Distributionally Robust Stochastic Optimization with Wasserstein Distance

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Stochastic programming is a powerful approach for decision-making under uncertainty. Unfortunately, the solution may be misleading if the underlying distribution of the involved random parameters is not known exactly. In this paper, we study distributionally robust stochastic programming (DRSP) in which the decision hedges against the worst possible distribution that belongs to an ambiguity set, which comprises all distributions that are close to some reference distribution in terms of the Wasserstein distance. We derive a tractable reformulation of the DRSP problem by constructing the worst-case distribution explicitly via the first-order optimality condition of the dual problem. Using the precise structure of the worst-case distribution, we show that the DRSP can be approximated by robust programs to arbitrary accuracy. Then we apply our results to a variety of stochastic optimization problems, including the newsvendor problem, two-stage linear program, worst-case value-at-risk analysis, point processes control and distributionally robust transportation problems.

Key words: distributionally robust optimization; Wasserstein ambiguity set; data-driven; Mirror-Prox algorithm
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1. Introduction

In decision making under uncertainty, a large class of problems can be modeled as stochastic programming. In its classical form, it is formulated as the minimization of an expected value function

$$\min_{x \in X} \mathbb{E}_\mu[\Psi(x, \xi)],$$

where $x$ is the decision variable with (deterministic) feasible region $X \subset \mathbb{R}^n$, the random element $\xi$ has distribution $\mu$ supported on a Polish space $\Xi$ and $\Psi : X \times \Xi \rightarrow \mathbb{R}$. We refer to Shapiro et al. [42] for a thorough study of stochastic programming.

One major criticism of stochastic programming is the assumption of the awareness of the exact probability distribution $\mu$. After all, in real-world problem $\mu$ can never be obtained exactly, while deviation from $\mu$ may results in bad decisions. This motivates the notion of distributionally robust stochastic programming (DRSP) as follows. Suppose one can construct a family $\mathcal{M}$ of probability distributions containing all conceivable distributions that are close to the unknown underlying distribution. Then the decision maker finds an optimal solution which hedges against the worst-case expected value among these distributions by solving the minimax stochastic programming

$$\min_{x \in X} \sup_{\mu \in \mathcal{M}} \mathbb{E}_\mu[\Psi(x, \xi)].$$ (1)

Such an approach has its root in von Neumann’s game theory and has been used in many fields such as inventory management (Scarf et al. [41], Gallego and Moon [19]), statistical decision analysis (Berger [7]), as well as stochastic programming (Záčková [52], Shapiro and Kleywegt [43]). Recently due to the development of Big Data, it regains attention in Operations Research and sometimes is called data-driven stochastic programming or ambiguous stochastic programming. The idea is that, in many practical applications, the historical observations of $\xi$, with or without noise, are available to help decision makers infer the relevant distributions. A good choice of
ambiguity set $\mathcal{M}$ should take into account both the practical meaning and the tractability of the resulting optimization problem. In general, constructing the ambiguity set $\mathcal{M}$ has two ways — moment-based and statistical-distance-based methods.

**Moment-based ambiguity set.** In this approach, the ambiguity contains a family of distributions of which moments satisfies certain conditions, such as fixed mean and variance (Scarf et al. [41], Delage and Ye [15], Popescu [38], Zymler et al. [54]). It has been shown that in many cases the resulting optimization problem can be reformulated as a conic quadratic or semi-definite program. Moreover, the inner maximization problem of (3) can be reformulated as a linear/ conic quadratic/ semidefinite program. Therefore, it has been appealing to many researchers. Nevertheless, we point out that $\phi$-divergence may not be favorable in some settings. Let us consider the following example.

**Statistical-based ambiguity set.** The other way is by considering a set of probability measures that are close, in the sense of statistical distance, to some reference distribution $\nu$. Let $I_N$ be the standard $N$-simplex $I_N := \{ (p_0, \ldots, p_N) \in \mathbb{R}^{n+1} : \sum_{i=0}^{N} p_i = 1, \ p_i \geq 0, \forall i \}$.

### Table 1. Examples of $\phi$-divergence

<table>
<thead>
<tr>
<th>Divergence</th>
<th>Kullback-Leibler</th>
<th>Burg entropy</th>
<th>$\chi^2$-distance</th>
<th>Modified $\chi^2$</th>
<th>Hellinger</th>
<th>Total Variation</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\phi(t)$, $t \geq 0$</td>
<td>$t \log t - t + 1$</td>
<td>$-\log t + t - 1$</td>
<td>$\frac{1}{2} (t-1)^2$</td>
<td>$(t-1)^2$</td>
<td>$(\sqrt{t} - 1)^2$</td>
<td>$</td>
</tr>
<tr>
<td>$I_N (p, q)$</td>
<td>$\sum p_i \log \left( \frac{p_i}{q_i} \right)$</td>
<td>$\sum q_i \log \left( \frac{q_i}{p_i} \right)$</td>
<td>$\sum \frac{(p_i-q_i)^2}{p_i}$</td>
<td>$\sum \frac{(p_i-q_i)^2}{q_i}$</td>
<td>$\sum (\sqrt{p_i} - \sqrt{q_i})^2$</td>
<td>$\sum</td>
</tr>
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</table>
Example 1. Suppose there is an underlying true image (Figure (1b)). The decision maker is not aware of it but only has a vintage-style observation (Figure (1a)). In fact, Figure (1a) is obtained from Figure (1b) by a low-contrast intensity transformation (cf. Gonzalez and Woods [23]), in which the black pixels become brighter and the white pixels become darker. In terms of the histogram, the observation $q$ is obtained by shifting the true histogram $p_{true}$ inwards. On the other hand, pathological image (Figure (1c)) with histogram $p_{pathol}$ is too dark to see the details. Suppose the decision maker constructs an ambiguity set $\mathcal{M}$ by considering all distributions whose Kullback-Leibler divergence to the observation $q$ is less than or equal to $\theta$. Since $I_{\phi_{KL}}(q, p_{true}) \gg I_{\phi_{KL}}(q, p_{pathol})$, when the radius $\theta$ is chosen such that the pathological image (Figure (1c)) is excluded, the true image (Figure (1b)) is also excluded, thus the robustification of decisions cannot be achieved. Conversely, when the radius $\theta$ is large enough to include the true image (Figure (1b)), it has to include pathological image (Figure (1c)) as well, and then the decision may be over-conservative due to hedging against irrelevant distributions.

The reason for such inconsistency between perceptual (dis)similarity of images and the Kullback-Leibler distance of their histograms is that $\phi$-divergence ignores the metric structure in the histogram, namely, the space $\Xi$. More specifically, in the previous example, $\Xi = \{0, 1, \ldots, 255\}$ represents the 8-bit gray-scale levels. The smaller absolute difference between two points $\xi, \xi' \in \Xi$, the perceptually closer in color they are. While in the definition (2) of $\phi$-divergence $I_{\phi}(p, q)$, only the relative ratio $p_i/q_i$ for the same gray-scale level $i$ is compared, but the distance between different gray-scale levels is not reflected. This phenomenon has been observed in the field of computer science especially in image retrieval (Rubner et al. [40], Ling and Okada [31]), in which histogram comparison is used for extracting features of graphs, such as gray-scale, heat-map and texture. It has been pointed out in Rubner et al. [40] that many $\phi$-divergence focus on bin-by-bin comparison of the histograms, and fail to capture the (dis)similarity across the bins and are sensitive to the bin-size.

The second drawback of using $\phi$-divergence ambiguity set is the well-known fact that when $\lim_{t \to \infty} \phi(t)/t = \infty$, such as $\phi_{KL}$, $\phi_{M\chi^2}$, the $\phi$-divergence ambiguity set fails to include sufficiently
many relevant distributions. In fact, since \( 0 \phi(p_i/0) = p_i \lim_{t \to \infty} \phi(t)/t = \infty \) for all \( p_i > 0 \), the ambiguity set defined by \( \phi \)-divergence does not include any distribution which is not absolutely continuous with respect to the reference distribution \( q \). For example, suppose \( \Xi = \{0,1\}^s \) and the reference distribution \( q \) is supported on \( \{\xi_i\}_{i=1}^N \subset \Xi \). Then the ambiguity set defined by Kullback-Leibler divergence only contains distributions that are supported on the given \( N \) data points \( \{\hat{\xi}_i\}_{i=1}^N \). Unless \( N \) is large relative to \( 2^s \), the ambiguity set is far from rich enough to robustify the decision.

For \( \phi \)-divergence with \( \lim_{t \to \infty} \phi(t)/t < \infty \), such as \( \phi_0, \phi_{\lambda^2}, \phi_h, \phi_{tv} \), we have another potential problem which relates to the behavior of the worst-case distribution. Define \( I_0 := \{1 \leq i \leq N : q_i > 0\} \). Let us assume \( X \) is a singleton, i.e., \( \Psi(x, \xi) = \Psi(\hat{\xi}_i) \), so that we focus on the inner maximization of problem (3). Without loss of generality let us assume \( \Psi(\hat{\xi}_1) < \cdots < \Psi(\hat{\xi}_N) \). Then according to Ben-Tal et al. [5], Bayraksan and Love [3], the worst-case distribution satisfies

\[
\begin{align}
\frac{p_i^*}{q_i} & \in \partial \phi^*\left( \frac{\Psi(\hat{\xi}_i) - \mu^*}{\lambda^*} \right), & \forall i \in I_0, \\
p_j^* & = 0, & \forall j \notin I_0 \cup \{N\}, \\
p_N^* & = \begin{cases} 1 - \sum_{i \in I_0} p_i^*, & \text{if } \mu^* = \Psi(\hat{\xi}_N) - \lambda^* \lim_{t \to \infty} \phi(t)/t, \\ 0, & \text{if } \mu^* > \Psi(\hat{\xi}_N) - \lambda^* \lim_{t \to \infty} \phi(t)/t, \end{cases}
\end{align}
\]  

for some \( \lambda^* \geq 0 \) and \( \mu^* \geq \Psi(\hat{\xi}_N) - \lambda^* \lim_{t \to \infty} \phi(t)/t \). When \( \lim_{t \to \infty} \phi(t)/t < \infty \), although the ambiguity set is allowed to contain distributions that are not absolutely continuous with respect to the reference, (4b) suggests that the support of the worst-case distribution and that of the reference distribution can differ by at most one point \( \hat{\xi}_N \). If \( p_N > 0 \), (4c) suggests that the probability mass is moved away from all scenarios in \( I_0 \) to the worst scenario \( \hat{\xi}_N \). This is called “popping” behavior in Bayraksan and Love [3]. Note that in many applications where the support of \( \xi \) is unknown, given the historical observations, the choice for the underlying space \( \Xi \) may be arbitrary, so the worst-case behavior heavily depends on the specification of \( \Xi \) and the shape of function \( \Psi \). We refer to Section 4.1 for a numerical illustration.

In the above discussion we assume that both the underlying distribution and the reference are discrete distributions. Now let us consider, for example, the underlying distribution is continuous. Given the \( N \) observed data, to avoid some drawbacks of \( \phi \)-divergence discussed before, one may be willing to modify the discrete reference distribution into continuous one via parametric or nonparametric density estimation (see Ben-Tal et al. [5], Jiang and Guan [27] for more details). However, such manipulation unnecessarily introduces further ambiguity and the phenomenon in previous examples is not fully eliminated.

In this paper, we advocate the use of Wasserstein ambiguity set and our main focus is to provide a tractable reformulation for the problem

\[
\min_{x \in X} \sup_{\mu \in \mathcal{M}} \mathbb{E}_\mu[\Psi(x, \xi)]
\]

with

\[
\mathcal{M} := \{\mu \in \mathcal{P} : W_p(\mu, \nu) \leq \theta\},
\]

where \( \mathcal{P} \) is the set of Borel probability distributions on \( \Xi \), \( \nu \) is the reference measure, \( \theta \in [0, \infty) \) is the radius of the Wasserstein ball, representing in some sense the confidence level or the degree of risk aversion, and \( W_p \) is the Wasserstein distance of order \( p \in [1, \infty) \) defined by

\[
W_p^p(\mu, \nu) = \min_{\gamma} \{\mathbb{E}_\gamma[d^p(\xi, \zeta)] : \gamma \text{ has marginal distributions } \mu, \nu\},
\]

where \( d \) is the metric on \( \Xi \). A more formal and general definition of Wasserstein distance for probability measures on a Polish space in Section 2. For now let us roughly view the joint distribution
\(\gamma\) as certain transportation plan that splits and moves mass from \(\mu\) to \(\nu\), so the Wasserstein distance is the minimal expected total distance over all transportation plans. (This is the reason why Wasserstein distance is also called Earth Mover’s distance in computer science literature.) We note that by definition, Wasserstein distance explicitly exploit the metric structure \(d\) on \(\Xi\) as opposed to \(\phi\)-divergence discussed in Example 1.

**Example 2 (Revisiting Example 1).** Let us compute the Wasserstein distance between histograms in Example 1. Roughly speaking, the cheapest way of transporting mass from \(q\) to \(p_{true}\) is by transporting mass near the boundary inwards. Then a large proportion of the total mass is transported within a small distance, so the total expected traveling distance is relatively small. In contrast, to compute \(W_1(q,p_{pathol})\), one has to transport all the mass on the right to the left. Such long traveling distance results in a large expected total distance, whence \(W_1(q,p_{pathol}) > W_1(q,p_{true})\). Therefore, using Wasserstein distance is consistent with the perceptual intuition.

Thanks to results on concentration inequalities with Wasserstein distance (cf. Bolley et al. [8], Fournier and Guillin [18]), it has been pointed out in Esfahani and Kuhn [17] that using Wasserstein ambiguity set provides good out-of-sample performance of the resulting DRSP. Moreover, as suggested in Carlsson et al. [12], the Wasserstein distance is a natural choice for certain transportation problems as it inherits the Euclidean structure if Euclidean metric \(d\) is used to define the ambiguity set.

Wasserstein distance is actually not new to Operations Researchers. Indeed, in 1942 together with the newborn Linear Programming theory Kantorovich [29], Leonid Kantorovich Kantorovich [28] tackled the Monge’s problem originally brought up in 1781 in the field of Optimal Transport, which can be viewed as a generalized transportation problem between two probability distributions. This seminal work also gives the name of Monge-Kantorovich distance, which is now known as \(W_1\)-Wasserstein distance. In the field of stochastic programming, Wasserstein distance has been used for multistage stochastic programming (see Pflug and Pichler [36] and reference therein) and Wasserstein ambiguity set is considered in Wozabal [51]. But only until recently, Esfahani and Kuhn [17], Zhao and Guan [53] show that under certain convexity and/or compactness assumptions, the DRSP with Wasserstein distance is tractable, in which problem (5) is reformulated as a finite convex program by deriving the dual of the inner supremum problem

\[
\sup_{\mu\in \mathcal{M}} \mathbb{E}_\mu[\Psi(x,\xi)].
\]

In this paper, we prove the duality using a constructive approach under general settings, thereby we obtain a precise structure of the worst-case distribution. The powerfulness is demonstrated by applying to various problems in stochastic optimization.

### 1.1. Main Contributions

- **General Settings.** The two most related work to ours are Esfahani and Kuhn [17], Zhao and Guan [53]. Both paper only consider the ambiguity set defined by a \(W_1\)-Wasserstein ball on a finite dimensional normed vector space and centered at some reference distribution with finite support. Moreover, additional technical assumptions are used in their work. For example, Esfahani and Kuhn [17] made certain convexity assumptions to derive the worst-case distribution, and Zhao and Guan [53] assumed the random parameter is compact-supported and has a density. In contrast, in our setting the reference measure \(\nu\) can be any general probability measure on a Polish space and Wasserstein distance of any order \(p\in [1,\infty)\) is considered, and the only assumption on \(\Psi(x,\xi)\) is upper semi-continuity with respect to \(\xi\).

- **Elementary Proof.** Our proof approach is elementary in the sense that we do not resort to tools from infinite dimensional convex programming as did in Esfahani and Kuhn [17], Zhao and Guan [53]. The proof is constructive, thereby we obtain a detailed representation for the worst-case distribution. Moreover, the proof method can be applied to other distributionally robust
problems, such as a large class of distributionally robust transportation problems considered in Carlsson et al. [12] (Section 4.6). The basic idea of the proof is deriving the weak dual and then using the first-order optimality condition of the dual program to construct a primal feasible solution which turns out to achieve the same objective value as the associated dual feasible solution. Except for some technicality in measure theory — which can be circumvented if only considering distribution with finite support as in Esfahani and Kuhn [17], Zhao and Guan [53] — such an approach is more straightforward than the method therein.

- **Insightful Structure of the Worst-case Distribution.** Thanks to its concise form, the worst-case distribution has intuitive meanings and establishes the connection between the classical Robust Programming and the DRSP. More specifically, we show that in the case of finite-supported reference distribution, the distributionally robust stochastic program can be approximated by robust programs to any accuracy, and such an approximation becomes exact when the objective function $\Psi(x, \xi)$ is concave in the uncertainty $\xi$.

- **Broad Applicability.** To illustrate the power of the duality results, we apply it to the newsvendor problem, two-stage linear programming, worst-case value-at-risk analysis, point processes control, and develop a version of Mirror-Prox algorithm for saddle-point problems to solve the distributionally robust stochastic programming, provided that the objective is concave in the uncertainty.

The rest of this paper is organized as follows. In Section 2, we review some main results on Wasserstein distance from the theory of optimal transport. Next we prove the duality results for the general reference distribution and distribution with finite support in Section 3.1 and 3.2 respectively. Then in Section 4, we apply the strong duality results and the structural description of the worst-case distribution to a variety of stochastic optimization problems. We conclude this paper in Section 5 and provide some lemmas and proofs in the technical appendix.

### 2. Preliminaries on Wasserstein Distance

In this section, we introduce notations and briefly outline some results on Wasserstein distance from the theory of optimal transport. For a more detailed discussion we refer the reader to Villani [47, 48].

Let $\Xi$ be a Polish space equipped with metric $d$. We denote by $\mathcal{P}(\Xi)$ the set of Borel probability distributions (measures) on $\Xi$, and by $\mathcal{P}_p(\Xi)$ the set of probability distributions on $\Xi$ with finite $p$-th moment:

$$\mathcal{P}_p(\Xi) := \{\mu \in \mathcal{P} : \int_\Xi d^p(\xi, \zeta_0) \mu(d\xi) < \infty \text{ for some } \zeta_0 \in \Xi\}.$$  

Note that from triangle inequality the above definition is not dependent on the particular choice of $\zeta_0$. Throughout this paper, we assume that $p \in [1, \infty)$. Given a Borel probability distribution $\nu \in \mathcal{P}$ and a Borel map $T : \Xi \to \Xi$, we say $\mu \in \mathcal{P}$ is the image distribution of $\nu$ by $T$, or $T$ transport $\nu$ onto $\mu$, denoted by $\mu = T_#\nu$, if

$$\mu(A) = \nu({\{\xi \in \Xi : T(\xi) \in A\}}), \ \forall \ \text{Borel set } A \subset \Xi.$$  

We define by

$$\Gamma(\mu, \nu) := \{\gamma \in \mathcal{P}(\Xi \times \Xi) : \pi^1_#\gamma = \mu, \pi^2_#\gamma = \nu\}$$  

the set of Borel probability distributions on $\Xi \times \Xi$ that has marginal distribution $\mu$ and $\nu$, where $\pi^i : \Xi \times \Xi \to \Xi$, $i = 1, 2$ are canonical projections. We call $\gamma \in \Gamma(\mu, \nu)$ is a transportation plan from $\mu$ to $\nu$. We denote by $C_b(\Xi)$ the space of continuous and bounded real functions on $\Xi$ and by $L^1(\nu)$ the $L^1$ space of $\nu$-measurable functions.

**DEFINITION 1 (WASSERSTEIN DISTANCE).** Let $p \in [1, \infty)$. The Wasserstein distance of order $p$ between $\mu, \nu \in \mathcal{P}_p(\Xi)$ is defined by

$$W^p_p(\mu, \nu) := \min_\gamma \left\{\int_{\Xi \times \Xi} d^p(\xi, \zeta) \gamma(d\xi, d\zeta) : \gamma \in \Gamma(\mu, \nu)\right\}.$$  

(8)
From Definition 1, Wasserstein distance between $\mu, \nu$ \{a metric (cf. Theorem 7.3 in Villani [47]). From Hölder’s inequality, it can be easily checked that
\[
\min_{\gamma_{ij} \geq 0} \left\{ \sum_{i,j} c_{ij} \gamma_{ij} : \sum_j \gamma_{ij} = q_i, \forall i, \sum_i \gamma_{ij} = p_j, \forall j \right\}.
\]
In particular, when $M = N$, it reduces to the classical assignment problem. By Birkhoff’s Theorem, there exists an $N$-permutation, denoted as $T_N$, such that $\gamma^0_{ij} := \frac{1}{N} \mathbf{1}_{\{j = T_N(i)\}}$ is the optimal transportation plan. In this case, the first marginal of $\gamma^0$ is $\mu^0 := \pi_{\#}^1 \gamma^0 = T_N \# \nu$, that is, $T_N$ transports $\nu$ onto $\mu^0$.

Example 3 suggests the name of “transporting map” $T$ and “transportation plan” $\gamma \in \Gamma(\mu, \nu)$. From Definition 1, Wasserstein distance between $\mu, \nu$ is the minimal transportation cost (in terms of $d^p$) among all transportation plans in $\Gamma(\mu, \nu)$. It can be shown that $W_p$ defined above is indeed a metric (cf. Theorem 7.3 in Villani [47]). From Hölder’s inequality, it can be easily checked that $p_1 \geq p_2 \geq 1$ implies that $W_{p_1} \geq W_{p_2}$ (cf. Section 7.1.2 in Villani [47]). By Kantorovich’s duality (Theorem 1.3 in Villani [47]),
\[
W_p^p(\mu, \nu) = \max_{u \in L^1(\mu), v \in L^1(\nu)} \left\{ \int_{\Xi} u(\xi) \mu(d\xi) + \int_{\Xi} v(\zeta) \nu(d\zeta) : u(\xi) + v(\zeta) \leq d^p(\xi, \zeta), \forall \xi, \zeta \in \Xi \right\}.
\]
Note that the set of functions under the maximum above can be replaced by $u, v \in C_b(\Xi)$. Particularly when $p = 1$, by Kantorovich-Rubinstein theorem (cf. Theorem 1.14 in Villani [47]), (9) can be simplified to
\[
W_1(\mu, \nu) = \max_{u \in C_b(\Xi)} \left\{ \int_{\Xi} u(\xi) d(\mu - \nu)(\xi) : u \text{ is 1-Lipschitz} \right\}.
\]
So for $L$-Lipschitz function $\Psi$, $|E_{\mu}[\Psi(\xi)] - E_{\nu}[\Psi(\xi)]| \leq L\theta$ for all $\mu' \in \mathcal{M} = \{\mu \in \mathcal{P}_p(\Xi) : W_p(\mu, \nu) \leq \theta\}$. The following lemma generalizes the above statement.

**Lemma 1.** Let $\Psi(\xi) : \Xi \rightarrow \mathbb{R}$. Suppose $\Psi(\xi)$ satisfies
\[
|\Psi(\xi) - \Psi(\zeta)| \leq \kappa(M + d^p(\xi, \zeta)) \tag{10}
\]
for some $\kappa > 0$, $M \geq 0$ and all $\xi, \zeta \in \Xi$. Then
\[
|E_{\mu}[\Psi(\xi)] - E_{\nu}[\Psi(\xi)]| \leq \kappa(M + \theta^p), \quad \forall \mu \in \mathcal{M}.
\]

We close the section by pointing out that one important feature of Wasserstein distance, that is, $W_p$ metrizes the weak convergence in $\mathcal{P}_p(\Xi)$ (cf. Theorem 6.9 in Villani [48]). In other words, let $\{\mu_k\}_{k=1}^{\infty}$ be a sequence of measures in $\mathcal{P}_p(\Xi)$ and $\mu \in \mathcal{P}_p(\Xi)$, then $\lim_{k \to \infty} W_p(\mu_k, \mu) = 0$ if and only if $\mu_k$ converges weakly in $\mathcal{P}_p(\Xi)$ to $\mu$. Therefore, convergence in the Wasserstein distance of order $p$ implies convergence up to $p$-th moment. We refer the reader to Chapter 6 of Villani [48] for a discussion on the advantages of Wasserstein distance over other distances that metrize the weak convergence, such as Prokhorov metric.
3. Tractable Reformulation via Duality

The DRSP problem (5) involves a supremum of expectations over infinitely many distributions, which makes it difficult to solve. In this section we develop a tractable reformulation of the inner problem (7) by deriving its strong dual, which turns out to be a finite dimensional program. To emphasize that $x$ is fixed if only the inner supremum is considered, we suppress $x$ subscript of $\Psi$, and thus (7) is rewritten as

$$[\text{Primal}] \quad v_P := \sup_{\mu \in \mathcal{P}} \left\{ \int_{\Xi} \Psi(x) \mu(d\xi) : W_p(\mu, \nu) \leq \theta \right\}. \quad (11)$$

where $p \in [1, \infty)$, $\theta \geq 0$, $\mu, \nu \in \mathcal{P} := \mathcal{P}_p(\Xi)$ and $\Psi(x) := \Psi(x, \xi) \in L^1(\nu)$. In Proposition 2, we will derive the (weak) dual program

$$[\text{Dual}] \quad v_D := \inf_{\lambda \geq 0} \left\{ \lambda \theta^p - \int_{\Xi} \inf_{\xi \in \Xi} \left[ \lambda d^p(\xi, \zeta) - \Psi(x(\xi) \right\} \nu(d\xi) \right\}. \quad (12)$$

Our main goal is to show the strong duality holds, i.e., $v_P = v_D$, and identify the condition for the existence of the worst-case distribution. We shall see that the existence of the worst-case distribution is related to the growth rate of $\Psi(x(\xi)$ as $\xi$ approaches to infinity. More specifically, for some fixed $\zeta_0 \in \Xi$, we define the growth rate $\kappa$ by

$$[\text{Growth Rate}] \quad \kappa := \limsup_{d(\xi, \zeta_0) \to \infty} \frac{\Psi(x(\xi) - \Psi(x(\zeta_0))}{d^p(\xi, \zeta_0) + 1}. \quad (13)$$

We note that the value of $\kappa$ actually does not depend on the particular choice of $\zeta_0$, as proved in Lemma 5 in the Appendix.

