Monoidal Cut Strengthening and Generalized Mixed-Integer Rounding for Disjunctive Programs

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Abstract
This article investigates cutting planes for mixed-integer disjunctive programs. In the early 1980s, Balas and Jeroslow presented monoidal disjunctive cuts exploiting the integrality of variables. For disjunctions arising from binary variables, it is known that these cutting planes are essentially the same as Gomory mixed-integer and mixed-integer rounding cuts. In this article, we investigate the relation of monoidal cut strengthening to other classes of cutting planes for general two-term disjunctions. In this context, we introduce a generalization of mixed-integer rounding cuts. We also demonstrate the effectiveness of monoidal disjunctive cuts via computational experiments on instances involving complementarity constraints.

1 Introduction

Let $n$ be a positive integer and $V = \{1, \ldots, n\}$. This article investigates cutting planes for problems involving a disjunction and mixed-integer conditions of the form

$$\bigvee_{j \in D} \left( \sum_{\nu \in V} d_{j\nu} x_\nu \geq d_{j0} \right), \quad x \in \mathbb{Z}^I \times \mathbb{R}^J,$$

where $D$ is a finite set, $V = I \cup J$, and the coefficients $d_{j\nu}$ and right hand sides $d_{j0}$ are assumed to be real valued.

Supposing that $x \geq 0$ and $d_{j0} > 0$ for $j \in D$, one can observe that the disjunctive inequality $\delta^T x \geq 1$ with

$$\delta_\nu \coloneqq \max \left\{ \frac{d_{j\nu}}{d_{j0}} : j \in D \right\} \quad \forall \nu \in V$$

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is valid for (1). This inequality has many applications, e.g., for disjunctions arising from binary variables (see Balas and Perregaard [5]), semi-continuous variables, complementarity constraints, or cardinality constraints (see de Farias et al. [8]).

The integrality information of variables can be exploited to strengthen the coefficients $\delta_\nu$, $\nu \in I$. Balas and Jeroslow [2] realized this with the help of an algebraic method called monoidal cut strengthening. For disjunctions $x_i \leq 0 \lor x_i \geq 1$ arising from binary variables, it is known that in terms of the simplex tableau, the monoidal strengthened cuts are equivalent to Gomory mixed-integer cuts, see Balas and Qualizza [3]. Gomory mixed-integer cuts in turn are essentially the same as mixed-integer rounding cuts, see Marchand and Wolsey [13]. However, Balas and Qualizza [3] stated that for general disjunctions, the relation of monoidal strengthened cuts to other classes of cutting planes is unexplored. In this article, we will close this gap for general two-term disjunctions by showing that monoidal strengthened cuts are generalized mixed-integer rounding cuts.

Disjunctive cuts are widely used in literature. Balas et al. [4] investigated disjunctive cuts for sets described by a single disjunction and an additional inequality system $Ax \leq b$. These cutting planes are separated by solving a (relatively large) cut generating linear program (CGLP), which lives in a lifted space. However, in practice, explicitly solving CGLP is often too time consuming. To work around with this, Balas and Perregaard [5] investigated a separation procedure for disjunctions modeled as binary variables, which solves CGLP implicitly by executing pivot operations in the simplex tableau of the original LP. Later, Kis [12] generalized this approach to two-term disjunctions and tested it computationally for disjunctions modeled as complementarity constraints. Balas and Perregaard used monoidal cut strengthening to improve their cutting planes. Moreover, monoidal cut strengthening is a subject in an article of Balas and Qualizza [3], who investigated an enhanced monoidal cut strengthening procedure. For disjunctions modeled as binary variables, the enhanced procedure sometimes yields cutting planes that are stronger than Gomory mixed-integer cuts. To the best of our knowledge, monoidal cut strengthening has not been investigated for complementarity constraints so far.

The contents of this article is organized as follows: In Section 2, we will review the monoidal cut strengthening method of Balas and Jeroslow. We will then investigate the relation of monoidal disjunctive cuts to other classes of cutting planes: We will first discuss the relation to Gomory mixed-integer cuts for disjunctions arising from integer variables (see Section 3), which so far only has been studied for disjunctions arising from binary variables by Balas and Qualizza [3]. Afterwards, in Section 4, we will introduce a generalized version of the mixed-integer rounding cut that strengthens an inequality of Nemhauser and Wolsey [14]. Our main result proves that for two-term disjunctions with $|D| = 2$, monoidal disjunctive cuts are special cases of gen-
eralized mixed-integer rounding cuts. Finally, in Section 5, we demonstrate
the effectiveness of monoidal cut strengthening on randomly generated in-
stances involving disjunctions modeled as complementarity constraints.

