Acc(x^k) \neq \emptyset$ *and every element of* $Acc(x^k)$ *is a point of* $S(f, \bar{C})$.

(b) If $Acc(x^k) \cap S^*(f, \bar{C}) \neq \emptyset$ *then* $\{x^k\}$ *converges to an element of* $S^*(f, \bar{C})$.

Proof. Consider true the first case $i)$.
From Proposition 4.2, **c**, $\{x^k\}$ is bounded, and therefore there exists a convergent subsequence and thus $Acc(x^k) \neq \emptyset$. Define $L = \{k_1, k_2, ..., k_j, ...\}$, then from above we obtain $\{x^{k_l}\} \to x^*$, from (3.4) and $g^l \in \partial_2 f(x^l, x^l)$ we have

$$f(x^l, x) \geq \left\langle g^l, x - x^l \right\rangle = \left\langle e^l, x - x^l \right\rangle - \lambda_l \left\langle \nabla_1 d(x^l, x^{l-1}), x - x^l \right\rangle, \tag{4.15}$$

for all $l \in L$ and for each $x \in \bar{C}$. In view of (4.9) and that $\{\lambda_l\}$ is bounded from above we have that $||e^l|| \to 0$. Then, as $\{x^l\}$ is bounded, we obtain $\left\langle e^l, x - x^l \right\rangle \to 0$. Thus, only is necessary to analyze the convergence of the sequence

$$-\lambda_l \left\langle \nabla_1 d(x^l, x^{l-1}), x - x^l \right\rangle .$$

From Definition 2.4 (**Iii**), we have

$$-\lambda_l \left\langle \nabla_1 d(x^l, x^{l-1}), x - x^l \right\rangle \geq \lambda_l \left[H(x, x^l) - H(x, x^{l-1}) \right] .$$

Fix $x \in \bar{C}$, we analyze two cases:
If $\{H(x, x^l)\}$ converges, then $\lambda_l \left[H(x, x^l) - H(x, x^{l-1}) \right] \to 0$, since $\{\lambda_l\}$ is bounded, from (4.15) and **H1**

$$f(x^*, x) \geq \limsup_{l \to \infty} f(x^l, x) \geq 0.$$

If $\{H(x, x^l)\}$ is not convergent, then the sequence is not monotonically decreasing and so there are infinite $l \in L$ such that $H(x, x^l) \geq H(x, x^{l-1})$. Let $\{l_j\} \subset L$ such that $H(x, x^{l_j}) \geq H(x, x^{l_j - 1})$, for all $j \in I\!N$, then from **H1** we obtain

$$f(x^*, x) \geq \limsup_{j \to \infty} f(x^{l_j}, x) \geq \limsup_{j \to \infty} \lambda_{l_j} \left[H(x, x^{l_j}) - H(x, x^{l_j - 1}) \right] \geq 0,$$

then in both cases we obtain

$$f(x^*, x) \geq 0.$$

implying that $x^* \in S(f, \bar{C})$. Assume that $x^* \in S^*(f, \bar{C})$
If the condition $i)$ is satisfied and using Proposition 4.2, **a)** and Proposition 2.1, we have $\{x^k\}$ converges to x^*.
Now we consider true the second case $ii)$.
From Proposition 4.3, **b.**, $\{x^k\}$ is bounded, so $Acc(x^k) \neq \emptyset$. Take a subsequence $\{x^{k_j}\}$ such that $x^{k_j} \to \bar{x}$, then $\bar{x} \in S(f, \bar{C})$, and from Definition 2.4 (**Ivii**), $H(\bar{x}, x^{k_l}) \to 0$. As $H(\bar{x}, x^k)$ is convergent, see Proposition 4.3, **a**, and the sequence $H(\bar{x}, x^{k_l})$ converges to zero we obtain that $H(\bar{x}, x^{k_j}) \to 0$. From Definition 2.4, (**Ivi**), we obtain that $x^{k_j} \to \bar{x}$ and due to the uniqueness of the limit we have $x^* = \bar{x}$. Thus, $\{x^k\}$ converges to x^*. ■

Theorem 4.2 *Suppose that assumptions* **H1**, **H2**, **H3**, **H4** *and* **H5** *are satisfied. If* $(d, H) \in \mathcal{F}_+(\bar{C})$ *satisfying the condition* (**Iviii**) *instead of* (**Ivii**), $0 < \lambda_k < \bar{\lambda}$, *for some* $\bar{\lambda} > 0$, *and one of the following condition is satisfied:*

i. the conditions (4.9)-(4.10) *are satisfied;*

ii. (d, H) *satisfies* **(Iix)** *and the condition* (4.9) *is satisfied;*

then

a1) $\{x^k\}$ *converges weakly to an element of* $S(f, \bar{C})$, *that is,* $Acc(x^k) \neq \emptyset$ *and every element of* $Acc(x^k)$ *is a point of* $S(f, \bar{C})$.

a2) If $Acc(x^k) \subset S^*(f, \bar{C})$ *then* $\{x^k\}$ *converges to an element of* $S^*(f, \bar{C})$.