3.1. General Reference Distribution

In this subsection, we consider $\nu$ being any general distribution. While a large amount of existing literature on DRSP focus on “data-driven” approach, where the reference distribution is chosen as the empirical distribution obtained from data, our result is not confined to this setting. Such generality broadens the applicability of the DRSP. For example, the result is useful when the reference distribution is some parametric distribution such as Gaussian distribution (Section 4.4), or some stochastic processes (Section 4.5).

To begin with, let us assume $\theta > 0$. We first consider $\kappa = \infty$.

**Proposition 1.** Suppose $\kappa = \infty$ and $\theta > 0$. Then $v_P = v_D = \infty$.

**Remark 1 (Choosing Wasserstein order $p$).** Let $\zeta_0 \in \Xi$. Define

$$p := \inf_{p \geq 1} \left\{ \lim_{d(\zeta, \zeta_0) \to \infty} \frac{\Psi(x(\zeta) - \Psi(x(\zeta_0))}{d^p(\zeta, \zeta_0) + 1} \right\}. \quad (14)$$

Proposition 1 suggests that the Wasserstein order $p$ should be at least greater than or equal to $p$. Noticing that in the definition (15) of $\Phi(\lambda, \zeta)$, beside $\lambda$, parameter $p$ also controls the extent of perturbation. When $p$ is much greater than $p$, only small perturbation is allowed. In both Esfahani and Kuhn [17], Zhao and Guan [53] only $p = 1$ is considered. By considering higher order $p$ in our analysis, we have more flexibility to choose the ambiguity set and control the degree of conservativeness based on the information of function $\Psi(x, \xi)$. Moreover, as will be seen in Section 4, some choice of $p$ may have some computational benefit.

Now let us consider $\kappa \in [0, \infty)$. The main idea of the proof is rather straightforward. We first derive the weak duality from the Lagrangian, which is a relatively simple routine. The resulting dual problem is a one-dimensional convex minimization problem since there is only one constraint in the primal problem (11). Then by exploiting the first-order optimality for the dual problem (12), we construct a primal feasible solution which yields the same value as the associated dual solution, and thus the strong duality follows. Let us first derive the weak duality.
Proposition 2 (Weak duality). Suppose $\kappa < \infty$. Then $v_P \leq v_D$.

Proof. Let us write the Lagrangian and apply minimax inequality which yields that

$$v_P = \sup_{\mu \in \mathcal{P}} \inf_{\lambda \geq 0} \left\{ \int_{\Xi} \Psi_x(\xi) \mu(d\xi) + \lambda(\theta^p - W^p_p(\mu, \nu)) \right\}$$

$$\leq \inf_{\lambda \geq 0} \left\{ \lambda \theta^p + \sup_{\mu \in \mathcal{P}} \left\{ \int_{\Xi} \Psi_x(\xi) \mu(d\xi) - \lambda W^p_p(\mu, \nu) \right\} \right\}.$$  \hspace{1cm} (14)

To provide an upper bound on $\sup_{\mu \in \mathcal{P}} \left\{ \int_{\Xi} \Psi_x(\xi) \mu(d\xi) - \lambda W^p_p(\mu, \nu) \right\}$, using the equivalent definition (9) of $W_p$, we obtain that

$$\sup_{\mu \in \mathcal{P}} \left\{ \int_{\Xi} \Psi_x(\xi) \mu(d\xi) - \lambda W^p_p(\mu, \nu) \right\}$$

$$= \sup_{\mu \in \mathcal{P}} \left\{ \int_{\Xi} \Psi_x(\xi) \mu(d\xi) - \lambda \sup_{u \in L^1(\mu), v \in L^1(\nu)} \left\{ \int_{\Xi} u(\xi) \mu(d\xi) + \int_{\Xi} v(\xi) v(d\xi) : \right\} \right\}$$

$$v(\xi) \leq \inf_{\xi \in \Xi} d^p(\xi, \zeta) - u(\xi), \forall \xi \in \Xi \right\}. \right.$$  \hspace{1cm} (15)

Set $u_\lambda := \Psi_x / \lambda$ for $\lambda > 0$, then $u_\lambda \in L^1(\mu)$ due to $\Psi_x \in L^1(\nu)$ and Lemma 1. Plugging $u_\lambda$ into the inner supremum for $u$, we obtain that for $\lambda > 0$,

$$\sup_{\mu \in \mathcal{P}} \left\{ \int_{\Xi} \Psi_x(\xi) \mu(d\xi) - \lambda W^p_p(\mu, \nu) \right\} \leq - \int_{\Xi} \left[ \inf_{\xi \in \Xi} \lambda d^p(\xi, \zeta) - \Psi_x(\xi) \right] \nu(d\zeta) = - \int_{\Xi} \Phi(\lambda, \zeta) \nu(d\zeta). \right.$$  \hspace{1cm} (16)

Note that the above inequality also holds for $\lambda = 0$, so combining it with (14) we finally have

$$v_P \leq \inf_{\lambda \geq 0} \left\{ \lambda \theta^p - \int_{\Xi} \Phi(\lambda, \zeta) \nu(d\zeta) \right\} = v_D. \right.$$  \hspace{1cm} (17)

As mentioned before, the inner infimum involved in the dual problem (12) can be viewed as a perturbation of $\zeta \supp \nu$, which plays an important role in the analysis of worst-case distribution. Let us define

$$\Phi(\lambda, \zeta) := \inf_{\xi \in \Xi} \lambda d^p(\xi, \zeta) - \Psi_x(\xi). \right.$$  \hspace{1cm} (18)

for every $\lambda \geq 0$ and $\zeta \in \Xi$. Observe that when $p = 2$ and $\lambda > 0$, $\Phi(\lambda, \zeta)$ is the classical Moreau-Yosida regularization (cf. Parikh and Boyd [35]) of $-\Psi_x(\xi)$ with parameter $1/\lambda$ at $\zeta$. The parameter $\lambda$ controls the extent to which the point $\xi$ is perturbed towards the supremum of $\Psi_x$ over $\Xi$. With a smaller value of $\lambda$, $\zeta$ is tending to the supremum of $\Psi_x$, while a larger value of $\lambda$ controls the perturbation within a small neighborhood of $\zeta$. We prepare some properties of $\Phi(\lambda, \zeta)$ for the proof of strong duality.

Lemma 2. Assume $\Psi_x$ is upper semi-continuous and $\kappa < \infty$. Then for any $\zeta \in \Xi$ and $\lambda > \kappa$, the following holds:

(i) The right-hand side of (15) admits a minimizer.

(ii) $\Phi(\lambda, \zeta)$ is non-decreasing and concave with respect to $\lambda$ and is continuous with respect to $\zeta$.

(iii) For any $\zeta \in \Xi$, define

$$\hat{T}_-(\lambda, \zeta) := \arg \min_{\xi} \left\{ d(\xi, \zeta) : \xi \in \arg \min_{\xi} \left\{ \lambda d^p(\xi, \zeta) - \Psi_x(\xi) \right\} \right\},$$

$$\hat{T}_+(\lambda, \zeta) := \arg \max_{\xi} \left\{ d(\xi, \zeta) : \xi \in \arg \min_{\xi} \left\{ \lambda d^p(\xi, \zeta) - \Psi_x(\xi) \right\} \right\}. \right.$$  \hspace{1cm} (19)

for every $\lambda \geq 0$ and $\zeta \in \Xi$. Observe that when $p = 2$ and $\lambda > 0$, $\Phi(\lambda, \zeta)$ is the classical Moreau-Yosida regularization (cf. Parikh and Boyd [35]) of $-\Psi_x(\xi)$ with parameter $1/\lambda$ at $\zeta$. The parameter $\lambda$ controls the extent to which the point $\xi$ is perturbed towards the supremum of $\Psi_x$ over $\Xi$. With a smaller value of $\lambda$, $\zeta$ is tending to the supremum of $\Psi_x$, while a larger value of $\lambda$ controls the perturbation within a small neighborhood of $\zeta$. We prepare some properties of $\Phi(\lambda, \zeta)$ for the proof of strong duality.
Then there exists a Borel measurable selection $T_- (\lambda, \zeta)$ (resp. $T_+ (\lambda, \zeta)$) of the multi-valued function $T_- (\lambda, \zeta)$ (resp. $T_+ (\lambda, \zeta)$) such that $d(T_-(\lambda, \zeta), \zeta)$ and $\Psi_x(T_-(\lambda, \zeta))$ (resp. $d(T_+(\lambda, \zeta), \zeta)$ and $\Psi_x(T_+(\lambda, \zeta))$) are left (resp. right) continuous with respect to $\lambda$.

(iv) $\Phi(\lambda, \zeta)/\lambda$ has both a left and right partial derivative

$$\partial_{\lambda^+} \left( \frac{\Phi(\zeta, \lambda)}{\lambda} \right) = \frac{\Psi_x(T_-(\lambda, \zeta))}{\lambda^2}, \quad \partial_{\lambda^+} \left( \frac{\Phi(\zeta, \lambda)}{\lambda} \right) = \frac{\Psi_x(T_+(\lambda, \zeta))}{\lambda^2}.$$

(v) Suppose $\Phi(\kappa, \zeta)$ is finite. $\lim_{\lambda \to \kappa^+} \Phi(\lambda, \zeta) = \Phi(\kappa, \zeta)$, and particularly when $\kappa = 0$, it holds that $\lim_{\lambda \to 0^+} \Psi_x(T_-(\lambda, \zeta)) = \sup_{\xi \in \Xi} \Psi_x(\xi)$.

(vi) There exists $D_\lambda > 0$ such that $d^p(T_+(\lambda, \zeta), \zeta) < D_\lambda$ for all $\zeta$.

Statement (i)(ii)(iv) are similar to the properties of Moreau-Yosida regularization. The $T_\pm(\lambda, \zeta)$ defined in (iii) represent the nearest and the furthest minimizer associated with $\Phi(\lambda, \zeta)$, which will help us construct the worst-case distribution. Their Borel measurability is only needed when deriving strong duality for general reference distribution, and can be avoided if we focus on finite-supported reference distributions. Property (v) will be used for the threshold case where the dual minimizer equal to $\kappa$. Finally, (vi) shows that the furthest distance between $\zeta$ and the minimizer of $\inf_{\xi \in \Xi} \lambda d^p(\xi, \zeta) - \Psi_x(\xi)$ are uniformly bounded with respect to $\zeta$.

We are now ready to show that the strong duality holds.

**Theorem 1 (Strong duality and worst-case distribution).** Let $\theta > 0$. Assume $\Psi_x$ is upper semi-continuous and $\kappa \in [0, \infty)$. Then the following holds:

(i) $v_p = v_D < \infty$.

(ii) The dual problem (12) always admits a minimizer $\lambda^*$. If there exists a dual minimizer $\lambda^* > \kappa$, then the distribution $\mu^* \in \mathcal{P}$ defined by

$$\mu^*(A) = \int_{\Xi} \left[ p(\zeta) \mathbb{1}_A(T_+(\lambda^*, \zeta)) + (1 - p(\zeta)) \mathbb{1}_A(T_-(\lambda^*, \zeta)) \right] \nu(d\zeta), \quad \forall \text{ Borel set } A \subset \Xi, \quad (17)$$

is primal optimal, where $T_+(\lambda^*, \zeta)$, $T_-(\lambda^*, \zeta)$ are defined in Lemma 2 and

$$p(\zeta) = \frac{\theta \nu - \int_{\Xi} \frac{\Phi(\lambda^*, \zeta)}{\lambda} \nu(d\zeta) - \Psi_x(T_-(\lambda^*, \zeta))}{\Psi_x(T_+(\lambda^*, \zeta)) - \Psi_x(T_-(\lambda^*, \zeta))} \mathbb{1}_{\{\Psi_x(T_+(\lambda^*, \zeta)) \neq \Psi_x(T_-(\lambda^*, \zeta))\}}.$$

(iii) If there exists a primal optimal solution $\mu^*$ satisfying $\mu^* = T_\# \nu$ for some Borel map $T^*$: $\text{supp } \nu \to \Xi$, that is, $\mu^*(A) = \nu(T^{-1}(A))$ for all Borel set $A$, then we have the equivalence

$$v_p = v_D = \sup_{\mu \in \mathcal{M}'} \mathbb{E}_{\nu}[\Psi_x(\xi)], \quad (18)$$

where

$$\mathcal{M}' := \left\{ \mu = T_\# \nu \mid T : \text{supp } \nu \to \Xi, \int_{\Xi} d^p(\xi, T(\xi)) \nu(d\xi) \leq \theta \right\}. \quad (19)$$

The existence of $T^*$ can be ensured, for example, when

- For $\xi \in \text{supp } \nu$, the problem $\inf_{\lambda^*} \lambda^* d^p(\xi, \zeta) - \Psi_x(\xi)$ has a unique minimizer.
- $\Xi$ is convex and $\Psi_x(\xi)$ is concave in $\xi$.

**Remark 2.** Combining Theorem 1(i) with Proposition 1, we have proved the strong duality with the only assumption of upper semi-continuity of $\Psi_x$, which can hardly be eliminated in order to guarantee the existence of the worst-case distribution. Moreover, the results holds for any reference distribution supported on a Polish space. Therefore we have established the result in a very general setting. In Esfahani and Kuhn [17], Zhao and Guan [53], strong duality is established only for $W_1$-Wasserstein ball centered at a reference distribution with finite support in finite-dimensional normed vector space, and additional assumptions are needed – convexity assumption of $\Psi_x(\xi)$ and $\Xi$ in Esfahani and Kuhn [17], and compactness of $\Xi$ and absolute continuity of $\mu$ in Zhao and Guan [53]. We will demonstrate the power of such generality in Section 4.4 and 4.5.
It follows from (21) that $p$.

Based on the optimality condition (21), we construct a primal solution $\partial$.

The first-order optimality condition reads and use Lemma 2(iv) to obtain that $\Psi$.

Note that $\Phi(\lambda, \zeta)$ helps us derive interesting results in many problems.

Remark 3 (Perturbation by splitting and transporting). (ii) also shows that when $\lambda_x > \kappa$, there exists a worst-case distribution $\mu^*$ which can be viewed as a perturbation of the reference $\nu$ in the following way. Each point $\zeta \in \text{supp } \nu$ (with infinitesimal probability mass) is split into two parts with weights $p(\zeta)$ and $1 - p(\zeta)$, each of which is then transported to the nearest and furthest minimizer $T_\pm(\lambda^*, \zeta)$ associated with $\Phi(\lambda^*, \zeta)$. As will be seen in Section 3.2, this structure helps us derive interesting results in many problems.

Proof of Theorem 1. In view of the weak duality (Proposition 2), it suffices to show that $v_p \geq v_D$. Set

$$h(\lambda) := \lambda \theta^p - \int_\Xi \Phi(\lambda, \zeta) \nu(d\zeta).$$

Note that $\Phi(\lambda, \zeta) = -\infty$ for all $\lambda < \kappa$. By Lemma 2(ii)(v), $h(\lambda)$ is the sum of a linear function $\lambda \theta^p$ and an extended real-valued convex function $-\int_\Xi \Phi(\lambda, \zeta) \nu(d\zeta)$ on $[\kappa, \infty)$, hence $h(\lambda)$ is also convex and continuous on $[\kappa, \infty)$. In addition, since $\Phi(\lambda, \zeta) \geq -\Psi_x(\zeta)$, it follows that $h(\lambda) \geq \lambda \theta^p - \int_\Xi \Psi_x(\zeta) \nu(d\zeta) \to \infty$ as $\lambda \to \infty$. Thus $h(\lambda)$ is a convex function on $[\kappa, \infty)$ which tends to $\infty$, so it admits a minimizer $\lambda^*$ in $[\kappa, \infty)$. Let us consider the following three cases.

- Case 1. There exists a minimizer $\lambda^* > \kappa$.

It follows that $v_D = v_P > -\infty$ and $\int_\Xi \Phi(\lambda^*, \zeta) \nu(d\zeta) = \lambda^* \theta^p - v_D < \infty$. We rewrite the dual objective as

$$h(\lambda) = \lambda \theta^p - \lambda \int_\Xi \Phi(\lambda, \zeta) \nu(d\zeta).$$

The first-order optimality condition reads $\partial_{\lambda^*} h(\lambda^*) \geq 0$ and $\partial_{\lambda_x} h(\lambda^*) \leq 0$, that is,

$$\theta^p - \int_\Xi \frac{\Phi(\lambda^*, \zeta)}{\lambda^*} \nu(d\zeta) - \lambda^* \partial_{\lambda^*} \left( \int_\Xi \frac{\Phi(\lambda^*, \zeta)}{\lambda^*} \nu(d\zeta) \right) \leq 0,$$

$$\theta^p - \int_\Xi \frac{\Phi(\lambda^*, \zeta)}{\lambda^*} \nu(d\zeta) - \lambda^* \partial_{\lambda^-} \left( \int_\Xi \frac{\Phi(\lambda^*, \zeta)}{\lambda^*} \nu(d\zeta) \right) \geq 0. \quad (20)$$

Observe that $d^p(T_+(\lambda^*, \zeta), \zeta) \leq D_{\lambda^*}$ by Lemma 2(vi), so we have $\Psi_x(T_+(\lambda^*, \zeta)) \leq -\Phi(\lambda^*, \zeta) + \lambda^* D_{\lambda^*}$. Observe also that $\Psi_x(T_+(\lambda^*, \zeta)) \geq \Psi_x(\zeta)$. Hence $\Psi_x(T_+(\lambda^*, \zeta)) \in L^1(\nu)$, and similarly for $\Psi_x(T_-(\lambda^*, \zeta))$. Thus we can apply dominated convergence theorem on the left-hand side of (20) and use Lemma 2(iv) to obtain that

$$\theta^p \leq \int_\Xi \frac{\Phi(\lambda^*, \zeta)}{\lambda^*} \nu(d\zeta) + \int_\Xi \frac{\Psi_x(T_+(\lambda^*, \zeta))}{\lambda^*} \nu(d\zeta),$$

$$\int_\Xi \frac{\Phi(\lambda^*, \zeta)}{\lambda^*} \nu(d\zeta) + \int_\Xi \frac{\Psi_x(T_-(\lambda^*, \zeta))}{\lambda^*} \nu(d\zeta) \leq \theta^p. \quad (21)$$

Based on the optimality condition (21), we construct a primal solution $\mu^*$ as follows. We define $p : \text{supp } \nu \to [0, 1]$ by

$$p(\zeta) = \frac{\theta^p - \int_\Xi \frac{\Phi(\lambda^*, \zeta)}{\lambda^*} \nu(d\zeta) - \Psi_x(T_+(\lambda^*, \zeta))}{\Psi_x(T_+(\lambda^*, \zeta)) - \Psi_x(T_-(\lambda^*, \zeta))} \mathbb{1}\{\Psi_x(T_+(\lambda^*, \zeta)) \neq \Psi_x(T_-(\lambda^*, \zeta))\}. \quad (22)$$

It follows from (21) that

$$\theta^p = \int_\Xi \frac{\Phi(\lambda^*, \zeta)}{\lambda^*} \nu(d\zeta) + \frac{1}{\lambda^*} \int_\Xi \left( p(\zeta) \Psi_x(T_+(\lambda^*, \zeta)) + (1 - p(\zeta)) \Psi_x(T_-(\lambda^*, \zeta)) \right) \nu(d\zeta). \quad (23)$$

Since $T_+(\lambda^*, \zeta)$ and $T_-(\lambda^*, \zeta)$ are Borel measurable,

$$\mu^*(A) = \int_\Xi \left( p(\zeta) \mathbb{1}_A(T_+(\lambda^*, \zeta)) + (1 - p(\zeta)) \mathbb{1}_A(T_-(\lambda^*, \zeta)) \right) \nu(d\zeta).$$
is well defined for every Borel measurable set $A \subset \Xi$ and thus $\mu^* \in \mathcal{P}$. Then we verify the feasibility and optimality of $\mu^*$. We compute

$$W_p^p(\mu^*, \nu) = \max_{u,v \in C_b(\Xi)} \left\{ \int_{\Xi} u(\xi)\mu^*(d\xi) + \int_{\Xi} v(\zeta)\nu(d\zeta) : u(\xi) + v(\zeta) \leq d^p(\xi, \zeta), \forall \xi, \zeta \in \Xi \right\}$$

$$= \max_{u,v \in C_b(\Xi)} \left\{ \int_{\Xi} (p(\xi)u(T_+(\lambda^*, \zeta)) + (1 - p(\xi))u(T_-(\lambda^*, \zeta)))\nu(d\zeta) + \int_{\Xi} v(\zeta)\nu(d\zeta) : u(\xi) + v(\zeta) \leq d^p(\xi, \zeta), \forall \xi, \zeta \in \Xi \right\}$$

$$\leq \int_{\Xi} p(\xi)d^p(T_+(\lambda^*, \zeta), \zeta)\nu(d\zeta) + (1 - p(\xi))d^p(T_-(\lambda^*, \zeta), \zeta)\nu(d\zeta)$$

$$= \int_{\Xi} p(\xi)\left( \frac{\Psi_x(T_+(\lambda^*, \zeta))}{\lambda^*} + \frac{\Phi(\lambda^*, \zeta)}{\lambda^*} \right)\nu(d\zeta)$$

$$+ \int_{\Xi} (1 - p(\xi))\left( \frac{\Psi_x(T_-(\lambda^*, \zeta))}{\lambda^*} + \frac{\Phi(\lambda^*, \zeta)}{\lambda^*} \right)\nu(d\zeta)$$

$$= \theta^p,$$

where the last inequality follows from (23). Hence, $\mu^*$ is primal feasible. Meanwhile, we have

$$v_p \geq \int_{\Xi} \Psi_x(\xi)\mu^*(d\xi) = \int_{\Xi} (p(\xi)\Psi_x(T_+(\lambda^*, \zeta)) + (1 - p(\xi))\Psi_x(T_-(\lambda^*, \zeta)))\nu(d\zeta)$$

$$= \lambda^* \left( \theta^p - \int_{\Xi} \frac{\Phi(\lambda^*, \zeta)}{\lambda^*}\nu(d\zeta) \right) = v_D.$$ 

Therefore we obtain the result.

- Case 2. $\lambda^* = 0$ is the unique dual minimizer.