2 Monoidal Cut Strengthening

Monoidal cut strengthening is a method that was introduced by Balas and
Jeroslow [2] to improve the classical disjunctive cut $\delta^T x \geq 1$ with $\delta_\nu$ given
by (2). The method intends to strengthen the cut coefficients $\delta_\nu$ belonging
to integer variables $\nu \in I$ by finding optimal solutions over a monoid $M := \{m \in \mathbb{Z}^D : \sum_{j \in D} m_j \geq 0\}$. Although monoidal cut strengthening can be
applied to to general disjunctions of the form (1), it is usually investigated
in the context of binary variables. In the upcoming part of this article, we
discuss them in a more general setting.

We first review monoidal cut strengthening in the following two theo-
rems:

**Theorem 2.1** ([2, 3]). Let be given a problem $P$ involving a disjunction (1),
where $x \geq 0$ and $d_{j0} > 0$ for $j \in D$. Suppose that for all $j \in D$, a finite
upper bound $u_j > 0$ of $d_{j0} - \sum_{\nu \in V} d_{j\nu} x_{\nu}$ is known. Then the inequality
$\delta^T x \geq 1$ with

$$
\delta_\nu := \begin{cases} 
\min_{m \in M} \max \left\{ \frac{d_{i\nu} + u_i m_i}{d_{j0}}, j \in D \right\}, & \text{if } \nu \in I, \\
\delta_\nu, & \text{otherwise,}
\end{cases}
$$

(3)

is valid for $P$.

An optimal solution to (3) can be computed algorithmically with a com-
plexity that is linear in $|D|$, see [2]. Nevertheless, there does not exist a
general closed-form expression for the coefficients $\delta_\nu$, $\nu \in I$. This makes
it difficult to recognize a general correspondence of monoidal strengthened
disjunctive cuts to other classes of cutting planes. However, for the special
case of two-term disjunctions, i.e., $|D| = 2$, the coefficients $\delta_\nu$, $\nu \in I$, are
given by the following explicit formula.

**Theorem 2.2** ([2]). Let $D = \{i, k\}$. Then the coefficients of (3) can be
determined as

$$
\delta_\nu = \min \left\{ \frac{d_{i\nu} - u_i [m_{i\nu}^*]}{d_{i0}}, \frac{d_{k\nu} + u_k [m_{k\nu}^*]}{d_{k0}} \right\} \quad \forall \nu \in I,
$$

where

$$
m_{i\nu}^* := \frac{d_{i\nu} d_{k0} - d_{k\nu} d_{i0}}{u_i d_{k0} + u_k d_{i0}}.
$$

Note that $u_i [m_{i\nu}^*]$ and $u_k [m_{i\nu}^*]$ tend to zero if $u_k$ and $u_i$ tend to infinity,
respectively. In order to generate strong cutting planes, the bounds $u_i$ and $u_k$
should be small.
3 Relation to Gomory Mixed-Integer Cuts

Balas and Qualizza [3] showed that for binary variables $x_i \in \{0, 1\}$ modeled as $x_i \leq 0 \lor x_i \geq 1$, $x_i \in [0, 1]$, monoidal strengthened disjunctive cuts are equivalent to Gomory mixed-integer (GMI) cuts if they are both generated from the simplex tableau. For more general disjunctions $x_i \leq \kappa \lor x_i \geq \kappa + 1$ with $\kappa \in \mathbb{Z}$, which belong to integer variables $x_i \in \mathbb{Z}$, the equivalence depends on $u$. With a slight modification of the proof of Balas and Qualizza, we will derive the following result:

**Proposition 3.1.** Every GMI cut is a monoidal strengthened disjunctive cut of Theorem 2.1.

**Proof.** Let $x^*$ be a basic feasible solution of a linear program (LP)

$$\min_{x \in \mathbb{R}^n} \{c^T x : Ax = b, \ x \geq 0\}$$

with $c \in \mathbb{R}^n$, $b \in \mathbb{R}^m$, and $A \in \mathbb{R}^{m \times n}$. We denote by $B$ and $N$ the indices of the basic and nonbasic variables of $x^*$, respectively. Suppose that $x$ has to satisfy mixed-integer conditions $x \in \mathbb{Z}^I \times \mathbb{R}^J$. We generate the GMI cut from a single row

$$x_{B(k)} = \pi_{k0} - \sum_{\nu \in N} \pi_{k\nu} x_\nu$$

(4)

of the simplex tableau corresponding to some integer variable $x_{B(k)}$, where $B(k)$ is the $k$th entry of $B$, the coefficients $\pi_{k\nu}$ are entries of the matrix $A_B^{-1} A$, and $\pi_{k0} = (A_B^{-1} b)_k$.