Proof. Consider true the first case i).
From Proposition 4.2, **c**), $\{x^k\}$ is bounded, and therefore $Acc(x^k) \neq \emptyset$. Take a susequence $\{x^{k_j}\}$, such that $x^{k_j} \to \bar{x}$. from the proof of Theorem 4.1 we obtain that $\bar{x} \in S(f, \bar{C})$. From Proposition 4.2, **a,** $\{x^k\}$ is $H-$quasi-Féjer covergent to $S^*(f, \bar{C})$ and if we suppose $Acc(x^k) \subset S^*(f, \bar{C})$ then from Proposition 2.2 we obtain the result.
Now we consider true the second case ii).
from Proposition 4.3, **b**, $\{x^k\}$ is bounded, $Acc(x^k) \neq \emptyset$. Take a subsequence $\{x^{k_j}\}$, such that $x^{k_j} \to \bar{x}$, mimicking the proof of Theorem 4.1 we obtain that $\bar{x} \in S(f, \bar{C})$. If $Acc(x^k) \subset S^*(f, \bar{C})$, let $\bar{x}$ and x^* two cluster points of $\{x^k\}$ with $x^{k_j} \to \bar{x}$ and $x^{k_l} \to x^*$, as $\bar{x}, x^* \in S^*(f, \bar{C})$, $\{H(\bar{x}, x^k)\}$ and $\{H(x^*, x^k)\}$ converge. We analyzed the following three possibilities.
If x^* and $\bar{x}$ belong to $bd(C)$ and suppose that $\bar{x} \neq x^*$, then from assumption (**Iviii**), $H(x^*, x^{k_j}) \to +\infty$, which contradict the convergence of $\{H(x^*, x^k)\}$, then we should have $\bar{x} = x^*$.
If x^* and $\bar{x}$ belong to C, from continuity of $H(.,.)$ in C we have $H(x^*, x^{k_l}) \to 0$. As $\{H(x^*, x^k)\}$ converges then $H(x^*, x^{k_j}) \to 0$. Using the condition (**Ivi**) we have $x^{k_j} \to x^*$, thus $\bar{x} = x^*$.
Now assume that $x^* \in C$ and $\bar{x} \in bd(C)$. Then, using the same argument as the last case we have that $\bar{x} = x^*$, which is a contradiction, so this case is not possible. ■

5 Pseudomonotone Case

In this section substituting the quasimonotone condition of f, see **H3**, by the pseudomonotone ones we will obtain the convergence of the sequence without the assumption **H5** but assumming that the set $S(f, \bar{C})$ is nonempty.

Along this section we substitute the assumptions **H3** and **H5**, of the previous section, by the following conditions:

Assumption H3'. f is pseudomonotone.

Assumption H6. $S(f, \bar{C}) \neq \emptyset$.

Proposition 5.1 *Under the assumptions* **H2**, **H3'**, **H4** *and* **H6** *with* $(d, H) \in \mathcal{F}(\bar{C})$, *and for all* $x^* \in S(f, \bar{C})$. *Then we have*

$$H(x^*, x^k) \leq H(x^*, x^{k-1}) + \frac{1}{\lambda_k} \left\langle e^k, x^k - x^* \right\rangle. \tag{5.16}$$

Proof. Given $x^* \in S(f, \bar{C})$ then $f(x^*, w) \geq 0, \forall w \in C$ now consider $w = x^k$ as f is pseudomonotone we get $f(x^k, x^*) \leq 0$. Mimicking the proof of the Proposition 4.1 we get (5.16). ■

The following convergence test will be very similar to quaimonotone case, which was developed in the previous section.