In this case, $v_D = g(0) = \sup_{\xi \in \Xi} \Psi_x(\xi) < \infty$. For any $\lambda > 0$, since zero is the unique minimizer, we have that

$$\lambda \theta^p - \int_{\Xi} \left( \inf_{\xi \in \Xi} \lambda d^p(\xi, \zeta) - \Psi_x(\xi) \right)\nu(d\zeta) > \sup_{\xi \in \Xi} \Psi_x(\xi), \forall \lambda > 0.$$ 

Dividing $\lambda$ on both sides and rearranging terms, we obtain

$$\theta^p > \int_{\Xi} \left( \inf_{\xi \in \Xi} d^p(\xi, \zeta) - \frac{\Psi_x(\xi)}{\lambda} \right)\nu(d\zeta) + \sup_{\xi \in \Xi} \frac{\Psi_x(\xi)}{\lambda}, \forall \lambda > 0. \quad (24)$$ 

We next show that $\{\mu_\lambda\}_\lambda$ is a sequence of feasible distributions approaching to optimality. Using the notations in Lemma 2(iii), for each $\lambda > 0$, we define

$$\mu_\lambda := \nu(\{\zeta : T_-(\lambda, \zeta) \subset A\}), \text{ for any Borel set } A \subset \Xi.$$ 

From Lemma 2(iii) we know that $T_-(\lambda, \zeta)$ is measurable and thus $\mu_\lambda \in \mathcal{P}(\Xi)$. In addition, we compute

$$W_p^p(\mu_\lambda, \nu) = \max_{u,v \in C_b(\Xi)} \left\{ \int_{\Xi} u(\xi)\mu_\lambda(d\xi) + \int_{\Xi} v(\zeta)\nu(d\zeta) : u(\xi) + v(\zeta) \leq d^p(\xi, \zeta), \forall \xi, \zeta \in \Xi \right\}$$

$$= \max_{u,v \in C_b(\Xi)} \left\{ \int_{\Xi} u(T_-(\lambda, \zeta)) + v(\zeta)\nu(d\zeta) : u(\xi) + v(\zeta) \leq d^p(\xi, \zeta), \forall \xi, \zeta \in \Xi \right\}$$

$$\leq \int_{\Xi} d^p(T_-(\lambda, \zeta), \zeta)\nu(d\zeta).$$
But by construction, \( d^p(T_-(\lambda, \zeta), \zeta) = \frac{\Psi_\varepsilon(T_-(\lambda, \zeta))}{\lambda} + \Phi(\lambda, \zeta) \), thereby
\[
W_p^p(\mu_\lambda, \nu) \leq \int_\Xi \left( \frac{\Psi_\varepsilon(T_-(\lambda, \zeta))}{\lambda} + \Phi(\lambda, \zeta) \right) \nu(d\zeta) \\
\leq \int_\Xi \left( \sup_{\xi \in \Xi} \frac{\Psi_\varepsilon(\xi)}{\lambda} + \Phi(\lambda, \zeta) \right) \nu(d\zeta) \\
< \theta^p,
\]
where the last inequality follows from (24). Hence \( \mu_\lambda \) is a feasible solution. Meanwhile, by Lemma 2(\( v \)), we conclude that
\[ v_p \geq - \lim_{\lambda \to 0^+} \int_\Xi \Psi_\varepsilon(\xi) \mu_\lambda(d\xi) = - \lim_{\lambda \to 0^+} \int_\Xi \Psi_\varepsilon(T_-(\lambda, \zeta)) \nu(d\zeta) = \sup_{\xi \in \Xi} \Psi_\varepsilon(\xi) = v_D. \]

\[ \text{• Case 3.} \quad \lambda^* = \kappa > 0 \text{ is the unique dual minimizer.} \]

In this case,
\[ v_D = \kappa \theta^p - \int_\Xi \Phi(\kappa, \zeta) \nu(d\zeta) < \lambda \theta^p - \int_\Xi \Phi(\lambda, \zeta) \nu(d\zeta), \forall \lambda > \kappa. \]
Thus \( v_D < \infty \). Rearranging terms and using the fact that \( \Phi(\kappa, \zeta) \leq \kappa d^p(T_-(\lambda, \zeta), \zeta) - \Psi_\varepsilon(T_-(\lambda, \zeta)) \) for any \( \lambda > \kappa \), we obtain that
\[ (\lambda - \kappa) \theta^p > \int_\Xi [\Phi(\lambda, \zeta) - \Phi(\kappa, \zeta)] \nu(d\zeta) \geq \int_\Xi (\lambda - \kappa) d^p(T_-(\lambda, \zeta), \zeta) \nu(d\zeta), \]
or, \( \int_\Xi d^p(T_-(\lambda, \zeta), \zeta) \nu(d\zeta) < \theta^p \) for any \( \lambda > \kappa \). Choose any compact set \( E \subset \Xi \) with \( \nu(E) > 0 \) and let \( \xi \in \arg \max_{\xi \in E} \Psi_\varepsilon(\xi) \), which is well-defined since \( \Psi_\varepsilon \) is upper semi-continuous. For any \( \epsilon \in (0, \kappa) \) and \( R > 0 \), there exists \( \zeta_\epsilon > 0 \), such that \( d^p(\zeta_\epsilon, \xi) > R \) and
\[ \Psi_\varepsilon(\zeta_\epsilon) - \Psi_\varepsilon(\xi) > (\kappa - \epsilon)(d^p(\zeta_\epsilon, \xi) + \text{diam}(E) + 1), \]
where \( \text{diam}(E) := \max_{\xi, \xi' \in E} d(\xi, \xi') \). It follows that
\[ \Psi_\varepsilon(\zeta_\epsilon) - \Psi_\varepsilon(\xi) > (\kappa - \epsilon)(d^p(\zeta_\epsilon, \xi) + 1), \quad \forall \xi \in E. \]

Similar as before, we are going to define a sequence of distributions approaching to optimality. To this end, we define a map \( T_\varepsilon : \Xi \to \Xi \) by
\[ T_\varepsilon(\zeta) := \zeta_\epsilon \mathbb{1}_{\{\zeta \in E\}} + \zeta \mathbb{1}_{\{\zeta \in \Xi \setminus E\}}. \tag{25} \]
Note that we can choose sufficiently large \( R \) such that \( \int_E d^p(\zeta_\epsilon, \zeta) \nu(d\zeta) > \theta^p \). Thus we define \( \mu_{\lambda, \varepsilon} \) by
\[ \mu_{\lambda, \varepsilon}(A) := \left( 1 - p_{\lambda, \varepsilon} \right) \nu(\{\zeta : T_-(\lambda, \zeta) \in A\}) + p_{\lambda, \varepsilon} \nu(\{\zeta : T_\varepsilon(\zeta) \in A\}), \quad \text{for any Borel set } A \subset \Xi, \]
where \( p_{\lambda, \varepsilon} := \frac{\theta^p - \int_\Xi d^p(T_-(\lambda, \zeta), \zeta) \nu(d\zeta)}{\int_\Xi d^p(T_-(\lambda, \zeta), \zeta) \nu(d\zeta) - \int_\Xi d^p(T_-(\lambda, \zeta), \zeta) \nu(d\zeta)} \). Then \( \mu_{\lambda, \varepsilon} \) is primal feasible since
\[ W_p^p(\mu_{\lambda, \varepsilon}, \nu) \leq (1 - p_{\lambda, \varepsilon}) \int_\Xi d^p(T_-(\lambda, \zeta), \zeta) \nu(d\zeta) + p_{\lambda, \varepsilon} \int_E d^p(\zeta_\epsilon, \zeta) \nu(d\zeta) = \theta^p, \]
and that
\[ v_p \geq \int_\Xi \Psi_x(\zeta) \mu_{\lambda, \epsilon}(\zeta) \]
\[ = (1 - p_{\lambda, \epsilon}) \int_\Xi \Psi_x(T_-(\lambda, \zeta)) \nu(d\zeta) + p_{\lambda, \epsilon} \left[ \int_E \Psi_x(\zeta) \nu(d\zeta) + \int_{E \setminus \Xi} \Psi_x(\zeta) \nu(d\zeta) \right] \]
\[ \geq (1 - p_{\lambda, \epsilon}) \int_\Xi \lambda \Phi(T_-(\lambda, \zeta), \zeta) \nu(d\zeta) - (1 - p_{\lambda, \epsilon}) \int_\Xi \Phi(\lambda, \zeta) \nu(d\zeta) \]
\[ + p_{\lambda, \epsilon}(\kappa - \epsilon) \int_E d^p(\zeta, \zeta) \nu(d\zeta) + p_{\lambda, \epsilon}(\kappa - \epsilon) \int_\Xi \Psi_x(\zeta) \nu(d\zeta) + p_{\lambda, \epsilon} \int_{E \setminus \Xi} \Psi_x(\zeta) \nu(d\zeta) \]
\[ \geq (\kappa - \epsilon) \theta \epsilon - (1 - p_{\lambda, \epsilon}) \int_\Xi \Phi(\lambda, \zeta) \nu(d\zeta) + p_{\lambda, \epsilon} \int_\Xi \Psi_x(\zeta) \nu(d\zeta). \]

Note that \( p_{\lambda, \epsilon} \to 0 \) as \( R \to \infty \) and by Lemma 2(v), \( \Phi(\lambda, \zeta) \to \Phi(\kappa, \zeta) \) as \( \lambda \to \kappa^+ \). Hence letting \( \epsilon \to 0, R \to \infty \) and \( \lambda \to \kappa^+ \), we have \( v_p \geq \theta \epsilon \epsilon - \int_\Xi \Phi(\kappa, \zeta) \nu(d\zeta) = v_p \). Therefore we have shown (i) and (ii).

Now to prove (iii), suppose \( \mu^* = T^\#_p \nu \) for some Borel map \( T^*: \text{supp } \nu \to \Xi \) is a primal optimal solution. From Definition 1 and the fact that \( (T \times \text{id}) \# \nu \in \Gamma(\mu, \nu) \), we have \( \mathcal{W}' \subset \mathcal{W} \) and thus
\[ i \geq \sup_{\mu \in \mathcal{W}'} E_{\mu}[\Psi_x(\xi)]. \]
But at optimality \( W^p(\mu^*, \nu) \leq \int_\Xi d^p(\xi, T^*(\xi)) \nu(d\xi) \leq \theta \epsilon \), whence \( \mu^* \in \mathcal{W}' \) and \( i \leq \sup_{\nu \in \mathcal{W}'} E_{\mu^*}[\Psi_x(\xi)]. \) Therefore, (19) is proved.

Suppose for \( \xi \in \text{supp } \nu \), the problem \( \inf_\xi \lambda^* d^p(\xi, \zeta) - \Psi_x(\xi) \) has a unique minimizer. Setting \( T^*(\zeta) := \arg\min_\xi \lambda^* d^p(\xi, \zeta) - \Psi_x(\xi) \) and using the similar argument in Case 1, we can show that \( \mu^* = T^*(\zeta) \# \nu \) is feasible and optimal.

Suppose \( \Xi \) is a convex metric space and \( \Psi_x(\xi) \) is concave, we have \( \kappa < \infty \). If \( \lambda^* > \kappa \), set
\[ T^*(\zeta) := p^* T_-(\lambda^*, \zeta) + (1 - p^*) T_+(\lambda^*, \zeta), \forall \zeta \in \text{supp } \nu, \]
where
\[ p^* := \frac{\theta \epsilon - \int_\Xi \Psi_x(T_-(\lambda^*, \zeta)) \nu(d\zeta) - \int_\Xi \Psi_x(T_+(\lambda^*, \zeta)) \nu(d\zeta)}{\int_\Xi \Psi_x(T_+(\lambda^*, \zeta)) \nu(d\zeta) - \int_\Xi \Psi_x(T_-(\lambda^*, \zeta)) \nu(d\zeta)}, \]
provided that the denominator is nonzero, otherwise we set \( p^* = 1 \). By concavity \( T^*(\zeta) \) is also an optimal solution of the right-hand side of (15). Then using the same argument as Case 1 above we can show that the distribution \( \mu^* = T^\#_p \nu \) is a primal optimal solution. If \( \lambda^* = \kappa = 0 \), then in Case 2 above, there will be a sequence of distributions \( T_{\lambda, \#} \nu \) converges to optimality as \( \lambda \to \kappa^+ \). If \( \lambda^* = \kappa > 0 \), then define
\[ T_{\lambda, \epsilon}(\zeta) := (1 - p_{\lambda, \epsilon}) T_-(\lambda, \zeta) + p_{\lambda, \epsilon} T_+(\lambda, \zeta), \]
where \( p_{\lambda, \epsilon} := \frac{\theta \epsilon}{\int_\Xi d^p(T_{\lambda, \epsilon} \# \nu)(d\zeta)}. \) Similar to Case 3 above, we can verify that \( \mu_{\lambda, \epsilon} := T_{\lambda, \epsilon} \# \nu \) approach to optimal as \( \lambda \to \kappa^+ \) and \( \epsilon \to 0 \). Therefore we complete the proof. \( \square \)

**Remark 4 (Existence of worst-case distribution).** (ii) shows that the existence of worst-case distribution is essentially determined by the growth parameter \( \kappa \). When \( \lambda^* > \kappa \), a worst-case distribution always exists; otherwise the worst-case distribution may not exist but there exists a sequence of distributions approaching to the optimal value. From the proof of Case 2, 3, a necessary condition for \( \lambda^* > \kappa \) is
\[ \int_\Xi d^p(T_-(\lambda, \zeta), \zeta) \nu(d\zeta) < \theta \epsilon \]
for all \( \lambda > \kappa \). Hence, unless \( \text{supp } \nu \subset \arg\min_\xi \lambda d^p(\xi, \zeta) - \Psi_x(\xi) \) for all \( \lambda > \kappa \), or \( \Psi_x(\xi) \leq \kappa d^p(\xi, \zeta) - \Psi_x(\xi) \) for all \( \zeta \in \text{supp } \nu \) and \( \xi \in \Xi \), there is sufficiently small \( \theta \) such that the worst-case distribution exists. We also note that Example 1 in Esfahani and Kuhn [17] corresponds to the threshold case \( \lambda^* = \kappa = 1 \). Comparing to Corollary 4.7 in Esfahani and Kuhn [17], we here provide a complete description of the condition for the existence of a worst-case distribution.
Remark 5. In the above three cases, we repeatedly use the same proof idea. That is, we derive the first-order optimality condition, based on which we construct a distribution or a sequence of distributions that is proven to be optimal or approaching to optimal. We will use this idea to prove the strong duality for another class of problems in Section 4.

We close this section by consider the degenerate case \( \theta = 0 \). Note that when \( \kappa = \infty \), we may not have strong duality. For example, let \( \nu = \delta_{i_0} \) for some \( \xi_0 \in \Xi \). Then \( W_p(\mu, \nu) = 0 \) implies that \( \mu = \nu \), and thus \( v_p = \Psi_x(\xi_0) \). However, \( \Phi(\lambda, \xi_0) = \inf_{\xi \in \Xi} \lambda d^p(\xi, \xi_0) - \Psi_x(\xi) = -\infty \) for any \( \lambda \geq 0 \), so \( v_D = \infty \). Nevertheless, when \( \kappa < \infty \), we still have strong duality.

**Proposition 3.** Suppose \( \theta = 0 \) and \( \kappa < \infty \). Then \( v_p = v_D \).

**Proof.** Since \( \Psi_x \) is upper semi-continuous and \( \kappa < \infty \), there exists \( \tilde{\kappa} < \infty \) such that \( \Psi_x(\xi) \leq \tilde{\kappa}(d^p(\xi, \xi_0) + 1) \) for all \( \xi \in \Xi \). Then by Lemma 1, \( v_p = \int_{\Xi} \Psi_x(\xi) \nu(d\xi) \). Since \( \Phi(\lambda, \zeta) \leq -\Psi_x(\zeta) \) for all \( \zeta \), we have that
\[
v_D = \inf_{\lambda \geq 0} \left\{ -\int_{\Xi} \Phi(\lambda, \zeta) \nu(d\zeta) \right\} \geq \int_{\Xi} \Psi_x(\zeta) \nu(d\zeta) = v_p.
\]

On the other hand, by Moreau’s theorem (cf. Theorem 1.25 in Rockafellar and Wets [39]),
\[
\lim_{\lambda \to \infty} \Phi(\lambda, \zeta) = -\Psi_x(\zeta) \quad \text{for all} \quad \zeta \in \Xi.
\]
Moreover, since \( d^p(\xi, \xi_0) \leq (d(\xi, \zeta) + d(\zeta, \xi_0))^p \leq 2^{p-1}(d^p(\xi, \zeta) + d^p(\zeta, \xi_0)) \), we have
\[
\Phi(\lambda, \zeta) \geq \inf_{\xi \in \Xi} \lambda d^p(\xi, \zeta) - \tilde{\kappa}(d^p(\xi, \xi_0) + 1)
\geq \inf_{\xi \in \Xi} (\lambda - 2^{p-1}\tilde{\kappa}) d^p(\xi, \zeta) - 2^{p-1}\tilde{\kappa}(d^p(\xi, \xi_0) + 1).
\]
Hence when \( \lambda > 2^{p-1}\tilde{\kappa} \), we have \( \Phi(\lambda, \zeta) \geq -2^{p-1}\tilde{\kappa}(d^p(\zeta, \xi_0) + 1) \). Together with \( \Phi(\lambda, \zeta) \leq -\Psi_x(\zeta) \), we have shown that \( \Phi(\lambda, \zeta) \) is dominated by some function in \( L^1(\nu) \). Therefore, applying dominated convergence we have \( v_D \leq -\lim_{\lambda \to \infty} \int_{\Xi} \Phi(\lambda, \zeta) \nu(d\zeta) = \int_{\Xi} \Psi_x(\zeta) \nu(d\zeta) = v_p \).

### 3.2. Finite-supported Reference Distribution

In this subsection, we restrict attention to the case when the reference distribution \( \nu = \frac{1}{N} \sum_{i=1}^N \delta_{\hat{\xi}^i} \) for some \( \hat{\xi}^i \in \Xi, \ i = 1, \ldots, N \). This occurs when the decision maker collects \( N \) observations that constitute an empirical distribution.

**Corollary 1 (Data-driven DRSP).** Let \( \nu = \frac{1}{N} \sum_{i=1}^N \delta_{\hat{\xi}^i} \). Suppose \( \theta > 0 \). Assume \( \Psi_x \) is upper semi-continuous. Then the following holds:

(i) The primal problem (11) has a strong dual program
\[
\min_{\lambda \geq 0} \left\{ \lambda \theta^p + \frac{1}{N} \sum_{i=1}^N \max_{\xi \in \Xi} \left[ \Psi_x(\xi) - \lambda d^p(\xi, \hat{\xi}^i) \right] \right\},
\]
which always admits a minimizer \( \lambda^* \).

(ii) When \( \lambda^* > \kappa \), there exists a worst-case distribution which is supported on at most \( N + 1 \) points and has the form
\[
\mu^* = \frac{1}{N} \sum_{i \neq i_0} \delta_{\hat{\xi}^i} + \frac{p^*}{N} \delta_{\xi_0^+} + \frac{1-p^*}{N} \delta_{\xi_0^-},
\]
where \( 1 \leq i_0 \leq N, \ p^* \in [0, 1], \ \xi_0^+ := T_+(\lambda^*, \hat{\xi}^i), \) and \( \xi_i^+ \in \{T_+(\lambda^*, \hat{\xi}^i), T_-(\lambda^*, \hat{\xi}^i)\} \) for all \( i \neq i_0 \).

(iii) Suppose there exists \( L, M \geq 0 \) such that \( |\Psi_x(\xi) - \Psi_x(\zeta)| < M + Ld(\xi, \zeta) \) for all \( \xi, \zeta \in \Xi \). Let \( K \) be any positive integer and define the robust program
\[
v_K := \sup_{(\xi^k)_{i,k} \in \mathbb{M}_K} \frac{1}{NK} \sum_{i=1}^N \sum_{k=1}^K \Psi_x(\xi^k),
\]
where

\[ \mathcal{M}_K := \left\{ (\xi_{ik})_{i,k} : \frac{1}{NK} \sum_{i=1}^{N} \sum_{k=1}^{K} d^p(\xi_{ik}, \tilde{\xi}) \leq \theta^p, \; \xi_{ik} \in \Xi, \forall i,k \right\} \quad (28) \]

Then \( v_K \uparrow \sup_{\mu \in \mathfrak{M}} \mathbb{E}_\mu[\Psi_x(\xi)] \) as \( K \to \infty \). In particular, if \( \lambda^* = \kappa = 0 \), \( v_1 = \sup_{\mu \in \mathfrak{M}} \mathbb{E}_\mu[\Psi_x(\xi)] \), else if \( \lambda^* > \kappa \), it holds that

\[ v_K \leq \sup_{\mu \in \mathfrak{M}} \mathbb{E}_\mu[\Psi_x(\xi)] \leq v_K + \frac{M + LD^*_\lambda}{NK}, \]

where \( D^*_\lambda \) is the constant defined in Lemma 2(vi).

(iv) Assume \( \kappa < \infty \). When \( \Xi \) is convex and \( \Psi_x \) is concave, (26) is further reduced to

\[ \sup_{\xi \in \Xi} \left\{ \frac{1}{N} \sum_{i=1}^{N} \Psi_x(\xi_i) : \frac{1}{N} \sum_{i=1}^{N} d^p(\xi_i, \tilde{\xi}_i) \leq \theta \right\} \quad (29) \]

Remark 6. (i)(iv) are similar to the results in Esfahani and Kuhn [17] and Zhao and Guan [53]. In (ii) we provide a more detailed form of the worst-case distribution which further implies (iii), suggesting that any distributionally robust stochastic programming (11) with ambiguous set \( \mathfrak{M} \) can be approximated by a robust program with uncertainty set \( \mathfrak{M}_K \), which is a subset of \( \mathfrak{M} \) which contains distributions supported on \( NK \) points with equal probability. Such robust-program approximation expands the tractability of DRSP problems. We will show some examples in Section 4.

Remark 7 (Hedging against noise in the data value). Given \( N \) historical data, as pointed out in Remark 3, the structure (27) of the worst-case distribution means that, each observation is perturbed to \( \xi_i \pm \sigma^* \) according to Bernoulli distribution \( \text{Bernoulli}(p_i \pm \sigma^*) \). From this point of view, using Wasserstein ambiguity set has the ability to hedge against the noise of the data values.

Remark 8 (Choosing metric \( d \) and Total Variation metric). We have discussed that \( \lambda \) and \( p \) controls the extent of perturbation by means of \( \Phi(\lambda, \zeta) \), and here we pointed out the metric \( d \) also plays a similar role. For example, when \( d \) is the metric induced by \( l_q \)-norm, large \( q \) implies larger allowable perturbation. As another example, By choosing the discrete metric \( d(\xi, \zeta) = 1_{\{\xi \neq \zeta\}} \) on \( \Xi \), the Wasserstein distance is equal to Total Variation distance (Gibbs and Su [21]), which can be used for the situation where the distance of perturbation does not matter and provides a rather conservative decision. In this case, suppose \( \theta \) is chosen such that \( N \theta \) is an integer, then there is no fractional point in (27) and the problem is reduced to (29). Note that in both Esfahani and Kuhn [17], Zhao and Guan [53], only the Euclidean space with norm-induced metric is considered. Whereas our results hold generally for any complete and separable metric space. This provided more flexibility to apply the reformulation and worst-case analysis to a broader class of problems, such as \( \nu \) being the distribution of continuous random vectors or even stochastic processes.

Proof of Corollary 1. (i) (iv) follows directly from Theorem 1 and Proposition 1. To prove (ii), by Theorem 1(ii), when \( \lambda^* > \kappa \), there exists a worst-case distribution which is supported on at most \( 2N \) points and has the form

\[ \mu^* = \frac{1}{N} \sum_{i=1}^{N} p^{i+}\delta_{\xi_i^+} + p^{i-}\delta_{\xi_i^-}, \quad (30) \]
where $p^{i \pm} \geq 0$, $p^i + p^i = 1$ and $\xi_{e}^{i \pm} = T_{\pm}(\lambda^*, \hat{\xi}^i)$. Therefore, we obtain the following (possibly non-convex) reformulation of (11):

$$
\max_{\xi^{i \pm} \in \mathbb{P}^{i \pm} \geq 0} \left\{ \frac{1}{N} \sum_{i=1}^{N} \Psi_x(\xi^i) + \frac{1}{N} \sum_{i=1}^{N} \left[ p^i(\Psi_x(\xi^{i+}) - \Psi_x(\hat{\xi}^i)) + p^i(\Psi_x(\xi^{i-}) - \Psi_x(\hat{\xi}^i)) \right] : \right. \\
\left. \frac{1}{N} \sum_{i=1}^{N} p^i d^p(\xi^{i+}, \hat{\xi}^i) + p^i d^p(\xi^{i-}, \hat{\xi}^i) \leq \theta^p, p^i + p^i = 1, \forall i \right\},
$$

Given $\xi^{i \pm}$ for all $i$ and by the assumption on $\Xi$, the problem

$$
\max_{0 \leq p^i \leq 1} \left\{ \frac{1}{N} \sum_{i=1}^{N} p^i(\Psi_x(\xi^{i+}) - \Psi_x(\hat{\xi}^i)) + (1 - p^i)(\Psi_x(\xi^{i-}) - \Psi_x(\hat{\xi}^i)) : \right. \\
\left. \frac{1}{N} \sum_{i=1}^{N} p^i d^p(\xi^{i+}, \hat{\xi}^i) + (1 - p^i)d^p(\xi^{i-}, \hat{\xi}^i) \leq \theta^p, p^i \geq 0 \right\}
$$

is a linear program and has an optimal solution which has at most one fractional point. Thus there exists a worst-case distribution which is supported on at most $N + 1$ points, and has the form

$$
\mu^* = \frac{p^*}{N} \delta_{\xi_{i_0}^+} + \frac{1 - p^*}{N} \delta_{\xi_{i_0}^-} + \frac{1}{N} \sum_{i \neq i_0} \delta_{\xi_i},
$$

where $i_0 \in \{1, \ldots, N\}$, $\xi_{i_0}^{+} = T_{\pm}(\lambda^*, \hat{\xi}^{i_0})$, and $\xi_i \in \{T_{\pm}(\lambda^*, \hat{\xi}^i), T_{\pm}(\lambda^*, \hat{\xi}^i)\}$ for all $i \neq i_0$.

