Mathematical Programms with Equilibrium Constraints: A sequential optimality condition, new constraint qualifications and algorithmic consequences. *

Alberto Ramos †

April 30, 2016

Abstract

Mathematical programs with equilibrium (or complementarity) constraints, MPECs for short, is a difficult class of constrained optimization problems. The feasible set has a very special structure and violates most of the standard constraint qualifications (CQs). Thus, the standard KKT conditions are not necessary satisfied by minimizers and the convergence assumptions of many standard methods for solving constrained optimization problems are not fulfilled. This makes it necessary, both from a theoretical and numerical point of view, to consider suitable optimality conditions, tailored CQs, and specially designed algorithms for solving MPECs. In this paper, we present a new sequential optimality condition useful for the convergence analysis for several relaxation methods for solving MPECs. We also introduce a companion CQ for M-stationary that is weaker than the recently introduced MPEC relaxed constant positive linear dependence (MPEC-RCPLD). Relations between the old and new CQs as well as the algorithmic consequences will be discussed.

Key words: mathematical programs with complementarity constraints; constraints qualifications; KKT-points; stationary points; strong stationarity; M-stationarity; C-stationarity; inexact relaxation methods; inexact regularization methods.

1 Introduction

We consider mathematical programs with complementarity (or equilibrium) constraints, MPECs for short. These are mathematical programs of the form

\[
\begin{align*}
\text{minimize} & \quad f(x) \\
\text{subject to} & \quad g_j(x) \leq 0 \quad \forall j \in \mathcal{P} := \{1, \ldots, p\} \\
& \quad h_i(x) = 0 \quad \forall i \in \mathcal{E} := \{1, \ldots, q\} \\
& \quad 0 \leq H_i(x) \perp G_i(x) \geq 0 \quad \forall i \in \mathcal{M} := \{1, \ldots, m\}
\end{align*}
\]  

(1.1)

where \( f, \ h_i, \ g_j, \ H_i, \ G_i : \mathbb{R}^n \to \mathbb{R} \) are continuously differentiable functions. The notation \( 0 \leq u \perp v \geq 0 \) for two vectors \( u \) and \( v \) in \( \mathbb{R}^n \) is a shortcut for \( u \geq 0, v \geq 0 \) and \( \langle u, v \rangle = 0 \).

MPECs form an important class of optimization problems. The MPEC has its origin in bi-level programming, [16] and appears naturally in various applications of the Stackelberg game in economic sciences. MPECs also play an important role in many others fields, such as engineering design, robotics, multilevel game and transportation science. For further details, see [16] [29] [34].

*This work was supported by CNPq (Grant 454798/2015-6).
†Instituto de Matemática Pura e Aplicada (IMPA), Estrada Dona Castorina 110, Rio de Janeiro, RJ, CEP 22460-20, Brazil e-mail: aramos@impa.br.
Mathematical Programms with Equilibrium Constraints

MPECs are known to be difficult constrained optimization problems. The main problem, both from a theoretical and a numerical point of view, comes from the complementarity constraints. In fact, many of the standard CQs as the linear independence CQ (LICQ) and the Mangasarian Fromovitz CQ (MFCQ) are violated at any feasible point. The only known standard CQ applicable in the context of MPECs is the Guignard CQ, see [19], which is the weakest CQ for nonlinear mathematical programming problems; see [20]. In the absence of CQs, the Karush-Kuhn-Tucker (KKT) conditions may not hold at minimizer (even in the case, where all constraint functions are linear) and the convergence assumptions for mostly all standard methods for the solution of constrained optimization problems are not satisfied.

For this reason, several notion of stationarity designed for MPECs have emerged over the years. For instance, we have the strong stationarity and the Clarke-stationarity (C-stationarity) introduced in [37]. It is known that the notion of strong stationarity is equivalent to the KKT conditions of the MPEC (1.1), e.g. [19]. As consequence, the strong stationarity is a necessary optimality condition only under strong assumptions. C-stationarity, by the other hand, is weaker than strong stationarity and a necessary optimality condition under very mild conditions. Using Mordukhovich’s limiting calculus, [31], another stationarity concept emerged, called M-stationarity, see for instance [32, 33, 41]. M-stationarity is a stronger than C-stationarity and it can be shown that it is also a necessary optimality condition under the same assumptions as C-stationarity. Others stationary concepts in the literature as A-stationary [17] and Bouligand-stationarity (B-stationary) [29, 30] will be not discussed in this paper.

In order to ensure that a local minimizer of the MPEC (1.1) is stationary in one of the above senses, we need conditions on the analytic representation of the feasible set, the so-called CQs. In view of many standard CQs do not hold for MPECs, a whole zoo of tailored MPEC-CQs has been developed over the years, most of them analogues of standard CQs for nonlinear mathematical programs, c.f. [10, 28, 26, 23, 12]. In the other hand, since standard CQs fail, we may search for strong optimality conditions valid independently of any CQ. For nonlinear mathematical problems, for short NLPs, a useful concept is the notion of sequential optimality conditions, [2, 4, 7, 10]. Sequential optimality conditions are genuine optimality conditions, independent of the fulfillment of any CQ. There are strong optimality conditions in the sense that they imply the KKT conditions, under weaker CQs and more important, they provide a theoretical tools to justify stopping criteria for several NLP solvers. This property makes the sequential optimality conditions a useful tool for improve the global convergence analysis of several NLP methods under weak assumptions, [7, 6].

In this paper we introduce a new sequential optimality condition suitable for MPECs, called MPEC-AKKT. We will show that MPEC-AKKT is a truly optimality condition, strong in the sense that it implies the M-stationarity under weaker assumption and under mild assumptions several relaxation methods generate sequences whose limit points satisfy that condition. We also introduce a companion CQ for M-stationarity, based on the cone continuity property (CCP) introduced in [7], called MPEC-CCP. MPEC-CCP is strictly weaker that the recently introduced MPEC-RCPLD [22] and can replace it in the global convergence analysis of certain inexact relaxation methods under mild assumptions.

The paper is organized in the following way: Section 2 we set our standard notation, basic definitions and stationarity concepts for MPECs. In Section 3 we will introduce a new sequential optimality condition and a companion CQ. We also discussed the relation of this sequential optimality condition with the standard sequential optimality condition for NLPs, the AKKT condition, [2, 31]. Relationship between old and new CQs for M-stationarity will be discussed in the Section 4. We will pay special attention for the MPEC-RCPLD, MPEC-Abadie CQ and MPEC-quasinormality (see the Section 3 for definitions). Finally in the Section 5 we will use the MPEC-AKKT to improve the convergence analysis for several relaxation schemes under standard assumptions. Conclusions are given in Section 6.
2 Preliminaries and basic assumptions

We first give the notation that will be used in the paper. Our notation is standard in optimization and variational analysis; cf. [36, 11]. \( \mathbb{R}^n \) stands for the \( n \)-dimensional real Euclidean space, \( n \in \mathbb{N} \). \( \mathbb{R}_+ \) the set of positive scalars, \( \mathbb{R}_- \) the set of negative numbers, \( a^+ = \max \{0, a\} \), the positive part of \( a \in \mathbb{R} \). We use \( \langle \cdot, \cdot \rangle \) to denote the Euclidean inner product, \( || \cdot || \) the associated norm. For every \( a \in \mathbb{R}^n \), the support of \( a \) is given by \( \text{supp}(a) := \{ i : a_i \neq 0 \} \). Given a set-valued mapping (multifunction) \( \Gamma : \mathbb{R}^n \rightrightarrows \mathbb{R}^d \), the sequential Painlevé-Kuratowski outer limit of \( \Gamma(z) \) as \( z \to z^* \) is denoted by

\[
\limsup_{z \to z^*} \Gamma(z) := \{ w^* \in \mathbb{R}^d : \exists \ (z^k, w^k) \to (z^*, w^*) \text{ with } w^k \in \Gamma(z^k) \}. \tag{2.1}
\]

We say that \( \Gamma \) is outer semicontinuous (osc) at \( z^* \) if

\[
\limsup_{z \to z^*} \Gamma(z) \subset \Gamma(z^*). \tag{2.2}
\]

The sequential Painlevé-Kuratowski inner limit of \( \Gamma(z) \) as \( z \to z^* \) is denoted by

\[
\liminf_{z \to z^*} \Gamma(z) := \{ w^* \in \mathbb{R}^d : \forall z^k \to z^*, \ \exists w^k \to w^* \text{ with } w^k \in \Gamma(z^k) \}. \tag{2.3}
\]

We say that \( \Gamma \) is inner semicontinuous (isc) at \( z^* \) if

\[
\Gamma(z^*) \subset \liminf_{z \to z^*} \Gamma(z). \tag{2.4}
\]

Given the set \( X \subset \mathbb{R}^n \), the symbol \( z \xrightarrow{X} z^* \) means that \( z \to z^* \) with \( z \in X \). We denote by \( \text{cl} X \) the closure of \( X \), and for \( \text{conv} X \) the convex hull of \( X \). For a cone \( K \subset \mathbb{R}^n \), its polar (negative dual) is \( K^\circ := \{ v \in \mathbb{R}^n | \langle v, k \rangle \leq 0 \text{ for all } k \in K \} \). In this case, we always have \( K^{\circ\circ} = \text{cl} \text{ conv} K \). If in addition, we assume that \( K \) is a closed convex cone, we have \( K^{\circ\circ} = K \). The notation \( o(t) \) means any real function \( \phi(t) \) such that \( \limsup_{t \to 0} t^{-1} \phi(t) = 0 \) and \( O(t) \) stands for any function \( \phi(t) \) such that \( \phi(t) \leq Mt \) for some scalar \( M > 0 \). For a given \( X \subset \mathbb{R}^n \) and \( z^* \in X \), define the (Bouligand-Severi) tangent/contingent cone to \( X \) at \( z^* \) by

\[
T_X(z^*) := \limsup_{t \downarrow 0} \frac{X - z^*}{t} = \{ d \in \mathbb{R}^n : \exists t_k \downarrow 0, d_k \to d \text{ with } z^* + t_k d_k \in X \}. \tag{2.5}
\]

The (Fréchet) regular normal cone to \( X \) at \( z^* \in X \) is

\[
\hat{N}_X(z^*) := \{ w \in \mathbb{R}^n : \langle w, z - z^* \rangle \leq o(\|z - z^*\|) \text{ for } z \in X \}. \tag{2.6}
\]

The (Mordukhovich) limiting normal cone to \( X \) at \( z^* \in X \) is defined by

\[
N_X(z^*) := \limsup_{z \rightharpoonup z^*} \hat{N}_X(z). \tag{2.7}
\]

The Clarke’s normal cone to \( X \) at \( z^* \in X \) is defined by

\[
N^C_X(z^*) := \text{cl conv} N_X(z^*). \tag{2.8}
\]

We always have \( \hat{N}_X(z) \subset N_X(z) \subset N^C_X(z) \) for all \( z \in X \). This inclusion may be strict, see for instance, Proposition [2.1]. When \( X \) is a closed convex set, all these normal cones coincide, [34].

In order to describe geometrically the complementary constraints, we define

\[
C := \{ (c_1, c_2) \in \mathbb{R}^2 : 0 \leq -c_1 \perp -c_2 \geq 0 \}. \tag{2.9}
\]

Observe that the complementary constraints, \( 0 \leq H_i(x) \perp G_i(x) \geq 0 \ \forall i \in M = \{1, \ldots, m\} \) is equivalent to say that \( (\{-H_1(x), -G_1(x)\}, \ldots, \{-H_m(x), -G_m(x)\}) \in C^m \). The minus signs are used only for convenience of our analysis. Straightforward calculations show that we can actually compute the tangent cone as well as the regular normal cone and the limiting normal cone, c.f. [18, 40].
Proposition 2.1. For every $c = (c_1, c_2) \in C$, we have

- The tangent cone

$$T_C((c_1, c_2)) := \left\{ d = (d_1, d_2) \in \mathbb{R}^2 : \begin{array}{ll}
d_1 = 0, d_2 \in \mathbb{R} & \text{if } c_1 = 0, c_2 < 0 \\
d_1 \in \mathbb{R}, d_2 = 0 & \text{if } c_1 < 0, c_2 = 0 \\
(d_1, d_2) \in C & \text{if } c_1 = 0, c_2 = 0
\end{array} \right\}; \quad (2.10)$$

- The regular normal cone

$$\tilde{N}_C((c_1, c_2)) := \left\{ d = (d_1, d_2) \in \mathbb{R}^2 : \begin{array}{ll}
d_1 \in \mathbb{R}, d_2 = 0 & \text{if } c_1 = 0, c_2 < 0 \\
d_1 = 0, d_2 \in \mathbb{R} & \text{if } c_1 < 0, c_2 = 0 \\
d_1 \geq 0, d_2 \geq 0 & \text{if } c_1 = 0, c_2 = 0
\end{array} \right\}; \quad (2.11)$$

- The limiting normal cone

$$N_C((c_1, c_2)) := \left\{ d = (d_1, d_2) \in \mathbb{R}^2 : \begin{array}{ll}
d_1 \in \mathbb{R}, d_2 = 0 & \text{if } c_1 = 0, c_2 < 0 \\
d_1 = 0, d_2 \in \mathbb{R} & \text{if } c_1 < 0, c_2 = 0 \\
\text{either } d_1d_2 < 0 \text{ or } d_1 > 0, d_2 > 0 & \text{if } c_1 = 0, c_2 = 0
\end{array} \right\}. \quad (2.12)$$

- The Clarke’s normal cone

$$N^C_C((c_1, c_2)) := \left\{ d = (d_1, d_2) \in \mathbb{R}^2 : \begin{array}{ll}
d_1 \in \mathbb{R}, d_2 = 0 & \text{if } c_1 = 0, c_2 < 0 \\
d_1 = 0, d_2 \in \mathbb{R} & \text{if } c_1 < 0, c_2 = 0 \\
d_1 \in \mathbb{R}, d_2 \in \mathbb{R} & \text{if } c_1 = 0, c_2 = 0
\end{array} \right\}. \quad (2.13)$$

The next lemma is a characterization of limiting normal cone. The proof follows [36, Proposition 6.41]

Lemma 2.2. Let $\Lambda := \mathbb{R}_+^p \times \mathbb{R}^q \times C^m$ and $z := (a, b, (c_1^1, c_2^1), \ldots, (c_1^m, c_2^m)) \in \Lambda$. We may write the limiting normal cone as

$$N_\Lambda(z) = \prod_{j=1}^p N_{\mathbb{R}_+}(a_j) \times \prod_{j=1}^p N_{(0)}(b_j) \times \prod_{i=1}^m N_C((c_1^i, c_2^i)). \quad (2.14)$$

Similar formula holds for the Clarke’s normal cone [15, Exercise 10.33]. We end this section with the following lemma, which is a variation of Carathéodory’s lemma.

Lemma 2.3. [5] Lemma 1] Suppose that $v = \sum_{i \in B} \alpha_i p_i + \sum_{j \in D} \beta_j q_j$ for some vectors $p_i, q_j \in \mathbb{R}^n$ with $i \in B, j \in D$, such that $\{p_i : i \in B\}$ is a linearly independent set and $\beta_j \neq 0$ for every $j \in D$. Then, there is a subset $D' \subset D$ and scalars $\alpha_i, \beta_j$ for all $i \in B, j \in D'$ with $\beta_j \beta_j > 0$, $j \in D'$ such that

- $v = \sum_{i \in B} \alpha_i p_i + \sum_{j \in D'} \beta_j q_j$ and $\{p_i, q_j : i \in B, j \in D'\}$ is a linearly independent set.

2.1 Mathematical programs with equilibrium constraints.

For NLPs, the KKT conditions are the most common notion of stationarity. In contrast to MPECs, several different stationarity concepts have emerged over the years. We will restrict ourselves to the most common ones. Before continue, in order to exploit the very special structure of the complementary constraints, we rewrite the MPEC problem [11] as a optimization problem with geometric constraints:

$$\text{minimize } f(x) \text{ subject to } F(x) \in \Lambda,$$  \hspace{1cm} (2.15)
where

\[ F(x) := (g(x), h(x), \Psi(x)) \quad \text{and} \quad \Psi(x) := ((-H_1(x), -G_1(x)), \ldots, (-H_m(x), -G_m(x))) \] (2.16)

with

\[ \Lambda := \mathbb{R}^p_* \times \{0\}^q \times \mathbb{C}^m \quad \text{and} \quad \mathcal{C} = \{(c_1, c_2) \in \mathbb{R}^2 : 0 \leq -c_1 \perp -c_2 \geq 0\} \] (2.17)

The minus signs are convenient for our analysis. Denote the feasible region of the optimization problem with geometric constraints (2.15) by \( \Omega := \{x \in \mathbb{R}^n : F(x) \in \Lambda\} \).

In order to proceed with the different stationary concepts, we will define some crucial index sets that will occur frequently in the subsequent analysis.

Since, we will deal with several MPECs, these index sets must be explicitly dependent of the feasible constraint sets. Consider a set of the form, \( \Lambda = \mathbb{R}^p_* \times \{0\}^q \times \mathbb{C}^m \) for some \( p, q, m \in \mathbb{N} \). Now, for every point, \( z := (a, b, -(c_1^1, c_2^1), \ldots, -(c_1^m, c_2^m)) \) in \( \Lambda = \mathbb{R}^p_* \times \{0\}^q \times \mathbb{C}^m \), we use the notation

\[
\mathcal{I}(z, \Lambda) := \{i \in \{1, \ldots, m\} : c_1^i = 0, c_2^i > 0\},
\]
\[
\mathcal{J}(z, \Lambda) := \{i \in \{1, \ldots, m\} : c_1^i = 0, c_2^i = 0\},
\]
\[
\mathcal{K}(z, \Lambda) := \{i \in \{1, \ldots, m\} : c_1^i > 0, c_2^i = 0\}.
\] (2.18)

When, \( \Lambda \) is clear for the context, we write \( \mathcal{I}(z) \), \( \mathcal{J}(z) \) and \( \mathcal{K}(z) \) instead of \( \mathcal{I}(z, \Lambda) \), \( \mathcal{J}(z, \Lambda) \) and \( \mathcal{K}(z, \Lambda) \) respectively. Given a feasible point \( x^* \in \Omega = \{x \in \mathbb{R}^n : F(x) \in \Lambda\} \), we let

\[
A(x^*, \Omega) := \{j \in \{1, \ldots, p\} : g_j(x^*) = 0\},
\]
\[
\mathcal{I}(x^*, \Omega) := \mathcal{I}(F(x^*), \Lambda) = \{i \in \{1, \ldots, m\} : H_i(x^*) = 0, G_i(x^*) > 0\},
\]
\[
\mathcal{J}(x^*, \Omega) := \mathcal{J}(F(x^*), \Lambda) = \{i \in \{1, \ldots, m\} : H_i(x^*) = 0, G_i(x^*) = 0\},
\]
\[
\mathcal{K}(x^*, \Omega) := \mathcal{K}(F(x^*), \Lambda) = \{i \in \{1, \ldots, m\} : H_i(x^*) > 0, G_i(x^*) = 0\}.
\] (2.19)

Similarly, when \( \Omega \) is clear in the context, we write \( A(x^*), \mathcal{I}(x^*), \mathcal{J}(x^*) \) and \( \mathcal{K}(x^*) \) instead of \( A(x^*, \Omega), \mathcal{I}(x^*, \Omega), \mathcal{J}(x^*, \Omega) \) and \( \mathcal{K}(x^*, \Omega) \) respectively. There is no risk of confusion, between \( \mathcal{I}(x) \) and \( \mathcal{I}(z) \) since we reserve the letter \( z \) for elements of \( \Lambda \). The same considerations for the other index sets.