We assume that $x_{B(k)}$ is currently at a fractional value $\pi_{k0} = x_{B(k)}^* \notin \mathbb{Z}$. Since the lower and upper bounds of $x_{B(k)}$ can be far apart from $\pi_{k0}$ (or even $\pm \infty$), one can observe that the upper bounds defined in Theorem 2.1 can be large, and hence the coefficient strengthening of this theorem can be weak. To work around with this, we write $x_{B(k)}$ as a combination $x_{B(k)} = z_{B(k)} - w_{B(k)} + y_{B(k)}$ of auxiliary variables, where $w_{B(k)} \geq 0$ and $z_{B(k)} \geq [\pi_{k0}]$ are integer and $y_{B(k)}$ is binary. As LP solutions of these variables, we use $w_{B(k)}^* := 0$, $z_{B(k)}^* := [\pi_{k0}]$, and $y_{B(k)}^* := f_0$, where we denote $f_0 := \pi_{k0} - [\pi_{k0}]$. Note that the bounds 0 and 1 of $y_{B(k)}$ are close to $f_0$ with a distance of at most 1. We introduce a new integer variable $s_{B(k)} := z_{B(k)} - [\pi_{k0}]$ and substitute (4) into $y_{B(k)} = x_{B(k)} + w_{B(k)} - s_{B(k)} - [\pi_{k0}]$ to obtain

$$y_{B(k)} = f_0 - \sum_{\nu \in N} \pi_{k\nu} x_\nu + w_{B(k)} - s_{B(k)}.$$

(5)

In the following, $w_{B(k)}$ and $s_{B(k)}$ are considered as additional nonbasic variables that are currently at their lower bound 0. Inserting (5) into the dis-
junction $y_B(k) \leq 0 \lor y_B(k) \geq 1$ yields
\[ \sum_{\nu \in N} a_{k\nu} x_{\nu} - w_B(k) + s_B(k) \geq f_0 \]
\[ \lor - \sum_{\nu \in N} a_{k\nu} x_{\nu} + w_B(k) - s_B(k) \geq 1 - f_0. \] (6)

From $0 \leq y_B(k) \leq 1$, we deduce that both expressions $f_0 - \sum_{\nu \in N} a_{k\nu} x_{\nu} + w_B(k) - s_B(k)$ and $1 - f_0 - (\sum_{\nu \in N} a_{k\nu} x_{\nu} + w_B(k) - s_B(k))$ have 1 as valid upper bound. Let $N_I := N \cap I$ be the index set of nonbasic integer variables and $N_J := N \cap J$ the index set of nonbasic continuous variables. We apply Theorems 2.1 and 2.2 to (6) to obtain
\[ m^*_\nu = a_{k\nu} (1 - f_0) + a_{k\nu} f_0 = a_{k\nu} \quad \forall \nu \in N_I, \]
\[ \delta_\nu = \min \left\{ \frac{\bar{a}_{k\nu} - \lfloor m^*_\nu \rfloor}{f_0}, \frac{-\bar{a}_{k\nu} + \lceil m^*_\nu \rceil}{1 - f_0} \right\} \quad \forall \nu \in N_I, \]
\[ \overline{\delta}_\nu = \max \left\{ \frac{\bar{a}_{k\nu}}{f_0}, \frac{-\bar{a}_{k\nu}}{1 - f_0} \right\} \quad \forall \nu \in N_J. \]

Note that $\delta_\nu = 0$ if $\nu \in N_I$ and $\bar{a}_{k\nu} \in \mathbb{Z}$. By denoting $f_\nu := a_{k\nu} - \lfloor a_{k\nu} \rfloor$, we may write $\delta_\nu = \min \left\{ \frac{f_\nu - f_0}{f_0}, \frac{1 - f_\nu}{f_0} \right\}$, $\nu \in N_I$, even if $\bar{a}_{k\nu} \in \mathbb{Z}$. The cut coefficients of the integer variables $w_B(k)$ and $s_B(k)$ are zero, since they have an integer coefficient in (5). This yields the valid inequality
\[ \sum_{\nu \in N_I} \min \left\{ \frac{f_\nu}{f_0}, \frac{1 - f_\nu}{f_0} \right\} x_{\nu} + \sum_{\nu \in N_J} \max \left\{ \frac{\bar{a}_{k\nu}}{f_0}, \frac{-\bar{a}_{k\nu}}{1 - f_0} \right\} x_{\nu} \geq 1. \]