To prove (iii), let us first consider $\lambda^* \geq \beta$. By assumption on $\Psi_x$ we have $\kappa \leq L < \infty$. Observe that

$$
\xi^{ik} = \begin{cases} \\
\xi_{ik}, & \forall 1 \leq k \leq K, \forall i \neq i_0, \\
\xi_{ik}^{+}, & \forall 1 \leq k \leq \lfloor Kp^* \rfloor, i = i_0, \\
\xi_{ik}^{-}, & \forall \lfloor Kp^* \rfloor < k \leq N, i = i_0,
\end{cases}
$$

belongs to $\mathfrak{M}_K$. Since $|p^* - \lfloor Kp^* \rfloor / K| < 1 / K$, and $d^p(\xi_{i_0}^{\pm}, \hat{\xi}_{i_0}) \leq D_{\lambda^*}$ by Lemma 2(vi), it follows that

$$
|v_K - E_{\Psi^*}[\Psi_x(\xi)]| \leq \frac{1}{NK}|p^* - \lfloor Kp^* \rfloor / K| \cdot (\Psi_x(\xi_{i_0}^{+}) - \Psi_x(\xi_{i_0}^{-}))
\leq \frac{1}{NK}(\Psi_x(\xi_{i_0}^{+}) - \Psi_x(\hat{\xi}_{i_0}^{+}))
\leq \frac{M + LD_{\lambda^*}}{NK}
\leq \frac{M + LD_{\lambda^*}}{NK}.
$$

When $\lambda^* = \kappa = 0$, from Case 2 in the proof of Theorem 1, there exists a sequence of distributions that supported on $N$ points with equal probability, so $K = 1$ gives the exact reformulation. When $\lambda^* = \kappa > 0$, from Case 3 in the proof of Theorem 1, there exists a sequence of distributions $\{\mu_\lambda\}_{\lambda}$ that approaches to optimality as $\lambda \to \kappa^+$ and has the form

$$
\mu_\lambda = \frac{1}{N} \sum_{i=1}^{N} (1 - p_i^*) \delta_{\xi_i^+} + p_i^* \delta_{\xi_i^-},
$$

where $p_i^* \in (0, 1)$ and $\xi_i^+ = T_{\pm}(\lambda^*, \hat{\xi}^i)$, $\xi_i^- = T_{\pm}(\lambda^*, \hat{\xi}^i)$ (see (25) for the definition). For any $p_\lambda'$, let $K_\lambda'$ be such that $1 / K_\lambda' < p_\lambda' < 1 / (K_\lambda' - 1)$, and define

$$
\tilde{\mu}_\lambda = \frac{1}{N} \sum_{i=1}^{N} (1 - \frac{1}{K_\lambda'}) \delta_{\xi_i^+} + \frac{1}{K_\lambda'} \delta_{\xi_i^-}.
$$
Then using the same argument as in Case 3, we can show that \( \widetilde{\mu}_\lambda \) is a sequence of primal feasible solutions that approach to optimality. \( \square \)

**Example 4 (Piecewise Concave Objective).** Suppose \( p = 1 \) and \( \Psi_x(\xi) = \max_{1 \leq j \leq J} \Psi^j_x(\xi) \) as considered in Esfahani and Kuhn [17]. From the structure of worst-case distribution (30)(31), for each \( 1 \leq i \leq N \), there exists \( j_i^+, j_i^- \in \{1, \ldots, J\} \) such that \( \Psi^i_x(\xi_i^j) \) is perturbed to the pieces at which the \( j_i \)-th function \( \Psi^j_x \) achieves the maximum. So without decreasing the optimal value we can restrict the set \( \mathcal{M} \) to a smaller set
\[
\mu = \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{J} p_{ij} \delta_{\xi_{ij}} : \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{J} p_{ij} d(\xi_{ij}, \xi_{ij}^\hat{}) \leq \theta, \sum_{j=1}^{J} p_{ij} = 1, \forall i, p_{ij} \geq 0, \xi_{ij} \in \Xi, \forall i, j
\]
Replacing \( \xi_{ij} \) by \( \hat{\xi}_i^j + (\xi_{ij} - \hat{\xi}_i^j)/p_{ij} \), by positive homogeneity of norms and convexity-preserving property of perspective functions (cf. Section 2.3.3 in Boyd and Vandenberghe [10]), we obtain an equivalent convex program reformulation of (11):
\[
\sup_{p_{ij} \geq 0, \xi_{ij}} \left\{ \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{J} p_{ij} \Psi^j_x(\xi_i^j + (\xi_{ij} - \hat{\xi}_i^j)/p_{ij}) : \frac{1}{N} \sum_{i=1}^{N} \sum_{j=1}^{J} d(\xi_{ij}, \hat{\xi}_i^j) \leq \theta, \sum_{j=1}^{J} p_{ij} = 1, \forall i, \hat{\xi}_i^j \right\}
\]
Reformulation (33) follows from Theorem 1(i), and (34) can be obtained using the equivalent definition of Wasserstein distance in Example 3. \( \square \)
4. Applications In this section, we apply our duality results and the structural description of the worst-case distribution to a variety of stochastic optimization problems. In Section 4.1, we perform a numerical study on the newsvendor problem and show that using Wasserstein ambiguity set results in a reasonable worst-case distribution. Due to the close connection between DRSP and robust programming (Corollary 2(iii)(iv)), in Section 4.2 and 4.3, we illustrate the tractability of DRSPs via two-stage linear programming and DRSPs with objective concave in the uncertainty. Then in Section 4.4 and 4.5, we illustrate the great generality of strong duality results by studying worst-case Value-at-Risk problem and point processes control. Finally, we demonstrate the power of our constructive proof method by applying it to a class of distributionally robust transportation problems in Section 4.6.

4.1. Newsvendor Problems In this subsection, we perform a numerical study on distributionally robust newsvendor problems, with an emphasis on the worst-case distribution.

The newsvendor model is one of the most fundamental problems in operations management. The problem can then be formulated as

$$\min_{x \geq 0} \mathbb{E}_\mu [h(x - \xi)^+ + b(\xi - x)^+]$$

where $x$ is the decision variable for initial inventory level, $\xi$ is the random demand distribution, and $h, b$ represent respectively the overage and underage costs per unit. We assume that $b, h \geq 0$, and $\xi$ is supported on $\{0, 1, \ldots, B\}$ for some positive integer $B$. If the underlying distribution $\mu$ is known exactly, it is well-known that the set of optimal solutions is the whole interval of $[0, B]$. Now suppose we are given $N_i$ observations of historical demand $i$, $i = 1, \ldots, B$, and let $N = \sum_{i=0}^{B} N_i$ and $q_i := N_i / N$, $i = 1, \ldots, B$. The classical stochastic programming theory would suggest using Sample Average Approximation (SAA) method to solve the SAA counterpart

$$\min_{x \geq 0} \sum_{i=0}^{N} q_i [h(x - i)^+ + b(i - x)^+]$$

Then the empirical $\frac{b}{b+h}$-quantile gives an optimal solution $\hat{x} := \inf_{i \geq 0} \left\{ i : \sum_{j=0}^{i} q_j \geq \frac{b}{b+h} \right\}$. Since the problem is merely one-dimensional, SAA solution $\hat{x}$ should gives an accurate approximation provided that the sample size $N$ is not too small (Levi et al. [30]). On the other hand, in some practical settings, it is rather expensive to gather demand data. For example, a company is planning to introduce a new product of which the demand data is collected by either setting up pilot stores or simulating with complex systems. In such cases, with fewer data in hand, the decision maker may want to consider the distributionally robust approach by investigating the DRSP counterpart

$$\min_{x \geq 0} \mathbb{E}_{\mu} [h(x - \xi)^+ + b(\xi - x)^+]$$

Using Corollary 2, we obtain a convex programming reformulation of (35)

$$\min_{x, \lambda \geq 0} \left\{ \lambda \theta^p + \sum_{i=0}^{B} q_i y_i : y_i \geq \max \left[ h(x - i), b(i - x) \right] - \lambda |i - j|^p, \forall 0 \leq i, j \leq B \right\}.$$  (37)

In addition, given the optimal solution $(x^*, \lambda^*)$, the worst-case distribution can then be computed by solving

$$\max_{p^h_i, p^b_i \geq 0, \gamma_{ij} \geq 0} \left\{ \sum_{i=0}^{B} p^h_i h(x - i) + p^b_i b(i - x) : \sum_{i,j} \gamma_{ij} |i - j|^p \leq \theta^p, \sum_{j} \gamma_{ij} = p^h_i + p^b_i, \forall i, \sum_i \gamma_{ij} = q_j, \forall j \right\}.$$
We perform a numerical test of which setup is similar to Wang et al. [50] and Ben-Tal et al. [5]. We set \( b = h = 1, \bar{B} = 100, \) and \( N \in \{50, 500\} \) representing small and large data set. The data is then generated from Binomial(100, 0.5) and Geometric(0.1) truncated on [0, 100]. For a fair comparison, we estimate the radius of the ambiguity set such that it covers the underlying distribution with probability greater than 95%. For Burg entropy, we use the estimation developed in Wang et al. [50], while for Wasserstein distance, the radius is determined by a classical concentration inequality for Wasserstein distance developed in Bolley et al. [8] (see Appendix B for more details). Figure 2 shows the histograms of the worst-case distributions under the optimal initial inventory level determined by solving the DRSPs with Wasserstein and Burg entropy ambiguity set.

When the underlying distribution is Binomial, intuitively, the symmetry of Binomial distribution and \( b = h = 1 \) implies that the optimal initial inventory level is close to \( \bar{B}/2 = 50 \), and the corresponding worst-case distribution should be similar to a mixture distribution with two modes, representing high and low demand respectively. This intuition is consistent with the solid curves in Figure (2a)(2b) for which the Wasserstein distance is used to construct the ambiguity set. In addition, the tail distributions are smooth and reasonable for both small and large dataset. In contrast, if Burg entropy is used to define the ambiguity set (dashed curves in Figure (2a)(2b), the worst-case distribution is less realistic. First, the worst-case distribution is not symmetric and there is a large spike on the boundary 100, corresponding to the “popping” behavior mentioned in Section 1. Especially when the dataset is small, the spike is huge, which makes the solution too conservative.

When the underlying distribution is Geometric, conceivably, the worst-case distribution should have one spike for low demand and a heavy tail for high demand. By using Wasserstein distance,
we observe reasonable worst-case distributions (solid curves in Figure (2c)(2d)). While using Burg entropy (dashed curves in Figure (2c)(2d)), the worst-case distribution has disconnected support and unrealistic spikes on the boundary. In the case of Geometric distribution, the tail behaviors are even more problematic. Indeed, for distribution with unbounded support, we have much flexibility to specify the truncation threshold $\bar{B}$. Since the spike only appears on the boundary, the tail distribution is very sensitive to our choice of $\bar{B}$. Hence, the conclusion for this numerical test is that Wasserstein ambiguity set is likely to yield a more reasonable, robust and realistic worst-case distribution.

4.2. Two-stage Linear Programming In Corollary 1(iii)(iv) we established the close connection between the DRSP problem and robust programming. More specifically, we show that every DRSP problem can be approximated by robust programs with rather high accuracy, which significantly enlarges the applicability of the DRSP problem. To illustrate this point, in this subsection we show the tractability of the two-stage DRSP problem.

Consider the two-stage distributionally robust stochastic programming

$$\min_{x \in \mathbb{X}} c^\top x + \sup_{\mu \in \mathcal{M}} \mathbb{E}_\mu[\Psi(x, \xi)],$$

where $\Psi(x, \xi)$ is the optimal value of the second-stage problem

$$\min_{y \in \mathbb{R}^m} \{ q(\xi)^\top y : T(\xi)x + W(\xi)y \leq h(\xi) \},$$

and

$$q(\xi) = q^0 + \sum_{l=1}^s \xi_l q^l, \quad T(\xi) = T^0 + \sum_{l=1}^s \xi_l T^l, \quad W(\xi) = W^0 + \sum_{l=1}^s \xi_l W^l, \quad h(\xi) = h^0 + \sum_{l=1}^s \xi_l h^l.$$

We assume $p = 2$ and $\Xi = \mathbb{R}^s$ with Euclidean distance $d$. In general, the two-stage problem (38) is hard to solve. Indeed, let $N = 1$, $\nu = \delta_{\hat{\xi}}$, and $\alpha_l > \beta_l > 0$, $l = 1, \ldots, s$ be given constants. Consider a special case of problem (38) as follows.

$$\min_{x \in \mathbb{X}} c^\top x + \sup_{\mu \in \mathcal{M}} \mathbb{E}_\mu \left[ \min_{y_{l}^\pm \geq 0} \left\{ \sum_{l=1}^s \alpha_l y_l^+ - \beta_l y_l^- : y_l^+ + y_l^- = \xi_l - x_l, \forall l \right\} \right].$$

(40)

Note that

$$\sup_{\mu \in \mathcal{M}} \mathbb{E}_\mu \left[ \min_{y_{l}^\pm \geq 0} \left\{ \sum_{l=1}^s \alpha_l y_l^+ - \beta_l y_l^- : y_l^+ + y_l^- = \xi_l - x_l, \forall l \right\} \right] = \sup_{\mu \in \mathcal{M}} \mathbb{E}_\mu \left[ \sum_{l=1}^s [\alpha_l (\xi_l - x_l)^+ - \beta_l (x_l - \xi_l)^+] \right].$$

Recall the definition (15) of $\Phi$, we have for any $\lambda > 0$, the right-hand side of

$$\Phi(\lambda, \hat{\xi}) = \inf_{\xi \in \Xi} \left\{ \lambda \|\xi - \hat{\xi}\|_2^2 - \sum_{l=1}^s [\alpha_l (\xi_l - x_l)^+ - \beta_l (x_l - \xi_l)^+] \right\}$$

has a unique solution. When $\lambda = 0$, $\Phi(0, \hat{\xi})$ is $-\infty$. Hence by Theorem 1(iii), problem (40) is equivalent to

$$\min_{x \in \mathbb{X}} c^\top x + \sup_{\xi : \|\xi - \hat{\xi}\|_2 \leq \theta} \mathbb{E}_\mu \left[ \min_{y_{l}^\pm \geq 0} \left\{ \sum_{l=1}^s \alpha_l y_l^+ - \beta_l y_l^- : y_l^+ + y_l^- = \xi_l - x_l \right\} \right].$$

(41)
It has been shown in Minoux [32] that problem (41) is strongly NP-hard. As a consequence, the two-stage linear DRSP (38) is strongly NP-hard even when only the right-hand side vector \( h(\xi) \) is uncertain.

However, we are going to show that with tools from robust programming, we are able to obtain a tractable approximation of (38). Let \( \mathcal{M}_1 := \{ (\xi^1, \ldots, \xi^N) \in \Xi^N : \frac{1}{N} \sum_{i=1}^{N} ||\xi^i - \hat{\xi}||_2^2 \leq \theta^2 \} \). Using Theorem 1(ii) with \( K = 1 \), we obtain an adjustable-robust-programming approximation

\[
\begin{align*}
\min_{x \in X} \left\{ c^T x + \sup_{(\xi^i)_i \in \mathcal{M}_1} \frac{1}{N} \sum_{i=1}^{N} \Psi(x, \xi^i) \right\} \\
= \min_{x \in X, t \in \mathbb{R}} \left\{ t : c^T x + \frac{1}{N} \sum_{i=1}^{N} q(\xi^i)^T y(\xi^i), \forall (\xi^i)_i \in \mathcal{M}_1, \right. \\
\left. T(\xi)x + W(\xi)y(\xi) \leq h(\xi), \forall \xi \in \bigcup_{i=1}^{N} \{ \xi' \in \Xi : ||\xi' - \hat{\xi}||_2 \leq \theta \sqrt{N} \} \right\},
\end{align*}
\]

where the second set of inequalities follows from the fact that \( T(\xi)x + W(\xi)y(\xi) \leq h(\xi) \) should hold for all realizations of distribution \( \frac{1}{N} \sum_{i=1}^{N} \delta_{\xi^i} \). Although problem (42) is still intractable in general, there has been a substantial literature on different approximations to problem (42). One popular approach is to consider the so-called affinely adjustable robust counterpart (AARC) as follows. We assume that \( y \) is an affine function of \( \xi \):

\[
y(\xi) = y^0 + \sum_{l=1}^{s} \xi_ly^l, \forall \xi \in \bigcup_{i=1}^{N} B^i,
\]

for some \( y^0, y^l \in \mathbb{R}^m \), where \( B^i := \{ \xi' \in \Xi : ||\xi' - \hat{\xi}||_2 \leq \theta \sqrt{N} \} \). Then the AARC of (42) is

\[
\begin{align*}
\min_{x \in X, t \in \mathbb{R}} & \quad t : c^T x + \frac{1}{N} \sum_{i=1}^{N} \left( q^0 + \sum_{l=1}^{s} \xi^l q^l \right)^T \left( y^0 + \sum_{l=1}^{s} \xi^l y^l \right) - t \leq 0, \forall (\xi^i)_i \in \mathcal{M}_1, \\
& \quad \left( T^0 + \sum_{l=1}^{s} \xi^l T^l \right)x + \left( W^0 + \sum_{l=1}^{s} \xi^l W^l \right) \left( y^0 + \sum_{l=1}^{s} \xi^l y^l \right) - \left( h^0 + \sum_{l=1}^{s} h^l \xi^l \right) \leq 0, \forall \xi \in \bigcup_{i=1}^{N} B^i
\end{align*}
\]

Set \( \zeta_{il} := \xi^l_i - \hat{\xi}^l_i \) for \( i = 1, \ldots, N \) and \( l = 1, \ldots, s \). In view of \( \mathcal{M}_1 \), \( \zeta \) belongs to the ellipsoidal uncertainty set

\[
\mathcal{U}_z = \{ (\zeta_{il})_{i,l} : \sum_{i,l} \zeta_{il}^2 \leq N \theta^2 \}.
\]

Set \( z = [x; t; \{ y^l \}^l_{l=0}] \), and define

\[
\begin{align*}
\alpha_0(z) := & \left[ c^T x + (q^0 + \sum_{l=1}^{s} \hat{\xi}^l q^l)^T (y^0 + \sum_{l=1}^{s} \hat{\xi}^l y^l) - t \right], \\
\beta_0^{il}(z) := & - \left[ (q^0 + \sum_{l=1}^{s} \hat{\xi}^l q^l)^T y^l + q^l^T (y^0 + \sum_{l=1}^{s} \hat{\xi}^l y^l) \right], \forall 1 \leq i \leq N, 1 \leq l, l' \leq s.
\end{align*}
\]

Then the first set of constraints in (43) is equivalent to

\[
\alpha_0(z) + 2 \sum_{i,l} \beta_0^{il}(z) \zeta_{il} + \sum_{i} \sum_{l,l'} \Gamma_0^{(i,l,l')}(z) \zeta_{il} \zeta_{il'} \geq 0, \forall (\zeta_{il})_{i,l} \in \mathcal{U}_z.
\]
It follows from Theorem 4.2 in Ben-Tal et al. [6] that (44) takes place if and only if there exists \( \lambda_0 \geq 0 \) such that

\[
(\alpha_0(z) - \lambda_0)v^2 + 2v \sum_{i,l} \beta^l_i(z)w_{il} + \sum_i \sum_{l,l'} (\Gamma_0^{(i,l')} w_{i,l'}) + \frac{\lambda_0}{N\theta^2} \sum_{i,l} w_{il}^2 \geq 0, \quad \forall v \in \mathbb{R}, \forall w_{il} \in \mathbb{R}, \forall i, l.
\]

Or in matrix form,

\[
\exists \lambda_0 \geq 0 : \begin{pmatrix} \Gamma_0 \oplus I_N + \frac{\lambda_0}{N\theta^2} \cdot I_{sN} & \text{vec}(\beta_0) \\ \text{vec}^T(\beta_0) & \alpha_0(z) - \lambda_0 \end{pmatrix} \succeq 0,
\]

where \( I_N \) (resp. \( I_{sN} \)) is \( N \) (resp. \( sN \)) dimensional identity matrix, \( \oplus \) is the Kronecker product of matrices and vec is the vectorization of a matrix.

Now we reformulate the second set of constraints in (43). For all \( 1 \leq i \leq N, 1 \leq j \leq m \) and \( 1 \leq l, l' \leq s \), we set

\[
\begin{align*}
\alpha_{ij}(z) &:= -[(T_j^0 + \sum_{l=1}^s \hat{\xi}_j^l T_j^l) x + (W_j^0 + \sum_{l=1}^s \hat{\xi}_j^l W_j^l) (y^0 + \sum_{l=1}^s \hat{\xi}_j^l y^l)] - (h_j^0 + \sum_{l=1}^s \hat{\xi}_j^l h_j^l), \\
\beta_{ij}^l(z) &:= \frac{[T_j^l x + (W_j^0 + \sum_{l'=1}^s \hat{\xi}_j^{l'} W_j^{l'}) y^l + W_j^l (y^0 + \sum_{l'=1}^s \hat{\xi}_j^{l'} y^{l'}) - h_l]}{2}, \\
\Gamma_{j}^{(i,l')} (z) &:= -\frac{W_j^l y^{l'} + W_j^{l'} y^l}{2}.
\end{align*}
\]

Let \( \eta^i := \xi - \hat{\xi}^i \) for \( 1 \leq i \leq N \). Then the second set of constraints in (43) is equivalent to

\[
\alpha_{ij}(z) + 2\beta_{ij}^l(z) \eta^i + \eta^i \Gamma_j(z) \eta^j \geq 0, \quad \forall \eta^i \in \{ \eta^i \in \mathbb{R}^s : ||\eta^i||_2 \leq \theta \sqrt{N} \}, \forall 1 \leq i \leq N, 1 \leq j \leq m.
\]

Again by Theorem 4.2 in Ben-Tal et al. [6] we have further equivalence

\[
\exists \lambda_{ij} \geq 0 : \begin{pmatrix} \Gamma_j(z) + \frac{\lambda_{ij}}{N\theta^2} \cdot I_s & \beta_{ij}^l(z) \\
\beta_{ij}^l(z)^T & \alpha_{ij}(z) - \lambda_{ij} \end{pmatrix} \succeq 0, \quad \forall 1 \leq i \leq N, 1 \leq j \leq m.
\]

Combining (45) and (46) we obtain the following result.

**Proposition 4.** An exact reformulation of the AARC of (42) is given by

\[
\min_{x \in X, \lambda_0, \lambda_{ij} \geq 0, i=1,...,s, j=1,...,m} \{ t : (45), (46) holds \}. \tag{47}
\]

Since (42) is a fairly good approximation of the original two-stage DRSP problem (38), the semidefinite program (47) provides a tractable approximation of (38).

### 4.3. Concave objective

In Corollary 1(iii), we pointed out that problem (7) with finite-supported reference distribution and concave \( \Psi_z \) is equivalent to a robust optimization problem (29) with uncertainty set

\[
\left\{ (\xi^1, \ldots, \xi^N) \in \Xi^N : \frac{1}{N} \sum_{i=1}^N d^p(\xi^i, \hat{\xi}^i) \leq \theta^p \right\},
\]

which describes that the average distance (measured in \( d^p \)) of the noisy data \( \hat{\xi}^i \) from the “true” one is controlled by \( \theta^p \). In some situations, such reformulation yields to an efficiently computed program.
As an example, let us consider $\Psi(x, \xi)$ being an affine function of $x$, i.e.,

$$\Psi(x, \xi) = a^\top x + b,$$

(48)

where $\xi = [a; b]$ and let $\xi^i = [a_i; b_i]$ and $\hat{\xi} = [\hat{a}_i; \hat{b}_i]$, $i = 1, \ldots, I$. Assume the metric $d$ is induced by some norm $|| \cdot ||_q$. Set $u = (u_1, \ldots, u_N)$ and

$$\mathcal{U} = \left\{ u : \frac{1}{N} \sum_{i=1}^N ||u_i||^p \leq \theta^p \right\}.$$

Then by (29), problem (5) is equivalent to

$$\min_{x \in X, t \in \mathbb{R}} \left\{ t : \frac{1}{N} \sum_{i=1}^N (a_i[u]^\top x + b_i[u]) \leq t, \quad \left( a_i[u] \atop b_i[u] \right) = \left( \hat{a}_i \atop \hat{b}_i \right) + u_i, \forall u \in \mathcal{U} \right\}.$$

We have the following results.

**Corollary 3 (Affinely-perturbed objective).** With the above setup, problem (5) is equivalent to

$$\min_{x \in X, t \in \mathbb{R}} \left\{ t : \frac{1}{N} \sum_{i=1}^N (\hat{a}_i[u]^\top x + \hat{b}_i[u]) + \theta ||x||_q \leq t \right\}.$$

Now we turn to a general concave objective. With the help of reformulation (29), we can apply the Mirror-Prox algorithm to solve the convex-concave saddle point problem

$$\min_{x \in X} \max_{(\xi^1, \ldots, \xi^N) \in Y} \frac{1}{N} \sum_{i=1}^N \Psi(x, \xi^i),$$

(49)

where

$$Y = \left\{ (\xi^1, \ldots, \xi^N) \in \Xi^N : \frac{1}{N} \sum_{i=1}^N d^p(\xi^i, \hat{\xi}^i) \leq \theta^p \right\}.$$

In the following, we briefly describe and set up the algorithm. For a detailed description, we refer the reader to Nemirovski [33]. For ease of notation, set $y := (\xi^1, \ldots, \xi^N)$. We assume that $\Xi$ is a separable Hilbert space such with the metric $d$ induced from some inner product $\langle \cdot, \cdot \rangle$. Set $\Xi_i$ to be the translated space of $\Xi$ under translation mapping $\xi \mapsto \xi - \hat{\xi}^i$, then a natural norm on $\Xi_i$ is given by

$$||\xi||_{\Xi_i} := d(\xi, \hat{\xi}^i), \quad \forall \xi \in \Xi_i.$$

On the product space $\Xi^N := \prod_{i=1}^N \Xi_i$, we define a norm $|| \cdot ||_Y$ by

$$||y||_Y = ||(\xi^1, \ldots, \xi^N)||_Y := \left( \sum_{i=1}^N ||\xi^i||_{\Xi_i}^p \right)^{1/p}, \quad \forall \xi^i \in \Xi_i, \quad i = 1, \ldots, N.$$

We introduce the distance generating function

$$\omega_Y(y) = \omega_Y(\xi^1, \ldots, \xi^N) := \frac{1}{m \gamma} \sum_{i=1}^N ||\xi^i||_{\Xi_i}^m,$$

where $m, \gamma$ are chosen later such that $\omega_Y$ is strongly convex with modulus 1 with respect to $|| \cdot ||_Y$. We also assume that there exists a norm $|| \cdot ||_X$ on $X$ and a distance generating function $\omega_X(\cdot)$ which
is continuous and strongly convex with modulus 1 with respect to \( \| \cdot \|_X \), and admits a continuous selection \( \omega'(x) \) of subgradients. Let \( \Theta_{X} := \sup_{x \in X} \omega_X(x) - \inf_{x \in X} \omega_X(x) \).

