A feasible rounding approach for mixed-integer nonlinear optimization problems

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Abstract We introduce a new technique to generate good feasible points of mixed-integer nonlinear optimization problems. It makes use of the so-called inner parallel set of the relaxed feasible set, which was employed in O. Stein, Error bounds for mixed integer linear optimization problems, Mathematical Programming, Vol. 156 (2016), 101–123, as well as O. Stein, Error bounds for mixed integer nonlinear optimization problems, Optimization Letters, 2016, DOI 10.1007/s11590-016-1011-y, to bound the error occurring when an optimal point of the relaxed problem is rounded to the next integer point.

On the contrary, in the present paper we show that efficiently solving certain purely continuous optimization problems over the inner parallel set and rounding their optimal points leads to feasible points of the original mixed-integer problem. For their objective function values we present computable a-priori and a-posteriori bounds on the deviation from the optimal value, as well as a computable certificate for the consistency of the inner parallel set.

Numerical examples for large scale knapsack problems illustrate that our method is able to outperform standard software. A post processing step to our approach further improves the results.

Keywords Rounding · granularity · grid relaxation retract · global error bound · knapsack problem

Mathematics Subject Classification (2000) 90C11 · 90C10 · 90C31 · 90C30

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1 Introduction

In this paper we suggest a construction method for good feasible points of mixed-integer nonlinear optimization problems of the form

\[
\text{MIP}_h: \min_{(x,y) \in \mathbb{R}^n \times h\mathbb{Z}^m} f(x,y) \quad \text{s.t.} \quad g_i(x,y) \leq 0, \quad i \in I
\]

with an appropriate mesh size parameter \( h > 0 \), a nonempty and closed set \( D \subseteq \mathbb{R}^n \times \mathbb{R}^m \), a finite index set \( I = \{1, \ldots, p\} \) with \( p \in \mathbb{N} \), real-valued functions \( f, g_i, i \in I \), defined on \( D \), and “lattice box constraints” with \( y^b_h \in h(\mathbb{Z} \cup \{-\infty\})^m \) and \( y^u_h \in h(\mathbb{Z} \cup \{+\infty\})^m \).

Problems of the type \( \text{MIP}_h \) appear when the effect of different granularities of discrete decision variables is modeled, and were first systematically studied in \([30,31]\) for the mixed-integer linear and nonlinear cases, respectively. We will denote the feasible set of \( \text{MIP}_h \) by \( M_h \). Throughout this article we will assume that the problem \( \text{MIP}_h \) is solvable, we denote its optimal value by \( v_h \), and any optimal point by \((x^*_h, y^*_h)\).

For the time being, that is, until we shall specify which values of the mesh size parameter \( h \) are appropriate for our approach at the end of Section 2, one may think of \( h \) to be fixed, for example to \( h = 1 \).

The functions \( g_i : D \rightarrow \mathbb{R}, \ i \in I \), are assumed to satisfy global Lipschitz conditions with respect to \( y \) uniformly in \( x \). To be more specific, for any \( x \in \mathbb{R}^n \) we define the set

\[
D(x) := \{y \in \mathbb{R}^m | (x,y) \in D\}
\]

and denote by

\[
\text{pr}_x D := \{x \in \mathbb{R}^n | D(x) \neq \emptyset\}
\]

the orthogonal projection of \( D \) to the “\( x \)-space” \( \mathbb{R}^n \). Then the functions \( g_i, i \in I \), are assumed to satisfy Lipschitz conditions with respect to the \( \ell_\infty \)-norm on the fibers \( \{x\} \times D(x) \), independently of the choice of \( x \in \text{pr}_x D \): for all \( i \in I \) there exists some \( L^i_x \geq 0 \) such that for all \( x \in \text{pr}_x D \) and all \( y^1, y^2 \in D(x) \) we have

\[
|g_i(x,y^1) - g_i(x,y^2)| \leq L^i_x \| (x,y^1) - (x,y^2) \|_\infty = L^i_x \| y^1 - y^2 \|_\infty
\]

(see \([31]\) for a brief discussion of this condition). We allow for vanishing Lipschitz constants to cover trivial cases.