The index set \( A(x^*) \) is set of active inequalities and the index sets \( \mathcal{I}(x^*), \mathcal{J}(x^*) \) and \( \mathcal{K}(x^*) \) form a partition for every feasible point \( x^* \in \Omega \), i.e. for every feasible point \( x^* \in \Omega \), the sets \( \mathcal{I}(x^*), \mathcal{J}(x^*) \) and \( \mathcal{K}(x^*) \) are disjoint whose union is \( \{1, \ldots, m\} \). The set \( \mathcal{J}(x^*) \) is called the bi-active set.

Now, we are now able to define the next stationarity concepts for MPECs.

**Definition 2.1.** Let \( x^* \) be a feasible point for the MPEC (1.1). Suppose that there are multipliers \( \mu \in \mathbb{R}^p_* \), \( \lambda \in \mathbb{R}^q \), \( u \in \mathbb{R}^m \) and \( v \in \mathbb{R}^m \) with \( \text{supp}(\mu) \subset A(x^*) \) such that

\[
\nabla f(x^*) + \sum_{j=1}^p \mu_j \nabla g_j(x^*) + \sum_{i=1}^q \lambda_i \nabla h_i(x^*) - \sum_{i=1}^m u_i \nabla H_i(x^*) - \sum_{j=1}^m v_j \nabla G_j(x^*) = 0.
\] (2.20)

Then, \( x^* \) is said to be

(a) **Strongly stationary (S-stationary)**, if \( u_i = 0, i \in \mathcal{K}(x^*), v_j = 0, i \in \mathcal{I}(x^*) \) and \( u_i \geq 0, v_i \geq 0 \) for all \( i \in \mathcal{J}(x^*) \);

(b) **M-stationary**, if \( u_i = 0, i \in \mathcal{K}(x^*), v_j = 0, i \in \mathcal{I}(x^*) \) and either \( u_i > 0, v_i > 0 \) or \( u_i v_i = 0 \) for all \( i \in \mathcal{J}(x^*) \);

(c) **C-stationary**, if \( u_i = 0, i \in \mathcal{K}(x^*), v_j = 0, i \in \mathcal{I}(x^*) \) and \( u_i v_i \geq 0 \) for all \( i \in \mathcal{J}(x^*) \);

(d) **Weakly stationary**, if \( u_i = 0, i \in \mathcal{K}(x^*) \) and \( v_j = 0, i \in \mathcal{I}(x^*) \).
The different notions differ, basically, in how the multipliers $u_i$ and $v_i$ act over the bi-active set $\mathcal{J}(x^*)$. Clearly, these stationary concepts coincide when the bi-active set is an empty set. It is also clear from the definitions that the following chain of implications holds:

$$
\text{S-stationary} \Rightarrow \text{M-stationary} \Rightarrow \text{C-stationary} \Rightarrow \text{weak stationary}.
$$

These stationary concepts can be stated in a geometric way, using the Proposition 2.1 and Lemma 2.2:

**Proposition 2.4.** Let $x^*$ be a feasible point for the MPEC (1.1). We have the following statements:

(a) **S-stationary** is equivalent to $0 \in \nabla f(x^*) + \nabla F(x^*)^\top \tilde{N}_\Lambda(F(x^*))$;

(b) **M-stationary** is equivalent to $0 \in \nabla f(x^*) + \nabla F(x^*)^\top N_\Lambda(F(x^*))$;

(c) **C-stationary** is equivalent to $0 \in \nabla f(x^*) + \nabla F(x^*)^\top \tilde{N}_C \Lambda(F(x^*))$, where $\tilde{N}_\Lambda(z)$ represents the set $N_{\mathbb{R}^p}(a) \times N_{\{0\}}(b) \times \prod_{i=1}^m (\tilde{N}_C((c_i, c_{i+1})) \cup -\tilde{N}_C((c_i, c_{i+1})))$, for $z = (a, b, c) \in \mathbb{R}^p \times \{0\}^q \times \mathbb{C}^m$;

(d) **Weakly stationary** is equivalent to $0 \in \nabla f(x^*) + \nabla F(x^*)^\top N^C_\Lambda(F(x^*))$.

## 3 Sequential optimality condition and a new CQ

This section is devoted to the study of sequential optimality conditions suitable for MPECs. We will introduce a new sequential optimality condition called MPEC-AKKT. Relations with the AKKT condition (a standard sequential optimality condition for NLPs) are discussed. We also present a new CQ for M-stationarity. The relationship with other MPEC-CQs and algorithmic consequences will be discussed in the next sections.

### 3.1 Sequential optimality conditions for geometric constraints

First, we consider the role of sequential optimality condition for NLPs. Consider the feasible set

$$
X := \{ x \in \mathbb{R}^n : g_j(x) \leq 0, j \in \{1, \ldots, p\}, h_i(x) = 0, i \in \{1, \ldots, q\} \}.
$$

Consider the NLP: minimize $f(x)$ subject to $x \in X$. Since, it is usually not possible to “solve” the NLPs exactly; most of the standard NLPs method stops when the KKT conditions are satisfied approximately, although KKT conditions are not necessary satisfied by minimizers. The Approximate KKT (AKKT) condition justifies theoretically this practice. [2, 35].
**Definition 3.1.** The Approximate-Karush-Kuhn-Tucker (AKKT) condition is said to hold at \( x^* \in \mathbb{R}^n \) if there are sequences \( \{x^k\} \subset \mathbb{R}^n \), \( \{\lambda^k\} \subset \mathbb{R}^q \), \( \{\mu^k\} \subset \mathbb{R}^p_+ \) and \( \{\varepsilon_k\} \subset \mathbb{R}_+ \), such that \( x^k \to x^* \), \( \varepsilon_k \to 0^+ \),

\[
\|h(x^k)\| \leq \varepsilon_k, \quad \|\max\{0, g(x^k)\}\| \leq \varepsilon_k, \quad (3.1)
\]

\[
\|\nabla f(x^k) + \sum_{j=1}^{p} \mu^k_j \nabla g_j(x^k) + \sum_{i=1}^{q} \lambda^k_i \nabla h_i(x^k)\| \leq \varepsilon_k, \quad (3.2)
\]

and

\[
\mu^k_j = 0 \text{ if } g_j(x^k) < -\varepsilon_k. \quad (3.3)
\]

We can re-state the AKKT condition saying that for some \( x^k \to x^* \), \( \{\lambda^k\} \subset \mathbb{R}^q \) and \( \{\mu^k\} \subset \mathbb{R}^p_+ \) with \( \text{supp}(\mu^k) \subset A(x^*) \), we have \( \nabla f(x^k) + \sum_{j \in A(x^*)} \mu^k_j \nabla g_j(x^k) + \sum_{i=1}^{q} \lambda^k_i \nabla h_i(x^k) \to 0 \).

Certainly, we can used different \( \varepsilon_k \) for the each different parts of the definition of AKKT. To keep, the notation simple, we decided to take the same \( \varepsilon_k \) for all the parts. We also note that the notion of AKKT condition is independent of this choice.

**Remark 1.** Each point \( x^k \) is called of \( \varepsilon_k \)-stationarity point. Several variants of the AKKT condition have been proposed in the literature. They basically differ in the ways that they treat the complementarity conditions. For instance, if additionally to \( (3.1), (3.2) \) and \( (3.3) \) we require \( \sum_{i=1}^{q} |\lambda^k_i h_i(x^k)| + \sum_{j=1}^{p} |\mu^k_j g_j(x^k)| \leq O(\varepsilon_k) \), we have the complementary AKKT condition (CAKKT) introduced in [8]. If instead of \( \sum_{i=1}^{q} |\lambda^k_i h_i(x^k)| + \sum_{j=1}^{p} |\mu^k_j g_j(x^k)| \leq O(\varepsilon_k) \) we only require \( \sum_{j=1}^{p} |\mu^k_j g_j(x^k)| \leq O(\varepsilon_k) \) we obtain a notion equivalent to the notion of \( \varepsilon \)-stationarity considered in [28].

The AKKT condition justifies the stopping criteria for several NLP methods. In fact, it attempts to catch a property of several NLPs solvers: NLPs solvers are devised to find a primal sequence and approximate multipliers for which the KKT residual goes to zero. The AKKT condition, as other sequential optimality conditions, shares three important properties. First, it is a true necessary optimality condition, independently of the fulfillment of any CQ [2]. Second, it is strong, in the sense that it implies necessary optimality condition as “KKT or not CQ” for weak CQ as MFCQ, RCPLD or CPG, see [3 4 7]. Third, there are many algorithms that generate sequences whose limit points satisfy it. For instance, in the case of AKKT, we have that some augmented Lagrangian methods [1] [10], some Sequential Quadratic Programming (SQP) algorithms [35], interior point methods [13] and inexact restoration methods. Those methods generate primal sequences \( \{x^k\} \) with approximate multipliers \( \{\mu^k, \lambda^k\} \) for a given error tolerance \( \{\varepsilon_k\} \) for which \( (3.1), (3.2) \) and \( (3.3) \) are fulfilled, see [4]. This makes possible improve the global converge analysis of those methods under weaker assumptions, [1] [8] [9] [10]. The sequence \( \{x^k\} \) is called an AKKT sequence and we say that these methods generate AKKT sequences.

Motivated by AKKT, we introduce a sequential optimality condition suitable for the convergence analysis of optimization problems with geometric constraints of the form (2.15).

**Definition 3.2.** We say the Approximate stationarity condition holds for the feasible point \( x^* \) for the problem (2.15), if there are sequences \( \{x^k\}, \{\lambda^k\} \) and \( \{\gamma^k\} \) such that \( x^k \to x^* \), \( z^k \to F(x^*) \), \( z^k \in A \)

\[
\nabla f(x^k) + \nabla F(x^k)^\top \gamma^k \to 0 \quad \text{and} \quad \gamma^k \in N_A(z^k). \quad (3.4)
\]

If we write \( \gamma^k \) as \( (\mu^k, \lambda^k, u^k, v^k) \in \mathbb{R}^p_+ \times \mathbb{R}^q \times \mathbb{R}^m \times \mathbb{R}^m \). Then, using the explicit form of \( N_A(z^k) \), (Lemma 2.2), the expression (3.4) can be written, for \( k \) sufficiently large, as

\[
\nabla f(x^k) + \sum_{j \in A(x^*)} \mu^k_j \nabla g_j(x^k) + \sum_{i \in E} \lambda^k_i \nabla h_i(x^k) - \sum_{i \in I(\varepsilon) \cup J(z^*)} u^k_i \nabla H_i(x^k) - \sum_{j \in K(z^*) \cup J(z^*)} v^k_j \nabla G_j(x^k) \to 0 \quad (3.5)
\]

where \( \text{supp}(\mu^k) \subset A(x^*) \), \( \text{supp}(u^k) \subset I(z^k) \cup J(z^*) \), \( \text{supp}(v^k) \subset K(z^k) \cup J(z^*) \) and either \( u^k_\ell v^k_\ell = 0 \) or \( u^k_\ell > 0, v^k_\ell > 0 \) for each \( \ell \in J(z^k) \).
Remark 1. Let \(\delta > 0\) such that \(f(x^*) \leq f(x)\) for every \(x \in \Omega \cap B(x^*, \delta)\) and \(\rho_k \uparrow \infty\). For each \(k\), consider the problem

\[
\text{minimize } f(x) + \frac{1}{2}\|x - x^* + z - F(x^*)\|^2 + \frac{1}{2}\rho_k\|F(x) - z\|^2 \text{ subject to } (x, z) \in U.
\]

where \(U := \{(x, z) \in \mathbb{R}^n \times \Lambda : \|x - x^* + z - F(x^*)\| \leq \delta\}\). Let \((x^k, z^k)\) be a global solution of this subproblem \((3.6)\), which is well defined by the compactness of \(U\) and continuity of the functions. We will show that the sequence \(\{(x^k, z^k)\}\) converges to \((x^*, F(x^*))\). In fact, due to the optimality, we have

\[
f(x^k) + \frac{1}{2}\|x^k - x^* + z^k - F(x^*)\|^2 + \frac{1}{2}\rho_k\|F(x^k) - z^k\|^2 \leq f(x^*)
\]

(3.7)

Let \((\hat{x}, \hat{z})\) be a limit point of \(\{(x^k, z^k)\}\). It follows from (3.7) that \(\|F(x^k) - z^k\| \rightarrow 0\) and so \(F(\hat{x}) = \hat{z}\). As consequence \(\hat{x}\) is a feasible point. Furthermore, by (3.7), we also have \(\|\hat{x} - x^* + \hat{z} - F(x^*)\|^2 \leq 2(f(x^*) - f(\hat{x}))\). But, since \(f(x^*) \leq f(\hat{x})\) we conclude that \(\hat{x} = x^*\) and that \(\{(x^k, z^k)\}\) converge since it has a unique limit point, namely, \((x^*, F(x^*))\).

Now, for each \(k\) sufficiently large, we have \(\|(x, z) - (x^*, F(x^*))\| < \delta\) and then, by the optimality of \((x^k, z^k)\), [36 Theorem 6.12], we obtain

\[
0 = x^k + \nabla f(x^k) + \nabla F(x^k)^T \gamma^k \text{ with } \gamma^k \in N_\Lambda(z^k),
\]

(3.8)

where \(r^k := -\nabla F(x^k)^T (F(x^*) - z^k) + (x^k - x^*)\) and \(\rho_k := \rho_k(F(x^k) - z^k) + (F(x^*) - z^k)^T\). From the continuity, we get \(r^k \rightarrow 0\). Thus, by (3.8), we have that \(x^*\) is a MPEC-AKKT point.

\[\square\]

Remark 3. The results above can be carry out for arbitrary geometrical constraints, where \(\Lambda\) is a closed set and \(F\) is a continuously differentiable function.

Remark 4. Note that the AKKT condition holds at \(x^*\) for the mathematical problem \((1.1)\) iff there is a sequence \(x^k \in \mathbb{R}^n\) with \(x^k \rightarrow x^*\) such that

\[
\nabla f(x^k) + \rho_k \nabla g_j(x^k) + \sum_{i \in A(x^k)} \lambda^i_k \nabla h_i(x^k) - \sum_{i : h_i(x^*) = 0} u^i_k \nabla H_i(x^k) - \sum_{j : G_j(x^*) = 0} v^j_k \nabla G_j(x^k) + \sum_{i=1}^m \rho^i_k \nabla (G_i(x^k)H_i(x^k)) \rightarrow 0
\]

(3.9)
for some approximate multipliers \((\mu^k, \lambda^k, u^k, v^k, \rho^k) \in \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m\) with \(\text{supp}(\mu^k) \subset A(x^*)\), \(\text{supp}(u^k) \subset \{i \in \{1, \ldots, m\} : H_i(x^*) = 0\}\) and \(\text{supp}(v^k) \subset \{i \in \{1, \ldots, m\} : G_i(x^*) = 0\}\). We observe that (3.9) and (3.5) differ in the way how we deal with the complementarity constraints.

The problem (1.1) can be viewed as an optimization problem with geometrical constraints (2.15). Thus, each local minimizer of (1.1) is, in fact, a MPEC-AKKT point. In the other hand, (1.1) can also be see as a NLP. So, a local minimizer is also an AKKT point. Furthermore, if we try to solve (1.1) by using NLP algorithms (as augmented lagrangian methods or some SQP methods) we obtain a sequence of iterates such that every feasible limit point satisfies the AKKT condition (a nontrivial optimality condition). We can consider this fact as a possible reason why, in general, NLP algorithms are not applied in MPECs. Thus, since AKKT and MPEC-AKKT are both true optimality conditions, one with clear algorithmically implications, the exact relation between MPEC-AKKT and AKKT becomes relevant.

First, we observe that MPEC-AKKT does not imply AKKT as the following example shows.

**Example 3.1.** (MPEC-AKKT does not imply AKKT) Consider in \(\mathbb{R}^2\), the function \(f(x_1, x_2) := x_1\), the point \(x^* := (0, 0)\) and the complementary constraints given by the functions \(H_1(x_1, x_2) := x_2 \exp(-x_1 x_2),\) \(G_1(x_1, x_2) := \exp(x_1 x_2)\) and \(H_2(x_1, x_2) := -x_2 \exp(x_1 x_2),\) \(G_2(x_1, x_2) := \exp(-x_1 x_2)\). Clearly, \(x^*\) is a feasible point. Now, we will show that \(x^*\) is not an AKKT point. Indeed, the AKKT condition implies that \(\nabla f(x^*_1, x^*_2) = u^*_1 \nabla H_1(x^*_1, x^*_2) - u^*_2 \nabla H_2(x^*_1, x^*_2) + \rho^*_1 \nabla (H_1 G_1)(x^*_1, x^*_2) + \rho^*_2 \nabla (H_2 G_2)(x^*_1, x^*_2) \rightarrow (0, 0)\) for some sequence \(x^k = (x^k_1, x^k_2) \rightarrow (0, 0)\) and approximate multipliers \(u^k_1, u^k_2 \geq 0, \rho^*_1, \rho^*_2 \in \mathbb{R}\). See Remark 4. Since \((H_1 G_1)(x_1, x_2) = x_2\) and \((H_2 G_2)(x_1, x_2) = -x_2\) from the AKKT condition, we must get \(1 - u^k_1 (-x_2 \exp(-x_1 x_2)) - u^k_2 (-x_2 \exp(x_1 x_2)) \rightarrow 0\) which is impossible because \(u^k_1, u^k_2 \geq 0\). A continuation, we will show that MPEC-AKKT holds at \(x^*\). Take \(x^k_1 := 0, x^k_2 := 1/k, z^k := (-0, 1, -0, 1)\).

Clearly, we have \((x^k_1, x^k_2) \rightarrow (0, 0), z^k \rightarrow F(x^*) = ((-0, 1, -0, 1)), \mathcal{I}(z^k) = \{1, 2\}, \mathcal{K}(z^k) = \emptyset\) and \(\mathcal{J}(z^k) = \emptyset\). Define \(u^k_1 := -((x^k_1)^2 \exp(-x^k_1 x^k_2) + (x^k_2)^2 \exp(x^k_1 x^k_2))^{-1}, u^k_2 := u^k_1, \) \(v^k_1 := 0\). By calculations, \(\nabla f(x^k_1, x^k_2) - u^k_1 \nabla H_1(x^k_1, x^k_2) - u^k_2 \nabla H_2(x^k_1, x^k_2) = (0, 0)\). Thus, expression (3.5) holds at \(x^*\) and hence \(x^*\) is a MPEC-AKKT point.

Under additional assumptions, MPEC-AKKT implies AKKT as the next theorem shows.

**Theorem 3.2.** Let \(x^*\) be a MPEC-AKKT point and \(\{(x^k, z^k, \gamma^k)\} \) be a MPEC-AKKT sequence associated to \(x^*\) with \(\gamma^k = (\mu^k, \lambda^k, (u^k_1, v^k_1), \ldots, (u^k_m, v^k_m))\) and \(z^k := (a^k, b^k, -c^k) \in \Lambda\). If, we assume that

1. \(u^k_j \geq 0, v^k_j \geq 0\) for all \(j \in \mathcal{J}(z^k)\) and;
2. The sequence \(\{\max\{0, \max_{j \in \mathcal{I}(z^k)}(-u^k_j/c^k_j), \max_{j \in \mathcal{K}(z^k)}(-v^k_j/c^k_j)\}\}_{k \in \mathbb{N}}\) has a subsequence bounded.