On the product space \( Z := X \times Y \), we define a norm

\[
\| z \|_Z := \sqrt{\frac{1}{\Theta_X^2} \| x \|_X^2 + \frac{1}{\Theta_Y^2} \| y \|_Y^2}
\]

for any \( z \in Z \). It can be easily checked that

\[
\| z \|_{Z,*} = \sqrt{\Theta_X^2 \| x \|_{X,*}^2 + \Theta_Y^2 \| y \|_{Y,*}^2}
\]

defines the dual norm. Suppose there exists \( L_{11}, L_{12}, L_{21}, L_{22}, M_{11}, M_{12}, M_{21}, M_{22} \geq 0 \) such that for any \( x, x' \in X, \xi, \xi' \in \Xi \),

\[
\begin{aligned}
\| \partial_x \Psi(x, \xi) - \partial_x \Psi(x', \xi) \|_{X,*} &\leq L_{11} \| x - x' \|_X + M_{11}, \\
\| \partial_x \Psi(x, \xi) - \partial_x \Psi(x, \xi') \|_{X,*} &\leq L_{12} \| \xi - \xi' \|_\Xi + M_{12}, \\
\| \partial_{\xi} \Psi(x, \xi) - \partial_{\xi} \Psi(x', \xi) \|_{Y,*} &\leq L_{21} \| x - x' \|_X + M_{21}, \\
\| \partial_{\xi} \Psi(x, \xi) - \partial_{\xi} \Psi(x', \xi') \|_{Y,*} &\leq L_{22} \| \xi - \xi' \|_Y + M_{22}.
\end{aligned}
\]  

(50)

Set

\[
L := \sqrt{2\Theta_X^2 L_{11}^2 + 2\Theta_X^2 \Theta_Y^2 (L_{12}^2 + L_{21}^2) + 2\Theta_Y^2 L_{22}^2},
\]

\[
M := \sqrt{2\Theta_X^2 M_{11}^2 + 2\Theta_Y^2 M_{21}^2 + 2\Theta_X^2 M_{22}^2 + 2\Theta_Y^2 M_{22}^2}.
\]  

(51)

Define the vector field

\[
F(z) := \frac{1}{N} \left[ \sum_{i=1}^{N} \nabla_x \Psi(x, \xi^i); \{ -\nabla_{\xi} \Psi(x, \xi^i) \}_{i=1}^{N} \right] \quad z \in X \times Y.
\]

(52)

It follows that from Lemma 6 in the Appendix that

\[
\| F(z) - F(z') \|_{Z,*} \leq L \| z - z' \|_Z + M.
\]

Set

\[
\omega(z) := \frac{1}{\Theta_X} \omega_X(x) + \frac{1}{\Theta_Y} \omega_Y(y),
\]

then \( \omega \) is a distance generating function compatible to \( \| \cdot \|_Z \), and \( \Theta := \sup_{z \in Z} \omega(z) - \inf_{z \in Z} \omega(z) = 1 \).

Suppose there is a first-order oracle which computes \( F(z) \) on each call. The accuracy of a candidate solution \( (x, y) \in X \times Y \) is characterized by

\[
\epsilon_{sad}(x, y) := \max_{(\xi^1, \ldots, \xi^N) \in Y} \frac{1}{N} \sum_{i=1}^{N} \Psi(x, \xi^i) - \min_{x' \in X} \frac{1}{N} \sum_{i=1}^{N} \Psi(x', \xi^i).
\]

Given \( N \) and \( \theta \), the lower bound on oracle complexity of (49) using first-order methods is \( O(M/\sqrt{t}) \) when \( L = 0 \) and \( O(L/t) \) when \( M = 0 \), and the Mirror-Prox algorithm can achieve the lower bound up to a constant. The Mirror-Prox algorithm is shown in Algorithm 1.

**Proposition 5.** Assume (50) holds and let \( L, M \) be defined in (51). Set

\[
m = \begin{cases}
1 + \frac{1}{\ln N}, & p = 1 \\
\frac{1}{p}, & 1 < p \leq 2 \\
2, & p > 2
\end{cases}, \quad \gamma = \begin{cases}
\frac{N}{\ln N}, & p = 1 \\
\frac{\theta^{pN^{-1}}}{(p - 1)\theta^{2\gamma N^2/\gamma^2}}, & 1 < p \leq 2 \\
\frac{N^{2/p - 1}}{p > 2}, & p > 2
\end{cases}.
\]  

(53)
Algorithm 1: Mirror-Prox Algorithm

1: \( z_1 := (x_1, y_1) \in X \times Y \).
2: \( w_t = \text{Prox}(\gamma_t F(z_t)) \), \( z_{t+1} = \text{Prox}(\gamma_t F(w_t)) \), where \( \text{Prox}(\gamma_t F(z_t)) := \arg \min_{z \in Z} [V(z, z_t) + \langle \gamma_t F(z_t), z \rangle ] \) and \( V(z, z') := \omega(z) - \omega(z') - \langle \nabla \omega(z'), z \rangle \).
3: \( z^t = \frac{\sum_{\tau=1}^t \gamma_{\tau}w_{\tau}}{\sum_{\tau=1}^t \gamma_{\tau}} \).

Let \( \gamma_{\tau}, 1 \leq \tau \leq t \) satisfies
\[
\gamma_{\tau}(F(z_{\tau}) - F(w_{\tau}), w_{\tau} - z_{\tau+1}) - V(z_{\tau}, w_{\tau}) - V(w_{\tau}, z_{\tau+1}) \leq 0,
\]
which is definitely satisfied when \( \gamma_{\tau} \in (0, \frac{1}{\sqrt{2N}}) \), or \( \gamma_{\tau} \in (0, 1/L) \) when \( M = 0 \). Then it holds that
\[
\epsilon_{\text{sad}}(z^t) \leq \frac{1 + M^2 \sum_{\tau=1}^t \gamma_{\tau}^2}{\sum_{\tau=1}^t \gamma_{\tau}}, \quad (54)
\]
and
\[
\Theta_Y \leq \begin{cases} 
\epsilon \theta^2 N^2 \ln N / (1+1/p) \ln N, & p = 1 \\
\theta^p N^2, & 1 < p \leq 2, \\
\theta^2 N^2, & p > 2.
\end{cases} \quad (55)
\]

Note that if \( \gamma_{\tau} = O(1/L) \), by (51)(54) we have \( \epsilon_{\text{sad}}(z^t) = O(\Theta_Y^2) \). Thus according to (55), \( \epsilon_{\text{sad}}(z^t) = O(\theta^4 N^4) \) when \( p > 1 \) and \( \epsilon_{\text{sad}}(z^t) = O(\theta^4 N^4 \ln^2 N) \) when \( p = 1 \). We argue that the complexity of the algorithm may not be very large as it appears. For one thing, we only use DRSP formulation when \( N \) is not very large, since otherwise the Sample Average Approximation method provides a relatively accurate solution. For another thing, the radius \( \theta \) goes to zero as \( N \) goes to infinity. Therefore, we believe that the Mirror-Prox algorithm (Algorithm 1) is useful in many practical applications.

4.4. Worst-case Value-at-Risk

In the strong duality results (Theorem 1), the reference distribution can be any measure on a Polish space. Such generality greatly enhances the applicability of our results. As an example, in this subsection we study the worst-case Value-at-Risk with Wasserstein ambiguity set.

Value-at-risk is a popular risk measure in financial industry. Given a random variable \( Z \) and \( \alpha \in (0, 1) \), the value-at-risk \( \text{VaR}_\alpha[Z] \) of \( Z \) with respect to measure \( \nu \) is defined by
\[
\text{VaR}_\alpha[Z] := \inf \{ t : \mathbb{P}_{\nu} \{ Z \leq t \} \geq 1 - \alpha \}. \quad (56)
\]

Note that the \( \text{VaR}_\alpha[Z] \leq 0 \) is equivalent to the chance constraint \( \mathbb{P}\{Z \leq 0\} \geq 1 - \alpha \). In the spirit of Ghaoui et al. [20], we consider the following worst-case VaR problem. Suppose we are given a portfolio consisting of \( n \) assets with allocation weight \( w \) satisfying \( \sum_{i=1}^n w_i = 1 \) and \( w \geq 0 \). Let \( \xi_i \) be the (random) return rate of asset \( i, i = 1, \ldots, n \), and \( r = \mathbb{E}[\xi] \) the vector of the expected return rate. Assume the metric \( d \) associated with the set \( \mathcal{M} \) in (6) is induced by the infinity norm \( \| \cdot \|_\infty \) on \( \mathbb{R}^n \). The worst-case VaR with respect to the set of probability distributions \( \mathcal{M} \) is defined as
\[
\text{VaR}_\alpha^{\text{wc}}(w) := \min_q \left\{ q : \inf_{\mu \in \mathcal{M}} \mathbb{P}\{ -w^\top \xi \leq q \} \geq 1 - \alpha \right\}. \quad (57)
\]

For simplicity we assume \( p = 1 \).

The following lemma shows the continuity of the probability bound with respect to the boundary \( \partial C \) of a Borel set \( C \).
LEMMA 3. Let $C$ be a Borel set and $\theta > 0$. Then

$$\inf_{\mu \in \mathcal{M}} \mu(C) = \min_{\mu \in \mathcal{M}} \mu(\text{int}(C)).$$

Hence by Lemma 3, the distributionally robust chance constraint is equivalent to its strict counterpart $\min_{\mu \in \mathcal{M}} \mathbb{P}_\mu\{ -w^\top \xi < b \} \geq 1 - \alpha$.

PROPOSITION 6. Let $\theta > 0$, $\alpha \in (0, 1)$, $w \in \{ w' \in \mathbb{R}^n : \sum_{i=1}^n w'_i = 1, w'_i \geq 0 \}$. Define

$$\beta_0 := \min \left( 1, \frac{(\alpha - \nu \{ \xi : -w^\top \xi > \text{VaR}_\alpha[-w^\top \xi]\})(q - \text{VaR}_\alpha[-w^\top \xi])^p}{\theta^p - \mathbb{E}_\nu[(q + w^\top \xi)^p \mathbb{1}_{\{ -w^\top \xi > \text{VaR}_\alpha[-w^\top \xi]\}}]} \right).$$

Then

$$\inf_{\mu \in \mathcal{M}} \mathbb{P}_\mu\{ -w^\top \xi \leq q \} \geq 1 - \alpha \iff \mathbb{E}_\nu\left[ ((q + w^\top \xi)^+) + \mathbb{1}_{\{ -w^\top \xi > \text{VaR}_\alpha[-w^\top \xi]\}} \right] + \beta_0 \mathbb{E}_\nu\left[ ((q + w^\top \xi)^+) \cdot \mathbb{1}_{\{ -w^\top \xi = \text{VaR}_\alpha[-w^\top \xi]\}} \right] \geq \theta.$$ 

In particular, suppose $p = 1$, when $\nu$ is a continuous distribution or $\nu = \frac{1}{N} \sum_{i=1}^N \delta_{\xi_i}$ and $\alpha N$ is an integer,

$$\inf_{\mu \in \mathcal{M}} \mathbb{P}_\mu\{ -w^\top \xi \leq q \} \geq 1 - \alpha \iff -\alpha \text{CVaR}_\alpha[-(q + w^\top \xi)^+] \leq -\theta^p,$$

where $\text{CVaR}_\alpha[Z]$ is the conditional value-at-risk defined by $\text{CVaR}_\alpha[Z] := \inf_{t \in \mathbb{R}}\{ t - \alpha^{-1} \mathbb{E}_\nu[Z - t]^+ \}$.

**Example 5 (Worst-case VaR with Gaussian reference distribution).** Suppose $\nu \sim N(\mu, \Sigma)$ and consider $p = 1$. It follows that $-w^\top \xi \sim N(-w^\top \mu, w^\top \Sigma w)$ and $\text{VaR}_\alpha[-w^\top \xi] = -w^\top \mu + \sqrt{w^\top \Sigma w} \Phi^{-1}(1 - \alpha)$. By Proposition 6, $\text{VaR}_\alpha^{wc}[-w^\top \xi]$ is the minimal $q$ such that (see Figure 3)

$$\frac{1}{\sqrt{2\pi w^\top \Sigma w}} \int_{\text{VaR}_\alpha[-w^\top \xi]}^q (q - y)e^{-\frac{(y + w^\top \mu)^2}{2w^\top \Sigma w}} \, dy \geq \theta.$$ 

Let $q'$ be the solution of the equation

$$q'[\Phi(q') - (1 - \alpha)] - \frac{1}{\sqrt{2\pi}} \left[ e^{-\frac{q'^2}{2}} - e^{-\frac{q'^2}{2}} \right] = \theta,$$

where $\Phi$ is the cumulative distribution function of standard normal distribution. Then $\text{VaR}_\alpha^{wc}[-w^\top \xi] = q'\sqrt{w^\top \Sigma w} - w^\top \mu$. Note that the one-dimensional nonlinear equation (58) can be solved efficiently via any root-finding algorithm.
4.5. On/Off Process Control In this subsection, we consider a distributionally robust process control problem where the reference distribution $\nu$ is a point process. In such a problem, the decision maker controls a two-state (on/off) system given an exogenous demand process. When the system is switched on, it meets any arriving demand, each of which contributes 1 dollar profit, but the cost of maintaining on-state is $c$ dollar per unit time. When the system is off, there is no cost, but it cannot meet any demand so there is no profit either. The decision maker chooses a control policy so that the overall profit during a finite time horizon is maximized. Such problem is a prototype of many problems in sensor network and revenue management. In many practical settings, the demand process is not known exactly. Instead, the decision maker has i.i.d. historical sample paths, which constitute an empirical point process. Using the empirical point process, the classical Sample Average Approximation method yields a degenerate control policy, in which the system is switched off and no demand is met. Due to such degeneracy and instability of SAA method, we resort to the distributionally robust approach.

Before providing a formal mathematical definition of the problem, we remark that we can extend Definition 1 of Wasserstein distance to any Borel measures $\mu, \nu \in \mathcal{B}(\Xi)$ provided that $\nu(\Xi) < \infty$. In fact, when $\nu(\Xi) = 0$, by Lemma 8, if $\mu(\Xi) > 0$, then $W_p(\mu, \nu) = \infty$, with corresponding (8) infeasible. Thus the ambiguity set $\mathfrak{M}$ only contains zero measure $\nu$ itself. When $\nu(\Xi) > 0$, without loss of generality, we can always replace $\nu$ by $\nu/\nu(\Xi)$ such that it is a Borel probability measure and the strong duality developed in Section 3 remains valid. Therefore, in the remaining of this section, when we write $W_p$ distance between two Borel measures, we use the above extended definition.

We set up the problem as follows. We scale the finite time horizon to $[0, 1]$. Let $\Xi = \{\sum_{t=1}^{m} \delta_{\xi_t} : m \in \mathbb{Z}_+, \xi_t \in [0, 1], t = 1, \ldots, m\}$ be the space of finite counting measures on $[0, 1]$. We assume the metric $d$ on $\Xi$ satisfies the following conditions:

1) For any $\tilde{\eta} = \sum_{t=1}^{m} \delta_{\tilde{\xi}_t}$ and $\eta = \sum_{t=1}^{m} \delta_{\xi_t}$, where $m$ is a positive integer and $\{\xi_t\}_{t=1}^{m}, \{\tilde{\xi}_t\}_{t=1}^{m} \subset [0, 1]$, it holds that
   \[ d(\eta, \tilde{\eta}) = W_1(\eta, \tilde{\eta}) = \sum_{t=1}^{m} |\xi_t - \tilde{\xi}_t|, \]  
   (59)

   where $\xi_t$ (resp. $\tilde{\xi}_t$) are the order statistics of $\xi_t$ (resp. $\tilde{\xi}_t$).

2) For any Borel set $C \subset [0, 1]$ and $\tilde{\theta} \geq 0$, and $\tilde{\eta} = \sum_{t=1}^{m} \delta_{\tilde{\xi}_t}$, where $m$ positive integer and $\{\tilde{\xi}_t\}_{t=1}^{m} \subset [0, 1]$, it holds that
   \[ \inf_{\eta \in \Xi} \left\{ \eta(C) : d(\eta, \tilde{\eta}) = \tilde{\theta} \right\} \geq \inf_{\tilde{\eta} \in \mathcal{B}([0,1])} \left\{ \tilde{\eta}(C) : W_1(\tilde{\eta}, \tilde{\eta}) \leq \tilde{\theta} \right\}, \]  
   (60)

3) The metric space $(\Xi, d)$ is Polish, i.e., it is a complete and separable metric space.

We note that condition (59) is only imposed on $\eta, \tilde{\eta} \in \Xi$ such that $\eta([0,1]) = \tilde{\eta}([0,1])$. Possible choices for $d$ can be

\[ d\left(\sum_{t=1}^{l} \delta_{\xi_t}, \sum_{t=1}^{m} \delta_{\xi_t}\right) = \sum_{t=1}^{\min(m,l)} |\xi_t - \tilde{\xi}_t| + |m - l|, \]

or

\[ d\left(\sum_{t=1}^{l} \delta_{\xi_t}, \sum_{t=1}^{m} \delta_{\xi_t}\right) := \begin{cases} \max(m, l), & l \neq m, \\ \sum_{t=1}^{\min(m,l)} |\xi_t - \tilde{\xi}_t|, & l = m > 0, \\ 0, & l = m = 0. \end{cases} \]

Such metric is similar to the one developed in Barbour and Brown [1], Chen and Xia [14].

Given the metric $d$, the point processes on $[0, 1]$ are then defined by the set $\mathcal{P}(\Xi)$ of Borel probability measures on $\Xi$. For simplicity, we choose the distance between two point processes $\mu, \nu \in \mathcal{P}(\Xi)$ to be $W_1(\mu, \nu)$ as defined in (8). Suppose we are given $N$ sample paths $\hat{\eta}^i = \sum_{t=1}^{m_i} \delta_{\hat{\xi}_t}$, $i = 1, \ldots, N$, where $m_i$ is some nonnegative integer and $\hat{\xi}_t \in [0, 1]$ for all $i, t$. Then the reference
distribution \( \nu = \frac{1}{N} \sum_{i=1}^{N} \delta_{\eta^i} \), and the ambiguity set \( \mathfrak{M} = \{ \mu \in \mathcal{P}(\Xi) : W_i(\mu, \nu) \leq \theta \} \). Denote by \( X \) the set of all functions \( x : [0, 1] \to \{0, 1\} \) such that \( x^{-1}(1) \) is a Borel set, where \( x^{-1}(1) := \{ t \in [0, 1] : x(t) = 1 \} \). The decision maker is looking for a policy \( x \in X \) which maximizes the total revenue by solving the problem

\[
v^* := \sup_{x \in X} \left\{ v(x) : = -c \int_0^1 x(t) dt + \inf_{\mu \in \mathfrak{M}} \mathbb{E}_{\nu \sim \mu} [\eta(x^{-1}(1))] \right\}. \tag{61}
\]

We first investigate the structure of the optimal policy. Let \( \text{int}(x^{-1}(1)) \) be the interior of the set \( x^{-1}(1) \) on the space \([0, 1]\) with canonical topology (and thus \( 0, 1 \in \text{int}([0, 1]) \)).

**Lemma 4.** For any \( \nu \in \mathcal{P}(\Xi) \) and policy \( x \), it holds that

\[
\inf_{\mu \in \mathfrak{M}} \mathbb{E}_{\nu \sim \mu} [\eta(x^{-1}(1))] = \inf_{\rho \in \mathcal{P}(\mathcal{B}([0, 1]) \times \Xi)} \left\{ \mathbb{E}_{(\tilde{\eta}, \tilde{\eta}) \sim \rho} [\eta(\text{int}(x^{-1}(1))) : \mathbb{E}_{(\tilde{\eta}, \tilde{\eta}) \sim \rho} [W_1(\tilde{\eta}, \tilde{\eta})] \leq \theta, \pi_{\#} = \nu \right\}.
\]

Suppose \( \nu = \frac{1}{N} \sum_{i=1}^{N} \delta_{\eta^i} \) with \( \eta^i = \sum_{i=1}^{m_i} \delta_{\xi^i} \). Then there exists a positive integer \( M \) such that

\[
v^* = \sup_{\sum_{j=1}^{M} \mathbb{I}_{[x_{j-}, x_{j+}]} \leq 1} \left\{ v \left( \sum_{j=1}^{M} \mathbb{I}_{[x_{j-}, x_{j+}]} \right) : = -c \sum_{j=1}^{M} (x_{j+} - x_{j-}) + \inf_{\mu \in \mathfrak{M}} \mathbb{E}_{\nu \sim \mu} [\eta(\cup_{j=1}^{M} [x_{j-}, x_{j+}])] \right\}.
\]

**Note** that by (75),

\[
\inf_{\mu \in \mathfrak{M}} \mathbb{E}_{\nu \sim \mu} [\eta(x^{-1}(1))] = \inf_{\gamma \in \mathcal{P}(\Xi^2)} \left\{ \mathbb{E}_{(\eta, \eta) \sim \gamma} [\eta(x^{-1}(1))] : \mathbb{E}_{\gamma} [d(\eta, \eta)] \leq \theta, \pi_{\#} \gamma = \nu \right\}.
\]

Hence (62) shows that without changing the optimal value, we can replace \( d \) by \( W_1 \) in the constraint above, and enlarge the set of joint distributions from \( \mathcal{P}(\Xi^2) \) to \( \mathcal{P} (\mathcal{B}([0, 1]) \times \Xi) \). Moreover, (63) shows that it suffices to consider the set of policies of which the duration of on-state is a finite disjoint union of intervals with positive length. We next show that given a policy \( \{x_{j+}\}_{j=1}^{M} \), the computation of worst-case point process reduces to a linear program. For every \( 1 \leq i \leq N \) and \( 1 \leq t \leq m_i \), if \( \xi^i_t \in \cup_{j=1}^{M} [x_{j-}, x_{j+}] \), we set \( j^i_t \in \{1, \ldots, M\} \) be such that \( \xi^i_t \in [x_{j^i_t-}, x_{j^i_t+}] \), otherwise we set \( j^i_t = 0 \). In addition, we set \( x_0 \) be any real number.

**Proposition 7.** Let \( v(\{x_{j+}\}_{j=1}^{M}) \) be defined in (63). Then it holds that

\[
v \left( \sum_{j=1}^{M} \mathbb{I}_{[x_{j-}, x_{j+}]} \right) = \sum_{j=1}^{M} -c(x_{j+} - x_{j-}) + \frac{1}{N} \sum_{i=1}^{N} \mathbb{I}_{[x_{j-}, x_{j+}]}(\xi^i_t) \]

\[
+ \min_{p_{\pm} \in [0, 1]} \left\{ -\frac{1}{N} \sum_{i=1}^{N} \sum_{t=1}^{m_i} (p^i_{t-} + p^i_{t+}) : \frac{1}{N} \sum_{i=1}^{N} \sum_{1 \leq t \leq m_i} (p^i_{t-} | x_{j^i_t-} - \xi^i_t| + p^i_{t+} | x_{j^i_t+} - \xi^i_t|) \leq \theta, \right. \]

\[
\sum_{j=1}^{M} (p^i_{t-} + p^i_{t+}) \leq 1, \forall i, t \text{ such that } j^i_t > 0 \right\}.
\]

Moreover, the above linear program can be solved by a greedy algorithm (see Algorithm 2), and there exists a worst-case point process that has the form

\[
\mu^* (x) = \frac{1}{N} \sum_{i=1}^{N} \delta_{\eta^i} + \frac{p^*}{N} \delta_{\eta^{0+}} + \frac{(1 - p^*)}{N} \delta_{\eta^{0-}},
\]

where \( i_0 \in \{1, \ldots, N\} \), \( \eta^i \in \Xi \) for all \( i \neq i_0 \) and \( \eta^{0+} \in \Xi \), and \( \eta^i([0, 1]) = \eta^i([0, 1]) \) for all \( i \neq i_0 \) and \( \eta^{0+} ([0, 1]) = \eta^{0+} ([0, 1]) \).
Replacing $p_i$ by $p_i p_i^t + (1 - p_i) \tilde{p}_i^t$, and noticing that at optimality, $p_i^t > 0$ only if $\xi_i \in [x_i, x_{i+1}]$, and at most one of $\{p_i^t\}_{i,t}$ can be fractional, we obtain the result. □

We finally remark that in this process control problem, the space $\Xi$ is infinite dimensional and non-convex, which violates the assumptions made in Esfahani and Kuhn [17], Zhao and Guan [53]. Hence, our generality of the strong duality result indeed enlarges the class of tractable DRSP problems comparing to their work.
Algorithm 2 Greedy Algorithm
1: for $i = 1: M$ do
2:     for $t = 1: m_i$ do
3:         if $\hat{\xi}_t \notin \bigcup_{j=1}^{M} [x_{j-}, x_{j+}]$ then
4:             $d_t^i \leftarrow +\infty$.
5:         $p_{t,i}^* \leftarrow 0$, $\forall j = 1, \ldots, M$.
6:     else
7:         Find $j_t^i \in \{j\pm\}_{j=1}^{M}$ such that $d_t^i := |x_{j_t^i} - \hat{\xi}_t| = \min_{1 \leq j \leq M} \min(|x_{j-} - \hat{\xi}_t|, |x_{j+} - \hat{\xi}_t|)$.
8:     end if
9: end for
10: Sort $\{d_t^i\}_{1 \leq i \leq N, 1 \leq t \leq m_i}$ in increasing order, denoted by $d_t^{i(1)} \leq d_t^{i(2)} \leq \ldots \leq d_t^{i(\sum_{i=1}^{N} m_i)}$.
11: $\bar{\theta} \leftarrow 0$, $k \leftarrow 1$.
12: while $\bar{\theta} < \theta$ do
13:     $p_{t,k}^* \leftarrow \min \left(1, (\theta - \bar{\theta})/d_t^{i(k)}\right)$.
14:     $p_{t,k}^* \leftarrow 1$, $\forall j \in \{j\pm\}_{j=1}^{M} \setminus \{j_{t(k)}^i\}$.
15:     $k \leftarrow k + 1$.
16: end while

4.6. Distributionally Robust Transportation Problems
In this subsection, we demonstrate the powerfulness of our proof method by applying the same idea to a generic class of distributionally robust transportation problems.