This inequality is equivalent to the *Gomory mixed-integer (GMI) cut*
\[ \sum_{\nu \in N_I} \frac{f_\nu}{f_0} x_{\nu} + \sum_{\nu \in N_I} \frac{1 - f_0}{f_0} x_{\nu} + \sum_{\nu \in N_I, f_\nu > f_0} \frac{\bar{a}_{k\nu}}{f_0} x_{\nu} - \sum_{\nu \in N_I, \bar{a}_{k\nu} < 0} \frac{\bar{a}_{k\nu}}{1 - f_0} x_{\nu} \geq 1 \] (7)

that was first established by Gomory [10]. Adding (7) to the LP cuts off $x^*$ because $x^*_N = 0$. \hfill \Box

A natural question is whether monoidal cut strengthening can be traced back to known theory in mixed-integer programming for other disjunctions than $x_i \leq \kappa \lor x_i \geq \kappa + 1$ with $\kappa \in \mathbb{Z}$. This is studied in the next section.
4 Relation to Mixed-Integer Rounding

The mixed-integer rounding (MIR) procedure was introduced by Nemhauser and Wolsey [14] in 1988. The idea is to exploit the integrality information of variables via rounding. Later, Wolsey [17] explicitly defined the MIR inequality. This inequality has many applications in mixed-integer programming, e.g., for the derivation of flow cover inequalities, see Marchand and Wolsey [13]. In this section, we introduce a generalized mixed-integer rounding inequality. We will show that for two-term disjunctions with $|D| = 2$, the monoidal disjunctive cuts of Theorem 2.1 are special cases of generalized mixed-integer rounding inequalities.

Given $\lambda \in \mathbb{R}$ we denote $\lambda^+ := \max\{0, \lambda\}$. The (classical) MIR inequality is given as:

**Lemma 4.1** ([17]). Consider the mixed-integer set

$$S = \{x \in \mathbb{Z}_+^I \times \mathbb{R}_+^J : \sum_{\nu \in V} \alpha_{\nu} x \leq \beta\}.$$  

Define $f_0 := \beta - \lfloor \beta \rfloor$ and $f_{\nu} := \alpha_{\nu} - \lfloor \alpha_{\nu} \rfloor$ for $\nu \in I$. Then the MIR inequality

$$\sum_{\nu \in I} \left(\lfloor \alpha_{\nu} \rfloor + \frac{(f_{\nu} - f_0)^+}{1 - f_0} \right) x_{\nu} + \frac{1}{1 - f_0} \sum_{\nu \in J, \alpha_{\nu} < 0} \alpha_{\nu} x_{\nu} \leq \lfloor \beta \rfloor$$

is valid for $\text{conv}(S)$.

Marchand and Wolsey [13] showed that the GMI inequality, which is generated from a single simplex tableau row, can be derived from the MIR inequality. In order to show a similar result for cutting planes that are generated from two rows of the simplex tableau, we need to generalize Lemma 4.1 to the mixed-integer set

$$S' = \{x \in \mathbb{Z}_+^I \times \mathbb{R}_+^J : \sum_{\nu \in V} \alpha_{1\nu} x \leq \beta^1, \sum_{\nu \in V} \alpha_{2\nu} x \leq \beta^2\},$$

which is constrained by two inequalities. Define

$$\Delta_0 := \beta^2 - \beta^1, \quad \Delta_{\nu} := \alpha_{2\nu} - \alpha_{1\nu} \quad \forall \nu \in I,$$

$$g_0 := \Delta_0 - \lfloor \Delta_0 \rfloor, \quad g_{\nu} := \Delta_{\nu} - \lfloor \Delta_{\nu} \rfloor \quad \forall \nu \in I.$$  

(8)

Nemhauser and Wolsey [14] showed that the inequality

$$\sum_{\nu \in I} |\Delta_{\nu}| x_{\nu} + \frac{1}{1 - g_0} \left(\sum_{\nu \in I} \alpha_{1\nu} x_{\nu} \right.$$  

$$\left. + \sum_{\nu \in J} \min\{\alpha_{1\nu}, \alpha_{2\nu}\} x_{\nu} - \beta^1\right) \leq \lfloor \Delta_0 \rfloor$$

(9)

is valid for $\text{conv}(S')$. One can derive a stronger inequality if $g_{\nu} > g_0$ for some $\nu \in I$ as follows.
Lemma 4.2. The generalized mixed-integer rounding (GMIR) inequality

\[ \sum_{\nu \in I} \left( |\Delta_\nu| + \frac{(g_\nu - g_0)^+}{1 - g_0} \right) x_\nu + \frac{1}{1 - g_0} \left( \sum_{\nu \in I} \alpha_{1,\nu}^1 x_\nu \right. \\
+ \sum_{\nu \in J} \min \{ \alpha_{1,\nu}^1, \alpha_{2,\nu}^2 \} x_\nu - \beta_1 \left. \right) \leq |\Delta_0| \]

is valid for \( \text{conv}(S') \).