Then, \(x^*\) is an AKKT point for the problem (1.1) considered as a NLP.

**Proof.** Denote by \(\rho_k := \max\{0, \max_{j \in \mathcal{I}(z^k)}(-u^k_j/c^k_j), \max_{j \in \mathcal{K}(z^k)}(-u^k_j/c^k_j)\}\). Clearly \(\rho_k \geq 0\). Assume without loss of generality (after take an adequate subsequence) that \(\{\rho_k\}\) itself is bounded. To show, that under the boundedness of \(\{\rho_k\}\), MPEC-AKKT implies AKKT, we only focus on the complementary part of (3.5). Define

\[
\omega^k := - \sum_{i \in \mathcal{I}(z^k) \cup \mathcal{J}(z^k)} u^k_i \nabla H_i(x^k) - \sum_{j \in \mathcal{K}(z^k) \cup \mathcal{J}(z^k)} v^k_j \nabla G_j(x^k). \tag{3.10}
\]

Let us recall that the sets \(\mathcal{I}(z^k), \mathcal{K}(z^k)\) and \(\mathcal{J}(z^k)\) are a partition of \(\{1, \ldots, m\}\). Take \(\bar{u}^k_j := u^k_j + \rho_k c^k_j \geq 0, j \in \mathcal{I}(z^k),\) \(\bar{v}^k_j := v^k_j, j \in \mathcal{J}(z^k)\) and \(\bar{v}^k_j := v^k_j + \rho_k c^k_j \geq 0, j \in \mathcal{K}(z^k),\) \(\bar{v}^k_j := v^k_j, j \in \mathcal{J}(z^k)\). Note that \(\mathcal{I}(z^k) \cup \mathcal{J}(z^k) = \{i : c^k_{1i} = 0\}\) and \(\mathcal{K}(z^k) \cup \mathcal{J}(z^k) = \{j : c^k_{2j} = 0\}\). Thus, replacing into (3.10), we obtain that

\[
- \sum_{i \in \mathcal{I}(z^k) \cup \mathcal{J}(z^k)} u^k_i \nabla H_i(x^k) - \sum_{j \in \mathcal{K}(z^k) \cup \mathcal{J}(z^k)} v^k_j \nabla G_j(x^k). \tag{3.11}
\]
Let $k$ be the approximate multipliers associated with $x^*$. Let $\impliedby$ be the AKKT point, we only rest to show that $\bar{u}^k_i \geq 0$ for every $i$ such that $H_i(x^*) = 0$ and $\tilde{v}^k_i \geq 0$ for every $i$ such that $G_i(x^*) = 0$, see Remark [ Remark 4. But, this is true, since for $k$ large enough, we have that $\{i \in \{1, \ldots, m\} : c^{k}_i = 0\} \subseteq \{i \in \{1, \ldots, m\} : H_i(x^*) = 0\}$ and also we have that $\{i \in \{1, \ldots, m\} : c^{k}_i = 0\} \cap \{i \in \{1, \ldots, m\} : G_i(x^*) = 0\}$.

By the other hand, it is not true, that AKKT always implies MPEC-AKKT. See the next example.

**Example 3.2.** (AKKT does not imply MPEC-AKKT) Consider $f(x_1, x_2) := -x_2, H_1(x_1, x_2) := x_1, G_1(x_1, x_2) := x_2$ and $x^* := (0, 1)$. Clearly, $x^*$ is a feasible point. Now, we will see that AKKT holds at $x^*$. Take $x^k = (x^k_1, x^k_2) := (1/k, 1)$, $\rho_k := k, \lambda_1 := k$ and $\lambda_2 := 0$. By straightforward calculations, $\nabla f(x^k) - \lambda_1\nabla H_1(x^k) - \lambda_2\nabla G_1(x^k) + \rho_k \nabla (H_i(x^k)G_1(x^k)) = 0$. Thus, AKKT holds at $x^*$. However, MPEC-AKKT fails at $x^*$. In fact, since there is only one complementary constraint and $I(x^*) = \{1\}$, (3.5) holds iff $\nabla f(x^k) - u^k\nabla H_1(x^k) = (0, -1) - u^k(1, 0) = -(u^k, 1) \rightarrow (0, 0)$ for some $u^k \in \mathbb{R}$, which is impossible. Thus, MPEC-AKKT fails.

Under some assumptions, AKKT implies MPEC-AKKT as the next theorem shows.

**Theorem 3.3.** Let $x^*$ be an AKKT point. If there is a feasible AKKT sequence, that is, there exist an AKKT sequence $\{x^k\}$ with $x^k \in \Omega$ for all $k \in \mathbb{N}$ such that $u^k_i H_i(x^k) = 0$ and $v^k_i G_i(x^k) = 0$ hold for every $k \in \mathbb{N}$ and for every $i \in \{1, \ldots, m\}$. Then, $x^*$ is a MPEC-AKKT point.

**Proof.** Let $\{x^k\}$ be the feasible sequence with $x^k \rightarrow x^*$ satisfying the hypothesis. Let $(\mu^k, \lambda^k, u^k, v^k, \rho^k)$ be the approximate multipliers associated with $\{x^k\}$. Now, we focus on the complementarity part. Put

$$\omega^k := -\sum_{\{i : H_i(x^k) = 0\}} u^k_i \nabla H_i(x^k) - \sum_{\{i : G_i(x^k) = 0\}} v^k_i \nabla G_i(x^k) + \sum_{i=1}^{m} \rho^k_i \nabla (H_i(x^k)G_i(x^k))$$

(3.14)

where $(u^k, v^k, \mu^k) \in \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^m$ with supp$(u^k) \subset \{i : H_i(x^*) = 0\}, \text{supp}(v^k) \subset \{i : G_i(x^*) = 0\}, \langle u^k, H(x^k) \rangle = 0$ and $\langle v^k, G(x^k) \rangle = 0$. Define $z^k := F(x^k) \in G$. Thus,

$$\sum_{i=1}^{m} \rho^k_i \nabla (H_i(x^k)G_i(x^k)) = \sum_{i \in \mathcal{I}(z^k)} u^k_i G_i(x^k) \nabla H_i(x^k) + \sum_{i \in \mathcal{K}(z^k)} v^k_i H_i(x^k) \nabla G_i(x^k).$$

(3.15)

Furthermore, since $u^k_i H_i(x^k) = 0, v^k_i G_i(x^k) = 0, \forall i \in \{1, \ldots, m\}$, we obtain that $u^k_i = 0, i \in \mathcal{K}(z^k)$ and $v^k_i = 0, i \in \mathcal{I}(z^k)$. Evenmore, for $k$ sufficient large, $\mathcal{I}(z^k) = \mathcal{I}(z) \cap \{i : H_i(x^*) = 0\}$ and $\mathcal{K}(z^k) = \mathcal{K}(z) \cap \{i : G_i(x^*) = 0\}$. Then, (3.14) can be written as

$$\omega^k = -\sum_{\{i : H_i(x^*) = 0\}} \bar{u}^k_i \nabla H_i(x^k) - \sum_{\{i : G_i(x^*) = 0\}} \bar{v}^k_i \nabla G_i(x^k)$$

(3.16)
where \((\bar{u}^k, \bar{v}^k) \in \mathbb{R}^m \times \mathbb{R}^n\) with supp\((\bar{u}^k) \subset \{i : H_i(x^*) = 0\}\) and supp\((\bar{v}^k) \subset \{i : G_i(x^*) = 0\}\) are defined as follows

\[
\bar{u}_i^k := \begin{cases} 
  u_i^k - \rho_i^k G_i(x^k) & \text{if } i \in \{i : H_i(x^*) = 0\} \cap \mathcal{I}(z^k) \\
  u_i^k & \text{if } i \in \{i : H_i(x^*) = 0\} \cap \mathcal{J}(z^k)
\end{cases}
\]

(3.17)

\[
\bar{v}_i^k := \begin{cases} 
  v_i^k - \rho_i^k H_i(x^k) & \text{if } i \in \{i : G_i(x^*) = 0\} \cap \mathcal{K}(z^k) \\
  v_i^k & \text{if } i \in \{i : G_i(x^*) = 0\} \cap \mathcal{J}(z^k)
\end{cases}
\]

(3.18)

Note that \(\bar{u}_i^k \geq 0, \bar{v}_i^k \geq 0\) for \(\ell \in \mathcal{J}(z^k)\). Thus, the sequence \(\{x^k\}\) with the approximate multipliers \((\mu^k, \lambda^k, \bar{u}^k, \bar{v}^k)\) is a MPEC-AKKT sequence.

Recently Andreani et al [7] introduced a new CQ intimately related with the AKKT condition, called CCP. This CQ serves as the most accurate measure of strength of the sequential optimality condition AKKT, in fact, under CCP, every AKKT point is actually a KKT point and when CCP fails, it is possible to find an AKKT point which is not a stationary (i.e. KKT point) as the proof of the [7] Theorem 3.2 shows. Unfortunately, as others standard CQs, CCP may not hold for MPECs. So, when we try to solve MPECs problems using NLPs methods, such methods (as the augmented lagrangian methods, for instance) can generate an AKKT point, accepted as possible solution, but which is not a stationary KKT-point. Motivated by CCP, [7], we define the next MPEC-type CCP condition.

**Definition 3.3.** Let \(x^* \in \Omega\). We says that the MPEC-Cone Continuity Property (MPEC-CCP) holds if set-valued mapping \(\mathbb{R}^n \times \Lambda \ni (x, z) \mapsto \nabla F(x)^\top N_A(z)\) is outer semicontinuous at the point \((x^*, F(x^*))\), i.e.

\[
\limsup_{(x, z) \to (x^*, F(x^*))} \nabla F(x)^\top N_A(z) \subset \nabla F(x^*)^\top N_A(F(x^*)).
\]

(3.19)

The cone \(\nabla F(x)^\top N_A(z)\) can considered as a perturbation of the cone \(\nabla F(x^*)^\top N_A(F(x^*))\) around the point \(x^*\). Since, \(\limsup_{z \to z^*} \nabla F(x^*)^\top N_A(F(x^*)) = N_A(z^*)\), each element of \(\limsup_{(x, z) \to (x^*, F(x^*))} \nabla F(x)^\top N_A(z)\) can be approximate by elements in \(\nabla F(x^*)^\top \nabla A(\hat{z}^k)\) for some appropriate sequence \((x^k, \hat{z}^k) \to (x^*, F(x^*))\). Thus, we have the next lemma

**Lemma 3.4.** We always have

\[
\limsup_{(x, z) \to (x^*, F(x^*))} \nabla F(x)^\top N_A(z) = \limsup_{(x, z) \to (x^*, F(x^*))} \nabla F(x^*)^\top \nabla A(z).
\]

(3.20)

From [7] we have that CCP is equivalent to state that every AKKT point is, in fact, a KKT point. A similar result we have for MPEC-CCP. The precise statement is given in the next theorem.

**Theorem 3.5.** MPEC-CCP holds iff every MPEC-AKKT point is actually a M-stationary point.

**Proof.** Let us show first that, if MPEC-CCP holds, the sequential MPEC-AKKT condition implies the M-stationarity condition independently of the objective function. Let \(f\) be an objective function such that the sequential MPEC-AKKT condition holds at \(x^*\), then there are sequences \(\{x^k\} \to x^*, z^k \to F(x^*)\) and \(\{\gamma^k\} \in N_A(z^k)\) such that

\[
r^k := \nabla f(x^k) + \nabla F(x^k)^\top \gamma^k \to 0.
\]

(3.21)

Define \(\omega^k := \nabla F(x^k)^\top \gamma^k\), we see that \(\omega^k \in \nabla F(x^k)^\top N_A(z)\) and \(\omega^k = r^k - \nabla f(x^k)\). By the continuity of \(\nabla f(x)\) and \(r^k \to 0\), we get

\[
-\nabla f(x^*) = \lim_{(x, z) \to (x^*, F(x^*))} \nabla F(x)^\top N_A(z) \subset \nabla F(x^*)^\top N_A(F(x^*)),
\]

where the last inclusion follows from the MPEC-CCP. Therefore, \(-\nabla f(x^*)\) belongs to \(\nabla F(x^*)^\top N_A(F(x^*))\), which is equivalent, by Proposition [2.4]b, to say that \(x^*\) is a M-stationary point.
Now, let us prove that, if the sequential MPEC-AKKT condition implies the M-stationarity for every objective function, then MPEC-CCP holds. Let \( \omega^* \in \limsup_{(x,z) \to (x^*,F(x^*))} \nabla F(x)^\top N_A(z) \), by definition of outer limit, there are sequences \( \{x^k\}, \{z^k\}, \{\gamma^k\} \) and \( \{\omega_k\} \) such that \( x^k \to x^* \), \( z^k \to F(x^*) \), \( \omega_k \to \omega^* \) and \( \omega_k = \nabla F(x^k)^\top \gamma^k \) where \( \gamma^k \in N_A(z^k) \). Define \( f(x) = -\langle \omega^*, x \rangle \) for all \( x \in \mathbb{R}^n \). Then the MPEC-AKKT condition holds at \( x^* \) for this function, since \( \nabla f(x^k) + \omega_k = -\omega^* + \omega_k \to 0 \). So by hypothesis \( x^* \) is a M-stationary point, that is, \( -\nabla f(x^*) = \omega^* \in \nabla F(x^*)^\top N_A(F(x^*)) \).

We will show that MPEC-CCP is a CQ for M-stationary. For this purpose we need the following next lemmas. The first one is a characterisation of regular normal cone.

**Lemma 3.6.** [36, Theorem 6.11] Let \( \bar{x} \in \Omega \). For every \( v \in T^*_\Omega(\bar{x}) \), there exists a smooth function \( \phi \) such that \( \nabla \phi(\bar{x}) = v \) and attains its global minimum relative to \( \Omega \) uniquely at \( \bar{x} \).

**Theorem 3.7.** We always have

\[
N_\Omega(x^*) \subset \limsup_{(x,z) \to (x^*,F(x^*))} \nabla F(x)^\top N_A(z) \tag{3.23}
\]

If in addition, MPEC-CCP holds at the basis point \( x^* \), then \( N_\Omega(x^*) \subset \nabla F(x^*)^\top N_A(F(x^*)) \).

**Proof.** Let \( \omega \in N_\Omega(x^*) \), so by definition of normal cone \( (2.7) \) there are sequences \( \{x^k\} \in \Omega \), \( \{v^k\} \) such that

\[
x^k \to_k x^* \quad v^k \to_k \omega \quad \text{and} \quad v^k \in T^*_\Omega(x^k).
\]

Using the Lemma 3.6 for each \( v^k \in T^*_\Omega(x^k) \), we have a smooth function \( \phi_k \) such that \( -\nabla \phi_k(x^k) = v^k \) and attains its global minimum relative to \( \Omega \) uniquely at \( x^k \). Now, since MPEC-AKKT is an optimality condition, by Theorem 3.7, there are sequences \( \{x^{k,s}\}, \{z^{k,s}\}, \{v^{k,s}\} \) and \( \{\gamma^{k,s}\} \) satisfying \( x^{k,s} \to_s x^k \), \( z^{k,s} \to_s F(x^k) \), \( v^{k,s} := -\nabla \phi_k(x^{k,s}) \to_s v^k \) and

\[
- v^{k,s} + \nabla F(x^{k,s})^\top \gamma^{k,s} \to_s 0 \quad \text{with} \quad \gamma^{k,s} \in N_A(z^{k,s}). \tag{3.24}
\]

Thus, for each \( k \in \mathbb{N} \), there exists \( s(k) \) such that:

- \( \|x^k - x^{k,s(k)}\| + \|F(x^k) - z^{k,s(k)}\| < 1/2^k \);
- \( \|v^k - \nabla F(x^{k,s(k)})^\top \gamma^{k,s(k)}\| < 1/2^k \) with \( \gamma^{k,s(k)} \in N_A(z^{k,s(k)}) \).

Clearly, we have found sequences such that \( x^{k,s(k)} \to_k x* \), \( z^{k,s(k)} \to_k F(x^*) \) and \( \nabla F(x^{k,s(k)})^\top \gamma^{k,s(k)} \to_k \omega \) with \( \nabla F(x^{k,s(k)})^\top \gamma^{k,s(k)} \in \nabla F(x^{k,s(k)})^\top N_A(z^{k,s(k)}) \). Thus, \( \omega \in \limsup_{(x,z) \to (x^*,F(x^*))} \nabla F(x)^\top N_A(z) \).

If in addition, we suppose that MPEC-CCP holds at \( x^* \), we obtain \( N_\Omega(x^*) \subset \nabla F(x^*)^\top N_A(F(x^*)) \).

As the consequence of the Theorem 3.7 we get

**Corollary 3.8.** If \( x^* \) is a local minimizer and MPEC-CCP holds at \( x^* \). Then, \( x^* \) is M-stationary.

**Proof.** Let \( x^* \) be a local minimizer of \( (2.15) \) for a smooth objective function \( f \). Due to the optimality, [36, Theorem 6.12], we have that \( 0 \in \nabla f(x^*) + N_\Omega(x^*) \) but since MPEC-CCP holds at \( x^* \), we get \( 0 \in \nabla f(x^*) + \nabla F(x^*)^\top N_A(F(x^*)) \) which is equivalent to state that \( x^* \) is a M-stationary point, by Proposition 2.4 (b).

### 4 Relationship between old and new MPEC-CQs

Whenever new CQs are introduced in the literature, we must show the relationship with other CQs. In this section we show the relationship between the recently proposed MPEC-CQ and the other MPEC-CQs. We will focus on MPEC-RCPLD, MPEC-Abadie CQ and MPEC-quasinormality.
4.1 MPEC-RCPLD and MPEC-CCP

Here we will show that MPEC-RCPLD implies MPEC-CCP. Furthermore, we will prove that MPEC-CCP is strictly weaker than MPEC-RCPLD. First some notations. Given a point \( x \in \mathbb{R}^n \) and three index subsets \( I_1 \subset \{1, \ldots, q\} \), \( I_2 \subset \{1, \ldots, m\} \) and \( I_3 \subset \{1, \ldots, m\} \), following [21], we define

\[
G(x; I_1, I_2, I_3) := \{ \nabla h_i(x), \nabla H_i(x), \nabla G_j(x) : i \in I_1, i \in I_2, j \in I_3 \}.
\]

We denote by \( \text{span} \ G(x; I_1, I_2, I_3) \) the linear subspace generated by \( G(x; I_1, I_2, I_3) \). Now, we proceed with the definition of MPEC-RCPLD.