While the Lipschitz continuity of the functions \( g_i \) with respect to the \( \ell_\infty \)-norm will be of special interest below, the function \( f : D \rightarrow \mathbb{R} \) is assumed to satisfy a Lipschitz condition on \( D \) with respect to some arbitrary norm whose appropriate choices will be motivated below: there exists some \( L^f \geq 0 \) such that for all \( (x^1,y^1), (x^2,y^2) \in D \) we have

\[
|f(x^1,y^1) - f(x^2,y^2)| \leq L^f \| (x^1,y^1) - (x^2,y^2) \|.
\]

Below, to make subproblems like the continuous relaxation of \( \text{MIP}_h \) numerically tractable, we shall impose additional assumptions like convexity of the set \( D \) and the functions \( f, g_i, i \in I \), but introduce them only where necessary. While the purely
integer case \((n = 0)\) is included in our analysis (with \(D(x) := D\)), we will assume \(m, p > 0\) throughout this article.

The idea to treat \(\text{MIP}_h\) by first solving its purely continuous NLP relaxation
\[
\text{MIP}_h : \quad \min_{(x,y) \in \mathbb{R}^n \times \mathbb{R}^m} f(x,y) \quad \text{s.t.} \quad g_i(x,y) \leq 0, \ i \in I, \ (x,y) \in D, \ y_h^i \leq y \leq y_h^i
\]
and then round the \(y\)-part of the obtained optimal point \((\hat{y}_h^i, \hat{y}_h^i)\) to the next mesh vector \(\tilde{y}_h^i\) is well-known, but the appearing optimality and feasibility errors were studied only recently in [30] for the linear and in [31] for the nonlinear case.

In the present paper, on the other hand, we employ some ideas introduced in [30, 31] in order to construct a feasible point of \(\text{MIP}_h\) by an exact and efficient method under suitable assumptions. To estimate its quality, we shall bound the deviation of its objective function value from the optimal value \(v_h\) of \(\text{MIP}_h\) by explicitly computable expressions.

In applications, such small bounds may lead to the decision to accept the feasible point as “close enough to optimal”. Otherwise, a good feasible point may be used, for example, to initialize an appropriate branch-and-bound or branch-and-cut method with a small upper bound on the optimal value, or to start a local search heuristic there (cf., e.g., [9]).

Example 1.1 We shall specify our subsequent results to the case of a mixed-integer linear optimization problem
\[
\text{MILP}_h : \quad \min_{(x,y) \in \mathbb{R}^n \times \mathbb{R}^m} c^T x + d^T y \quad \text{s.t.} \quad Ax + By \leq b
\]
with vectors \(c \in \mathbb{R}^n, d \in \mathbb{R}^m, b \in \mathbb{R}^p\), a \((p,n)\)-matrix \(A\), a \((p,m)\)-matrix \(B\), and a polyhedral set \(D \subseteq \mathbb{R}^n \times \mathbb{R}^m\). Denoting by \(\alpha_i^x, i = 1, \ldots, p, \beta_i^y\), the rows of \(A\) and \(B\), respectively, we obtain \(g_i(x,y) = \alpha_i^x x + \beta_i^y y - b_i, i \in I\). This setting slightly differs from the one in [30], which is further discussed in [31].

Clearly, global Lipschitz conditions hold for all involved functions, and the best possible Lipschitz constant for \(f(x,y) = c^T x + d^T y\) on \(\mathbb{R}^n \times \mathbb{R}^m\), that is, the Lipschitz modulus
\[
\sup_{(x_1,y_1) \neq (x_2,y_2)} \frac{|f(x_1,y_1) - f(x_2,y_2)|}{||(x_1,y_1) - (x_2,y_2)||}
\]
is easily seen to coincide with the dual norm of \((c,d)\), so that we may put
\[
L^f := ||(c,d)||^* := \max \{c^T x + d^T y ||(x,y)|| \leq 1\}.
\]
Under Assumption 2.8 below, \(L^f\) also is the best possible Lipschitz constant for \(f\) on the set \(D\). Likewise, we may choose
\[
L^\beta_i := \|\beta_i\|_1 = ||\beta_i||_1, \quad i \in I.
\]
Of course, the Lipschitz constant for any of the box constraints equals 1. \(\square\)
For an explicit nonlinear example for $MIP_h$ we refer to [31, Ex. 1.2].