Proof. The proof is a modification of a proof from Nemhauser and Wolsey [14] showing the validity of Inequality (9).

\[ \sum_{\nu \in V} \alpha_{1,\nu}^1 x_\nu \leq \beta_1 \quad \text{and} \quad \sum_{\nu \in J} \min \{ \alpha_{1,\nu}^1, \alpha_{2,\nu}^2 \} x_\nu \leq \beta_k, \]

for \( k \in \{1, 2\} \), is valid. The inequality for \( k = 2 \) can be written as

\[ \sum_{\nu \in I} (\alpha_{2,\nu}^2 - \alpha_{1,\nu}^1) x_\nu - s \leq \beta^2 - \beta^1, \]

where

\[ s := - \sum_{\nu \in I} \alpha_{1,\nu}^1 x_\nu - \sum_{\nu \in J} \min \{ \alpha_{1,\nu}^1, \alpha_{2,\nu}^2 \} x_\nu + \beta^1. \]

From the inequality for \( k = 1 \), we deduce that \( s \geq 0 \). Therefore, Inequality (11) is in a form suitable for application to Lemma 4.1 and we obtain the validity of (10).

From the proof of Lemma 4.2, it follows that every GMIR inequality is an MIR inequality. The converse is also correct, since the GMIR inequality reduces to the MIR inequality if \( \alpha^1 = 0 \) and \( \beta^1 = 0 \).

The GMIR inequality allows for a new perspective on monoidal strengthened disjunctive cuts:

Theorem 4.3. Let \( D = \{i, k\} \). Then the cut derived from monoidal cut strengthening (see Theorem 2.1) is a GMIR inequality. 

Proof. We consider the two-term disjunction

\[ \sum_{\nu \in V} d_{i,\nu} x_\nu \geq d_{i,0} \lor \sum_{\nu \in V} d_{k,\nu} x_\nu \geq d_{k,0}, \]

where \( x \geq 0 \) and \( d_{i,0}, d_{k,0} > 0 \). Suppose that \( u_i \) and \( u_k \) with \( u_i, u_k > 0 \) are valid upper bounds of \( d_{i,0} - \sum_{\nu \in V} d_{i,\nu} \) and \( d_{k,0} - \sum_{\nu \in V} d_{k,\nu} \), respectively. We introduce an auxiliary binary variable \( z \in \{0, 1\} \) to transform (12) into