**Definition 4.1.** Let \( x^* \) be a feasible point and \( E' \subset E, I' \subset I(x^*), K' \subset K(x^*) \) be index sets such that \( G(x^*; E', I(x^*), K(x^*)) \) is a basis for span \( G(x^*; E, I(x^*), K(x^*)) \). We say that MPEC relaxed constant positive linear dependence CQ (MPEC-RCPLD) holds at \( x^* \) iff there is a \( \delta > 0 \) such that

1. \( G(x; E, I(x^*), K(x^*)) \) has the same rank for each \( x \in B(x^*, \delta) \);
2. For each \( A' \subset A(x^*) \) and \( J' \subset J(x^*) \), if there are multipliers \( \{\lambda, \mu, u, v\} \) which are not all zero with \( \mu_j \geq 0 \) for each \( j \in A' \), and either \( u_\ell v_\ell = 0 \) or \( u_\ell > 0, v_\ell > 0 \) for each \( \ell \in J(x^*) \), such that

\[
\sum_{i \in A'} \mu_i \nabla g_i(x^*) + \sum_{j \in E'} \lambda_j \nabla h_j(x^*) + \sum_{i \in I(x^*) \cup J'} u_i \nabla H_i(x^*) + \sum_{j \in K(x^*) \cup K'} v_j \nabla G_j(x^*) = 0. \quad (4.1)
\]

Then, \( \{G(x; E', I' \cup J', K' \cup K'), \nabla g_j(x) : j \in A'\} \) is linearly independent for every \( x \in B(x^*, \delta) \).

Now, we will continue with the next theorem.

**Theorem 4.1.** MPEC-RCPLD implies MPEC-CCP.

**Proof.** Let \( \omega^* \) be an element of \( \limsup_{(x^*, \cdot) \to (x^*, F(x^*))} \nabla F(x)^\top N_A(z) \). Then, by definition of outer limit, there are sequences \( \{x^k\}, \{z^k\}, \{\omega^k\} \) and \( \{\gamma^k\} \) such that \( x^k \to x^*, z^k \to F(x^*) \) and \( \omega^k \to \omega^* \) with \( \gamma^k := DF(x^k)^\top \gamma^k \) and \( \gamma^k \in N_A(z^k) \). Furthermore, by Lemma 3.4, there is no loss of generality, if we assume \( \gamma^k \in \widehat{N}_A(z^k) \).

Denote \( \gamma^k \) by \( (\mu^k, \lambda^k, (u^k_1, v^k_1), \ldots, (u^k_m, v^k_m)) \). Then, for \( k \) sufficiently large, we have

\[
\omega^k = \sum_{j \in A(x^*)} \mu^k_j \nabla g_j(x^k) + \sum_{i \in E(x^*)} \lambda^k_i \nabla h_i(x^k) - \sum_{i \in I(x^*) \cup J(x^*)} u^k_i \nabla H_i(x^k) - \sum_{j \in K(x^*) \cup K(x^*)} v^k_j \nabla G_j(x^k) \quad (4.2)
\]

where \( \mu^k \in \mathbb{R}_+, \supp(\mu^k) \subset A(x^*) \) and \( u^k_\ell \geq 0, v^k_\ell \geq 0 \) for each \( \ell \in J(x^*) \), since \( \gamma^k \in \widehat{N}_A(z^k) \). Taking an adequate subsequence, there is no loss of generality, if we assume that \( I(x^*) \subset I(x^k) \) and \( K(x^*) \subset K(z^k) \) hold for every \( k \in \mathbb{N} \). Now, we decompose each \( \omega^k \) into two parts \( \omega^k_1 \) and \( \omega^k_2 \), where

\[
\omega^k_1 := \sum_{i \in E(x^*)} \lambda^k_i \nabla h_i(x^k) - \sum_{i \in I(x^*)} u^k_i \nabla H_i(x^k) - \sum_{j \in K(x^*)} v^k_j \nabla G_j(x^k)
\]

and

\[
\omega^k_2 := \sum_{j \in A(x^*)} \mu^k_j \nabla g_j(x^k) - \sum_{i \in (I(x^*) \setminus I(x^*))) \cup (J(x^*) \setminus J(x^*))} u^k_i \nabla H_i(x^k) - \sum_{j \in (K(x^*) \setminus K(x^*))) \cup (K(x^*) \setminus K(x^*))} v^k_j \nabla G_j(x^k).
\]

Now, take index sets \( E' \subset E, I' \subset I(x^*) \) and \( K' \subset K(x^*) \) such that \( G(x^*; E', I(x^*), K(x^*)) \) is a basis of span \( G(x^*; E, I(x^*), K(x^*)) \). By MPEC-RCPLD, we get that the set \( G(x; E', I(x'), K') \) is a basis of span \( G(x; E, I(x), K(x^*)) \) for every \( x \) near \( x^* \), in particular for \( x = x^k \). Thus, we can write \( \omega^k_1 \) as

\[
\omega^k_1 = \sum_{i \in E'} \lambda^k_i \nabla h_i(x^k) - \sum_{i \in I'} u^k_i \nabla H_i(x^k) - \sum_{j \in K'} v^k_j \nabla G_j(x^k) \quad (4.3)
\]
for some $(\hat{\lambda}^k, \hat{\mu}^k, \hat{\nu}^k) \in \mathbb{R}^q \times \mathbb{R}^m \times \mathbb{R}^m$ with $\text{supp}(\hat{\lambda}^k) \subset \mathcal{E}'$, $\text{supp}(\hat{\mu}^k) \subset \mathcal{I}'$ and $\text{supp}(\hat{\nu}^k) \subset \mathcal{K}'$. Now, using Lemma 2.3 for each $k \in \mathbb{N}$, we find index subsets $A'(k) \subset A(x^*)$, $\mathcal{I}_+(k) \subset \mathcal{I}(x^*) \setminus \mathcal{I}(x^*)$, $\mathcal{K}_+(k) \subset \mathcal{K}(x^*) \setminus \mathcal{K}(x^*)$, $\mathcal{J}_H(k)$, $\mathcal{J}_G(k) \subset \mathcal{J}(x^*)$ such that

$$\omega^k = \sum_{j \in A'} \tilde{\mu}_j \nabla g_j(x^k) - \sum_{i \in \mathcal{I}_+(k) \cup \mathcal{J}_H(k)} \tilde{u}_i \nabla H_i(x^k) - \sum_{j \in \mathcal{K}_+(k) \cup \mathcal{J}_G(k)} \tilde{v}_j \nabla G_j(x^k) \tag{4.4}$$

for some multipliers $(\tilde{\mu}^k, \tilde{u}^k, \tilde{v}^k) \in \mathbb{R}^p \times \mathbb{R}^m \times \mathbb{R}^m$ with $\text{supp}(\tilde{\mu}^k) \subset A'(k)$, $\text{supp}(\tilde{u}^k) \subset \mathcal{I}_+(k) \cup \mathcal{J}_H(k)$, $\text{supp}(\tilde{v}^k) \subset \mathcal{K}_+(k) \cup \mathcal{J}_G(k)$, and $\tilde{u}_i^k, \tilde{v}_j^k \geq 0$ for all $i \in \mathcal{J}(z^k)$ such that for each $k \in \mathbb{N}$, the vectors

$$(\tilde{g}(x^k), \mathcal{I}' \cup \mathcal{I}_+(k) \cup \mathcal{J}_H(k), \mathcal{K}' \cup \mathcal{K}_+(k) \cup \mathcal{J}_G(k)) \text{ and } \{\nabla g_j(x^k) : j \in A'(k)\} \text{ are linearly independent.} \tag{4.5}$$

Since there are only a finite number of index subset, we can assume, after taken an adequate subsequence, that $A'(k), \mathcal{I}_+(k), \mathcal{K}_+(k), \mathcal{J}_H(k)$ and $\mathcal{J}_G(k)$ are all independent of $k$. Denote them by $A', \mathcal{I}_+, \mathcal{K}_+, \mathcal{J}_H$ and $\mathcal{J}_G$ respectively. Substituting (4.4) and (4.3) into (4.2), we get

$$\omega^k = \sum_{j \in A'} \tilde{\mu}_j \nabla g_j(x^k) + \sum_{i \in \mathcal{I}'} \tilde{\lambda}_i \nabla h_i(x^k) - \sum_{i \in \mathcal{I} \cup \mathcal{I}_+ \cup \mathcal{J}_H} \tilde{u}_i \nabla H_i(x^k) - \sum_{j \in \mathcal{K} \cup \mathcal{K}_+ \cup \mathcal{J}_G} \tilde{v}_j \nabla G_j(x^k) \tag{4.6}$$

where the multipliers $(\tilde{\mu}^k, \tilde{\lambda}^k, \tilde{u}^k, \tilde{v}^k) \in \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^m \times \mathbb{R}^m$ are given by

$$\begin{align*}
\tilde{\mu}_j^k &:= \tilde{\mu}_j^k (j \in A'), \\
\tilde{\lambda}_i^k &:= \tilde{\lambda}_i^k (i \in \mathcal{I}'), \\
\tilde{u}_i^k &:= \tilde{u}_i^k (i \in \mathcal{I}_+) \\
\tilde{v}_j^k &:= \tilde{v}_j^k (j \in \mathcal{K}), \\
\tilde{v}_j^k &:= \tilde{v}_j^k (j \in \mathcal{K}_+ \cup \mathcal{J}_G)) \tag{4.7}
\end{align*}$$

with $\text{supp}(\tilde{\mu}^k) \subset A'$, $\text{supp}(\tilde{\lambda}^k) \subset \mathcal{I}'$, $\text{supp}(\tilde{u}^k) \subset \mathcal{I}_+ \cup \mathcal{I}_+ \cup \mathcal{J}_H$ and $\text{supp}(\tilde{v}^k) \subset \mathcal{K} \cup \mathcal{K}_+ \cup \mathcal{J}_G$ with $\tilde{u}_i^k, \tilde{v}_j^k \geq 0$ for all $i \in \mathcal{J}(z^k)$. Furthermore, we can see that

$$\gamma^k := (\mu^k, \lambda^k, (u_1^k, v_1^k), \ldots, (u_m^k, v_m^k)) \in N_{\Lambda}(z^k). \tag{4.8}$$

Now, the sequence $\{(\tilde{\mu}^k, \tilde{\lambda}^k, \tilde{u}^k, \tilde{v}^k)\}$ has a bounded subsequence, otherwise, dividing the expression (4.6) by $M_k := \|\mu^k, \lambda^k, u^k, v^k\|$ and considering an adequate convergent subsequence of $M_k^{-1}(\tilde{\mu}^k, \tilde{\lambda}^k, \tilde{u}^k, \tilde{v}^k)$, says $\{(\mu, \lambda, u, v)\}$, we obtain that

$$\sum_{j \in A'} \tilde{\mu}_j \nabla g_j(x^*) + \sum_{i \in \mathcal{I}'} \tilde{\lambda}_i \nabla h_i(x^*) - \sum_{i \in \mathcal{I} \cup \mathcal{I}_+ \cup \mathcal{J}_H} \tilde{u}_i \nabla H_i(x^*) - \sum_{j \in \mathcal{K} \cup \mathcal{K}_+ \cup \mathcal{J}_G} \tilde{v}_j \nabla G_j(x^*) = 0 \tag{4.9}$$

where $\{(\mu, \lambda, u, v)\}$ is a non zero vector. Furthermore, we have that $\mu \geq 0$, $\text{supp}(\mu) \subset A'$ and either $\tilde{u}_i \tilde{v}_i = 0$ or $\tilde{u}_i > 0, \tilde{v}_i > 0$ for each $i \in \mathcal{J}(x^*)$ as consequence of the next inclusion

$$(\mu, \lambda, (u_1, v_1), \ldots, (u_m, v_m)) \in \limsup_{k \to \infty} N_{\Lambda}(z^k) \subset \limsup_{z \to F(x^*)} N_{\Lambda}(z) \subset N_{\Lambda}(F(x^*)), \tag{4.10}$$

which follows from the outer semicontinuity of $N_{\Lambda}$, [30 Proposition 6.6]. Now, the expressions (4.5) and (4.9) are not compatible with the MPEC-RCPLD assumption. Thus, $\{(\tilde{\mu}^k, \tilde{\lambda}^k, \tilde{u}^k, \tilde{v}^k)\}$ has a convergent subsequence. Now, assume, without loss of generality, that the sequence $\{(\tilde{\mu}^k, \tilde{\lambda}^k, \tilde{u}^k, \tilde{v}^k)\}$ itself converges to some vector $(\mu, \lambda, u, v)$. Furthermore, by (4.8) and the outer semicontinuity of the normal cone, we get that $(\mu, \lambda, (u_1, v_1), \ldots, (u_m, v_m))$ is in $N_{\Lambda}(F(x^*))$ Taking limit in (4.6) we get

$$\omega = \sum_{j \in A'} \tilde{\mu}_j \nabla g_j(x^*) + \sum_{i \in \mathcal{I}'} \tilde{\lambda}_i \nabla h_i(x^*) - \sum_{i \in \mathcal{I} \cup \mathcal{I}_+ \cup \mathcal{J}_H} \tilde{u}_i \nabla H_i(x^*) - \sum_{j \in \mathcal{K} \cup \mathcal{K}_+ \cup \mathcal{J}_G} \tilde{v}_j \nabla G_j(x^*), \tag{4.10}$$
where \( \bar{\mu}_j \in \mathbb{R}_+ \), \((j \in A')\) and either \( \bar{u}_j \bar{v}_\ell = 0 \) or \( \bar{u}_j > 0, \bar{v}_\ell > 0 \) for each \( \ell \in J(x^*) \). Moreover, \( \text{supp}(\bar{\mu}) \subset A' \), \( \text{supp}(\bar{\lambda}) \subset \mathcal{E} \), \( \text{supp}(\bar{u}) \subset \mathcal{I} \cup \mathcal{J}_{\mu} \) and \( \text{supp}(\bar{v}) \subset \mathcal{K} \cup \mathcal{K}_+ \cup \mathcal{J} \). From (4.10) and since \( \mathcal{I} \cup \mathcal{I}_+ \) and \( \mathcal{K} \cup \mathcal{K}_+ \) are disjoint index sets, we conclude that \( \omega^* \) is an element of \( \nabla F(x^*) \top N_\Lambda(F(x^*)) \). Thus, MPEC-CCP holds at \( x^* \).

MPEC-CCP is strictly weaker than MPEC-RCPLD as the next example shows.

**Example 4.1.** (MPEC-CCP does not imply MPEC-RCPLD) Consider in \( \mathbb{R}^2 \), the point \( x^* = (0, 0) \) and the constraint system with complementary constraints defined by the functions

\[
\begin{align*}
g_1(x_1, x_2) &= x_1 + x_2; \\
g_2(x_1, x_2) &= -x_1 - x_2; \\
g_3(x_1, x_2) &= x_1^2 + x_2^2; \\
0 &\leq H_1(x_1, x_2) := x_1 \perp G_1(x_1, x_2) := x_2 \geq 0.
\end{align*}
\]

In this case, we have that MPEC-RCPLD does not hold, since \( \nabla g_3(x_1, x_2) \) is positive linearly dependent at \( x^* = (0, 0) \), but not in any neighborhood of \( x^* \). By the other hand, it is not difficult to see that MPEC-CCP holds at \( x^* \). In fact, it follows from \( \nabla F(x^*) \top N_\Lambda(F(x^*)) = \mathbb{R}^2 \).

### 4.2 MPEC-CCP and MPEC-Abadie CQ

MPEC-CQs come from several different approaches and in many cases, clarifying the relations among them is difficult. Here, we show the MPEC-CCP implies the MPEC-Abadie CQ under certain assumption. Now, we continue with the definition of MPEC-Abadie CQ.

**Definition 4.2.** We say that the MPEC-Abadie CQ (MPEC-ACQ) holds at \( x^* \in \Omega \) iff \( T_\Omega(x^*) = L_\Omega(x^*) \), where \( L_\Omega(x^*) := \{ d \in \mathbb{R}^n : \nabla F(x^*) \top d \in T_\Lambda(F(x^*)) \} \).

Using the Proposition 2.1 \( L_\Omega(x^*) \) can be written as:

\[
L_\Omega(x^*) = \left\{ d \in \mathbb{R}^n : \begin{array}{l}
\quad \nabla g_j(x^*) \top d \leq 0, j \in A(x^*); \\
\quad \nabla h_j(x^*) \top d = 0, j \in \mathcal{E} \\
\quad \nabla H_j(x^*) \top d = 0, j \in \mathcal{I}(x^*); \\
\quad \nabla G_j(x^*) \top d = 0, j \in \mathcal{K}(x^*) \\
\quad 0 \leq \nabla H_j(x^*) \top d \perp \nabla G_j(x^*) \top d \geq 0, j \in J(x^*)
\end{array} \right\}.
\]

Note that always \( T_\Omega(x^*) \subset L_\Omega(x^*) \). Now, we proceed with the introduction of the set-valued mapping \( \mathbb{R}^n \times \Lambda \ni (x, z) \Rightarrow L_\Omega(x, z) \) where \( L_\Omega(x, z) \) is given by

\[
L_\Omega(x, z) := \left\{ d \in \mathbb{R}^n : \begin{array}{l}
\quad \nabla g_j(x) \top d \leq 0, j \in A(x^*); \\
\quad \nabla h_j(x) \top d = 0, j \in \mathcal{E} \\
\quad \nabla H_j(x) \top d = 0, j \in \mathcal{I}(z); \\
\quad \nabla G_j(x) \top d = 0, j \in \mathcal{K}(z) \\
\quad 0 \leq \nabla H_j(x) \top d \perp \nabla G_j(x) \top d \geq 0, j \in J(z)
\end{array} \right\}.
\]

The mapping \( L_\Omega(x, z) \) coincides \( L_\Omega(x^*) \) when \( (x, z) = (x^*, F(x^*)) \). Thus, \( L_\Omega(x, z) \) can be considered as a perturbation of \( L_\Omega(x^*) \). Now, we continue analyzing the relations between \( \nabla F(x) \top \tilde{N}_\Lambda(z) \), \( L_\Omega^2(z, x) \) and \( \nabla F(x) \top N_\Lambda(z) \). Since \( \tilde{N}_\Lambda(z) = T_\Omega(z) \), [24] Theorem 6.28(a)], a simple inspection shows that \( \nabla F(x) \top \tilde{N}_\Lambda(z) \subset L_\Omega^2(z, x) \). By the other hand, adapting the proof of [21] Theorem 3.3, we can see that \( L_\Omega^2(z, x) \subset \nabla F(x) \top N_\Lambda(z) \). We summarize these results in the next proposition.

**Proposition 4.2.** We always have \( \nabla F(x) \top \tilde{N}_\Lambda(z) \subset L_\Omega^2(z, x) \subset \nabla F(x) \top N_\Lambda(z) \), \forall \( (x, z) \in \mathbb{R}^n \times \Lambda \).

From Lemma 3.4 we see that \( \limsup_{(x, z) \to (x^*, F(x^*))} \nabla F(x) \top N_\Lambda(z) = \limsup_{(x, z) \to (x^*, F(x^*))} L_\Omega^2(z, x) \).

The next theorem can be seen as a primal version of Theorem 3.7.
Theorem 4.3. We always have
\[ \liminf_{(x,z)\to(x^*,F(x^*))} L^0_\Omega(x,z) \subset T_\Omega(x^*). \]

Proof. By Theorem 3.7, we have \( N_\Omega(x^*) \subset \limsup_{(x,z)\to(x^*,F(x^*))} \nabla F(x)^\top N_\Lambda(z). \) Then, we get
\[
[ \limsup_{(x,z)\to(x^*,F(x^*))} L^0_\Omega(x,z)]^0 = [ \limsup_{(x,z)\to(x^*,F(x^*))} \nabla F(x)^\top N_\Lambda(z)]^0 \subset N^0_\Omega(x^*) \subset T_\Omega(x^*),
\]
where the last inclusion follows from [36] Theorems 6.28(b) and 6.26. Since \( L^0_\Omega(x,z) \) is a closed convex cone, we can use the duality theorem [36] Theorem 1.1.8 to get
\[
[ \limsup_{(x,z)\to(x^*,F(x^*))} L^0_\Omega(x,z)]^0 = \liminf_{(x,z)\to(x^*,F(x^*))} L^0_\Omega(x,z).
\]
From the last equality, we obtain the desired result. \[ \square \]

Motivated by Theorem 1.3 we define the next property

Definition 4.3. Let \( x^* \) be a feasible point. We say that MPEC-Continuity of the Linearized Cone (MPEC-CLC) holds at \( x^* \) if the set-valued mapping \( L_\Omega(x,z) \) is isc at \( (x^*,F(x^*)) \), that is, \( L_\Omega(x^*) = L_\Omega(x^*,F(x^*)) \subset \liminf_{(x,z)\to(x^*,F(x^*))} L_\Omega(x,z) \).