Remark 1.2 As mentioned already in [30], the mesh dependence of $MIP_h$ might as well be modeled using the substitution $y = hz$ with $z \in \mathbb{Z}^m$, which leads to the problem

$$MIP'_h: \min_{(x,z) \in \mathbb{R}^n \times \mathbb{Z}^m} f(x, hz) \quad \text{s.t.} \quad g_i(x, hz) \leq 0, \quad i \in I$$

$$(x, hz) \in D$$

$$y^l_h \leq hz \leq y^u_h.$$ 

In particular, the mixed integer linear problem from Example 1.1 may then equivalently be stated as

$$MILP'_h: \min_{(x,z) \in \mathbb{R}^n \times \mathbb{Z}^m} \frac{1}{\pi} c^T x + d^T z \quad \text{s.t.} \quad \frac{1}{\pi} A x + B z \leq \frac{1}{\pi} b$$

$$(x, hz) \in D$$

$$z^l_h \leq z \leq z^u_h$$

with $z^l_h \in (\mathbb{Z} \cup \{-\infty\})^m$ and $z^u_h \in (\mathbb{Z} \cup \{+\infty\})^m$.

While we shall use this reformulation for our numerical illustrations in Section 5, the main advantage of the model without substitution of $y$ is the clear visibility of the granular nature of $y$. Moreover, for $y^l_h \equiv y^l \in (\{0\} \cup \{-\infty\})^m$ and $y^u_h \equiv y^u \in (\{0\} \cup \{+\infty\})^m$ (which covers, e.g., the case of nonnegativity constraints on $y$), the NLP relaxation $\hat{MIP}$ and, in particular, its feasible set $\hat{M}$ and its optimal value $\hat{v}$ are mesh independent (like in [30,31]). We emphasize, however, that our choice of the model is for notational convenience only, and that the alternative modeling would not affect our results below.

Since the construction of a feasible point is NP-hard even for mixed-integer linear optimization problems, many search heuristics are available in the literature, among them the feasibility pump ([1,12,13]), Undercover ([7]), relaxation enforced neighborhood search ([6]), diving strategies ([8]), and many others (see [5, Sec. 6] for a recent survey). Our approach does not fall into any of these categories. One may note some remote resemblance to MILP-based rounding ([25]) or to the feasibility pump, but after solving our auxiliary continuous NLPs we do not need to enforce integrality via solving an MILP, since our NLPs are set up in such a way that roundings of their optimal points are necessarily feasible for the original problem. In particular, we only have to solve purely continuous auxiliary problems.

The remainder of this article is structured as follows. In Section 2 we introduce our main construction, the inner parallel set, as well as the resulting feasible rounding approach (FRA) in two different versions, FRA-ROR and FRA-ROPR. Furthermore we state how the consistency of the inner parallel set may be checked algorithmically. Section 3 provides a-priori and a-posteriori bounds on the deviation from the optimal value $v_h$ of the objective function value of the feasible points generated by FRA-ROR and FRA-ROPR, respectively. Section 4 adds a post processing step which systematically tries to further improve the feasible points generated by FRA-ROR and FRA-ROPR. In Section 5 we present numerical results for large scale knapsack problems. They illustrate that our method is able to outperform standard software. Some final remarks conclude the article in Section 6.
2 The main construction

In the following we shall denote the feasible set of the relaxed problem $\tilde{MIP}_h$ by $\tilde{M}_h$, its optimal value by $\tilde{v}_h$ ($\leq v_h$), and any optimal point by $(\tilde{x}_h', \tilde{y}_h')$. The central object of our technique is the inner parallel set (in the sense of Minkowski) to $\tilde{M}_h$ with respect to $K = \{0\} \times B_\infty(0, \frac{1}{2})$ at distance $h$,

$$\tilde{M}_{-h} := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m | (x, y) + hK \subseteq \tilde{M}_h\},$$

where we put $B_\infty(0, \frac{1}{2}) := \{y \in \mathbb{R}^m | \|y\|_\infty \leq \frac{1}{2}\}$.

In the terminology of [30], the set $\tilde{M}_{-h}$ is also called grid relaxation retract. For the consistency of $\tilde{M}_{-h}$ see Proposition 2.9 below.

For any point $(x, y) \in \mathbb{R}^n \times \mathbb{R}^m$ we call $(\tilde{x}_h, \tilde{y}_h)$ a rounding if

$$\tilde{x}_h = x \quad \text{and} \quad \tilde{y}_h \in h\mathbb{Z}^m, \quad |(\tilde{y}_h)_j - y_j| \leq \frac{h}{2}, \quad j = 1, \ldots, m$$

hold, that is, $y$ is rounded componentwise to a point in the mesh $h\mathbb{Z}^m$, and $x$ remains unchanged. Note that a rounding does not have to be unique, and that any rounding of $(x, y)$ satisfies

$$(\tilde{x}_h, \tilde{y}_h) \in ((x, y) + hK) \cap (\mathbb{R}^n \times h\mathbb{Z}^m). \quad (2.1)$$

Our analysis bases on the following result which constitutes the central part of the proof of [31, Lem. 2.3]. We recall its brief proof here not only for completeness, but also to prepare an improvement for lattice box constraints which were not explicitly present in [31].