\[ d_{i,0} - \sum_{\nu \in V} d_{i,\nu} x_\nu \leq u_i (1 - z), \quad d_{k,0} - \sum_{\nu \in V} d_{k,\nu} x_\nu \leq u_k z. \]
Multiplying the first inequality with \( \gamma := \frac{d_{\kappa_0}}{u_i d_{\kappa_0} + u_k d_{\kappa_0}} > 0 \) and the second inequality with \( \mu := \frac{d_{\kappa_0}}{u_i d_{\kappa_0} + u_k d_{\kappa_0}} > 0 \) yields
\[
\gamma u_i z - \sum_{\nu \in V} \gamma d_{i\nu} x_\nu \leq \gamma (u_i - d_{\kappa_0}),
\]
\[
-\mu u_k z - \sum_{\nu \in V} \mu d_{k\nu} x_\nu \leq -\mu d_{k0}.
\]
We will apply Lemma 4.2 with
\[
\beta^1 = \gamma (u_i - d_{\kappa_0}), \quad \alpha^1_{\nu^*} = \gamma u_i, \quad \alpha^1_\nu = -\gamma d_{i\nu} \forall \nu \in V;
\]
\[
\beta^2 = -\mu d_{k0}, \quad \alpha^2_{\nu^*} = -\mu u_k, \quad \alpha^2_\nu = -\mu d_{k\nu} \forall \nu \in V,
\]
where \( \nu^* \) is the additional index belonging to \( z \). We use that \( \gamma u_i + \mu u_k = 1 \), \( \gamma d_{\kappa_0} = \mu d_{k0} \), and \( 0 < \gamma u_i \leq 1 \), to obtain the following values defined in (8)
\[
\Delta_0 = -\mu d_{k0} - \gamma (u_i - d_{\kappa_0}) = -\gamma u_i,
\]
\[
g_0 = -\gamma u_i - [\gamma u_i] = -\gamma u_i + 1 \geq 0,
\]
\[
\Delta_{\nu^*} = -\mu u_k - \gamma u_i = -1, \quad g_{\nu^*} = 0,
\]
\[
\Delta_\nu = -\mu d_{k\nu} + \gamma d_{i\nu} = m^*_{\nu}, \quad g_\nu = m^*_{\nu} - [m^*_{\nu}], \quad \forall \nu \in I,
\]
where \( m^*_{\nu} := \frac{d_{i\nu} d_{k0} - d_{k\nu} d_{\kappa_0}}{u_i d_{\kappa_0} + u_k d_{\kappa_0}} \) is defined as in Theorem 2.2. Taking into account that \( g_{\nu^*} - g_0 \leq 0 \), we derive from Lemma 4.2 the inequality
\[
- z + \sum_{\nu \in I} \left( [m^*_{\nu}] + \frac{(g_\nu - g_0)^+}{1 - g_0} \right) x_\nu + \frac{1}{1 - g_0} \left( \gamma u_i z - \sum_{\nu \in I} \gamma d_{i\nu} x_\nu \right)
\]
\[
- \sum_{\nu \in I} \max \{ \gamma d_{i\nu}, \mu d_{k\nu} \} x_\nu - \gamma (u_i - d_{i0}) \leq |\Delta_0|.
\]
Since \( |\Delta_0| = -1 \) and \( 1 - g_0 = \gamma u_i \), this inequality simplifies to
\[
- z + \sum_{\nu \in I} \left( [m^*_{\nu}] + \frac{(g_\nu - g_0)^+}{\gamma u_i} \right) x_\nu + z - \sum_{\nu \in I} \gamma d_{i\nu} x_\nu
\]
\[
- \sum_{\nu \in I} \max \left\{ \frac{d_{i\nu}}{u_i} \frac{\mu d_{k\nu}}{\gamma u_i} \right\} x_\nu - 1 + \frac{d_{i0}}{u_i} \leq -1,
\]
Multiplication with \( -\frac{u_i}{d_{i0}} < 0 \) and further simplification yields
\[
- \sum_{\nu \in I} \left( \frac{u_i}{d_{i0}} [m^*_{\nu}] + \frac{(g_\nu - g_0)^+}{\gamma d_{i0}} \right) x_\nu + \sum_{\nu \in I} \frac{d_{i\nu}}{d_{i0}} x_\nu
\]
\[
+ \sum_{\nu \in I} \max \left\{ \frac{d_{i\nu}}{d_{i0}} \frac{d_{k\nu}}{d_{k0}} \right\} x_\nu \geq 1.
\]
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Recall that \( m^*_\nu := \frac{d_{i\nu}d_{k0} - d_{\nu}d_{i0}}{u_{i\nu}d_{k0} + u_{\nu}d_{i0}} \) and \( \mu := \frac{d_{i0}}{u_{i\nu}d_{k0} + u_{\nu}d_{i0}} \). We consider the general case that \( g_\nu \leq g_0 \) is possible and use that \( \gamma u_i + \mu u_k = 1 \) to obtain

\[
\frac{u_i}{d_{i0}} \lfloor m^*_\nu \rfloor + \frac{1}{\gamma d_{i0}} (g_\nu - g_0)
\]

\[
= \frac{u_i}{d_{i0}} \lfloor m^*_\nu \rfloor + \frac{u_i d_{k0} + u_k d_{i0}}{d_{i0} d_{k0}} (m^*_\nu - \lfloor m^*_\nu \rfloor + \gamma u_i - 1)
\]

\[
= - \frac{u_k}{d_{k0}} \lfloor m^*_\nu \rfloor + \frac{u_i d_{k0} + u_k d_{i0}}{d_{i0} d_{k0}} (m^*_\nu - \mu u_k)
\]

\[
= - \frac{u_k}{d_{k0}} \lfloor m^*_\nu \rfloor + \frac{u_i d_{i0} - d_{k0} d_{i0}}{d_{i0} d_{k0}} - \frac{u_k}{d_{k0}}
\]

\[
= - \frac{u_k}{d_{k0}} \lfloor m^*_\nu \rfloor + \frac{d_{i\nu} - d_{i0}}{d_{i0}} \frac{d_{k0}}{d_{i0}} + \frac{d_{k\nu} - d_{k0}}{d_{k0}}.
\]