From the inclusion \( L_\Omega(x,z) \subset \text{cl conv } L_\Omega(x,z) = L^0_\Omega(x,z) \) and the Theorem 4.3, we conclude that if MPEC-CLC holds at \( x^* \), then \( L_\Omega(x^*) \subset T_\Omega(x^*) \). Thus, MPEC-CLC implies MPEC-ACQ and as consequence MPEC-CLC is a CQ for M-stationarity. The next Theorem shows that MPEC-CLC implies MPEC-CCP.

Theorem 4.4. MPEC-CLC always implies MPEC-CCP.

Proof. Indeed, from MPEC-CLC, we get
\[
L_\Omega(x^*) \subset \liminf_{(x,z)\to(x^*,F(x^*))} L_\Omega(x,z) \subset \liminf_{(x,z)\to(x^*,F(x^*))} L^0_\Omega(x,z).
\]
Thus, \( \liminf_{(x,z)\to(x^*,F(x^*))} L^0_\Omega(x,z) \subset L^0_\Omega(x^*) \subset \nabla F(x^*)^\top N_\Lambda(F(x^*)) \), where the last inclusion follows from Proposition 4.2. Now, using [36] Theorem 1.1.8, we conclude \( \limsup_{(x,z)\to(x^*,F(x^*))} L^0_\Omega(x,z) \subset \liminf_{(x,z)\to(x^*,F(x^*))} L^0_\Omega(x,z) \). But, as consequence of the Proposition 4.2 we obtain the next equality, \( \limsup_{(x,z)\to(x^*,F(x^*))} L^0_\Omega(x,z) = \liminf_{(x,z)\to(x^*,F(x^*))} \nabla F(x)^\top N_\Lambda(z). \) So, MPEC-CCP holds at \( x^* \). \[ \square \]

Remark 5. In the absence of complementary constraints, we always have \( N_\Lambda(z) = \tilde{N}_\Lambda(z) \) for every \( z \in \Lambda \). Then, as consequence of the Proposition 4.2 we always have \( L^0_\Omega(x,z) = \nabla F(x)^\top N_\Lambda(z) \). Furthermore, \( L^0_\Omega(x,z) = L_\Omega(x,z) \). Thus, the osc of \( \nabla F(x^*)^\top N_\Lambda(z) \) is, in fact, equivalent to the isc of \( L^0_\Omega(x,z) \). So, MPEC-CLC can be seen as a primal version of MPEC-CCP.

An interesting question is the exact relation between MPEC-RCPLD and MPEC-ACQ. Several partial answers have been provided in the literature. In [11] Example 4.1, the authors showed an example where MPEC-ACQ holds but not MPEC-RCPLD. By the other hand, in [14], the authors proved that in the absence of inequality constraints and the presence of only one complementary constraint, MPEC-RCPLD always implies MPEC-ACQ, c.f. [14] Theorem 3.3. A simple observation shows that under \( L^0_\Omega(x^*) = \nabla F(x^*)^\top N_\Lambda(F(x^*)) \), MPEC-CCP (and as consequence MPEC-RCPLD) is sufficient to guarantee the validity of MPEC-ACQ.

Proposition 4.5. If \( L^0_\Omega(x^*) = \nabla F(x^*)^\top N_\Lambda(F(x^*)) \) holds. Then, MPEC-CCP implies MPEC-ACQ.

Proof. In fact, if MPEC-CCP holds at \( x^* \), we have \( N_\Omega(x^*) \subset \nabla F(x^*)^\top N_\Lambda(F(x^*)) \). Now, by assumption, this implies that \( N_\Omega(x^*) \subset L^0_\Omega(x^*) \). By polarity theorem and [36] Theorems 6.28(b) and 6.26, we have
\[
L_\Omega(x^*) \subset \text{cl conv } L_\Omega(x^*) = L^0_\Omega(x^*) \subset N^0_\Omega \subset T_\Omega(x^*).
\]
Since, \( T_\Omega(x^*) \) is always included in \( L^0_\Omega(x^*) \), MPEC-ACQ holds. \[ \square \]
As we just see, in the absence of complementary constraints, we have $L_0^*(x^*) = \nabla F(x^*) \top N_A(F(x^*))$. So, MPEC-CCP (in this case, CCP) always implies Abadie CQ. In the other hand, MPEC-ACQ is not sufficient to guarantee MPEC-CCP, as the example 4.2 will show.

### 4.3 MPEC-CCP, MPEC-quasinormality and MPEC-pseudonormality

Now, we will proceed analyzing the relationship between MPEC-CCP, MPEC quasinormality and MPEC-pseudonormality. We will show that MPEC-CCP is independent of MPEC-pseudonormality and MPEC-quasinormality. Let us recall the definition of MPEC-quasinormality, [26].

**Definition 4.4.** We say that MPEC-quasinormality holds at $x^* \in \Omega$ if whenever

$$\sum_{j \in A(x^*)} \mu_j \nabla g_j(x^*) + \sum_{i \in \ell} \lambda_i \nabla h_i(x^*) - \sum_{i \in I(x^*) \cup J(x^*)} u_i \nabla H_i(x^*) - \sum_{j \in K(x^*) \cup J(x^*)} v_j \nabla G_j(x^*) = 0 \quad (4.13)$$

for some nonzero multipliers $\{(\mu, \lambda, u, v)\}$ with $\mu \in \mathbb{R}^p_+$, \text{supp$(\mu)$} \subset A(x^*), \text{supp$(u)$} \subset I(x^*) \cup J(x^*)$, \text{supp$(v)$} \subset K(x^*) \cup J(x^*) and either $u_i v_i = 0$ or $u_i > 0, v_i > 0$ for each $\ell \in J(x^*)$, there is no sequence $x^k \to x^*$, such that, for each $k \mu_j g_j(x^k) > 0$ when $\mu_j > 0$, $\lambda_i h_i(x^k) > 0$ if $\lambda_i > 0$, $-u_i H_i(x^k) > 0$ when $u_i > 0$ and $-v_j G_j(x^k) > 0$ if $v_j > 0$.

The definition of MPEC-pseudonormality is the following one.

**Definition 4.5.** We say that MPEC-pseudonormality holds at $x^* \in \Omega$ if whenever

$$\sum_{j \in A(x^*)} \mu_j \nabla g_j(x^*) + \sum_{i \in \ell} \lambda_i \nabla h_i(x^*) - \sum_{i \in I(x^*) \cup J(x^*)} u_i \nabla H_i(x^*) - \sum_{j \in K(x^*) \cup J(x^*)} v_j \nabla G_j(x^*) = 0 \quad (4.14)$$

for some nonzero multipliers $\{(\mu, \lambda, u, v)\}$ with $\mu \in \mathbb{R}^p_+$, \text{supp$(\mu)$} \subset A(x^*), \text{supp$(u)$} \subset I(x^*) \cup J(x^*)$, \text{supp$(v)$} \subset K(x^*) \cup J(x^*) and either $u_i v_i = 0$ or $u_i > 0, v_i > 0$, $\ell \in J(x^*)$, there is no sequence $x^k \to x^*$, such that $\sum_{j=1}^p \mu_j g_j(x^k) + \sum_{i=1}^q \lambda_i h_i(x^k) - \sum_{i=1}^m u_i H_i(x^k) - \sum_{j=1}^n v_j G_j(x^k) > 0$, for every $k \in \mathbb{N}$.

From [26], we know that MPEC-pseudonormality implies MPEC-quasinormality and MPEC-ACQ. Both conditions are sufficient to guarantee that every minimizer is, actually, a M-stationary point. When the basis point $x^*$ satisfies the strict complementarity condition, (i.e. $J(x^*)$ is an empty set), then MPEC-quasinormality also implies MPEC-ACQ. See [42, Theorem 3.1]. The next examples will show that MPEC-CCP is independent of the MPEC-pseudonormality and MPEC-quasinormality.

**Example 4.2.** (MPEC-pseudonormality does not imply MPEC-CCP) In $\mathbb{R}^2$, consider $x^* := (0,0)$ and the constraint system defined by the constraints

- $g_1(x_1, x_2) = -x_1$
- $g_2(x_1, x_2) = x_1 - x_2^2 x_1^2$
- $0 \leq H_1(x_1, x_2) := x_1 \perp G_1(x_1, x_2) := 1 \geq 0$

**Proof.** First, we will show that MPEC-pseudonormality holds at the basis point $x^*$. Assume that there are some nonzero multipliers $\{(\mu_1, \mu_2, u, v)\}$ such that $\mu_1 g_1(x^*) + \mu_2 g_2(x^*) - u H_1(x^*) = 0$. Thus, $u = \mu_2 - \mu_1$ with $\mu_1 \geq 0$ and $\mu_2 \geq 0$. Take any sequence $\{x^k = (x_1^k, x_2^k)\}$ with $x^k \to x^*$ if, for every $k \in \mathbb{N}$, we have $\mu_1 g_1(x^k) + \mu_2 g_2(x^k) - u H_1(x^k) > 0$. Then, the last inequality implies that $-\mu_2 (x_2^k)^2 (x_1^k)^2 > 0$ which is impossible. Thus, MPEC-pseudonormality holds at $x^*$. Now, we will show that MPEC-CCP fails. By some calculations, we have $\nabla F(x^*) \top N_A(F(x^*)) = \mathbb{R} \times \{0\}$. Define $x_1^k := 1/k$, $x_2^k := x_1^k$, $u^k := 0$, $\mu_2 := (2x_2^k)^{-1} \mu_1 := \mu_2(1 - 2x_1^k (x_2^k)^2)$, and $x^k := (0,0,(0,-1))$. Clearly, $(x_1^k, x_2^k) \to (0,0)$, $\mu_1^k, \mu_2^k \geq 0$. Denote $u^k := \mu_1^k g_1(x^k) + \mu_2^k g_2(x^k) - u^k H_1(x^k) = (0,-1)$ for all $k \in \mathbb{N}$. Thus, $(0,-1) \in \limsup_{(x_1, x_2) \to (x_1^k, x_2^k)} \nabla F(x^*) \top N_A(z)$ but $(0,-1)$ does not belong to $\nabla F(x^*) \top N_A(F(x^*)) = \mathbb{R} \times \{0\}$. So, MPEC CCP fails.

$\square$
Since MPEC-pseudonormality implies MPEC-ACQ, the above example also shows that MPEC-ACQ also does not imply MPEC-CCP.

**Example 4.3.** (MPEC-quasinormality fails but MPEC-CCP holds) Consider in $\mathbb{R}^2$, the point $x^* := (0, 0)$ and the constraint system given by

\[
\begin{align*}
g_1(x_1, x_2) &= -x_1 + x_2^2 \exp(x_1); \\
g_2(x_1, x_2) &= x_1 \exp(x_2); \\
0 \leq H_1(x_1, x_2) &= 1 \cup G_1(x_1, x_2) := x_2 \geq 0
\end{align*}
\]

**Proof.** First, we will show that MPEC-CCP holds at $x^*$. In fact, this follows from $\nabla F(x^*)^\top N_\Lambda(F(x^*)) = \mathbb{R}^2$. Now, to see that MPEC-quasinormality fails, take for each $k \in \mathbb{N}$, $x_k^1 := 1/k$, $x_k^2 := \sqrt{2} x_k^1 \exp(-x_k^1)$ and multipliers $\mu_1 := 1$, $\mu_2 := 1$ and $v := 0$. For that choice, $\mu_1 \nabla g_1(x^*) + \mu_2 \nabla g_2(x^*) - v \nabla G_1(x^*) = 0$. Evenmore, $\mu_2 g_2(x^k) = \mu_2 x_k^2 \exp(x_k^2) > 0$ and $\mu_1 g_1(x^k) = (x_k^2)^2 \exp(x_k^1)) = \mu_1 x_k^1 > 0$. Thus, MPEC-quasinormality cannot hold at $x^*$.

Observe that the above example also shows that MPEC-CCP does not implies MPEC-pseudonormality. From these examples, we get that MPEC-CCP is independent of MPEC-pseudonormality and MPEC-quasinormality.

An important MPEC-CQ is the following one introduced in [40] under a different name, namely, MPEC-no nonzero abnormal multiplier CQ (MPEC-NNAMCQ).

**Definition 4.6.** We say that MPEC generalized MFCQ (MPEC-GMFCQ) holds at $x^* \in \Omega$ if there are no nonzero multipliers $\{\mu, \lambda, u, v\}$ with $\mu \in \mathbb{R}^p_+$, $\text{supp}(\mu) \subset A(x^*)$, $\text{supp}(u) \subset I(x^*) \cup \mathcal{J}(x^*)$, $\text{supp}(v) \subset \mathcal{K}(x^*) \cup \mathcal{J}(x^*)$ and either $u_\ell v_\ell = 0$ or $u_\ell > 0$, $v_\ell > 0$ for each $\ell \in \mathcal{J}(x^*)$ such that

\[
\sum_{j \in A(x^*)} \mu_j \nabla g_j(x^*) + \sum_{i \in \mathcal{I}} \lambda_i \nabla h_i(x^*) - \sum_{i \in \mathcal{I}(x^*) \cup \mathcal{J}(x^*)} u_i \nabla H_i(x^*) - \sum_{j \in \mathcal{K}(x^*) \cup \mathcal{J}(x^*)} v_j \nabla G_j(x^*) = 0 \tag{4.15}
\]

Observe that MPEC-GMFCQ is equivalent to state that whenever $\nabla F(x^*)^\top \gamma = 0$ with $\gamma \in N_\Lambda(F(x^*))$ we have $\gamma = 0$, where $F$ is given by (2.16). Furthermore, it is not difficult to see that MPEC-GMFCQ is weaker than the MPEC-MFCQ condition; which state that there is no nonzero multipliers $\{\mu, \lambda, u, v\}$ such that (4.15) holds with $\mu \in \mathbb{R}^p_+$, $\text{supp}(\mu) \subset A(x^*)$, $\text{supp}(u) \subset I(x^*) \cup \mathcal{J}(x^*)$ and $\text{supp}(v) \subset \mathcal{K}(x^*) \cup \mathcal{J}(x^*)$.

Under MPEC-GMFCQ (and as consequence MPEC-MFCQ), we have that for every smooth objective function $f$, the set $\{\gamma \in N_\Lambda(F(x^*)) : \nabla f(x^*) + \nabla F(x^*)^\top \gamma = 0\}$ is bounded.

The Figure 2 shows the relations among several CQs for M-stationarity involved in this paper. For further informations about other MPEC-CQs, see [26, 22, 42, 21].

### 5 Algorithmic applications of MPEC-CCP

In this section, we will show that MPEC-CCP can be used in the convergence analysis for MPECs. The objective is to show that MPEC-CCP can replace other more stringent MPEC-CQs for M-stationarity in the assumptions to ensure the convergence of several relaxation methods under standard assumptions.

Since the complementarity constraints of the MPECs is the cause of difficulties from a numerical and theoretical point of view. A number of specially designed methods have been suggested to deal with it, with convergence under suitable assumptions. Among these MPEC-tailored schemes, the relaxation methods are one of the most important class of solution methods [38, 39, 27, 25]. The idea behind of all relaxation schemes is to get rid of the complementarity constraints by replacing these complementarity constraints in a suitable way such that the corresponding relaxed problem is an NLP problem with nicer properties than the original, and, then solve the relaxed problem by standard NLP methods. Usually, the
relaxed problem depends on certain parameter \( t > 0 \) which has to be driven to zero and when \( t = 0 \) we recover the feasible set of the MPEC.

We pay special attention to the \( \ell \)-relaxation method of Kanzow and Schwartz \cite{27} and the nonsmooth relaxation of Kadrani et al. \cite{25}. Both methods have strong convergence properties. In fact, under the weak MPEC-RCPLD, if \( x^k \) is a stationary point for the relaxed problem \( R(t_k) \), for every \( k \in \mathbb{N} \), the limit point \( x^* \) is a M-stationary point, \cite{24, 28}. When \( x^k \) is an approximate stationary point for \( R(t_k) \), instead of stationary point, weaker results are obtained. In this case, the limit point \( x^* \) may be weakly stationary or C-stationary, see \cite{28}.

### 5.1 \( \ell \)-relaxation of Kanzow and Schwartz

The relaxation scheme established by Kanzow and Schwartz in \cite{27} is given by

\[
\begin{align*}
\text{minimize} & \quad f(x) \\
\text{s.t.} & \quad g_j(x) \leq 0, \; \forall j \in \mathcal{P} = \{1, \ldots, p\} \\
& \quad h_i(x) = 0, \; \forall i \in \mathcal{E} = \{1, \ldots, q\} \\
& \quad 0 \leq H_i(x), 0 \leq G_i(x), \; \forall i \in \mathcal{M} = \{1, \ldots, m\} \\
& \quad \Phi_{KS}(H_i(x), G_i(x); t) \leq 0, \; \forall i \in \mathcal{M} = \{1, \ldots, m\}
\end{align*}
\]

(5.1)

where \( \Phi_{KS}(H_i(x), G_i(x); t) \) is defined as

\[
\Phi_{KS}(H_i(x), G_i(x); t) := \begin{cases} 
(H_i(x) - t)(G_i(x) - t) & \text{if } H_i(x) + G_i(x) \geq 2t, \\
-\frac{1}{2}((H_i(x) - t)^2 + (G_i(x) - t)^2) & \text{if } H_i(x) + G_i(x) < 2t.
\end{cases}
\]

(5.2)

When it is clear of the context, we use the notation \( \Phi_{KS}^i(x; t) \) instead of \( \Phi_{KS}(H_i(x), G_i(x); t) \). For a given \( t > 0 \), we use \( \text{NLP}^{KS}(t) \) to denote the mathematical program (5.1). The feasible set of \( \text{NLP}^{KS}(t) \) is given by the Figure 3.

By straightforward calculations, we have

\[
\nabla \Phi_{KS}(H_i(x), G_i(x); t) := \begin{cases} 
(H_i(x) - t)\nabla G_i(x) + (G_i(x) - t)\nabla H_i(x) & \text{if } H_i(x) + G_i(x) \geq 2t, \\
-(G_i(x) - t)\nabla G_i(x) - (H_i(x) - t)\nabla H_i(x) & \text{if } H_i(x) + G_i(x) < 2t.
\end{cases}
\]

(5.3)
where \( \Phi^K_S(x,t) = 0 \)

We improve the main convergence result for this relaxation \([24]\) by using the weaker MPEC-CCP instead of MPEC-CPLD.

**Theorem 5.1.** Let \( \{t_k\} \downarrow 0 \) and \( x^k \) be a KKT point of \( NLP^K_S(t_k) \) with \( x^k \to x^* \) such that MPEC-CCP holds in \( x^* \). Then, \( x^* \) is an M-stationary point for the MPEC \((1.1)\).