Suppose $\Xi = \mathbb{R}^2$ and let $A$ be a Borel probability measure on $\Xi$ with bounded support. In the famous paper of Beardwood et al. [4], it is shown that the length of the shortest traveling salesman tour through $N$ i.i.d. random points with density $fdA$ is asymptotically equal to $\beta \int_{\Xi} \sqrt{f}dA$ for some constant $\beta$. Since then, the idea of using continuous approximation to hard combinatorial problems is explored by many researchers, such as Stein tree problems (Hwang and Richards [26]), space-filling curves (Platzman and Bartholdi III [37], Bartholdi III and Platzman [2]), facility location (Haimovich and Rinnoy Kan [24]), and Steele’s generalization to sub-additive Euclidean functionals (Steele [44, 45]), which identifies a class of random processes whose limit equals to $\beta \int_{\Xi} f(s^{-1}/s)dA$ for some $\beta$, where $s$ is the dimension of $\Xi$.

Thanks to these results, Carlsson et al. [12] considers the continuous approximation of traveling salesman problem in a distributionally robust setting. More specifically, they solve the worst-case problem $\sup_{f \in \mathcal{M}_f} \int_{\Xi} \sqrt{f}dA$ in which the distribution induced from density function $f$ is assumed to lie in a Wasserstein ball $\mathcal{M}$. Using duality theory for convex functional, they are able to reformulate the problem and obtain a representation of the worst-case distribution.

In the same spirit, we consider a slightly general problem as follows. Define

$$\mathcal{A} := \{d\mu/dA : \mu \in \mathcal{M}, \mu \text{ is absolutely continuous w.r.t } A\},$$

where $\mathcal{M}$ is the Wasserstein ball defined in (6) with radius $\theta > 0$ and $d\mu/dA$ is the Radon-Nikodym derivative. We denote by $\mathcal{B}$ by the set of finite Borel probability measures on $\Xi$ that are absolutely continuous with respect to $A$ and by $\mathcal{P} \subset \mathcal{B}$ by the set of Borel probability measures on $\Xi$ that are absolutely continuous with respect to $A$. We use the overloaded notation $W_p(f, \nu)$ to represent the $W_p$ distance between the reference distribution $\nu$ and the distribution with density $f$ with respect to $A$. Let $\mathcal{L} : \mathbb{R} \rightarrow \mathbb{R}$ be a non-decreasing concave function, consider the problem

$$v_p = \sup_{f \in \mathcal{A}} \int_{\Xi} \mathcal{L} \circ f dA.$$  
(66)
Our goal is to derive the strong dual of (66) and obtain a representation for the worst-case distribution using the same proof method as in Section 3.1.

**Step 1.** Derive the weak duality.

We first derive the weak duality by writing the Lagrangian and applying the similar reasoning to the proof of Proposition 2, we have that

\[
v_p = \sup_{f \in \mathcal{B}} \inf_{\lambda \geq 0} \left\{ \int_{\Xi} \mathcal{L} \circ f dA + \lambda (\theta^p - W^p_p (f, \nu)) \right\}
\]

\[
= \sup_{f \in \mathcal{B}} \inf_{\lambda \geq 0} \left\{ \int_{\Xi} \mathcal{L} \circ f dA + \lambda (\theta^p - W^p_p (f, \nu)) \right\}
\]

\[
\leq \inf_{\lambda \geq 0} \left\{ \lambda \theta^p + \sup_{f \in \mathcal{B}} \left\{ \int_{\Xi} \mathcal{L} \circ f dA - \lambda \right\} \sup_{u, v \in C_b(\Xi)} \left\{ \int_{\Xi} u f dA + \int_{\Xi} v d\nu : u(\xi) \leq \inf_{\zeta \in \Xi} d^p(\xi, \zeta) - v(\zeta) \right\} \right\}
\]

\[
= \inf_{\lambda \geq 0} \left\{ \lambda \theta^p + \sup_{f \in \mathcal{B}} \left\{ \int_{\Xi} \mathcal{L} \circ f dA - \lambda \right\} \sup_{v \in C_b(\Xi) : f \in \mathcal{B}} v \sup_{d \nu = 0} \left\{ \int_{\Xi} \left[ \inf_{\zeta \in \Xi} d^p(\xi, \zeta) - v(\zeta) \right] f(\xi) A(d\xi) \right\} \right\}
\]

\[
\leq \inf_{\lambda \geq 0} \left\{ \lambda \theta^p + \sup_{f \in \mathcal{B}} \left\{ \int_{\Xi} \mathcal{L} \circ f(\xi) - \lambda \Phi_v(\xi) f(\xi) A(d\xi) \right\} \right\},
\]

where the first inequality follows from Lemma 8 and in the last inequality \( \Phi_v(\xi) := \inf_{\zeta \in \Xi} d^p(\xi, \zeta) - v(\zeta) \). Define

\[
\mathcal{L}^*(a) := \sup_{x \geq 0} \mathcal{L}(x) - ax, \quad a \in \mathbb{R},
\]

which can be viewed as the Legendre transform of concave function \( \mathcal{L} \). Thus \( \mathcal{L}^* \) is convex and we denote by \( \partial \mathcal{L}^*(a) \) its subdifferential at \( a \in \text{dom} \mathcal{L}^* \), where \( \text{dom} \mathcal{L}^* := \{ a \geq 0 : \mathcal{L}^*(a) < \infty \} \). It follows that

\[
v_p \leq \inf_{\lambda \geq 0} \left\{ \lambda \theta^p + \sup_{f \in \mathcal{B}} \left\{ \int_{\Xi} \mathcal{L} \circ f(\xi) - \lambda \Phi_v(\xi) f(\xi) A(d\xi) \right\} : \lambda \Phi_v(\xi) \in \text{dom} \mathcal{L}^*, \quad \forall \xi \in \text{supp} A \right\}
\]

\[
\leq \inf_{\lambda \geq 0} \left\{ \lambda \theta^p + \int_{\Xi} \mathcal{L}^*(\lambda \Phi_v(\xi)) A(d\xi) \right\}
\]

\[
= \inf_{\lambda \geq 0} \left\{ h_v(\lambda) \right\}
\]

\[
= v^*.
\]

**Step 2.** Show the existence of the dual minimizer.

Let \((\lambda_n, v_n)_n\) be a sequence approaching to \( v^* \). Since \( v_n \) are continuous and \( \text{supp} A \) is compact, by Arzela-Ascoli theorem, there exists a subsequence convergent to \( v^* \) on \( A \) and \( h_v(\lambda) \leq \lim_{n \rightarrow \infty} h_v(\lambda_n) \) for all \( \lambda \geq 0 \). Also note that \( \Phi_v(\xi) \geq 0 \) for all \( \xi \) such that \( \lambda \Phi_v(\xi) \in \text{dom} \mathcal{L}^* \), it follows that \( \mathcal{L}^*(\lambda \Phi_v(\xi)) \) is a non-increasing convex function, whence \( h_v(\lambda) \) is the sum of an increasing linear function and a non-increasing convex function and goes to infinity as \( \lambda \rightarrow \infty \). So \( h_v(\lambda) \) attains its minimum at some \( \lambda^* \in [0, \infty) \). Therefore we prove the existence of dual minimizer \((\lambda^*, v^*)\).

**Step 3.** Use first-order optimality to construct a primal solution.
We take the derivative and obtain the first-order optimality at \( \lambda^* \),

\[
\theta^p + \int_{\Xi} \partial_{\lambda-} \mathcal{L}^*(\lambda^* \Phi_{v^*}(\xi)) \Phi_{v^*}(\xi) A(d\xi) \leq 0,
\]
\[
\theta^p + \int_{\Xi} \partial_{\lambda+} \mathcal{L}^*(\lambda^* \Phi_{v^*}(\xi)) \Phi_{v^*}(\xi) A(d\xi) \geq 0.
\] (67)

We define

\[
f^*(\xi) := -\left[ p^* \partial_{\lambda-} \mathcal{L}^*(\lambda^* \Phi_{v^*}(\xi)) + (1 - p^*) \partial_{\lambda+} \mathcal{L}^*(\lambda^* \Phi_{v^*}(\xi)) \right], \quad \forall \xi \in \text{supp} \ A,
\] (68)

where

\[
p^* := \frac{\theta^p + \int_{\Xi} \partial_{\lambda+} \mathcal{L}^*(\lambda^* \Phi_{v^*}(\xi)) \Phi_{v^*}(\xi) A(d\xi) - \int_{\Xi} \partial_{\lambda-} \mathcal{L}^*(\lambda^* \Phi_{v^*}(\xi)) \Phi_{v^*}(\xi) A(d\xi)}{\int_{\Xi} \partial_{\lambda+} \mathcal{L}^*(\lambda^* \Phi_{v^*}(\xi)) \Phi_{v^*}(\xi) A(d\xi)},
\]

provided that the denominator is nonzero, otherwise we set \( p^* = 1 \). By definition of \( \mathcal{L}^* \), \( f \) is nonnegative. Also note that \( \mathcal{L}^* \) is convex, so \( f^* \) is measurable.

**Step 4.** Verify the feasibility and optimality.

We verify that \( f^*(\xi) \) is primal optimal. From the concavity of \( \mathcal{L} \), we have \( \mathcal{L}^*(f^*(\xi)) \geq p^* \mathcal{L}^*(-\partial_{\lambda-} \mathcal{L}^*(\lambda^* \Phi_{v^*}(\xi))) + (1 - p^*) \mathcal{L}^*(-\partial_{\lambda+} \mathcal{L}^*(\lambda^* \Phi_{v^*}(\xi))) \). Using (67), we have

\[
v_p \geq \int_{\Xi} \mathcal{L}^*(f^*(\xi)) A(d\xi)
\]
\[
\geq p^* \int_{\Xi} \mathcal{L}^*(-\partial_{\lambda-} \mathcal{L}^*(\lambda^* \Phi_{v^*}(\xi))) A(d\xi) + (1 - p^*) \int_{\Xi} \mathcal{L}^*(-\partial_{\lambda+} \mathcal{L}^*(\lambda^* \Phi_{v^*}(\xi))) A(d\xi)
\]
\[
= \lambda^* \theta^p + p^* \int_{\Xi} \left[ \mathcal{L}(-\partial_{\lambda-} \mathcal{L}^*(\lambda^* \Phi_{v^*}(\xi))) - \lambda^* \Phi_{v^*}(\xi) \partial_{\lambda-} \mathcal{L}^*(\lambda^* \Phi_{v^*}(\xi)) \right] A(d\xi)
\]
\[
+ (1 - p^*) \int_{\Xi} \left[ \mathcal{L}(-\partial_{\lambda+} \mathcal{L}^*(\lambda^* \Phi_{v^*}(\xi))) - \lambda^* \Phi_{v^*}(\xi) \partial_{\lambda+} \mathcal{L}^*(\lambda^* \Phi_{v^*}(\xi)) \right] A(d\xi)
\]
\[
\geq v_D,
\]

and that

\[
W_p(f^*, \hat{f}) = \max_{u, v \in C_b(\Xi); \int v d\nu = 0} \left\{ \int_{\Xi} u f^* dA : u(\xi) + v(\xi) \leq d^p(\xi, \zeta), \forall \xi \in \Xi \right\}
\]
\[
= \max_{u, v \in C_b(\Xi); \int v d\nu = 0} \left\{ \int_{\Xi} u f^* dA : u(\xi) \leq \Phi_{v^*}(\xi), \forall \xi \in \Xi \right\}
\]
\[
= \theta^p.
\]

Hence we conclude that there exists a worst-case distribution of the form (68). In particular, when \( \mathcal{L}(\cdot) = \sqrt{\cdot} \), we have \( \partial L^*(a) = \frac{1}{4\lambda}\sqrt{a} \), \( f^*(\xi) = \frac{1}{4\lambda^* \Phi_{v^*}(\xi)^2} \), and \( \lambda^* = \sqrt{\frac{1}{4\theta^p \Phi_{v^*}} dA} \). We remark that we obtain a slightly more compact form than that in Carlsson et al. [12].

**5. Conclusions** In this paper, we developed a constructive proof method to derive the dual reformulation of distributionally robust stochastic programming with Wasserstein distance under a general setting. Such approach allows us to obtain a precise structural description of the worst-case distribution, which connects the distributionally robust stochastic programming to classical robust programming. Based on our results, we obtain many theoretical and computational implications. For the future work, extensions to multi-stage distributionally robust stochastic programming will be explored.
Appendix A: Technical Lemmas and Proofs  The technical lemmas and proofs are presented in the order as they appear in the paper.

Proof of Lemma 1. Let $\gamma_0$ be the minimizer of (8). It follows that

$$\left| E(\Psi(\xi)) - E(\Psi(\zeta)) \right| \leq \int E(\Psi(\xi) - \Psi(\zeta)) \gamma_0(d\xi, d\zeta)$$

$$\leq \int E(1 + d^p(\xi, \zeta)) \gamma_0(d\xi, d\zeta)$$

$$= L(1 + W_p(\mu, \nu))$$

$$\leq L(1 + \delta).$$

LEMMA 5. Let $\kappa := \limsup_{\xi, \zeta \to \infty} \frac{\Psi_x(\xi) - \Psi_x(\zeta)}{d^p(\xi, \zeta) + 1}$ for some $\zeta_0 \in \Xi$. Then for any $\zeta \in \Xi$, it holds that

$$\limsup_{\xi, \zeta \to \infty} \frac{\Psi_x(\xi) - \Psi_x(\zeta)}{d^p(\xi, \zeta) + 1} = \kappa.$$

Proof of Lemma 5. Suppose that for some $\zeta \in \Xi$, it holds that $\kappa' := \limsup_{\xi, \zeta \to \infty} \frac{\Psi_x(\xi) - \Psi_x(\zeta)}{d^p(\xi, \zeta) + 1} < \kappa$. Then for any $\epsilon \in (0, \kappa - \kappa')$, there exists $R > 0$ such that for any $\xi$ with $d(\xi, \zeta) \geq R$, we have that

$$\Psi_x(\xi) - \Psi_x(\zeta_0) = \Psi_x(\xi) - \Psi_x(\zeta) + \Psi_x(\zeta) - \Psi_x(\zeta_0)$$

$$< (\kappa' + \epsilon)(d^p(\xi, \zeta) + 1) + [\Psi_x(\zeta) - \Psi_x(\zeta_0)].$$

It follows that

$$\limsup_{\xi, \zeta \to \infty} \frac{\Psi_x(\xi) - \Psi_x(\zeta_0)}{d^p(\xi, \zeta) + 1} \leq \limsup_{\xi, \zeta \to \infty} \frac{\Psi_x(\xi) - \Psi_x(\zeta_0)}{d^p(\xi, \zeta) + 1}$$

$$\leq \limsup_{\xi, \zeta \to \infty} \frac{(\kappa' + \epsilon)(d^p(\xi, \zeta) + 1) + [\Psi_x(\zeta) - \Psi_x(\zeta_0)]}{d^p(\xi, \zeta) - d^p(\zeta_0, \zeta) + 1}$$

$$= \kappa' + \epsilon < \kappa,$$

which is a contradiction. Therefore the result is obtained by switching the role of $\zeta$ and $\zeta_0$. We note that the above proof holds also for $\kappa = \infty$. □

Proof of Proposition 1. Choose any compact set $E \subset \Xi$ with $\nu(E) > 0$ and let $\bar{\xi} \in \arg\max_{\xi \in E} \Psi_x(\xi)$. By Lemma 5 it holds that

$$\limsup_{\xi, \zeta \to \infty} \frac{\Psi_x(\xi) - \Psi_x(\bar{\xi})}{d^p(\xi, \bar{\xi}) + 1} = \infty.$$ 

Thus for any $M, R > 0$, there exists $\zeta_M$ such that $d(\zeta_M, \bar{\xi}) > R$ and that

$$\Psi_x(\zeta_M) - \Psi_x(\bar{\xi}) > M(d^p(\zeta_M, \bar{\xi}) + \text{diam}(E) + 1),$$

where $\text{diam}(E) := \max_{\xi, \zeta \in E} d(\xi, \zeta')$. It follows that

$$\Psi_x(\zeta_M) - \Psi_x(\xi) > M(d^p(\zeta_M, \bar{\xi}) + 1), \quad \forall \xi \in E.$$ 

Thereby we define a map $T_M : \Xi \to \Xi$ by

$$T_M(\zeta) := \zeta_M \mathbb{1}_{\{\zeta \in E\}} + \zeta \mathbb{1}_{\{\zeta \in \Xi \backslash E\}}, \quad \forall \zeta \in \Xi,$$
Let \( \epsilon < \lambda \).

Then \( \mu_M := p_M T_M \# \nu + (1 - p_M) \nu \), where \( p_M := \frac{\rho^p}{\int_E d^p(x)}(\xi, \zeta) \nu(dx) \). Note that we can always make \( p_M \leq 1 \) by choosing sufficiently large \( R \).

Then \( \mu_M \) is primal feasible and

\[
v_P - \int_{\Xi} \Psi_x(\xi) \nu(d\xi) \geq p_M \int_{E} [\Psi_x(T_M(\zeta)) - \Psi_x(\zeta)] \nu(d\zeta) > p_M M \int_{E} d^p(T_M(\zeta), \zeta) \nu(d\zeta) = M \theta^p,
\]

which goes to \( \infty \) as \( M \to \infty \). On the other hand, for any \( \lambda \geq 0 \) and \( \zeta \in E \), \( \Phi(\lambda, \zeta) = -\infty \), whence \( v_D = \infty \). Therefore we prove that \( v_P = v_D \). \( \square \)

**Proof of Lemma 2.** (i) For any \( \zeta \in \Xi \) and \( \epsilon > 0 \), there exists \( R > 0 \) such that for any \( \xi \in \Xi \) with \( d(\xi, \zeta) > R \), it holds that

\[
\Psi_x(\xi) - \Psi_x(\zeta) \leq (\kappa + \epsilon) d^p(\xi, \zeta) + 1.
\]

Let \( \epsilon < \lambda - \kappa \), it follows that \( \lambda d^p(\xi, \zeta) - \Psi_x(\xi) > \Psi_x(\zeta) \geq \Phi(\lambda, \zeta) \) for all \( \xi \) with \( d^p(\xi, \zeta) > R \). Also note that \( -\Psi_x \) is lower semi-continuous, thus the minimum of the right-hand side of (15) admits a minimizer in \( \{ \xi : d(\xi, \zeta) \leq R \} \), and so the minimum \( \Phi(\lambda, \zeta) \) is finite-valued.

(ii) From (i) we know that \( \Phi(\lambda, \zeta) \) is finite-valued. Since \( d^p(\xi, \zeta) \) is non-negative, for each fixed \( \zeta \in \Xi \), \( \Phi(\lambda, \zeta) \) is the infimum of non-decreasing linear functions of \( \lambda \) and therefore it is non-decreasing and concave with respect to \( \lambda \). To show its continuity with respect to \( \zeta \), observe that

\[
|\Phi(\lambda, \zeta) - \Phi(\lambda, \zeta')| \leq \lambda |d^p(\xi, \zeta) - d^p(\xi, \zeta')| \leq \lambda d^p(\zeta, \zeta'),
\]

so we conclude that \( \Phi(\lambda, \zeta) \) is continuous in \( \zeta \).

(iii) For any fixed \( \lambda > \kappa \), by (i), \( \arg \min_{\xi \in \Xi} \{ \lambda d^p(\xi, \zeta) + \Psi_x(\xi) \} \) is non-empty and compact, thus the multi-valued function

\[
\hat{T}_-(\lambda, \zeta) := \arg \max_{\xi} \left\{ d(\xi, \zeta) : \xi \in \arg \min_{\xi \in \Xi} \{ \lambda d^p(\xi, \zeta) - \Psi_x(\xi) \} \right\}
\]

is well-defined. We claim that \( \hat{T}_-(\lambda, \zeta) \) is closed-valued. Indeed, let \( \{ \xi_k \}_k \subset \hat{T}_-(\lambda, \zeta) \) converge to \( \xi_0 \), from continuity of \( d \) and lower semi-continuity of \( \Psi_x \), it follows that \( \xi_0 \) minimizes \( \lambda d^p(\xi, \zeta) - \Psi_x(\xi) \) and \( \xi_0 \) also has largest distance to \( \zeta \) among the optimizers. We further claim that \( \hat{T}_-(\lambda, \zeta) \) is Borel measurable. Indeed, define the mapping \( f : \Xi \times (\Xi \times \mathbb{R}) \to \Xi \) defined by \( f(\xi, (\xi, \lambda)) := \lambda d^p(\xi, \zeta) - \Psi_x(\xi) \). Since \( \Psi_x(\xi) \) is upper semi-continuous, the epigraphical mapping \( epi \ f(\xi, \cdot) := \{(\xi, \lambda) : f(\xi, (\xi, \lambda)) \leq \alpha \} \) is closed-valued and Borel measurable, that is, \( f \) is a normal integrand (cf. Shapiro et al. [42], Castaing and Valadier [13]). Hence, the mapping \( P : \Xi \times (\Xi \times \mathbb{R}) \to \mathbb{R} \) defined by \( P(\zeta) := \arg \min_{\xi \in \Xi} f(\xi, (\xi, \lambda)) \) is Borel measurable (cf. Theorem 14.37 in Rockafellar and Wets [39] and Theorem 3.5 in Himmelberg et al. [25]). Then for any Borel subset \( A \subset \Xi \), its preimage \( P^{-1}(A) \) is Borel measurable. Note that \( T_\lambda^{-1}(A) \) is a closed subset of \( P^{-1}(A) \) and thus is also Borel.
measurable. Therefore $\bar{T}_-(\lambda, \zeta)$ is Borel measurable. Consequently, by Kuratowski–Ryll-Nardzewski measurable selection theorem (cf. Theorem 4.1 in Wagner [49]), $T_-(\lambda, \zeta)$ has a Borel measurable selection, denoted as $T_-(\lambda, \zeta)$.