Proposition 5.2 *Let* $(d, H) \in \mathcal{F}(\bar{C})$, *and suppose that the assumptions* **H2**, **H3'**, **H4**, *and* **H6** *are satisfied. If the following additional conditions are satisfied:*

$$\sum_{k=1}^{+\infty} \frac{\left\| e^k \right\|}{\lambda_k} < +\infty \tag{5.17}$$

$$\sum_{k=1}^{+\infty} \frac{\left|\langle e^k, x^k \rangle\right|}{\lambda_k} < +\infty \tag{5.18}$$

then

a). $\{x^k\}$ *is H-quasi-Fejér convergent to the set* $S(f, \bar{C})$*, it is,*

$$H\left(x^*, x^k\right) \leq H\left(x^*, x^{k-1}\right) + \epsilon^k,$$

for each $k \in I\!N$ *and all* $x^* \in S(f, \bar{C})$*, where* $\epsilon^k = \frac{1}{\lambda_k}\left(\|e^k\|\|x^*\| + |\langle e^k, x^k\rangle|\right)$ *with* $\sum_{k=1}^{+\infty} \epsilon^k < +\infty$.

b). $\{H(x^*, x^k)\}$ *converges for all* $x^* \in S(f, \bar{C})$.

c). $\{x^k\}$ *is bounded.*

Proof. It is similar to Proposition 4.2, in this case we consider $S(f, \bar{C})$ instead of $S^*(f, \bar{C})$.

Proposition 5.3 *If* **H3′** *is true,* $(d, H) \in \mathcal{F}\ (\bar{C})$*, and suppose that the assumptions* **H2**, **H4** *and* **H6** *are satisfied. If only the condition* (5.17) *is satisfied and suppose that the induced proximal distance* $H(.,.)$ *satisfies the following additional property:*

(Iix) *For each* $x \in \bar{C}$ *there exist* $\alpha(x) > 0$ *and* $c(x) > 0$ *such that:*

$$H(x, v) + c(x) \geq \alpha(x)||x - v||, \forall v \in C;$$

then

a. *For all* $x^* \in S(f, \bar{C})$*, we have*

$$H(x^*, x^k) \leq \left(1 + 2\frac{\left\|e^k\right\|}{\lambda_k \alpha(x^*)}\right) H(x^*, x^{k-1}) + 2\frac{\left\|e^k\right\| c(x^*)}{\lambda_k \alpha(x^*)},$$

for k *sufficiently large and therefore* $\{H(x^*, x^k)\}$ *converges.*

b. $\{x^k\}$ *is bounded.*

Proof. It is similar to Proposition 4.3. ∎

The following result is motivated from Theorem 9 of Langenberg and Tichatschke [33].

Theorem 5.1 *Suppose* **H2**, **H3′**, **H4** *and* **H6** *are satisfied. If* $(d, H) \in \mathcal{F}_+(\bar{C})$*,* $0 < \lambda_k < \bar{\lambda}$*, for some* $\bar{\lambda} > 0$*, and one of the following condition is satisfied:*

i). *The conditions* (5.17)-(5.18) *are satisfied.*

ii). (d, H) *satisfies* **(Iix)** *and only the condition* (5.17) *is satisfied.*

Then, $\left\{x^k\right\}$ *converges to a point of* $S(f, \bar{C})$.

Proof. Since Proposition 5.2 (for the condition $i)$) and Proposition 5.3 (for the condition $ii)$) assure that $\{x^k\}$ is bounded, let x^* be a cluster point of $\left\{x^k\right\}$ and $\left\{x^{k_j}\right\}$ be a subsequence which converges to x^*. Define $L := \{k_1, k_2, ..., k_j, ...\}$, then from above we obtain $\{x^l\}_{l\in L} \to x^*$. From (3.4) and $u^l \in \partial_2 f(x^l, x^l)$ we have

$$f(x^l, x) \geq \left\langle u^l, x - x^l \right\rangle = \left\langle e^l, x - x^l \right\rangle - \lambda_l \left\langle \nabla_1 d(x^l, x^{l-1}), x - x^l \right\rangle, \tag{5.19}$$

for all $l \in L$ and for each $x \in \bar{C}$. In view of (5.17) and that $\{\lambda_l\}$ is bounded from above we have that $||e^l|| \to 0$. Then, as $\{x^l\}$ is bounded, we obtain $\left\langle e^l, x - x^l \right\rangle \to 0$. Thus, only is necessary to analyze the convergence of the sequence

$$-\lambda_l \left\langle \nabla_1 d(x^l, x^{l-1}), x - x^l \right\rangle.$$

From Definition 2.4 (**Iii**), we have

$$-\lambda_l \left\langle \nabla_1 d(x^l, x^{l-1}), x - x^l \right\rangle \geq \lambda_l \left[H(x, x^l) - H(x, x^{l-1})\right].$$