**Proof.** We only need to show that \( x^* \) is a MPEC-AKKT point. For this, it will be sufficient to show that there is subsequence of \( \{x^k\} \) which is a MPEC-AKKT sequence. Since \( x^k \) is a stationary point for \( NLP^K_S(t_k) \), we have the KKT conditions:

\[
\nabla f(x^k) + \sum_{j=1}^p \mu_j^k \nabla g_j(x^k) + \sum_{i=1}^q \lambda_i^k \nabla h_i(x^k) - \sum_{i=1}^m u_i^k \nabla H_i(x^k) - \sum_{i=1}^m v_i^k \nabla G_i(x^k) + \sum_{i=1}^m \rho_i^k \Phi^K_S(x^k,t_k) = 0 \quad (5.4)
\]

with the constraints

\[
g_j(x^k) \leq 0 \quad \mu_j^k \geq 0 \quad \mu_j^k g_j(x^k) = 0 \quad \forall j \in \{1, \ldots, p\},
\]

\[
H_i(x^k) \geq 0 \quad u_i^k \geq 0 \quad u_i^k H_i(x^k) = 0 \quad \forall i \in \{1, \ldots, m\},
\]

\[
G_i(x^k) \geq 0 \quad v_i^k \geq 0 \quad v_i^k G_i(x^k) = 0 \quad \forall i \in \{1, \ldots, m\},
\]

\[
(H_i(x^k) - t_k)(G_i(x^k) - t_k) \leq 0 \quad \rho_i^k (H_i(x^k) - t_k)(G_i(x^k) - t_k) = 0 \quad \forall i \in \{1, \ldots, m\}. \quad (5.5)
\]

Using the gradient of \( \nabla \Phi^K_S(x^k,t_k) \), see (5.3), we obtain

\[
\nabla f(x^k) + \sum_{j=1}^p \mu_j^k \nabla g_j(x^k) + \sum_{i=1}^q \lambda_i^k \nabla h_i(x^k) - \sum_{i=1}^m \bar{u}_i^k \nabla H_i(x^k) - \sum_{i=1}^m \bar{v}_i^k \nabla G_i(x^k) = 0 \quad (5.6)
\]

where \( \bar{u}^k \) and \( \bar{v}^k \) are defined as follows

\[
\bar{u}_i^k := \begin{cases} 
  u_i^k - \rho_i^k (G_i(x^k) - t_k) & \text{if } H_i(x^k) + G_i(x^k) \geq 2t_k, \\
  u_i^k + \rho_i^k (H_i(x^k) - t_k) & \text{if } H_i(x^k) + G_i(x^k) < 2t_k.
\end{cases} \quad (5.7)
\]

\[
\bar{v}_i^k := \begin{cases} 
  v_i^k - \rho_i^k (H_i(x^k) - t_k) & \text{if } H_i(x^k) + G_i(x^k) \geq 2t_k, \\
  v_i^k + \rho_i^k (H_i(x^k) - t_k) & \text{if } H_i(x^k) + G_i(x^k) < 2t_k.
\end{cases} \quad (5.8)
\]

For each \( k \in \mathbb{N} \), define \( z^k := (g(x^*), h(x^*), -(H_1(x^*), G_1(x^*)), \ldots, -(H_m(x^*), G_m(x^*))). \) Clearly, we have \( \mathcal{I}(x^*) = \mathcal{I}(z^k), \mathcal{K}(x^*) = \mathcal{K}(z^k) \) and \( \mathcal{K}(x^*) = \mathcal{K}(z^k). \) Let us show that for \( k \) large enough, the multipliers
$\bar{u}^k$ and $\bar{v}^k$ conform the definition of MPEC-AKKT, that is, $\bar{u}^k_i = 0$ for $i \in K(z^k)$; $\bar{v}^k_i = 0$ for $i \in I(z^k)$ and either $\bar{u}^k_i \bar{v}^k_i = 0$ or $\bar{u}^k_i > 0$, $\bar{v}^k_i > 0$ for $i \in J(z^k)$.

First, we will show that $\bar{v}^k_i = 0, i \in I(z^k) = I(x^*)$ for $k$ large enough. For this purpose, we decompose the index set $I(x^*)$ into a partition of four subsets, namely: $I_1(x^*, k) := \{i \in I(x^*) : G_i(x^k) \geq t_k, H_i(x^k) \geq t_k\}$, $I_2(x^*, k) := \{i \in I(x^*) : G_i(x^k) \geq t_k, H_i(x^k) < t_k\}$, $I_3(x^*, k) := \{i \in I(x^*) : G_i(x^k) < t_k, H_i(x^k) \geq t_k\}$, $I_4(x^*, k) := \{i \in I(x^*) : G_i(x^k) < t_k, H_i(x^k) < t_k\}$. Furthermore, there is no loss of generality, possibly after taking an adequate subsequence, if we assume that each element of the partition is independent of $k$. Thus, we have the next partition

$$
\begin{align*}
I_1(x^*) &:= \{i \in I(x^*) : G_i(x^k) \geq t_k, H_i(x^k) \geq t_k\}, \\
I_2(x^*) &:= \{i \in I(x^*) : G_i(x^k) \geq t_k, H_i(x^k) < t_k\}, \\
I_3(x^*) &:= \{i \in I(x^*) : G_i(x^k) < t_k, H_i(x^k) \geq t_k\}, \\
I_4(x^*) &:= \{i \in I(x^*) : G_i(x^k) < t_k, H_i(x^k) < t_k\}.
\end{align*}
$$

Now, we proceed showing that $\bar{v}^k_i = 0, i \in I(z^k) = I(x^*)$. Since $G_i(x^k) > 0, i \in I(x^*)$, the sets $I_2(x^*)$ and $I_4(x^*)$ must be empty sets for $k$ large enough, otherwise taking limit in $G_i(x^k) < t_k$, we will get $G_i(x^k) = 0$, a contradiction. Now, take $i \in I_1(x^*)$, then $G_i(x^k) + \rho_i^k H_i(x^k) \geq 2t_k$ and from (5.9) we have $\bar{v}^k_i = u^k_i - \rho_i^k(H_i(x^k) - t_k)$. Since $G_i(x^k) > 0$, we have for $k$ large enough, that, $G_i(x^k) > t_k$. From the KKT conditions, we have $\bar{v}^k_i = 0$, and $\rho_i^k (G_i(x^k) - t_k) = 0$. Thus, we conclude that $\bar{v}^k_i = 0$ for every $i \in I_1(x^*)$. Carry out the same analysis for $i \in I_2(x^*)$, we get $\bar{v}^k_i = 0$. Therefore, we conclude that $\bar{v}^k_i = 0$ for every $i \in I(z^k) = I(x^*)$.

Now, similarly following the same arguments, we can conclude that $\bar{u}^k_i = 0$ for $i \in K(x^*)$. We continue analyzing the multipliers $\bar{u}^k_i$ and $\bar{v}^k_i$ for $i \in J(z^k)$. Following the same arguments decompose the index set $J(z^k)$ into a partition of four subsets, namely: $J_1(z^k) := \{i \in J(z^k) : G_i(x^k) \geq t_k, H_i(x^k) \geq t_k\}$, $J_2(z^k) := \{i \in J(z^k) : G_i(x^k) \geq t_k, H_i(x^k) < t_k\}$, $J_3(z^k) := \{i \in J(z^k) : G_i(x^k) < t_k, H_i(x^k) \geq t_k\}$ and $J_4(z^k) := \{i \in J(z^k) : G_i(x^k) < t_k, H_i(x^k) < t_k\}$, which we assume independent of $k$, maybe after taking an adequate subsequence. For each part, we will obtain that either $\bar{u}^k_i \bar{v}^k_i = 0$ or $\bar{u}^k_i > 0$, $\bar{v}^k_i > 0$, $i \in J(z^k) = J(x^*)$, for $k$ sufficiently large enough. We have the next cases:

- If $i \in J_1(z^k) = \{i \in J(z^k) : G_i(x^k) \geq t_k, H_i(x^k) \geq t_k\}$. Here, we see that $\bar{u}^k_i = u^k_i - \rho_i^k (G_i(x^k) - t_k)$ and $\bar{v}^k_i = u^k_i - \rho_i^k (H_i(x^k) - t_k)$. From the KKT conditions, since $t_k > 0$, we obtain that $\bar{u}^k_i = 0$ and $\bar{v}^k_i = 0$. Now, if $G_i(x^k) = t_k$ or $H_i(x^k) = t_k$, we have that $\bar{u}^k_i = 0$ or $\bar{v}^k_i = 0$ respectively. And, if $G_i(x^k) > t_k$ and $H_i(x^k) > t_k$, we get $\rho_i^k = 0$, $\bar{u}^k_i = 0$ and $\bar{v}^k_i = 0$. In both cases, we have $\bar{v}^k_i \bar{u}^k_i = 0$.

- If $i \in J_2(z^k) = \{i \in J(z^k) : G_i(x^k) \geq t_k, H_i(x^k) < t_k\}$. From the KKT conditions, we have that $\bar{v}^k_i = 0$, $\rho_i^k (G_i(x^k) - t_k) = 0$. If $H_i(x^k) + G_i(x^k) \geq 2t_k$, we have that $\bar{v}^k_i = -\rho_i^k (H_i(x^k) - t_k)$ and $\bar{u}^k_i = 0$. Now, if $H_i(x^k) + G_i(x^k) < 2t_k$, we obtain that $\bar{v}^k_i = \bar{u}^k_i = 0$. Thus, for $i \in J_2(z^k)$, we have that either $\bar{v}^k_i \bar{u}^k_i = 0$ or $\bar{u}^k_i > 0 \bar{v}^k_i > 0$.

- If $i \in J_3(z^k) = \{i \in J(z^k) : G_i(x^k) < t_k, H_i(x^k) \geq t_k\}$. By symmetry, we see that either $\bar{v}^k_i \bar{u}^k_i = 0$ or $\bar{u}^k_i > 0 \bar{v}^k_i > 0$.

- If $i \in J_4(z^k) = \{i \in J(z^k) : G_i(x^k) < t_k, H_i(x^k) < t_k\}$. From the KKT conditions, we see that $\rho_i^k = 0$. Thus, $\bar{u}^k_i = 0$ and $\bar{v}^k_i = 0$.

Thus, we conclude that $x^*$ is an MPEC-AKKT point. So, if MPEC-CCP holds at $x^*$, then $x^*$ is M-stationary point.

When we try to solve NLPs, we usually end up in an approximate KKT point, instead of a true KKT point. In fact, the stopping criteria for solving NLPs basically check whether an approximate KKT point has been found (in addition, maybe, to other criteria). So, we rarely end up in a KKT point. This fact has practical relevance. For example, the relaxation method of Kansow and Schwartz has stronger convergence.
properties: All limit points are M-stationary points, under weak CQ for M-stationary (as MPEC-CPLD, see [23, Theorem 3.3]), if each iterative is a KKT point. However, if we consider approximate KKT points instead of KKT points, we lost most of this advantage. Indeed, without additional assumptions, only convergence to C-stationary points or weakly stationary points can be obtained. From Kanzow and Schwartz [28, Theorem 13] we have the next result.

**Theorem 5.2.** Let \( t_k \downarrow 0, \, \varepsilon_k \downarrow 0, \{x^k\} \) be a sequence of \( \varepsilon_k \)-stationary points of \( \text{NLP}^{K,S}(t_k) \) with approximate multipliers \( (\mu^k, \lambda^k, \nu^k, \rho^k) \in \mathbb{R}^n_+ \times \mathbb{R}^q \times \mathbb{R}^m_+ \times \mathbb{R}^m_+ \), such that

\[
\max \{ |u^k_i H_i(x^k)|, |v^k_i G_i(x^k)|, |\mu^k_i \Phi^K_i(x^k, t_k)| : i \in \{1, \ldots, m\} \} \leq \varepsilon_k.
\]

Assume that \( x^k \to x^\ast \) with MPEC-MFCQ holding in \( x^\ast \in \Omega \). Then, \( x^\ast \) is a weak stationary point. Suppose further that \( \varepsilon_k = o(t_k) \) and there is a constant \( c > 0 \) such that, for all \( i \in J(x^\ast) \) and all \( k \) sufficiently large,

1. the iterates \((G_i(x^k), H_i(x^k))\) satisfy

\[
(G_i(x^k), H_i(x^k)) \notin \left[ ((t_k, (1 + c)t_k) \times ((1 - c)t_k, t_k)) \cup \left[ ((1 - c)t_k, t_k) \times (t_k, (1 + c)t_k) \right] \cup (t_k, (1 + c)t_k) \right]^{2} \cup \left( (t_k, (1 - c)t_k) \right) \ast (t_k, (1 + c)t_k). \tag{5.10}
\]

Then, \( x^\ast \) is a M-stationary point;

2. the iterates \((G_i(x^k), H_i(x^k))\) satisfy

\[
(G_i(x^k), H_i(x^k)) \notin \left[ ((t_k, (1 + c)t_k) \times ((1 - c)t_k, t_k)) \cup \left[ ((1 - c)t_k, t_k) \times (t_k, (1 + c)t_k) \right] \cup (t_k, (1 + c)t_k) \right]^{2} \cup \left( (t_k, (1 - c)t_k) \right) \ast (t_k, (1 + c)t_k). \tag{5.11}
\]

Then \( x^\ast \) is a C-stationary point of the MPEC.

We can improve the result under the MPEC-CPLD assumption. The precise statement is the following.

**Theorem 5.3.** Let \( t_k \downarrow 0, \, \varepsilon_k = o(t_k), \{x^k\} \) be a sequence of \( \varepsilon_k \)-stationary points of \( \text{NLP}^{K,S}(t_k) \) with approximate multipliers \( (\mu^k, \lambda^k, \nu^k, \rho^k) \in \mathbb{R}^n_+ \times \mathbb{R}^q \times \mathbb{R}^m_+ \times \mathbb{R}^m_+ \), such that

\[
\max \{ |u^k_i H_i(x^k)|, |v^k_i G_i(x^k)|, |\mu^k_i \Phi^K_i(x^k, t_k)| : i \in \{1, \ldots, m\} \} \leq \varepsilon_k.
\]

Assume that \( x^k \to x^\ast \) with MPEC-CPLD holding in \( x^\ast \in \Omega \). Suppose further that there is a constant \( c > 0 \) such that, for all \( i \in J(x^\ast) \) and all \( k \) sufficiently large,

1. the iterates \((G_i(x^k), H_i(x^k))\) satisfy

\[
(G_i(x^k), H_i(x^k)) \notin \left[ ((t_k, (1 + c)t_k) \times ((1 - c)t_k, t_k)) \cup \left[ ((1 - c)t_k, t_k) \times (t_k, (1 + c)t_k) \right] \cup (t_k, (1 + c)t_k) \right]^{2} \cup \left( (t_k, (1 - c)t_k) \right) \ast (t_k, (1 + c)t_k). \tag{5.12}
\]

Then, \( x^\ast \) is a M-stationary point.

**Proof:** We will proof that under the expression (5.12), the vector \( x^\ast \) is a MPEC-AKKT point. For this purpose, it will be sufficient to show that there is a subindex \( \mathcal{N} \subset \mathbb{N} \) such that \( \{x^k\}_{k \in \mathcal{N}} \) is a MPEC-AKKT sequence. Since \( x^k \) is an \( \varepsilon_k \)-stationary point for \( \text{NLP}^{K,S}(t_k) \), we have

\[
\| \nabla f(x^k) + \sum_{j=1}^{p} \mu^k_j \nabla g_j(x^k) + \sum_{i=1}^{q} \lambda^k_i \nabla h_i(x^k) - \sum_{i=1}^{m} u^k_i \nabla H_i(x^k) - \sum_{i=1}^{m} v^k_i \nabla G_i(x^k) + \sum_{i=1}^{m} \rho^k_i \Phi^K_i(x^k, t_k) \| \leq \varepsilon_k \tag{5.13}
\]
and
\[ g_j(x^k) \leq \varepsilon_k, \quad \mu_i^k \geq 0 \quad \left| \frac{1}{\mu_i} g_j(x^k) \right| \leq \varepsilon_k \quad \forall j \in \{1, \ldots, p\}, \]
\[ H_i(x^k) \geq -\varepsilon_k, \quad u_i^k \geq 0 \quad \left| \frac{1}{u_i} H_i(x^k) \right| \leq \varepsilon_k \quad \forall i \in \{1, \ldots, m\}, \]
\[ G_i(x^k) \geq -\varepsilon_k, \quad v_i^k \geq 0 \quad \left| \frac{1}{v_i} G_i(x^k) \right| \leq \varepsilon_k \quad \forall i \in \{1, \ldots, m\}, \]
\[ \Phi_i^{KS}(x^k, t_k) \leq \varepsilon_k, \quad \rho_i^k \geq 0 \quad \left| \frac{1}{\rho_i} \Phi_i^{KS}(x^k, t_k) \right| \leq \varepsilon_k \quad \forall i \in \{1, \ldots, m\}, \] (5.14)
Using the gradient of \( \nabla \Phi_i^{KS}(x^k, t_k) \), see (5.3), and for \( k \) large enough we obtain
\[ \| \nabla f(x^k) + \sum_{j \in A(x^k)} \mu_j^k \nabla g_j(x^k) + \sum_{i \in E} \lambda_i^k \nabla h_i(x^k) - \sum_{i=1}^m \tilde{u}_i^k \nabla H_i(x^k) - \sum_{i=1}^m \tilde{v}_i^k \nabla G_i(x^k) \| \leq \varepsilon_k \] (5.15)
where \( \tilde{u}_i^k \) and \( \tilde{v}_i^k \) are defined as follows
\[ \tilde{u}_i^k := \begin{cases} u_i^k - \rho_i^k (G_i(x^k) - t_k) & \text{if } H_i(x^k) + G_i(x^k) \geq 2t_k, \\ u_i^k + \rho_i^k (H_i(x^k) - t_k) & \text{otherwise} \end{cases} \] (5.16)
\[ \tilde{v}_i^k := \begin{cases} v_i^k - \rho_i^k (H_i(x^k) - t_k) & \text{if } H_i(x^k) + G_i(x^k) \geq 2t_k, \\ v_i^k + \rho_i^k (G_i(x^k) - t_k) & \text{otherwise} \end{cases} \] (5.17)
Define \( z^k := (g(x^k), h(x^k), -(H_1(x^k), G_1(x^k), \ldots, -(H_m(x^k), G_m(x^k))) \) for all \( k \in \mathbb{N} \). Clearly, we have \( I(z^k) = I(x^k), \ K(z^k) = K(x^k) \) and \( J(z^k) = J(x^k) \). Our aim is to find a subsequence \( N \subset \mathbb{N} \) and vectors \( \tilde{z}^k \) and \( \check{v}^k \) such that for every \( k \in N \), supp(\( \tilde{v}^k \)) \( \subset I(z^k) \cup J(z^k) \), supp(\( \check{v}^k \)) \( \subset K(z^k) \cup J(z^k) \), either \( \tilde{u}_i^k \tilde{v}_i^k = 0 \) or \( \tilde{u}_i^k > 0, \tilde{v}_i^k > 0 \) for \( \ell \in J(z^k) \) and
\[ \| \sum_{i=1}^m \tilde{u}_i^k \nabla H_i(x^k) + \sum_{i=1}^m \tilde{v}_i^k \nabla G_i(x^k) - \sum_{i=1}^m \check{v}_i^k \nabla H_i(x^k) - \sum_{i=1}^m \check{v}_i^k \nabla G_i(x^k) \| \rightarrow 0. \] (5.18)