Now we show that $d(T_-(\lambda, \zeta), \zeta)$ is left continuous in $\lambda$. Let $\{\lambda_n\}$ be a sequence convergent to $\lambda$ with $\lambda_n \in ((\lambda + \kappa)/2, \lambda)$ for all $n$. We claim that

$$
T_-(\lambda_n, \zeta) \subset \{\xi \in \Xi : d(T_-(\lambda, \zeta), \zeta) \leq d(\xi, \zeta) \leq R\}, \forall n.
$$

In fact, by (ii), $\Phi(\lambda', \zeta)$ has an upper bound on the compact set $[(\lambda + \kappa)/2, \lambda] \times \text{supp } \nu$. Then using the same argument as in (i), there exists $R_\lambda > 0$, such that for any $\xi \in \Xi$ with $d(\xi, \zeta_0) > R_\lambda$,

$$
\lambda d^p(\xi, \zeta) - \Phi(\lambda, \zeta) \geq \lambda d^p(\xi', \zeta), \quad \forall (\lambda', \zeta) \in [(\lambda + \kappa)/2, \lambda] \times \text{supp } \nu.
$$

Hence $d(T_-(\lambda_n, \zeta), \zeta) \leq R_\lambda$ for all $n$. On the other hand, for all $\xi \in \Xi$ with $d(\xi, \zeta) < d(T_-(\lambda, \zeta), \zeta)$,

$$
\Psi_x(T_-(\lambda, \zeta)) - \Psi_x(\xi) \geq \lambda [d^p(T_-(\lambda, \zeta), \zeta) - d^p(\xi, \zeta)] > \lambda_n [d^p(T_-(\lambda, \zeta), \zeta) - d^p(\xi, \zeta)],
$$
or,

$$
\lambda_n d^p(T_-(\lambda, \zeta), \zeta) - \Psi_x(T_-(\lambda, \zeta)) < \lambda_n d^p(\xi, \zeta) - \Psi_x(\xi), \quad \forall \xi \text{ with } d(\xi, \zeta) < d(T_-(\lambda, \zeta), \zeta).
$$

Therefore the claim holds. We then show that $d(T_-(\lambda_n, \zeta), \zeta)$ converges to $d(T_-(\lambda, \zeta), \zeta)$ as $n \to \infty$. Suppose on the contrary there exists $\delta > 0$ and a subsequence $\{\lambda_{n_k}\}$ such that $d(T_-(\lambda_{n_k}, \zeta), \zeta) \geq d(T_-(\lambda, \zeta), \zeta) + \delta$ for all $k$. By definition of $T_-(\lambda_{n_k}, \zeta)$,

$$
\lambda_{n_k} d^p(T_-(\lambda, \zeta), \zeta) - \Psi_x(T_-(\lambda, \zeta)) \leq \lambda_{n_k} d^p(T_-(\lambda_{n_k}, \zeta), \zeta) - \Psi_x(T_-(\lambda_{n_k}, \zeta)).
$$

From (69) and possibly passing to a subsequence we can assume $T_-(\lambda_{n_k}, \zeta)$ converges to some $\xi_0 \in \Xi$. Let $k \to \infty$ and by lower semi-continuity of $-\Psi_x$,

$$
\lambda d^p(\xi_0, \zeta) - \Psi_x(\xi_0) \leq \liminf_{k \to \infty} \lambda_{n_k} d^p(T_-(\lambda_{n_k}, \zeta), \zeta) - \Psi_x(T_-(\lambda_{n_k}, \zeta)) \leq \lambda d^p(T_-(\lambda, \zeta), \zeta) - \Psi_x(T_-(\lambda, \zeta)),
$$

thus we obtain that $\xi_0$ is a minimizer of $\inf_{\xi \in \Xi} \lambda d^p(\xi, \zeta) - \Psi_x(\xi)$, but $d(\xi_0, \zeta) \geq d(T_-(\lambda, \zeta), \zeta) + \delta$, which contradicts to that $T_-(\lambda, \zeta)$ is the minimizer with largest distance to $\zeta$. Therefore we have shown that $d(T_-(\lambda, \zeta), \zeta)$ converges to $d(T_-(\lambda, \zeta), \zeta)$ as $n \to \infty$, that is, $d(T_-(\lambda, \zeta), \zeta)$ is left continuous with respect to $\lambda$. Moreover, since $\Psi_x(T_-(\lambda, \zeta)) = \lambda d(T_-(\lambda, \zeta), \zeta) - \Phi(\lambda, \zeta)$ and $\Phi(\lambda, \zeta)$ is continuous with respect to $\lambda$, we conclude that $\Psi_x(T_-(\lambda, \zeta))$ is left continuous with respect to $\lambda$. Using a similar argument we can show the second part of (iii).

(iv) For any $0 < \epsilon < \lambda - \kappa$, by definition we have

$$
\frac{\Phi(\lambda - \epsilon, \zeta)}{\lambda - \epsilon} + \frac{\Psi_x(T_-(\lambda, \zeta))}{\lambda - \epsilon} \geq d^p(T_-(\lambda, \zeta), \zeta) = \Phi(\lambda, \zeta) + \frac{\Psi_x(T_-(\lambda, \zeta))}{\lambda},
$$

thereby

$$
\Psi_x(T_-(\lambda - \epsilon, \zeta)) \left(1 - \frac{1}{\lambda - \epsilon}\right) \leq \frac{\Phi(\lambda - \epsilon, \zeta)}{\lambda - \epsilon} - \frac{\Phi(\lambda, \zeta)}{\lambda},
$$

and similarly,

$$
\frac{\Phi(\lambda - \epsilon, \zeta)}{\lambda - \epsilon} - \frac{\Phi(\lambda, \zeta)}{\lambda} \leq \Psi_x(T_-(\lambda, \zeta)) \left(1 - \frac{1}{\lambda - \epsilon}\right).
$$

Dividing by $\epsilon$ on both sides of the above two inequalities and letting $\epsilon \to 0$, from left continuity of $T_-(\lambda, \zeta)$ it follows that

$$
\partial_{\lambda-} \left(\frac{\Phi(\lambda, \zeta)}{\lambda}\right) = \frac{\Psi_x(T_-(\lambda, \zeta))}{\lambda^2}.
$$
The expression of $\partial_{\lambda^+} \Phi(\zeta, \lambda)$ can be obtained similarly.

(v) For any $\epsilon > 0$, there exists $\xi_\epsilon$ such that $\kappa d^p(\xi_\epsilon, \zeta) - \Psi_x(\xi_\epsilon) < \Phi(\kappa, \zeta) + \epsilon/2$ and so for any $0 < \lambda < \frac{\epsilon}{2d^p(\xi_\epsilon, \zeta)},$

$$0 \leq \Phi(\lambda, \zeta) - \Phi(\kappa, \zeta) < (\lambda - \kappa)d^p(\xi_\epsilon, \zeta) + \epsilon/2 < \epsilon.$$ It follows that

$$\lim_{\lambda \to \kappa^+} \Phi(\lambda, \zeta) = \Phi(\kappa, \zeta).$$

In particular when $\kappa = 0$, $\Phi(0, \zeta) = -\sup \Psi_x(\zeta)$.

(vi) By definition we have $\lambda d^p(T_+(\lambda, \zeta), \zeta) - \Psi_x(T_+(\lambda, \zeta)) \leq -\Psi_x(\zeta), \text{ or } \Psi_x(T_+(\lambda, \zeta)) - \Psi_x(\zeta) \geq \lambda d^p(T_+(\lambda, \zeta), \zeta)$. On the other hand, since $\kappa < \infty$, it follows that for any $\epsilon > 0$, there exists $R_\epsilon > 1$, such that for any $\xi, \xi' \in \Xi$ with $d^p(\xi, \xi') > R_\epsilon$, it holds that $|\Psi_x(\xi) - \Psi_x(\xi')| < (\kappa + \epsilon)d^p(\xi, \xi')$. In particular, let $\epsilon := \lambda - \kappa$, $\Psi_x(\xi') < \Psi_x(\zeta) + \lambda(\xi', \zeta)$ for all $\xi'$ with $d^p(\xi', \zeta) > R_{\lambda - \kappa}$, therefore $d^p(T_+(\lambda, \zeta), \zeta) < D_\lambda := R_{\lambda - \kappa}$. □

Proof of Corollary 3. Note that $x$ is feasible if and only if

$$t \geq \max_{u \in D} \left\{ \frac{1}{N} \sum_{i=1}^{N} (a_i[u]^T x + b_i[u]) : \left( \begin{array}{c} a_i[u] \\ b_i[u] \end{array} \right) = \left( \begin{array}{c} \hat{a}_i \\ \hat{b}_i \end{array} \right) + u_i \right\}$$

$$= \frac{1}{N} \sum_{i=1}^{N} (\hat{a}_i^T x + \hat{b}_i) + \frac{1}{N} \max_{u \in D} \left\{ \sum_{i=1}^{N} u_i^T x \right\}$$

Then by Hölder’s inequality it follows that

$$\frac{1}{N} \max_{u \in D} \left\{ \sum_{i=1}^{N} u_i^T x \right\} \leq \frac{1}{N} \sum_{i=1}^{N} ||u_i||_q ||x||_{q^*} \leq \theta ||x||_{q^*},$$

where $1/q + 1/q^* = 1$, and equality holds at, for example, $u_i = \theta y$, $i = 1, \ldots, N$, where $y$ satisfies $y^T x = ||x||_{q^*}$. Hence $x$ is feasible if and only if

$$\frac{1}{N} \sum_{i=1}^{N} (\hat{a}_i[u]^T x + \hat{b}_i[u]) + \theta ||x||_{q^*} \leq t.$$ 

□

Lemma 6. Suppose $(50)$ holds and the constant $L, M$ are defined in $(51)$. Then for the vector field defined in $(52)$, it holds that $||F(z) - F(z')||_{z, \ast} \leq L ||z - z'||_z + M$ for all $z, z' \in Z$.

Proof of Lemma 6. The proof is a simple exercise in Calculus. Using condition $(50)$, we obtain the bound estimations

$$||F_1(x, y) - F_1(x', y)||_{x, \ast} \leq \frac{1}{N} \sum_{i=1}^{N} ||\partial_x \Psi(x, y^i + \tilde{\xi}^i) - \partial_x \Psi(x, y^i + \tilde{\xi}^i)||_{x, \ast} \leq L_{11} ||x - x'||_X + M_{11},$$

$$||F_1(x', y) - F_1(x', y')||_{X, \ast} \leq \frac{1}{N} \sum_{i=1}^{N} ||\partial_x \Psi(x', y^i + \tilde{\xi}^i) - \partial_x \Psi(x', y'^i + \tilde{\xi}^i)||_{x, \ast}$$

$$\leq \frac{1}{N} \sum_{i=1}^{N} L_{12} ||y^i - y'^i||_z + M_{12}$$

$$\leq \frac{L_{12}}{N} \left( \sum_{i=1}^{N} ||y^i - y'^i||_{p \Xi} \right)^{1/p} N^{1/q} + M_{12}$$

$$\leq L_{12} ||y - y'||_Y + M_{12},$$
\[
||F_2(x, y) - F_2(x', y)||_{Y,*} = N^{-1} \left( \sum_{i=1}^{N} \|\partial_x \Psi(x, y^i + \hat{\xi}) - \partial_x \Psi(x', y^i + \hat{\xi})\|_{\mathbb{E}}^q \right)^{1/q}
\]
\[
\leq N^{-1} \sum_{i=1}^{N} [L_{21} \|x - x'\|_X + M_{21}]
= L_{21} \|x - x'\|_X + M_{21},
\]
\[
||F_2(x', y) - F_2(x', y')||_{Y,*} = N^{-1} \left( \sum_{i=1}^{N} \|\partial_x \Psi(x', y^i + \hat{\xi}) - \partial_x \Psi(x', y'^i + \hat{\xi})\|_{\mathbb{E}}^q \right)^{1/q}
\]
\[
\leq N^{-1} \left( \sum_{i=1}^{N} [L_{22} ||y^i - y'^i||_{\mathbb{E}} + M_{22}] \right)^{1/q}
\leq N^{-1} L_{22} \left( \sum_{i=1}^{N} ||y^i - y'^i||_{\mathbb{E}}^q \right)^{1/q} + M_{22}
\leq N^{\max(1/q - 1/p, 0) - 1} L_{22} ||y - y'||_Y + M_{22}.
\]

Combining the above inequalities we have
\[
||F(z) - F(z')||_{Z,*} = \sqrt{||F_1(z) - F_1(z')||_{X,*}^2 + ||F_2(z) - F_2(z')||_{Y,*}^2}
\leq 2 \max \{ ||F_1(z) - F_1(z')||_{X,*}, ||F_2(z) - F_2(z')||_{Y,*} \}
\leq 4 \max \{ L_{11} ||x - x'||_X + M_{11}, L_{12} ||y - y'||_Y + M_{12}, L_{21} ||x - x'||_X + M_{21}, N^{\max(1/q - 1/p, 0) - 1} L_{22} ||y - y'||_Y + M_{22} \}.
\]

□

**Proof of Proposition 5.** According to Theorem 4.1 in Nemirovski [33], the results follows if we can provide an upper bound on \(\Theta_Y\). Let \(h = (h_1, \ldots, h_N) \in \Xi^N\) and \(m \leq 2\). The first-order directional derivative of \(\omega_Y\) at \(y\) along \(h\) is given by
\[
\gamma D \omega_Y(y)[h] = \sum_{i=1}^{N} ||\xi^i||_{\Xi_i^{-1}}^{-2} \langle \xi^i, h_i \rangle.
\]
The second-order directional derivative of \(\omega_Y\) at \(y\) along \([h, h]\) is given by
\[
\gamma D^2 \omega_Y(y)[h, h] = \sum_{i=1}^{N} ||\xi^i||_{\Xi_i^{-1}}^{-2} ||h_i||_{\Xi_i}^2 + (m - 2) \sum_{i=1}^{N} ||\xi^i||_{\Xi_i^{-1}}^{-4} \langle \xi^i, h_i \rangle^2
\geq (m - 1) \sum_{i=1}^{N} ||\xi^i||_{\Xi_i^{-1}}^{-2} ||h_i||_{\Xi_i}^2,
\]
where the inequality follows from Cauchy-Schwarz inequality and the condition \(m \leq 2\). On the other hand,
\[
||h||_{\mathbb{E}}^2 = \left( \sum_{i=1}^{N} ||h_i||_{\Xi_i}^p \right)^{2/p} \leq \left( \sum_{i=1}^{N} ||h_i||_{\Xi_i} \right)^2 \leq \left( \sum_{i=1}^{N} ||\xi||_{\Xi_i^{-1}}^{-2} ||h_i||_{\Xi_i}^2 \right) \left( \sum_{i=1}^{N} ||\xi||_{\Xi_i^{-1}}^{-2} \right),
\]
where the first inequality follows from \(p \geq 1\) and the second inequality follows from Cauchy-Schwarz inequality. Combining the above two inequalities yields
\[
||h||_{\mathbb{E}}^2 \leq \frac{\gamma}{m - 1} D^2 \omega_Y(y)[h, h] \left( \sum_{i=1}^{N} ||\xi||_{\Xi_i^{-1}}^{-2} \right).
\]
Noting that \( \sum_{i=1}^{N} \| \xi^i \|_{\Xi_i} \leq N \theta^p \), set \( t_i = \| \xi^i \|_{\Xi_i} \), whence \( \sum_{i=1}^{N} t_i \leq N \theta^p \). It follows from Hölder’s inequality that

\[
\sum_{i=1}^{N} \| \xi^i \|_{\Xi_i}^m \leq \theta^m N^{\max(m/p,1)}.
\] (71)

Hence \( \sum_{i=1}^{N} \| \xi^i \|_{\Xi_i}^{2-m} \leq \theta^{2-m} N^{\max(2-m/p,1)} \). Thus in view of (70), \( \omega_Y \) is strongly convex with modulus 1 with respect to \( \| \cdot \|_Y \) if

\[
\frac{\gamma}{m-1} \theta^{2-m} N^{\max(2-m/p,1)} = 1.
\] (72)

Now let us compute an upper bound of \( \Theta_Y \). According to (72), we have

\[
\Theta_Y = \max_{y \in Y} \omega_Y(y) - \min_{y \in Y} \omega_Y(y) \leq \frac{1}{m\gamma} \sum_{i=1}^{N} \| \xi^i \|_{\Xi_i} \leq \frac{1}{m(m-1)} \theta^{2-m} N^{\max(2-m/p,1)} \left( \sum_{i=1}^{N} \| \xi^i \|_{\Xi_i}^m \right).
\]

Using (71) we obtain that

\[
\Theta_Y \leq \frac{\theta^2}{m(m-1)} N^{\max(2-m/p,1)+\max(m/p,1)}.
\]

When \( p > 1 \), choosing \( m = \min(2,p) \), we have \( \Theta_Y \leq \frac{\theta^2}{m(m-1)} N^2 \); when \( p = 1 \), choosing \( m = 1 + \frac{1}{\ln N} \), we have \( \Theta_Y \leq e\theta^2 N^2 \ln N \). It can be easily checked that \( \gamma \) defined in (53) satisfies condition (72).

\[ \square \]

**Lemma 7.** Let \( C \) be a Borel set and has nonempty boundary \( \partial C \). Then for any \( \epsilon > 0 \), there exists a Borel map \( T_\epsilon : \partial C \to \Xi \setminus \text{cl}(C) \) such that \( d(\xi, T_\epsilon(\xi)) < \epsilon \) for all \( \xi \in \partial C \).

**Proof of Lemma 7.** Since \( \Xi \) is separable, \( \partial C \) has a dense subset \( \{ \xi^i \}_{i=1}^{\infty} \). For each \( \xi^i \), there exists \( \xi'^i \in \Xi \setminus \text{cl}(\Xi) \) such that \( \epsilon_i := 2d(\xi^i, \xi'^i) < \epsilon \). Thus \( \partial C = \bigcup_{i=1}^{\infty} B_{\epsilon_i}(\xi'^i) \), where \( B_{\epsilon_i}(\xi'^i) \) is the open ball centered at \( \xi'^i \) with radius \( \epsilon_i \). Define

\[
i^*(\xi) := \min_{i \geq 0} \{ i : \xi \in B_{\epsilon_i}(\xi^i) \}, \quad \xi \in \partial C;
\]

and

\[
T_\epsilon(\xi) := \xi^*_{i^*(\xi)}, \quad \xi \in \partial C.
\]

Then \( T_\epsilon \) satisfies the requirements in the lemma. \[ \square \]

**Proof of Lemma 3.** Since \( -1 \{ \xi \in \text{int}(C) \} \) is upper semi-continuous and binary-valued, Theorem 1 ensures the existence of the worst-case distribution of \( \min_{\mu \in \mathcal{M}} \mu(\text{int}(C)) \). Thus it suffices to show that for any \( \epsilon > 0 \), there exists \( \mu \in \mathcal{M} \) such that \( \mu(C) \leq \min_{\mu \in \mathcal{M}} \mu(\text{int}(C)) + \epsilon \).

Note that

\[
\min_{\mu \in \mathcal{M}} \mu(\text{int}(C))
\]

\[
= \min_{\gamma} \left\{ \int_{\Xi \times \text{int}(C)} \gamma(d\xi, d\zeta) : \pi_{\Xi_1}^1 \gamma = \nu, \int_{\Xi_2} d^p(\xi, \zeta) \gamma(d\xi, d\zeta) \leq \theta^p \right\}
\]

\[
= \nu(\text{int}(C)) - \max_{\gamma} \left\{ \int_{\text{int}(C) \times (\Xi \setminus \text{int}(C))} \gamma(d\xi, d\zeta) : \pi_{\Xi_1}^1 \gamma = \nu, \int_{\Xi_2} d^p(\xi, \zeta) \gamma(d\xi, d\zeta) \leq \theta^p \right\}.
\]

The second term suggests to find a transportation plan which transports mass from distribution \( \nu \) such that the total cost is no greater than \( \theta^p \) and that the total mass from \( \text{int}(C) \) to its complement \( \Xi \setminus \text{int}(C) \) is maximized. Since it is unnecessary to transport mass from \( \text{int}(C) \) further once it hits
\[ \partial C \text{, nor to transport mass from int}(C) \text{ to int}(C) \setminus \text{supp } \nu \text{, nor to transport mass in } \Xi \setminus \text{int}(C), \text{ therefore, there exists an optimal transportation plan } \gamma_0 \text{ such that} \]
\[ \text{supp } \gamma_0 \subset (\text{supp } \nu \times \text{supp } \nu) \cup ( (\text{supp } \nu \cap \text{int}(C)) \times \partial C). \]

We set \( \mu_0 := \pi_{\#}^\nu \gamma_0. \)

If \( \mu_0(\partial C) = 0, \) there is nothing to show, so we assume that \( \mu_0(\partial C) > 0. \) We first consider the case \( \nu(\text{int}(C)) = 0 \) (and thus \( \mu_0 \) can be chosen to be \( \nu \)). By Lemma 7, we can define a Borel map \( T_\varepsilon \) which maps each \( \xi \in \partial C \) to some \( \xi' \in \Xi \setminus \text{cl}(C) \) with \( d(\xi, \xi') < \varepsilon \in (0, \theta) \) and is an identity map elsewhere. We further define a distribution \( \mu_\varepsilon \) by
\[
\mu_\varepsilon(A) := \mu_0(A \setminus \partial C) + \mu_0\{\xi \in \partial C : T_\varepsilon(\xi) \in A\}, \text{ for all Borel set } A \subset \Xi.
\]
Then \( W_p(\mu_\varepsilon, \mu_0) \leq \varepsilon < \theta \) and \( \mu_\varepsilon(C) = \mu_0(\text{int}(C)). \)

Now let us consider \( \nu(\text{int}(C)) > 0. \) For any \( \varepsilon \in (0, \theta), \) we define a distribution \( \mu'_\varepsilon \) by
\[
\mu'_\varepsilon(A) := \mu_0(A \cap \text{int}(C)) + \frac{\varepsilon}{\theta} \left( \mu_0\{(A \cap \text{int}(C)) \times \partial C\} + \nu(\partial C) \right) \\
+ \left( 1 - \frac{\varepsilon}{\theta} \right) \mu_0\{\xi \in \partial C : T_\varepsilon(\xi) \in A\} + \mu_0(A \setminus \text{cl}(C)),
\]
for all Borel set \( A \subset \Xi. \) Then
\[
\mu'_\varepsilon(C) = \mu_0(\text{int}(C)) + \frac{\varepsilon}{\theta} [\mu_0(\partial C) - \nu(\partial C)] + 0 + 0 \\
\leq \mu_0(\text{int}(C)) + \frac{\varepsilon}{\theta}.
\]

Note that \( W_p^*(\mu_\varepsilon, \nu) = \int_{\text{int}(C) \times \partial C} d^p(\xi, \zeta) \gamma_0(d\xi, d\zeta), \) it follows that
\[
W_p(\mu_\varepsilon, \nu) \leq \int_{\text{int}(C) \times \partial C} d^p(\xi, \zeta) \gamma_0(d\xi, d\zeta) - \frac{\varepsilon}{\theta} \int_{\text{int}(C) \times \partial C} d^p(\xi, \zeta) \gamma_0(d\xi, d\zeta) + \left( 1 - \frac{\varepsilon}{\theta} \right) \varepsilon + 0 \\
\leq \theta.
\]

Hence the proof is completed. \( \Box \)

**Proof of Proposition 6.** Define \( C_w := \{ \xi : -w^\top \xi < q \} \) for all \( w. \) According to the structure of the worst-case distribution (Theorem 1(ii)(iii)), there exists a worst-case distribution \( \mu^* \) which attains the infimum \( \inf_{\mu \in \mathcal{M}} \mathbb{P}_{\mu}\{ -w^\top \xi < q \} \) and has the form \( (17). \) Let \( \lambda^* \) be the associated dual optimizer \( \lambda^*. \) Since \( 1_{C_w} \) is binary valued, by definition (Lemma 2(iii)) of \( T_+,(\lambda^*, \cdot) \) it holds that \( T_+,(\lambda^*, \zeta), \)
\( T_-(\lambda^*, \zeta) \in \{ \zeta, \inf_{\xi \in \Xi} C_w d(\xi, \zeta) \}. \) That is to say, for \( \zeta \in \text{supp } \nu, \) the worst-case distribution \( \mu^* \) either transports (probably with splitting) \( \zeta \) to its closest point in \( \Xi \setminus C_w, \) or just let it stay at \( \zeta. \)

In addition, we claim that there exists a worst-case distribution which satisfies, if \( \zeta \) is transported to \( \inf_{\xi \in \Xi} C_w d(\xi, \zeta), \) then all the points that closer to \( \Xi \setminus C_w \) than \( \zeta \) will be transported. This is simply because any other transportation plan that achieves the same objective has larger total distance of transportation.