Considering the case that \( g_\nu = m^*_\nu - \lfloor m^*_\nu \rfloor > g_0 \), it follows that \( m^*_\nu \notin \mathbb{Z} \) because \( g_0 \geq 0 \). Then \( 1 + \lfloor m^*_\nu \rfloor = \lceil m^*_\nu \rceil \) and with the help of (14), Inequality (13) can be reformulated as

\[
\sum_{\nu \in I} \frac{d_{i\nu} - u_i \lfloor m^*_\nu \rfloor}{d_{i0}} x_\nu + \sum_{\nu \in I} \frac{d_{k\nu} + u_k \lceil m^*_\nu \rceil}{d_{k0}} x_\nu
\]

\[+ \sum_{\nu \in I} \max \left\{ \frac{d_{i\nu}}{d_{i0}}, \frac{d_{k\nu}}{d_{k0}} \right\} x_\nu \geq 1.
\]

To show the assertion, it remains to prove that we have

\[
\min \left\{ \frac{d_{i\nu} - u_i \lfloor m^*_\nu \rfloor}{d_{i0}}, \frac{d_{k\nu} + u_k \lceil m^*_\nu \rceil}{d_{k0}} \right\}
\]

\[
= \begin{cases} 
\frac{d_{i\nu} - u_i \lfloor m^*_\nu \rfloor}{d_{i0}}, & \text{if } g_\nu \leq g_0, \\
\frac{d_{i\nu} - u_i \lceil m^*_\nu \rceil}{d_{i0}}, & \text{otherwise},
\end{cases}
\]

for \( \nu \in I \), see Theorems 2.1 and 2.2. If \( m^*_\nu \notin \mathbb{Z} \), then this result follows from the equality

\[
\frac{d_{i\nu} - u_i \lfloor m^*_\nu \rfloor}{d_{i0}} - g_\nu - g_0 = \frac{d_{k\nu} + u_k \lceil m^*_\nu \rceil}{d_{k0}}
\]

that we derive from (14) using that \( 1 + \lfloor m^*_\nu \rfloor = \lceil m^*_\nu \rceil \). Conversely, if \( m^*_\nu \in \mathbb{Z} \), then \( g_\nu = 0 \), and we obtain that

\[
\frac{g_\nu - g_0}{\gamma d_{i0}} = -1 + \gamma u_i = -\mu u_k = -\frac{d_{i0} u_k}{d_{k0} d_{i0}} = -\frac{u_k}{d_{k0}},
\]

and since \( \lfloor m^*_\nu \rfloor = \lceil m^*_\nu \rceil \), we derive from (14) that

\[
\frac{d_{i\nu} - u_i \lfloor m^*_\nu \rfloor}{d_{i0}} = \frac{d_{k\nu} + u_k \lceil m^*_\nu \rceil}{d_{k0}}.
\]

This completes the proof. \( \square \)
The converse of Theorem 4.3 is also correct: Every GMIR cut is an MIR cut (proof of Lemma 4.2), every MIR cut is a GMI cut (Cornuéjols and Li [7]), and every GMI cut is a monoidal disjunctive cut (Proposition 3.1). It remains an open question whether the results of this section can be generalized to arbitrary disjunctions including the case $|D| > 2$.

5 Computational Experiments

In this section, we report on computational experience with our implementation of monoidal cut strengthening. One example where disjunctions naturally occur are complementarity constraints of the form $x_i \cdot x_j = 0$ that can be modeled as $x_i \leq 0 \lor x_j \leq 0$ if $x_i, x_j \geq 0$. For the considered instances, the computational results will confirm that GMIR cuts are for complementarity constraints just as important as MIR cuts for mixed-integer problems. Our implementation uses the branch-and-cut solver presented in [9] with SCIP 3.2 [1, 15] as framework and CPLEX 12.6 as LP-solver. All experiments were run with a time limit of two hours on an Intel core i7-5820K CPU processor with 3.3 GHz and 15 MB cache.

5.1 Instances

The instances arise from an application in telecommunications engineering: Consider a multi-hop wireless network (see, e.g., Shi et al. [16]) represented by a digraph $D = (N, L)$. Here, $N$ denotes a set of base stations that are connected via a set of wireless links $L$. The goal is to find a maximum data flow from a source to a destination subject to link capacity, flow conservation, and interference constraints.