Now, similarly as the proof of Theorem 5.1, we decompose the index set \( I(x^k) \) into a partition of four subsets (which we assume independent of \( k \), after possibly taking an adequate subsequence), namely:
\[ I_1(x^k) := \{ i \in I(x^k) : G_i(x^k) \geq t_k, H_i(x^k) \geq t_k \}, \]
\[ I_2(x^k) := \{ i \in I(x^k) : G_i(x^k) \geq t_k, H_i(x^k) < t_k \}, \]
\[ I_3(x^k) := \{ i \in I(x^k) : G_i(x^k) < t_k, H_i(x^k) \geq t_k \}, \]
\[ I_4(x^k) := \{ i \in I(x^k) : G_i(x^k) < t_k, H_i(x^k) < t_k \}. \] (5.19)
We will show that there is a subsequence \( N \subset \mathbb{N} \), such that \( \tilde{v}_i^k \rightarrow 0 \) for every \( i \in I(x^k) = I(z^k) \). Thus, we can define \( \check{v}_i^k := 0 \) for all \( i \in I(x^k), k \in N \). Note that, \( \| \tilde{v}_i^k - \check{v}_i^k \| \rightarrow 0 \), \( \forall i \in I(x^k) \). Now, we will proceed to show that for each \( i \in I(x^k) = \{ i \in \{1, \ldots, m\} : H_i(x^* < 0, G_i(x^*) > 0 \} = I(z^k) \), we have \( \tilde{v}_i^k \rightarrow 0 \) for an adequate subsequence. Certainly, \( I_3(x^k) \) and \( I_4(x^k) \) must be empty sets for \( k \) large enough, since \( G_i(x^k) > 0 \). Now, suppose that \( i \in I_1(x^k) \). Thus, \( G_i(x^k) + H_i(x^k) \geq 2t_k \). Due to the \( \varepsilon_k \)-stationarity of \( x^k \), we have
\[ \check{v}_i^k G_i(x^k) \rightarrow 0 \] and
\[ \rho_i^k (G_i(x^k) - t_k)(H_i(x^k) - t_k) \rightarrow 0 \]
But, since \( G_i(x^k) \rightarrow G_i(x^*) > 0 \), we have that \( u_i^k \rightarrow 0, \rho_i^k (H_i(x^k) - t_k) \rightarrow 0 \) and as consequence \( \tilde{v}_i^k = v_i^k - \rho_i^k (H_i(x^k) - t_k) \rightarrow 0 \). Now, if \( i \in I_2(x^k) \). We have two cases: if there is a subsequence \( K_1 \subset \mathbb{N} \), such that \( G_i(x^k) + H_i(x^k) \geq 2t_k \) for \( k \in K_1 \). Here, as we just have seen, \( \tilde{v}_i^k \rightarrow K_1 \). In the other case, if there is a subsequence \( K_2 \subset \mathbb{N} \), such that \( G_i(x^k) + H_i(x^k) < 2t_k \) for \( k \in K_2 \), we have \( \tilde{v}_i^k = v_i^k + \rho_i^k (G_i(x^k) - t_k) \). But, from the \( \varepsilon_k \)-stationarity of \( x^k \), we get \( t_k \leq G_i(x^k) < 2t_k - H_i(x^k) \leq 2t_k + \varepsilon_k \) for \( k \in K_2 \), which is a contradiction, since \( G_i(x^k) \rightarrow G_i(x^*) > 0 \). So, in every case, there is subsequence \( N \subset \mathbb{N} \) such that \( \tilde{v}_i^k \rightarrow 0, \forall i \in I(x^k) \).
Following the same above arguments for $\mathcal{K}(x^*) = \mathcal{K}(z^k)$, given any subsequence of $\{x^k\}$, we can find another subsequence $\mathcal{N} \subset \mathbb{N}$ such that $\hat{u}_i^k \to 0$. Thus, we can define $\hat{u}_i^k := 0$ for all $i \in \mathcal{K}(z^k)$, $k \in \mathcal{N}$ with the property $\|\hat{u}_i^k - \hat{u}_i^k\| \to 0$, $\forall i \in \mathcal{K}(z^k) = \mathcal{K}(x^*)$.

Now, we will focus on the index subset $\mathcal{J}(z^k)$. In fact, we will find a subsequence $\mathcal{N} \subset \mathbb{N}$ and scalars $\hat{u}_i^k, \hat{v}_i^k$ for $i \in \mathcal{J}(z^k)$ such that either $\hat{u}_i^k \hat{v}_i^k = 0$ or $\hat{u}_i^k > 0$ and $\hat{v}_i^k > 0$ with $\|((\hat{u}_i^k, \hat{v}_i^k) - (\hat{u}_i^k, \hat{v}_i^k))\| \to 0$, $\forall i \in \mathcal{J}(z^k) = \mathcal{J}(x^*)$. For this purpose, we decompose the index set $\mathcal{J}(x^*)$ into a partition of four subsets, namely: $\mathcal{J}_1(x^*) := \{i \in \mathcal{J}(x^*) : G_i(x^k) \geq t_k, H_i(x^k) \geq t_k\}$, $\mathcal{J}_2(x^*) := \{i \in \mathcal{J}(x^*) : G_i(x^k) \geq t_k, H_i(x^k) < t_k\}$, $\mathcal{J}_3(x^*) := \{i \in \mathcal{J}(x^*) : G_i(x^k) < t_k, H_i(x^k) \geq t_k\}$ and $\mathcal{J}_4(x^*) := \{i \in \mathcal{J}(x^*) : G_i(x^k) < t_k, H_i(x^k) < t_k\}$. We have the next sub-cases.

- If $i \in \mathcal{J}_1(x^*) = \{i \in \mathcal{J}(x^*) : G_i(x^k) \geq t_k, H_i(x^k) \geq t_k\}$. Certainly, $G_i(x^k) + H_i(x^k) \geq 2t_k$. In that case, we obtain that $\hat{u}_i^k = u_i^k - \rho_i^k(H_i(x^k) - t_k)$ and $\hat{v}_i^k = v_i^k - \rho_i^k(G_i(x^k) - t_k)$. We will show that $\hat{u}_i^k \to 0$ or $\hat{v}_i^k \to 0$ for an adequate subsequence. First, since $H_i(x^k) \geq t_k, G_i(x^k) \geq t_k$, $|u_i^k H_i(x^k)| \leq o(t_k)$ and $|v_i^k G_i(x^k)| \leq o(t_k)$, we get that $u_i^k \to 0$ and $v_i^k \to 0$. Furthermore, from $H_i(x^k) \geq t_k$ and $G_i(x^k) \geq t_k$, there is a subsequence $K \subset \mathbb{N}$ such that $H_i(x_k)/t_k \to K \alpha$ and $G_i(x_k)/t_k \to K \beta$, for some scalars $\alpha, \beta \in [1, \infty]$. If $\alpha \neq 1$. Then,

$$|\rho_i^k(G_i(x^k) - t_k)(H_i(x^k) - t_k) - \rho_i^k\Phi^K_S(x_i^k, t_k)| \leq o(t_k) \to 0$$

implies that $\rho_i^k(G_i(x^k) - t_k) \to K \alpha$ and hence $\hat{u}_i^k = u_i^k - \rho_i^k(H_i(x^k) - t_k) \to K \alpha$. Similarly, if $\beta \neq 1$, we conclude that $\hat{v}_i^k \to K \beta$. Now, only rest to analyze when $\alpha = \beta = 1$. For this case, we get $(1 + c)u_i^k > H_i(x^k)$ for all $i \in \mathcal{J}_1(x^*)$, $i \notin \mathcal{J}_2(x^*)$ and $(1 + c)v_i^k > G_i(x^k)$ for all $i \in \mathcal{J}_1(x^*)$, $i \notin \mathcal{J}_3(x^*)$. If $i \in \mathcal{J}_1(x^*)$, then we have a subsequence $\mathcal{N} \subset \mathbb{N}$ such that $\hat{u}_i^k \to 0$ or $\hat{v}_i^k \to 0$ for all $i \in \mathcal{J}_1(x^*)$. 

- Take $i \in \mathcal{J}_2(x^*) = \{i \in \mathcal{J}(x^*) : G_i(x^k) \geq t_k, H_i(x^k) < t_k\}$. We will show that there is a subsequence $\mathcal{N} \subset \mathbb{N}$ and scalars $\hat{u}_i^k, \hat{v}_i^k$, such that $\|((\hat{u}_i^k, \hat{v}_i^k) - (\hat{u}_i^k, \hat{v}_i^k))\| \to 0$ and either $\hat{u}_i^k \hat{v}_i^k = 0$ or $\hat{u}_i^k > 0$ and $\hat{v}_i^k > 0$. Now, since $G_i(x^k) \geq t_k$, we conclude $\hat{u}_i^k \to 0$. Now, we have two alternatives, that $G_i(x^k) + H_i(x^k) \geq 2t_k$ holds for infinite many $k$ or $G_i(x^k) + H_i(x^k) < 2t_k$ holds for infinite many $k$. 

  - If $G_i(x^k) + H_i(x^k) \geq 2t_k$ holds for infinite many $k$. In this case, we get $\hat{v}_i^k = v_i^k - \rho_i^k(H_i(x^k) - t_k)$ and $\hat{u}_i^k = u_i^k - \rho_i^k(G_i(x^k) - t_k)$. From, the fact $-\varepsilon_k \leq H_i(x^k) < t_k$ and $G_i(x^k) \geq t_k$, there is a subsequence $K \subset \mathbb{N}$ such that $H_i(x_k)/t_k \to K \alpha$ and $G_i(x_k)/t_k \to K \beta$, for some scalars $\alpha \in [0, 1], \beta \in [1, \infty]$. If $\alpha \neq 1$. Then,

$$|\rho_i^k(G_i(x^k) - t_k)(H_i(x^k) - t_k) - \rho_i^k\Phi^K_S(x_i^k, t_k)| \leq o(t_k) \to 0$$

implies that $\rho_i^k(G_i(x^k) - t_k) \to K \alpha$. Now, we define $\hat{v}_i^k := \hat{u}_i^k$ and $\hat{u}_i^k := u_i^k$. Clearly, $\hat{u}_i^k \hat{v}_i^k \geq 0$ and $\|((\hat{u}_i^k, \hat{v}_i^k) - (\hat{u}_i^k, \hat{v}_i^k))\| = \rho_i^k(G_i(x^k) - t_k) \to K \alpha$. Now, if $\beta \neq 1$, we get $\rho_i^k(H_i(x^k) - t_k) \to 0$ and hence $\hat{v}_i^k \to K \beta$. So, define $\hat{v}_i^k := 0$ and $\hat{u}_i^k := u_i^k$. Here, $\hat{v}_i^k \hat{u}_i^k = 0$ and $\|((\hat{u}_i^k, \hat{v}_i^k) - (\hat{u}_i^k, \hat{v}_i^k))\| \to 0$. Now, only rest to analyze when $\alpha = \beta = 1$. For this case, we get $(1 + c)u_i^k < H_i(x^k) < t_k$ and $(1 + c)t_k > G_i(x^k) \geq t_k$ for $k \in K$ large enough. From $\hat{v}_i^k \geq 0$, the only possibility is $G_i(x^k) = t_k$. Thus, $\hat{u}_i^k = u_i^k \geq 0$. Now, we define $\hat{v}_i^k := \hat{v}_i^k$ and $\hat{u}_i^k := u_i^k$. Clearly, $\hat{u}_i^k \hat{v}_i^k > 0$ and $\|((\hat{u}_i^k, \hat{v}_i^k) - (\hat{u}_i^k, \hat{v}_i^k))\| = \rho_i^k(G_i(x^k) - t_k) \to 0$. 


Finally, we conclude, from all the analyzed cases, that there is a subsequence $x^k \to x^*$, and $G_i(x^k) \geq t_k$ for some scalars $\alpha \in [0, 1]$, $\beta \in [1, \infty]$. If $\alpha \neq 1$, the expression

$$\left| \rho_i^k(G_i(x^k) - t_k)(H_i(x^k) - 1) \right| \leq \frac{\rho_i^k(G_i(x^k) - t_k)^2 + (H_i(x^k) - t_k)^2}{2t_k} \to 0$$

implies that $\rho_i^k(G_i(x^k) - t_k) \to K$ and hence $v_i^k = \hat{v}_i^k + \rho_i^k(G_i(x^k) - t_k) \to K$. Now, we define $v_i^k := 0$ and $\hat{u}_i^k := \hat{u}_i^k$. Clearly, $\hat{v}_i^k \hat{u}_i^k = 0$ and $\|(v_i^k, \hat{v}_i^k) - (\hat{u}_i^k, \hat{v}_i^k)\| = \hat{v}_i^k \to K$. Now, if $\beta \neq 1$, we get

$$\left| \rho_i^k(G_i(x^k) - t_k)(H_i(x^k) - 1) \right| \leq \frac{\rho_i^k(G_i(x^k) - t_k)^2 + (H_i(x^k) - t_k)^2}{t_k} \to 0$$

which implies $\rho_i^k(G_i(x^k) - t_k) \to 0$ and hence $v_i^k \to K$. Now, define $v_i^k := 0$ and $\hat{u}_i^k := \hat{u}_i^k$. Hence, $\hat{v}_i^k \hat{u}_i^k = 0$ and $\|(v_i^k, \hat{v}_i^k) - (\hat{u}_i^k, \hat{v}_i^k)\| = \hat{v}_i^k \to K$.

- For $i \in J_4(x^*) = \{ i \in J(x^*) : G_i(x^k) < t_k, H_i(x^k) \geq t_k \}$, similarly, as item anterior, we can find a subsequence $K \subset N$ and points $\{ \hat{v}_i^k, \hat{v}_i^k \}$, such that either $\hat{v}_i^k \hat{v}_i^k = 0$ or $\hat{u}_i^k > 0$ and $\hat{v}_i^k > 0$. Furthermore, $\|(\hat{u}_i^k, \hat{v}_i^k) - (\hat{u}_i^k, \hat{v}_i^k)\| \to 0$.

- For $i \in J_4(x^*) = \{ i \in J(x^*) : G_i(x^k) < t_k, H_i(x^k) < t_k \}$, we obtain that $\hat{v}_i^k = u_i^k + \rho_i^k(H_i(x^k) - t_k)$ and $\hat{v}_i^k = u_i^k + \rho_i^k(G_i(x^k) - t_k)$. From $-\varepsilon_k \leq H_i(x^k) < t_k$ and $-\varepsilon_k \leq G_i(x^k) < t_k$, there is a subsequence $K \subset N$ such that $H_i(x^k)/t_k \to K$ and $G_i(x^k)/t_k \to K$, for some scalars $\alpha, \beta \in [0, 1]$. If $\alpha \neq 1$, then

$$\left| \rho_i^k(G_i(x^k) - t_k)(H_i(x^k) - 1) \right| \leq \frac{\rho_i^k(G_i(x^k) - t_k)^2 + (H_i(x^k) - t_k)^2}{2t_k} \to 0$$

implies that $\rho_i^k(G_i(x^k) - t_k) \to K$. Furthermore, from

$$\left| \rho_i^k(H_i(x^k) - t_k)(H_i(x^k) - 1) \right| \leq \frac{\rho_i^k(G_i(x^k) - t_k)^2 + (H_i(x^k) - t_k)^2}{2t_k} \to 0$$

we get $\rho_i^k(H_i(x^k) - t_k) \to K$. In this case, define $\hat{v}_i^k := v_i^k$ and $\hat{u}_i^k := u_i^k$. Clearly, $\hat{v}_i^k \geq 0$, $\hat{u}_i^k \geq 0$ and $\|(\hat{u}_i^k, \hat{v}_i^k) - (\hat{u}_i^k, \hat{v}_i^k)\|^2 = \rho_i^k(G_i(x^k) - t_k)^2 + \rho_i^k(H_i(x^k) - t_k)^2 \to K$. By symmetry, we obtain the same result if $\beta \neq 1$. Now, only rest analyze when $\alpha = \beta = 1$. In this case, we get $(1 - c)t_k < H_i(x^k) < t_k$ and $(1 - c)t_k < G_i(x^k) < t_k$ for $c > 0$ given by (5.12) and for $k \in K$ large enough, which is impossible by (5.12). Thus, from all the cases analyzed we can conclude that there is a subsequence $N \subset N$ and vector $\{ \hat{u}_i^k, \hat{v}_i^k \}$, such that $\|(\hat{u}_i^k, \hat{v}_i^k) - (\hat{u}_i^k, \hat{v}_i^k)\| \to K$ and either $\hat{v}_i^k \hat{v}_i^k = 0$ or $\hat{u}_i^k > 0$ and $\hat{v}_i^k > 0$ for $i \in J_4(x^*) = J_4(x^*)$.

Finally, we conclude, from all the analyzed cases, that there is a subsequence $N \subset N$ and points $\{ \hat{u}_i^k, \hat{v}_i^k \}$, $k \in N$, $i \in \{1, \ldots, m\}$, with $\hat{u}_i^k = 0$ for $i \in K(x^k)$, $\hat{v}_i^k = 0$ for $i \in I(x^k)$ and either $\hat{u}_i^k \hat{v}_i^k = 0$ or $\hat{u}_i^k > 0$ and $\hat{v}_i^k > 0$ for $i \in J(x^k)$ such that for all $i \in \{1, \ldots, m\}$, we have that $\|(\hat{u}_i^k, \hat{v}_i^k) - (\hat{u}_i^k, \hat{v}_i^k)\| \to N_0$ and $\sum_{i=1}^m \hat{u}_i^k \nabla H_i(x^k) + \sum_{i=1}^m \hat{v}_i^k \nabla G_i(x^k) - \sum_{i=1}^m \hat{u}_i^k \nabla H_i(x^k) - \sum_{i=1}^m \hat{v}_i^k \nabla G_i(x^k) \to N_0$. Thus, the subsequence $\{ x^k : k \in N \}$ with approximate multipliers $(\mu^k, \lambda^k, \hat{u}^k, \hat{v}^k)$ is a MPEC-AKKT sequence with $x^k \to N_0 x^*$. Since by hypothesis, MPEC-CCP holds at $x^*$, the point $x^*$ is a M-stationary point.
Remark 6. MPEC-MFCQ implies that the sequence of approximate multipliers \( \{(\mu^k, \lambda^k, u^k, v^k)\} \) is bounded. In fact, most of the CQs for M-stationary used in the convergence analysis of several MPECs algorithms, as MPEC-MFCQ, tries to bound or indirectly control the sequence of approximate multipliers. But, as we just see, we can obtain convergence to M-stationary points even if the sequence of approximate multipliers is unbounded. In the Theorem 5.1 or in the Theorem 5.3 (under (5.12)), we can guarantee convergence to M-stationary points with the less stringent MPEC-CCP without requiring boundedness of the multipliers.