Fix $x \in \bar{C}$, we analyze two cases:
If $\{H(x, x^l)\}$ converges, then $\lambda_l \left[H(x, x^l) - H(x, x^{l-1})\right] \to 0$, since $\{\lambda_l\}$ is bounded, and from (5.19) and **H1**

$$f(x^*, x) \geq \limsup_{l\to\infty} f(x^l, x) \geq 0.$$

If $\{H(x, x^l)\}$ is not convergent, then the sequence is not monotonically decreasing and so there are infinite $l \in L$ such that $H(x, x^l) \geq H(x, x^{l-1})$. Let $\{l_j\} \subset L$ such that $H(x, x^{l_j}) \geq H(x, x^{l_j-1})$, for all $j \in I\!N$, then to get **H1**

$$f(x^*, x) \geq \limsup_{j\to\infty} f(u^{l_j}, x) \geq \limsup_{j\to\infty} \lambda_{l_j} \left[H(x, x^{l_j}) - H(x, x^{l_j-1})\right] \geq 0,$$

we obtain

$$f(x^*, x) \geq 0,$$

impliying that $x^* \in S(f, \bar{C})$. If the condition $i)$ is satisfied, then (5.18) is true and using Proposition 5.2, **a)** and Proposition 2.1, we have $\{x^k\}$ converges to x^*.
Now, if the condition $ii)$ is satisfied, then (d, H) holds the condition **(Iix)**. Let $\bar{x}$ be another cluster point of $\{x^k\}$ where $x^{k_l} \to \bar{x}$, then $\bar{x} \in S(f, \bar{C})$, and from Definition 2.4 (**Ivii**), $H(\bar{x}, x^{k_l}) \to 0$. As $H(\bar{x}, x^k)$ is convergent, see Proposition 5.3, **a**, and the sequence $H(\bar{x}, x^{k_l})$ converges to zero we obtain that $H(\bar{x}, x^{k_j}) \to 0$. From Definition 2.4, (**Ivi**), we obtain that $x^{k_j} \to \bar{x}$ and due to the uniqueness of the limit we have $x^* = \bar{x}$. Thus, $\{x^k\}$ converges to x^*. ■

Theorem 5.2 *Suppose that f is a pseudomonotone and that the assumptions* **H4** *and* **H5** *are satisfied. If $(d, H) \in \mathcal{F}_+(\bar{C})$ satisfying the condition* (**Iviii**) *instead of* (**Ivii**), $0 < \lambda_k < \bar{\lambda}$, *for some $\bar{\lambda} > 0$, and one of the following condition is satisfied:*

i. the conditions (5.17)-(5.18) *are satisfied;*

ii (d, H) *satisfies* **(Iix)** *and the condition* (5.17) *is satisfied;*

then, $\left\{x^k\right\}$ converges to a point of $S(f, \bar{C})$.

Proof. *i*. If the conditions (5.17)-(5.18) are satisfied then from Proposition 5.2, **a)**, $\{x^k\}$ is H-quasi-Fejér convergent to $S(f, \bar{C})$. As any cluster point of $\{x^k\}$ belongs to $S(f, \bar{C})$, see the first part of the proof of Theorem 5.1, then using Proposition 2.2 we obtain the result.
ii. From Proposition 5.3, **b**, $\{x^k\}$ is bounded, mimicking the proof of Theorem 5.1 any cluster point belongs to $S(f, \bar{C})$. Let $\bar{x}$ and x^* two cluster points of $\{x^k\}$ with $x^{k_j} \to \bar{x}$ and $x^{k_l} \to x^*$, as $\bar{x}, x^* \in S(f, \bar{C})$, from Proposition 5.3, **a**, both $\{H(\bar{x}, x^k)\}$ and $\{H(x^*, x^k)\}$ converge. We analyze the three possibilities.
If x^* and $\bar{x}$ belong to $bd(C)$ and suppose that $\bar{x} \neq x^*$, then from assumption (**Iviii**), $H(x^*, x^{k_j}) \to +\infty$, which contradict the convergence of $\{H(x^*, x^k)\}$, then we should have $\bar{x} = x^*$.
If x^* and $\bar{x}$ belong to C, from continuity of $H(.,,)$ in C we have $H(x^*, x^{k_l}) \to 0$. As $\{H(x^*, x^k)\}$ converges then $H(x^*, x^{k_j}) \to 0$. Using the condition (**Ivi**) we have $x^{k_j} \to x^*$, thus $\bar{x} = x^*$.
Without lost of generality we can suppose that $x^* \in C$ and $\bar{x} \in bd(C)$. Then, using the same argument as the last case we have that $\bar{x} = x^*$, which is a contradiction, so this case is not possible. ∎

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