The model of Gupta and Kumar [11] makes use of the same channel that is split into subchannels for all data transmissions; e.g., by frequencies $t \in T$. The interference constraints state that a base station cannot use the same subchannel for more than one transmission and more than one reception. This can be modeled with the help of complementarity constraints.

Given $u \in N$ we denote by $\delta^+(u)$ and $\delta^-(u)$ the successors and predecessors of $u$ in $D$, respectively. Furthermore, we denote by $E$ the set of interfering links and by $x^t_\ell \in \mathbb{Z}$ the variable for the integer flow on link $\ell \in L$ at subchannel $t \in T$. Mathematically, the problem can be formulated as follows:

$$\max_{r \in \mathbb{R}} \quad r$$
$$\text{s.t.} \quad \sum_{t \in T} \left( \sum_{\ell \in \delta^+(u)} x^t_\ell - \sum_{\ell \in \delta^-(u)} x^t_\ell \right) = \beta_u \quad \forall u \in N,$$
$$0 \leq x^t_\ell \leq C_\ell, \quad x^t_\ell \in \mathbb{Z} \quad \forall \ell \in L, t \in T,$$
$$x^t_\ell \cdot x^t_{\ell'} = 0 \quad \forall \{\ell, \ell'\} \in E, t \in T.$$
Here, $\beta_u$ is $r$ if $u$ is the source, $-r$ if $u$ is the destination, and 0 otherwise. The capacities $C_\ell \in \mathbb{Z}_+$, $\ell \in L$, depend on the distance between two nodes and the maximum transmission power, see the formula in Shi et al. [16]. We generated 30 instances that were produced analogously to [9]. All instances consider $|N| = 80$ nodes that were located randomly on a $1000 \times 1000$ grid.

5.2 Experimental Results

We made tests with three different settings, whose explanation can be found in Table 1. Disjunctive cuts were generated directly from the simplex tableau for disjunctions $x_i \leq 0 \lor x_j \leq 0$ arising from complementarity constraints. If disjunctive cuts were turned on, we only separated them in the root node of the branch-and-bound tree. For all three settings, we additionally used bound inequalities that are described in [9]. We separated them with node-depth frequency 10. All other cutting plane separators of SCIP were turned off. In order to eliminate the influence of heuristics on the performance, we initialized the algorithm with precomputed optimal solutions. The remaining settings are set to their default in SCIP.

Table 2 shows the aggregated results of all 30 individual instances. Column “solved” lists the number of instances that could be solved within the time limit of two hours, column “cuts” the number of applied disjunctive cuts in arithmetic mean, and columns “nodes” and “time” the number of branching nodes and the CPU time after the solving process terminated both in shifted geometric mean. The shifted geometric mean of values $t_1, \ldots, t_n$ with shift value $\delta$ is defined as $\left( \prod_{i=1}^{n} (t_i + \delta) \right)^{1/n} - \delta$. For the shifted geometric mean, we used a shift value of 100 for the nodes and 10 for the CPU time.

The computational results clearly demonstrate that “disj-uns” outperforms “disj-off” and that “disj-str” outperforms “disj-uns” for the considered instances. Moreover, in direct comparison of “disj-off” and “disj-uns”, a Wilcoxon signed rank test, see [6], confirmed a statistically significant reduction in the time with a $p$-value of less than 0.05. If we compare “disj-off” and “disj-str”, the reduction is even more significant with $p < 0.005$.

Table 1: Settings for the separation of disjunctive cuts

<table>
<thead>
<tr>
<th>shortcut</th>
<th>explanation</th>
</tr>
</thead>
<tbody>
<tr>
<td>disj-off</td>
<td>no separation of disjunctive cuts</td>
</tr>
<tr>
<td>disj-uns</td>
<td>use unstrengthened disjunctive cuts (2)</td>
</tr>
<tr>
<td>disj-str</td>
<td>use strengthened disjunctive cuts (3)</td>
</tr>
</tbody>
</table>
Table 2: Computational experiments on all instances

<table>
<thead>
<tr>
<th>settings</th>
<th>solved</th>
<th>cuts</th>
<th>nodes</th>
<th>time</th>
</tr>
</thead>
<tbody>
<tr>
<td>disj-off</td>
<td>21</td>
<td>0.0</td>
<td>273617.0</td>
<td>1912.7</td>
</tr>
<tr>
<td>disj-uns</td>
<td>23</td>
<td>67.0</td>
<td>189894.0</td>
<td>1446.3</td>
</tr>
<tr>
<td>disj-str</td>
<td>24</td>
<td>65.4</td>
<td>149482.8</td>
<td>1118.6</td>
</tr>
</tbody>
</table>

References