5.2 The nonsmooth relaxation by Kadrani et al.

The relaxation scheme of Kadrani et al \[25\] is given by

\[
\begin{align*}
\text{minimize} & \quad f(x) \\
\text{subject to} & \quad g_j(x) \leq 0 \quad \forall j \in P = \{1, \ldots, p\} \\
& \quad h_i(x) = 0 \quad \forall i \in E = \{1, \ldots, q\} \\
& \quad 0 \leq H_i(x) + t, \quad 0 \leq G_i(x) + t \quad \forall i \in M = \{1, \ldots, m\} \\
& \quad \Phi^KDB(H_i(x), G_i(x); t) \leq 0 \quad \forall i \in M = \{1, \ldots, m\}
\end{align*}
\]

(5.20)

where \( \Phi^KDB(H_i(x), G_i(x); t) := (H_i(x) - t)(G_i(x) - t) \) and when it is clear of the context, we use the notation \( \Phi^KDB(x; t) \) instead of \( \Phi^KDB(H_i(x), G_i(x); t) \). The mathematical problem (5.20) is denoted by \( \text{NLP}^{KDB}(t) \). The Figure 4 shows the feasible set of \( \text{NLP}^{KDB}(t) \) for a given \( t > 0 \).

By straightforward calculations, we have

\[
\nabla \Phi^KDB(x; t) := (H_i(x) - t)\nabla G_i(x) + (G_i(x) - t)\nabla H_i(x) \quad \forall i \in \{1, \ldots, m\}.
\]

(5.21)

Using MPEC-CCP instead of MPEC-CPLD, we improve the result of Hoheisel et al \[24\]. The proof follows similar arguments as the Theorem 5.1.

**Theorem 5.4.** Let \( \{t_k\} \downarrow 0 \) and \( x^k \) be a KKT point of \( \text{NLP}^{KDB}(t_k) \) with \( x^k \rightarrow x^* \) such that MPEC-CCP holds in \( x^* \). Then, \( x^* \) is an M-stationary point.

When we replace the sequence of KKT points by a sequence of \( \varepsilon_k \)-stationary points, following a similar line of arguments as the Theorem 5.3 we have the next result.
Theorem 5.5. Let \( t_k \downarrow 0 \), \( \varepsilon_k = o(t_k) \), \( x^k \) be a sequence of \( \varepsilon_k \)-stationary points of \( \text{NL}P^\text{KDB} (t_k) \) with approximate multipliers \( (\mu^k, \lambda^k, u^k, \nu^k, \rho^k) \in \mathbb{R}_+^p \times \mathbb{R}^q \times \mathbb{R}_+^m \times \mathbb{R}_+^m \times \mathbb{R}_+^m \), such that

\[
\max \{ |u^k_i (H_i(x^k) + t_k)|, |v^k_i (G_i(x^k) + t_k)|, |\rho^k_i \Phi^\text{KDB} (x^k, t_k)| : i \in \{1, \ldots, m\} \} \leq \varepsilon_k.
\]

and \( x^k \to x^* \). Assume that \( x^* \) conforms MPEC-CCP. Suppose further that there is a constant \( c > 0 \) such that, for all \( i \in J(x^*) \) and all \( k \) sufficiently large,

1. the iterates \( (G_i(x^k), H_i(x^k)) \) satisfy

\[
(G_i(x^k), H_i(x^k)) \notin [(t_k, (1 + c)t_k) \times ((1 - c)t_k, t_k)] \cup [(1 - c)t_k, t_k) \times (t_k, (1+c)t_k)] \cup (t_k, (1+c)t_k)^2.
\]

Then, \( x^* \) is a \( M \)-stationary point.

Proof. In order to prove this, we will show that under the hypothesis \( \Box \), \( x^* \) is a MPEC-AKKT point. We will show that there is a subindex \( N \subset \mathbb{N} \) such that \( \{ x^k \}_{k \in N} \) is a MPEC-AKKT sequence. Since \( x^k \) is an \( \varepsilon_k \)-stationary point for \( \text{NL}P^\text{KDB} (t_k) \), we have

\[
\| \nabla f(x^k) + \sum_{j=1}^p \mu^k_j \nabla g_j(x^k) + \sum_{i=1}^q \lambda^k_i \nabla h_i(x^k) - \sum_{i=1}^m u^k_i \nabla H_i(x^k) - \sum_{i=1}^m v^k_i \nabla G_i(x^k) + \sum_{i=1}^m \rho^k_i \Phi^\text{KDB} (x^k, t_k) \| \leq \varepsilon_k
\]

and

\[
g_j(x^k) \leq \varepsilon_k, \quad \mu^k_j \geq 0 \quad |\mu^k_j g_j(x^k)| \leq \varepsilon_k \quad \forall j \in \{1, \ldots, p\},
\]

\[
H_i(x^k) + t_k \geq -\varepsilon_k, \quad u^k_i \geq 0 \quad |u^k_i (H_i(x^k) + t_k)| \leq \varepsilon_k \quad \forall i \in \{1, \ldots, m\},
\]

\[
G_i(x^k) + t_k \geq -\varepsilon_k, \quad v^k_i \geq 0 \quad |v^k_i (G_i(x^k) + t_k)| \leq \varepsilon_k \quad \forall i \in \{1, \ldots, m\},
\]

\[
\Phi^\text{KDB} (x^k, t_k) \leq \varepsilon_k, \quad \rho^k_i \geq 0 \quad |\rho^k_i \Phi^\text{KDB} (x^k, t_k)| \leq \varepsilon_k \quad \forall i \in \{1, \ldots, m\},
\]

Using the gradient of \( \nabla \Phi^\text{KDB} (x^k, t_k) \), see \( \Box \), and for \( k \) large enough we obtain

\[
\| \nabla f(x^k) + \sum_{j \in A(x^*)} \mu^k_j \nabla g_j(x^k) + \sum_{i \in \ell} \lambda^k_i \nabla h_i(x^k) - \sum_{i=1}^m \tilde{u}^k_i \nabla H_i(x^k) - \sum_{i=1}^m \tilde{v}^k_i \nabla G_i(x^k) \| \leq \varepsilon_k
\]

where \( \tilde{u}^k \) and \( \tilde{v}^k \) are defined as follows

\[
\tilde{u}^k_i := u^k_i - \rho^k_i (G_i(x^k) - t_k), \quad \forall i \in \{1, \ldots, m\}
\]

\[
\tilde{v}^k_i := v^k_i - \rho^k_i (H_i(x^k) - t_k), \quad \forall i \in \{1, \ldots, m\}
\]

Put \( z^k := (g(x^*), h(x^*), -(H_1(x^*), G_1(x^*)), \ldots, -(H_m(x^*), G_m(x^*))), k \in \mathbb{N} \). Clearly, \( \mathcal{I}(z^k) = \mathcal{I}(x^*) \), \( \mathcal{K}(z^k) = \mathcal{K}(x^*) \) and \( \mathcal{J}(z^k) = \mathcal{J}(x^*) \). Our aim is to find a subsequence \( N \subset \mathbb{N} \) and vectors \( \tilde{u}^k \) and \( \tilde{v}^k \) such that for every \( k \in N \), \( \text{supp}(\tilde{u}^k) \subset \mathcal{I}(z^k) \cup \mathcal{J}(z^k) \), \( \text{supp}(\tilde{v}^k) \subset \mathcal{K}(z^k) \cup \mathcal{J}(z^k) \), either \( \tilde{u}^k_\ell \tilde{v}^k_\ell = 0 \) or \( \tilde{u}^k_\ell > 0, \tilde{v}^k_\ell > 0 \) for \( \ell \in \mathcal{J}(z^k) \) and

\[
\lim_{k \in N} \sum_{i=1}^m \tilde{u}^k_i \nabla H_i(x^k) + \sum_{i=1}^m \tilde{v}^k_i \nabla G_i(x^k) - \sum_{i=1}^m \tilde{u}^k_i \nabla H_i(x^k) - \sum_{i=1}^m \tilde{v}^k_i \nabla G_i(x^k) = 0.
\]
Now, we will focus on the index subset \( J(\mathbf{z}^k) \). In fact, we will find a subsequence \( \mathcal{N} \subset \mathbb{N} \) and scalars \( \hat{u}_i^k, \hat{v}_i^k \) for \( i \in J(\mathbf{z}^k) \) such that either \( \hat{u}_i^k \hat{v}_i^k < 0 = 0 \) and \( \hat{v}_i^k > 0 \) with \( ||(\hat{u}_i^k, \hat{v}_i^k) - (\hat{u}_i^k, \hat{v}_i^k) || \rightarrow N_0, \forall i \in J(\mathbf{z}^k) = J(\mathbf{z}^k) \). For this purpose, we decompose the index set \( J(\mathbf{z}^k) \) into a partition of four subsets, namely: \( J_1(x^*) := \{ i \in J(\mathbf{z}^k) : G_i(x^k) > t_k, H_i(x^k) > t_k \}, J_2(x^*) := \{ i \in J(\mathbf{z}^k) : G_i(x^k) > t_k, H_i(x^k) \leq t_k \}, J_3(x^*) := \{ i \in J(\mathbf{z}^k) : G_i(x^k) \leq t_k, H_i(x^k) > t_k \} \) and \( J_4(x^*) := \{ i \in J(\mathbf{z}^k) : G_i(x^k) \leq t_k, H_i(x^k) \leq t_k \} \). There is no loss of generality, after possibly taking an adequate subsequence, if we assume that each element of the partition independent of \( k \). We have the next subcases:

- If \( i \in J_1(x^*) = \{ i \in J(\mathbf{z}^k) : G_i(x^k) > t_k, H_i(x^k) > t_k \} \). Then, there is a subsequence \( K \subset \mathbb{N} \) such that \( H_i(x^k)/t_k \rightarrow K \alpha \) and \( G_i(x^k)/t_k \rightarrow K \beta \), for some scalars \( \alpha, \beta \in [1, \infty) \). If \( \alpha \neq 1 \). Then, from \( \varepsilon_k = o(t_k) \) we get

\[
|u_i^k(H_i(x^k)/t_k) + 1| \leq o(t_k)/t_k \rightarrow 0 \text{ and } |\rho_i^k(G_i(x^k) - t_k)(H_i(x^k)/t_k) - 1| \leq o(t_k)/t_k \rightarrow 0.
\]

Thus, we get \( u_i^k \rightarrow 0, \rho_i^k(G_i(x^k) - t_k) \rightarrow 0 \) and hence \( \hat{u}_i^k = u_i^k - \rho_i^k(G_i(x^k) - t_k) \rightarrow 0 \). Similarly, if \( \beta \neq 1 \), from \( \| (\hat{u}_i^k, \hat{v}_i^k) - (\hat{u}_i^k, \hat{v}_i^k) \| \rightarrow N_0 \) and either \( \hat{u}_i^k \hat{v}_i^k = 0 \) or \( \hat{v}_i^k > 0 \) and \( \hat{u}_i^k = 0 \) and \( \hat{v}_i^k = 0 \). Now, from \( \varepsilon_k = o(t_k) \leq H_i(x^k) + t_k \), there is a subsequence \( K \subset \mathbb{N} \) such that \( H_i(x^k)/t_k \rightarrow K \alpha \) and \( G_i(x^k)/t_k \rightarrow K \beta \), for some scalars \( \alpha \in [-1, 1], \beta \in [1, \infty) \). If \( \alpha \neq 1 \). Then, from \( |u_i^k(H_i(x^k)/t_k) + 1| \leq o(t_k) \) and \( |\rho_i^k(G_i(x^k) - t_k)(H_i(x^k)/t_k) - 1| \leq o(t_k)/t_k \rightarrow 0 \), we get \( u_i^k \rightarrow 0, \rho_i^k(G_i(x^k) - t_k) \rightarrow 0 \) and hence \( \hat{u}_i^k = u_i^k - \rho_i^k(G_i(x^k) - t_k) \rightarrow 0 \). Similarly, if \( \beta \neq 1 \), from \( |v_i^k(G_i(x^k) + t_k) \leq o(t_k) \) and \( |\rho_i^k(G_i(x^k) - t_k)(H_i(x^k) - t_k) \leq o(t_k) | \rightarrow 0 \), we get \( v_i^k = v_i^k - \rho_i^k(H_i(x^k)) \rightarrow 0 \). Now, if \( \alpha = 1 \) and \( \beta = 1 \), we get, for \( k \) large enough, that \( t_k \leq H_i(x^k) > (1 - c)t_k \) and \( (1 + c)t_k > G_i(x^k) > t_k \). Thus, by expression (5.22), we must have \( H_i(x^k) = t_k \). Furthermore, by \( |v_i^k(G_i(x^k) + t_k) \leq o(t_k) \), we get \( v_i^k \rightarrow 0 \) and hence \( \hat{v}_i^k = v_i^k - \rho_i^k(H_i(x^k) - t_k) \rightarrow 0 \). Now, all the mentioned cases, there is subsequence \( \mathcal{N} \subset \mathbb{N} \), such that \( \hat{u}_i^k \rightarrow N_0 \) or \( \hat{v}_i^k \rightarrow N_0 \). So, when \( \hat{u}_i^k \rightarrow N_0 \), we define \( \hat{u}_i^k := 0, \hat{v}_i^k := \hat{v}_i^k, k \in \mathcal{N} \). And when \( \hat{v}_i^k \rightarrow N_0 \), we define \( \hat{u}_i^k := 0, \hat{v}_i^k := \hat{u}_i^k, k \in \mathcal{N} \). Clearly, we always have \( \| (\hat{u}_i^k, \hat{v}_i^k) - (\hat{u}_i^k, \hat{v}_i^k) \| \rightarrow N_0 \) and \( \hat{u}_i^k \hat{v}_i^k = 0 \) for all \( k \in \mathcal{N}, i \in J_2(x^*) \).

- For \( i \in J_3(x^*) = \{ i \in J(\mathbf{z}^k) : G_i(x^k) \leq t_k, H_i(x^k) \geq t_k \} \). Similarly, as item anterior, we can find a subsequence \( \mathcal{K} \subset \mathbb{N} \) and points \( \hat{u}_i^k, \hat{v}_i^k \), such that either \( \hat{u}_i^k \hat{v}_i^k = 0 = 0 \) and \( \hat{v}_i^k > 0 \). Furthermore, \( \| (\hat{u}_i^k, \hat{v}_i^k) - (\hat{u}_i^k, \hat{v}_i^k) \| \rightarrow N_0 \).

- For \( i \in J_4(x^*) = \{ i \in J(\mathbf{z}^k) : G_i(x^k) \leq t_k, H_i(x^k) \leq t_k \} \). From \( -\varepsilon_k - t_k \leq G_i(x^k) \leq t_k \), and \( -\varepsilon_k - t_k \leq G_i(x^k) \leq t_k \), there exists a subsequence \( \mathcal{K} \subset \mathbb{N} \) such that \( H_i(x^k)/t_k \rightarrow K \alpha \) and \( G_i(x^k)/t_k \rightarrow K \beta \), for some scalars \( \alpha, \beta \in [-1, 1] \). If \( \alpha \neq 1 \), from \( |u_i^k(H_i(x^k)/t_k) \leq o(t_k) \) and \( |\rho_i^k(G_i(x^k) - t_k)(H_i(x^k) - t_k) \leq o(t_k) | \rightarrow 0 \), we get \( u_i^k \rightarrow 0, \rho_i^k(G_i(x^k) - t_k) \rightarrow 0 \) and hence \( u_i^k \rightarrow 0 \). Similarly, if \( \beta \neq 1 \), from \( |v_i^k(G_i(x^k) + t_k) \leq o(t_k) \) and \( |\rho_i^k(G_i(x^k) - t_k)(H_i(x^k) - t_k) \leq o(t_k) | \rightarrow 0 \), we get \( v_i^k \rightarrow 0 \). Now, if \( \alpha = \beta = 1 \), we get, for \( k \) large enough, that \( t_k \geq H_i(x^k) > (1 - c)t_k \) and \( t_k \geq G_i(x^k) > (1 - c)t_k \). Here, we have the following subcases:
– If there are infinite \( k \in \mathbb{N} \) such that \( t_k > G_i(x^k) \). In this subcase, by expression \([5.22] \), we must have \( H_i(x^k) = t_k \). Now, by \( |v^k_i(G_i(x^k) + t_k)| \leq o(t_k) \), we get \( v^k_i \to 0 \) and hence \( \hat{v}^k_i = v^k_i - \rho^k_i(H_i(x^k) - t_k) = v^k_i \to 0 \).

– Now, if there are infinite \( k \) with \( t_k = G_i(x^k) \). From, \( |u^k_i(H_i(x^k) + t_k)| \leq o(t_k) \), we get \( u^k_i \to 0 \), and as consequence \( \hat{u}^k_i = u^k_i - \rho^k_i(G_i(x^k) - t_k) = u^k_i \to 0 \).

Thus, there is subsequence \( \bar{u}^k_i \to_{\mathcal{N}} 0 \) or \( \bar{v}^k_i \to_{\mathcal{N}} 0 \). When \( \bar{u}^k_i \to_{\mathcal{N}} 0 \), we define \( \hat{u}^k_i := 0, \hat{v}^k_i := \bar{v}^k_i, k \in \mathcal{N} \) and when \( \bar{v}^k_i \to_{\mathcal{N}} 0 \), define \( \hat{v}^k_i := 0, \hat{u}^k_i := \bar{u}^k_i, k \in \mathcal{N} \). Note that, we always have that \( \bar{u}^k_i \hat{v}^k_i = 0 \) for all \( k \in \mathcal{N}, i \in J_i(z^k) \) and \( \|(\hat{u}^k_i, \hat{v}^k_i) - (\bar{u}^k_i, \bar{v}^k_i)\| \to_{\mathcal{N}} 0 \).

Finally to conclude, we can see, from all the cases, that there is a subsequence \( \mathcal{N}' \subset \mathbb{N} \) and points \( \{\hat{u}^k_i, \hat{v}^k_i\} \), \( k \in \mathcal{N}', i \in \{1, \ldots, m\} \), with \( \hat{u}^k_i = 0 \) for \( i \in \mathcal{K}(z^k) \), \( \hat{v}^k_i = 0 \) for \( i \in \mathcal{I}(z^k) \) and either \( \hat{u}^k_i \hat{v}^k_i = 0 \) or \( \hat{u}^k_i > 0 \) and \( \hat{v}^k_i > 0 \) for \( i \in J_i(z^k) \) such that for all \( i \in \{1, \ldots, m\} \), we have that \( \|(\hat{u}^k_i, \hat{v}^k_i) - (\bar{u}^k_i, \bar{v}^k_i)\| \to_{\mathcal{N}' \to_x^*} 0 \) and \( \|\sum_{i=1}^m \hat{u}^k_i \nabla H_i(x^k) + \sum_{i=1}^m \hat{v}^k_i \nabla G_i(x^k) - \sum_{i=1}^m \hat{u}^k_i \nabla H_i(x^k) - \sum_{i=1}^m \hat{v}^k_i \nabla G_i(x^k)\| \to_{\mathcal{N}' \to_x^*} 0 \). Thus, the subsequence \( \{x^k : k \in \mathcal{N}'\} \) with approximate multipliers \( (\hat{\mu}^k_i, \hat{\lambda}^k_i, \hat{\mu}^k_i, \hat{\nu}^k_i) \) is a MPEC-AKKT sequence with \( x^k \to_{\mathcal{N}' \to_x^*} x^* \). Thus, since by hypothesis MPEC-CCP holds at \( x^* \), the point \( x^* \) is a M-stationary point.

\[ \square \]

6 Conclusions

We presented a new sequential optimality condition for MPECs. This condition and its companion CQ for M-stationarity, can be used, to prove convergence to M-stationary points for several relations schemes, under weak assumptions, even in the case, where the set of multipliers is unbounded. We hope that, this kind of analysis can be useful in the development and derivation of new methods for solving MPECs with strong convergence properties.

References