With this in mind, let \( \gamma^* \) be the optimal transport plan between \( \nu \) and \( \mu^*. \) and let
\[
t^* := \text{ess sup}_{\zeta \in \text{supp } \nu} \left\{ \min_{\xi \in \Xi \setminus C_w} d(\xi, \zeta) : \zeta \neq T_+,(\lambda^*, \zeta) \right\}.
\]
So \( t^* \) is the longest distance of transportation among all the points that are transported. (We note that infinity is allowed in the definition of \( t^* \), however, as will be shown, this violates the probability bound.) Then \( \mu^* \) transports all the points in \( \text{supp } \nu \cap \{ \xi : q - t^* < -w^\top \xi < b \}, \) and possibly a fraction of mass \( \beta^* \in [0, 1] \) in \( \text{supp } \nu \cap \{ \xi : -w^\top \xi = q - t^* \}. \) Also note that by Hölder's
inequality, the $d$-distance between two hyperplanes $\{\xi : -w^\top \xi = s\}$ and $\{\xi : -w^\top \xi = s'\}$ equals to $|s - s'|/\|x\|_1 = |s - s'|$. Using the above characterization, let us define a probability measure $\nu_w$ on $\mathbb{R}$ for any $w$ by

$$\nu_w\{(-\infty, s)\} := \nu\{\xi : -w^\top \xi < s\}, \forall s \in \mathbb{R},$$

then using the changing of measure, the total distance of transportation can be computed by

$$\int_{(\Xi \setminus C_w) \times C_w} d^p(\xi, \zeta) \gamma^*(d\xi, d\zeta) = \int_{(q - t^*)^+} (q - s)^p \nu_w(ds) + \beta^* \nu_w(\{q - t^*\}) t^p \leq \theta^p. \quad (73)$$

On the other hand, using property of marginal expectation and the characterization of $\gamma^*$,

$$\mu^*(C_w) = \int_{C_w \times \Xi} \gamma^*(d\xi, d\zeta)$$

$$= \nu(C_w) - \int_{(\Xi \setminus C_w) \times C_w} \gamma^*(d\xi, d\zeta)$$

$$= 1 - \nu_w([q, \infty)) - \beta^* \nu_w(\{q - t^*\}) + \nu_w(\{q - t^*, q\})$$

$$= 1 - \nu_w(q - t^*, \infty) - \beta^* \nu_w(\{q - t^*\}).$$

Thereby the condition $\inf_{\mu \in \mathcal{M}} \mu(C_w) \geq 1 - \alpha$ is equivalent to

$$\beta^* \nu_w(\{q - t^*\}) + \nu_w(q - t^*, \infty) \leq \alpha. \quad (74)$$

Now consider the quantity

$$J := \int_{(\text{VaR}_\alpha[-w^\top \xi])^+} (q - s)^p \nu_w(ds) + \beta_0 \nu_w(\{\text{VaR}_\alpha[-w^\top \xi]\}) (q - \text{VaR}_\alpha[-w^\top \xi])^p - \theta^p.$$

If $J < 0$, due to the monotonicity in $t^*$ of the right-hand side of (73), either $q - t^* < \text{VaR}_\alpha[-w^\top \xi]$ or $q - t^* = \text{VaR}_\alpha[-w^\top \xi]$ and $\beta > \beta_0$. But in both cases (74) is violated. On the other hand if $J \geq 0$, again by monotonicity, either $q - t^* > \text{VaR}_\alpha[-w^\top \xi]$, or $q - t^* > \text{VaR}_\alpha[-w^\top \xi]$ and $\beta^* \leq \beta_0^*$ and thus (74) is satisfied. □

**Lemma 8.** For any Borel measure $\mu$ with $\mu(\text{supp } \nu) \neq 1$, it holds that

$$\max_{u \in L^1(\mu), v \in L^1(\nu)} \left\{ \int_{\Xi} u(\xi) \mu(d\xi) + \int_{\Xi} v(\zeta) \nu(d\zeta) : u(\xi) + v(\zeta) \leq d^p(\xi, \zeta), \forall \xi, \zeta \in \Xi \right\} = \infty.$$

**Proof of Lemma 8.** Let $(u_0, v_0)$ be any feasible solution to the above maximization problem and define $u_t(\xi) := u(\xi) + t$ and $v_t(\zeta) := v(\zeta) - t$ for any $t \in \mathbb{R}$, $\xi \in \Xi$, and $\zeta \in \text{supp } \nu$. Then it follows that $u_t(\xi) + v_t(\zeta) \leq d^p(\xi, \zeta)$ and

$$\int_{\Xi} u_t(\xi) \mu(d\xi) + \int_{\Xi} v_t(\zeta) \nu(d\zeta) = \int_{\Xi} u_0(\xi) \mu(d\xi) + \int_{\Xi} v_0(\zeta) \nu(d\zeta) + t[\mu(\text{supp } \nu) - 1].$$

Since $\mu(\text{supp } \nu) \neq 1$,

$$\sup_{t \in \mathbb{R}} \left\{ \int_{\Xi} u_t(\xi) \mu(d\xi) + \int_{\Xi} v_t(\zeta) \nu(d\zeta) \right\} = \infty,$$

which proves the lemma. □
Proof of Lemma 4. If $\theta = 0$, clearly $v^* \leq \frac{1}{N} \sum_{i=1}^{N} m_i$. Let $\hat{X} := \{\hat{\xi}_i : i = 1, \ldots, N, t = 1, \ldots, m_i\}$. Suppose the elements in $\hat{X}$ can be sorted in increasing order by $\hat{\xi}_1 < \cdots < \hat{\xi}_M$, where $M = \text{card}(\hat{X})$. Then for any $\epsilon > 0$, let $x_{j+} = \xi_{(j)} + \epsilon/(2M)$. Then $v(\sum_{j=1}^{M} \mathbb{1}_{[x_{j-}, x_{j+}]} ) = -ce + \frac{1}{N} \sum_{i=1}^{N} m_i$. Let $\epsilon \to 0$ we obtain (63). So in the sequel we assume $\theta > 0$.

Observe that
\[
\inf_{\mu \in \mathcal{M}} \mathbb{E}_{\eta \sim \mu} [\eta(x^{-1}(1))] = \inf_{\mu \in \mathcal{P}(\Xi^2)} \left\{ \mathbb{E}_{\eta \sim \mu} [\eta(x^{-1}(1))] : \min_{\gamma \in \Gamma(\mu, \nu)} \mathbb{E}_\gamma [d(\eta, \hat{\eta})] \leq \theta \right\} 
= \inf_{\gamma \in \mathcal{P}(\Xi^2)} \left\{ \mathbb{E}_{\eta \sim \gamma} [\eta(x^{-1}(1))] : \mathbb{E}_\gamma [d(\eta, \hat{\eta})] \leq \theta, \pi_{\hat{\eta}} = \nu \right\}. \tag{75}
\]

For any $\gamma \in \mathcal{P}(\Xi^2)$, denote by $\gamma_{\hat{\eta}}$ the conditional distribution of $\bar{\theta} := d(\eta, \hat{\eta})$ given $\hat{\eta}$, and by $\gamma_{\hat{\eta}, \bar{\theta}}$ the conditional distribution of $\eta$ given $\hat{\eta}$ and $\bar{\theta}$. Using tower property of conditional probability, we have that for any $\gamma \in \mathcal{P}(\Xi^2)$ with $\pi_{\hat{\eta}} = \nu$,
\[
\mathbb{E}_{\eta \sim \gamma} [\eta(x^{-1}(1))] = \mathbb{E}_{\hat{\eta} \sim \nu} \left[ \mathbb{E}_{\hat{\eta} \sim \gamma_{\hat{\eta}}} \left[ \mathbb{E}_{\eta \sim \gamma_{\hat{\eta}, \bar{\theta}}} [\eta(x^{-1}(1))] \right] \right],
\]
and
\[
\mathbb{E}_\gamma [d(\eta, \hat{\eta})] = \mathbb{E}_{\hat{\eta} \sim \nu} \left[ \mathbb{E}_{\eta \sim \gamma_{\hat{\eta}}} [\bar{\theta}] \right].
\]

Note that the right-hand side of the equation above does not depend on $\gamma_{\hat{\eta}, \bar{\theta}}$. Thereby (75) can be reformulated as
\[
\inf_{\mu \in \mathcal{M}} \mathbb{E}_{\eta \sim \mu} [\eta(x^{-1}(1))] = \inf_{\gamma_{\hat{\eta}}} \left\{ \mathbb{E}_{\eta \sim \gamma_{\hat{\eta}}} \left[ \mathbb{E}_{\hat{\eta} \sim \gamma_{\hat{\eta}}} \left[ \mathbb{E}_{\eta \sim \gamma_{\hat{\eta}, \bar{\theta}}} [\eta(x^{-1}(1))] \right] \right] : \mathbb{E}_{\hat{\eta} \sim \nu} \left[ \mathbb{E}_{\eta \sim \gamma_{\hat{\eta}}} [\bar{\theta}] \right] \leq \theta \right\} 
= \inf_{\gamma_{\hat{\eta}}} \left\{ \mathbb{E}_{\eta \sim \gamma_{\hat{\eta}}} \left[ \inf_{\gamma_{\hat{\eta}, \bar{\theta}}} \mathbb{E}_{\eta \sim \gamma_{\hat{\eta}, \bar{\theta}}} [\eta(x^{-1}(1))] \right] : \mathbb{E}_{\hat{\eta} \sim \nu} \left[ \mathbb{E}_{\eta \sim \gamma_{\hat{\eta}}} [\bar{\theta}] \right] \leq \theta \right\}, \tag{76}
\]
where the second equality follows from interchangeability principle (cf. Theorem 14.60 in Rockafellar and Wets [39]).

\[
\inf_{\gamma_{\hat{\eta}}} \left\{ \mathbb{E}_{\eta \sim \gamma_{\hat{\eta}}} \left[ \inf_{\gamma_{\hat{\eta}, \bar{\theta}}} \mathbb{E}_{\eta \sim \gamma_{\hat{\eta}, \bar{\theta}}} [\eta(x^{-1}(1))] \right] : \mathbb{E}_{\hat{\eta} \sim \nu} \left[ \mathbb{E}_{\eta \sim \gamma_{\hat{\eta}}} [\bar{\theta}] \right] \leq \theta \right\} = 0.
\]

We claim that
\[
\inf_{\gamma_{\hat{\eta}}} \left\{ \mathbb{E}_{\eta \sim \gamma_{\hat{\eta}}} \left[ \inf_{\gamma_{\hat{\eta}, \bar{\theta}}} \mathbb{E}_{\eta \sim \gamma_{\hat{\eta}, \bar{\theta}}} [\eta(x^{-1}(1))] \right] : \mathbb{E}_{\hat{\eta} \sim \nu} \left[ \mathbb{E}_{\eta \sim \gamma_{\hat{\eta}}} [\bar{\theta}] \right] \leq \theta \right\} = \inf_{\rho \in \mathcal{P}([0,1] \times \Xi)} \left\{ \mathbb{E}_{\eta \sim \rho} \left[ \int_{[0,1]} (\eta(x^{-1}(1))) \right] : \mathbb{E}_{\eta \sim \rho} [W_1(\eta, \hat{\eta})] \leq \theta, \pi_{\hat{\eta}} = \nu \right\}. \tag{77}
\]

Indeed, let $\rho$ be any feasible solution of the right-hand side of (77). We denote by $\rho_{\hat{\eta}}$ the conditional distribution of $\bar{\theta} := W_1(\hat{\eta}, \hat{\eta})$ given $\hat{\eta}$ and by $\rho_{\hat{\eta}, \bar{\theta}}$ the conditional distribution of $\eta$ given $\hat{\eta}$ and $\bar{\theta}$. When $\hat{\eta} = 0$ or $\bar{\theta} = 0$, set $\gamma_{\hat{\eta}} = \delta_0$ and $\gamma_{\hat{\eta}, \bar{\theta}} = \bar{\eta} = \hat{\eta}$, that is, we choose $\gamma_{\hat{\eta}}$ and $\gamma_{\hat{\eta}, \bar{\theta}}$ be such that $\eta = \hat{\eta}$. When $\hat{\eta} \neq 0$ and $\bar{\theta} > 0$, applying Corollary 1 and Lemma 3 to the problem $\min_{\eta \in \mathcal{B}([0,1])} \{ \hat{\eta}(x^{-1}(1)) : W_1(\hat{\eta}, \eta) \leq \bar{\theta} \}$, we have that for any $\epsilon > 0$, there exists $1 \leq i_0 \leq \hat{\eta}([0,1]), p_{\hat{\eta}, \bar{\theta}} \in [0,1]$, and
\[
\hat{\eta} = \sum_{i=1}^{\hat{\eta}([0,1])} \delta_{x_i} + p_{\hat{\eta}, \bar{\theta}} \delta_{x_{i_0}} + (1 - p_{\hat{\eta}, \bar{\theta}}) \delta_{x_{i_0}},
\]
where \( \xi_i \in [0, 1] \) for all \( i \neq i_0 \) and \( \xi^{\pm}_{i_0} \in [0, 1] \), such that \( \eta(x^{-1}(1)) \leq \epsilon + \min_{\tilde{\eta} \in \mathcal{B}([0, 1])} \{ \eta(\text{int}(x^{-1}(1))) : W_1(\tilde{\eta}, \eta) \leq \tilde{\theta} \} \). Define

\[
\eta^+_{\tilde{\eta}, \tilde{\theta}} := \sum_{i=1}^{\tilde{\eta}(\{0, 1\})} \delta_{\xi_i} + \delta_{\xi^{\pm}_{i_0}},
\]

it follows that \( \eta^+_{\tilde{\eta}, \tilde{\theta}}([0, 1]) = \tilde{\eta}([0, 1]) \), and

\[
p_{\tilde{\eta}, \tilde{\theta}} \eta^+_{\tilde{\eta}, \tilde{\theta}}(x^{-1}(1)) + (1 - p_{\tilde{\eta}, \tilde{\theta}}) \eta^-_{\tilde{\eta}, \tilde{\theta}}(x^{-1}(1)) \leq \epsilon + \min_{\tilde{\eta} \in \mathcal{B}([0, 1])} \{ \eta(\text{int}(x^{-1}(1))) : W_1(\tilde{\eta}, \eta) \leq \tilde{\theta} \}, \tag{78}
\]

and

\[
p_{\tilde{\eta}, \tilde{\theta}} W_1(\eta^+_{\tilde{\eta}, \tilde{\theta}}, \hat{\eta}) + (1 - p_{\tilde{\eta}, \tilde{\theta}}) W_1(\eta^-_{\tilde{\eta}, \tilde{\theta}}, \hat{\eta}) \leq \tilde{\theta}. \tag{79}
\]

Define \( \tilde{\gamma}_{\tilde{\eta}} \) and \( \tilde{\gamma}_{\tilde{\eta}, \tilde{\theta}} \) by

\[
\tilde{\gamma}_{\tilde{\eta}}(C) := \int_{0}^{\infty} \left[ p_{\tilde{\eta}, \tilde{\theta}} I \{ W_1(\eta^+_{\tilde{\eta}, \tilde{\theta}}, \tilde{\eta}) \in C \} + (1 - p_{\tilde{\eta}, \tilde{\theta}}) I \{ W_1(\eta^-_{\tilde{\eta}, \tilde{\theta}}, \tilde{\eta}) \in C \} \right] \rho_{\tilde{\eta}}(d\tilde{\theta}), \quad \forall \text{ Borel set } C \subset [0, \infty),
\]

and

\[
\tilde{\gamma}_{\tilde{\eta}, \tilde{\theta}}(A) := \int_{0}^{\infty} \int_{\Xi} \left[ p_{\tilde{\eta}, \tilde{\theta}} I \{ \eta^+_{\tilde{\eta}, \tilde{\theta}} \in A, W_1(\eta^+_{\tilde{\eta}, \tilde{\theta}}, \tilde{\eta}) = \tilde{\theta} \} + (1 - p_{\tilde{\eta}, \tilde{\theta}}) I \{ \eta^-_{\tilde{\eta}, \tilde{\theta}} \in A, W_1(\eta^-_{\tilde{\eta}, \tilde{\theta}}, \tilde{\eta}) = \tilde{\theta} \} \right] \rho_{\tilde{\eta}, \tilde{\theta}}(d\eta) \rho_{\tilde{\eta}}(d\tilde{\theta}), \quad \forall \text{ Borel set } A \subset \Xi.
\]

We verify that \( \{ \tilde{\gamma}_{\tilde{\eta}} \}_{\tilde{\eta} \in \mathcal{B}([0, 1])}, \{ \tilde{\gamma}_{\tilde{\eta}, \tilde{\theta}} \}_{\tilde{\eta} \in \mathcal{B}([0, 1]), \tilde{\theta} \in \Xi} \) is a feasible solution to the left-hand side of (77). By condition (59), we have \( d(\eta^+_{\tilde{\eta}}, \tilde{\eta}) = W_1(\eta^+_{\tilde{\eta}}, \tilde{\eta}) \), hence (79) implies that \( p_{\tilde{\eta}, \tilde{\theta}} d(\eta^+_{\tilde{\eta}}, \tilde{\eta}) + (1 - p_{\tilde{\eta}, \tilde{\theta}}) d(\eta^-_{\tilde{\eta}}, \tilde{\eta}) \leq \tilde{\theta} \). Then taking expectation on both sides,

\[
\mathbb{E}_{\eta \sim \nu} \left[ \mathbb{E}_{\tilde{\eta} \sim \gamma_{\tilde{\eta}}}[\theta] \right] = \int_{0}^{\infty} \int_{0}^{\infty} \left[ p_{\tilde{\eta}, \tilde{\theta}} d(\eta^+_{\tilde{\eta}}, \tilde{\eta}) + (1 - p_{\tilde{\eta}, \tilde{\theta}}) d(\eta^-_{\tilde{\eta}}, \tilde{\eta}) \right] \rho_{\tilde{\eta}}(d\tilde{\theta}) \rho_{\tilde{\eta}}(d\eta) = \mathbb{E}_{\eta \sim \nu} \left[ \mathbb{E}_{\tilde{\eta} \sim \gamma_{\tilde{\eta}}}[\theta] \right] \leq \theta,
\]

hence \( \{ \tilde{\gamma}_{\tilde{\eta}} \}_{\tilde{\eta}} \) is feasible. Similarly, taking expectation on both sides of (78), we have that \( \mathbb{E}_{\eta \sim \nu} \left[ \mathbb{E}_{\tilde{\eta} \sim \gamma_{\tilde{\eta}}}[\mathbb{E}_{\eta \sim \gamma_{\eta}}[\eta(x^{-1}(1))]] \leq \epsilon + \mathbb{E}_{\eta \sim \nu}[\eta(x^{-1}(1))]. \right. \) Let \( \epsilon \to 0 \), we obtain that

\[
\inf_{\{ \gamma_{\tilde{\eta}} \}_{\tilde{\eta}}} \left\{ \mathbb{E}_{\tilde{\eta} \sim \gamma_{\tilde{\eta}}} \left[ \mathbb{E}_{\eta \sim \gamma_{\eta}}[\eta(x^{-1}(1))] \right] : \mathbb{E}_{\tilde{\eta} \sim \gamma_{\tilde{\eta}}}[\theta] \leq \theta \right\} \leq \inf_{\rho \in \mathcal{P}(\mathcal{B}([0, 1])) \times \Xi} \left\{ \mathbb{E}_{\eta \sim \gamma_{\eta}}[\eta(x^{-1}(1))] : \mathbb{E}_{\eta \sim \gamma_{\eta}}[W_1(\hat{\eta}, \eta)] \leq \theta, \, \pi_{\tilde{\theta}}^{\pi_{\tilde{\eta}}} \theta = \nu \right\}.
\]

To show the opposite direction, observe that \( \inf_{\rho \in \mathcal{P}(\mathcal{B}([0, 1])) \times \Xi} \left\{ \mathbb{E}_{\eta \sim \gamma_{\eta}}[\eta(x^{-1}(1))] : \mathbb{E}_{\eta \sim \gamma_{\eta}}[\theta] \leq \theta \right\} \leq \inf_{\rho \in \mathcal{P}(\mathcal{B}([0, 1])) \times \Xi} \left\{ \mathbb{E}_{\eta \sim \gamma_{\eta}}[\eta(x^{-1}(1))] : \mathbb{E}_{\eta \sim \gamma_{\eta}}[\theta] \leq \theta \right\} \)

\[
= \inf_{\{ \gamma_{\tilde{\eta}} \}_{\tilde{\eta}}} \left\{ \mathbb{E}_{\tilde{\eta} \sim \gamma_{\tilde{\eta}}} \left[ \mathbb{E}_{\eta \sim \gamma_{\eta}}[\eta(x^{-1}(1))] \right] : \mathbb{E}_{\tilde{\eta} \sim \gamma_{\tilde{\eta}}}[\theta] \leq \theta \right\} \quad \text{ (80)}.
\]
Let \( \{ \gamma_0, \eta_0, \bar{\gamma}, \bar{\theta} \} \) be a feasible solution of the right-hand side of (80). Then the joint distribution \( \tilde{\rho} \in \mathcal{P}(\mathcal{B}([0,1]) \times \Xi) \) defined by

\[
\tilde{\rho}(B) := \int_{\mathbb{R}} \int_0^\infty 1_{\{ \eta, \bar{\theta} \in \pi^1(B) \}} \gamma_0(d\bar{\theta}) \nu(d\eta), \quad \forall \text{ Borel set } B \subset \mathcal{B}([0,1]) \times \Xi.
\]

is a feasible solution of the right-hand side of (77). By condition (60), we have that

\[
\inf_{\eta \in \Xi} \{ \eta(x^{-1}(1)) : d(\eta, \bar{\theta}) = \hat{\theta} \} \geq \inf_{\eta \in \mathcal{B}([0,1])} \left\{ \eta(\text{int}(x^{-1}(1))) : W_1(\bar{\eta}, \bar{\theta}) \leq \hat{\theta} \right\},
\]

and thus \( \mathbb{E}_{\tilde{\eta} \sim \gamma_0} \left[ \inf_{\eta} \{ \eta(x^{-1}(1)) : d(\eta, \bar{\theta}) = \hat{\theta} \} \right] \geq \mathbb{E}_{\tilde{\eta}, \tilde{\theta}}[\tilde{\eta}(\text{int}(x^{-1}(1)))]. \) Therefore we prove the opposite direction and (77) holds. Together with (76), we obtain that

\[
\inf_{\rho \in \mathcal{S}} \mathbb{E}_{\eta \sim \mu} \eta(x^{-1}(1)) = \inf_{\rho \in \mathcal{S}} \left\{ \mathbb{E}_{\tilde{\eta}, \tilde{\theta}}[\tilde{\eta}(\text{int}(x^{-1}(1)))] : W_1(\tilde{\eta}, \tilde{\theta}) \leq \theta, \ \pi_{\tilde{\eta}}^\top \rho = \nu \right\}.
\]

It then follows that it suffices to only consider policy \( x \) such that \( x^{-1}(1) \) is an open set. Then by strong duality (Theorem 1), the problem \( \min_{\bar{\theta} \in \mathcal{B}([0,1])} \left\{ \tilde{\eta}(\text{int}(x^{-1}(1))) : W_1(\bar{\eta}, \bar{\theta}) \leq \hat{\theta} \right\} \) admits a worst-case distribution \( \tilde{\eta}, \tilde{\theta} \) and let \( \lambda, \theta \) be the associated dual optimizer. Recall \( \tilde{X} = \{ \xi_i : i = 1, \ldots, N, t = 1, \ldots, m_i \} \). We claim that it suffices to further restrict attention to those policies \( x \) such that each connected component of \( x^{-1}(1) \) contains at least one point in \( \tilde{X} \). Indeed, suppose there exists a connected component \( C_0 \) of \( x^{-1}(1) \) such that \( C_0 \cap \tilde{X} = \emptyset \). Then for every \( \zeta \in \text{supp} \tilde{\eta} \), recall that

\[
T_{\pm}(\lambda, \theta, \zeta) = \min_{\xi \in [0,1]} \left[ I_{x^{-1}(1)}(\xi) + |\xi - \zeta| \right], \quad \text{so } T_{\pm}(\lambda, \theta, \zeta) \notin C_0, \quad \text{and thus } \tilde{\eta}(x^{-1}(1)) = \tilde{\eta}(x^{-1}(1) \setminus C_0).
\]

Hence, in view of (61), \( x' := \{ x^{-1}(1) \setminus C_0 \} \) achieves a higher objective value \( v(x') \) than \( v(x) \) and thus \( x \) cannot be optimal. We finally conclude that there exists \( \{ x_j \}_{j=1}^M \), where \( M \leq \text{card}(\tilde{X}) \), such that

\[
v(x) = -c \sum_{j=1}^M (x_{j+} - x_{j-}) + \inf_{\mu \in \mathcal{S}} \mathbb{E}_{\eta \sim \mu} \eta \{ \bigcup_{j=1}^M [x_{j-}, x_{j+}] \}.
\]

\[\square\]

**Appendix B: Selecting radius \( \theta \)** We mainly use a classical result on Wasserstein distance from Bolley et al. [8]. Let \( \nu_N \) be the empirical distribution of \( \xi \) obtained from the underlying distribution \( \nu_0 \). In Theorem 1.1 (see also Remark 1.4) of Bolley et al. [8], it is shown that \( \mathbb{P} \{ W_1(\nu_N, \nu_0) > \theta \} \leq C(\theta) e^{-\frac{\lambda^2}{2} N \theta^2} \) for some constant \( \lambda \) dependent on \( \xi \), and \( C(\theta) \) dependent on \( \theta \). Since their result holds for general distributions, we here simply it for our purpose and explicitly compute the constants \( \lambda \) and \( C \). For a more detailed analysis, we refer the reader to Section 2.1 in Bolley et al. [8].

Noticing that by assumption \( \xi \in [0, \bar{B}] \), the truncation step in their proof is no longer needed, thus the probability bound (2.12) (see also (2.15)) is reduced to

\[
\mathbb{P} \{ W_1(\nu_N, \nu_0) > \theta \} \leq \max \left( 8 e \frac{\bar{B}}{\delta}, 1 \right) \left[ N(\frac{1}{4}) \right] \frac{\mathcal{N}(\frac{1}{4})}{\mathcal{N}(\delta - \delta)} e^{-\frac{\lambda^2}{2} N(\theta - \delta)^2}
\]

for some constant \( \lambda > 0, \delta \in (0, \theta) \), where \( e \) is the natural logarithm, and \( \mathcal{N}(\frac{1}{4}) \) is the minimal number of balls need to cover the support of \( \xi \) by balls of radius \( \delta / 2 \) and in our case, \( \mathcal{N}(\frac{1}{4}) = \bar{B} / \delta \). Now let us compute \( \lambda \). By Theorem 1.1, \( \lambda \) is the constant appeared in the Talagrand inequality

\[
W_1(\mu, \nu_0) \leq \sqrt{\frac{2}{\lambda} I_{\phi}(\mu, \nu_0)},
\]
where the Kullback-Leibler divergence of $\mu$ with respect to $\nu$ is defined by $I_{kl}(\mu, \nu) = +\infty$ if $\mu$ is not absolutely continuous with respect to $\nu$, otherwise $I_{kl}(\mu, \nu) = \int f \log f \, d\nu$, where $f$ is the Radon-Nikodym derivative $d\mu/d\nu$. Corollary 4 in Bolley and Villani [9] shows that $\lambda$ can be chosen as

$$\lambda = \left[ \inf_{\zeta_0 \in \Xi, \alpha > 0} \frac{1}{\alpha} \left( 1 + \log \int e^{\alpha d^2(\xi, \zeta_0)} \nu(d\xi) \right) \right]^{-1},$$

which can be estimated from data. Finally, we obtain a concentration inequality

$$\Pr\{W_1(\nu_N, \nu_0) > \theta\} \leq \max \left( 8e^{-\frac{\bar{B}}{\delta}}, 1 \right) \frac{\theta}{\delta} e^{-\frac{\delta}{8} N(\theta - \delta)^2}. \quad (81)$$

In the numerical experiment, we choose $\delta$ to make the right-hand side of (81) as small as possible, and $\theta$ is chosen such that the right-hand side of (81) is equal to 0.05.

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**References**