**Lemma 2.1** For any $h > 0$ and any point $(x, y) \in \tilde{M}_{-h}$, any of its roundings $(\tilde{x}_h, \tilde{y}_h)$ lies in $M_h$.

**Proof** From (2.1) and the definition of the inner parallel set $\tilde{M}_{-h}$ we obtain $(\tilde{x}_h, \tilde{y}_h) \in \tilde{M}_h \cap (\mathbb{R}^n \times h\mathbb{Z}^m) = M_h$. \(\square\)

For the algorithmic employment of Lemma 2.1 we need a functional description at least of a subset of $\tilde{M}_{-h}$. To this end, we define the set

$$T_{-h} := \{(x, y) \in D_{-h} | g_i(x, y) + L_{g_i} \frac{b_i}{2} \leq 0, \ i \in I, \ y'_u + \frac{h}{2}e \leq y \leq y'_l - \frac{h}{2}e\}$$

with the inner parallel set of $D$ with respect to $K$ at distance $h$,

$$D_{-h} := \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m | (x, y) + hK \subseteq D\},$$

and with the all ones vector $e \in \mathbb{R}^m$. The next result was used to show [31, Lem. 2.4]. Again, we repeat its proof and include the explicit presence of lattice box constraints.

**Lemma 2.2** For any $h > 0$ we have $T_{-h} \subseteq \tilde{M}_{-h}$. 


Proof In the case \( T_{-h} = \emptyset \) the assertion trivially holds. Otherwise, let \( (x, y) \in T_{-h} \).
We have to show
\[
(x, y + h\eta) \in \hat{M}_h = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m | g_i(x, y) \leq 0, \ i \in I, \ (x, y) \in D, \ y_i^L \leq y \leq y_i^U\}
\]
for all \( \eta \in B_\infty \left( 0, \frac{1}{2} \right) \).
In fact, \( T_{-h} \subseteq D_{-h} \subseteq D \) yields \( x \in \text{pr}_D \), so that our Lipschitz assumption on the functions \( g_i, i \in I \), implies for any \( i \in I \) and any \( \eta \in B_\infty \left( 0, \frac{1}{2} \right) \)
\[
g_i(x, y + h\eta) - g_i(x, y) \leq L_{\text{Lips}}^i h \|\eta\|_\infty \leq L_{\text{Lips}}^i \frac{h}{2}.
\]
From the definition of \( T_{-h} \) we thus obtain
\[
g_i(x, y + h\eta) \leq g_i(x, y) + L_{\text{Lips}}^i \frac{h}{2} \leq 0.
\]
Since all box constraints carry Lipschitz constants with value 1, an analogous reasoning leads to
\[
y_i^L \leq y + h\eta \leq y_i^U.
\]
Finally, \( (x, y) \in D_{-h} \) and \( (0, \eta) \in K \) imply \( (x, y + h\eta) = (x, y) + h(0, \eta) \in D \), so that altogether we have shown \( (x, y + h\eta) \in \hat{M}_h \).

Lemma 2.2 immediately give rise to the next result.

Lemma 2.3 For any \( h > 0 \) any rounding of any point from \( T_{-h} \) lies \( M_h \).

For the announced improved treatment of lattice box constraints, observe that in view of \( y_i^L \in h(\mathbb{Z} \cup \{-\infty\})^m \) and \( y_i^U \in h(\mathbb{Z} \cup \{+\infty\})^m \) the constraints \( y_i^L \leq y \leq y_i^U \) in the description of the original feasible set \( M_h \) may be relaxed to \( y_i^L - \frac{1}{2}e \leq y \leq y_i^U + \frac{1}{2}e \) without changing \( M_h \). Hence, although the relaxed feasible set then changes to
\[
\tilde{M}_h = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m | g_i(x, y) \leq 0, \ i \in I, \ (x, y) \in D, \ y_i^L - \frac{1}{2}e \leq y \leq y_i^U + \frac{1}{2}e\},
\]
and the inner parallel set accordingly to \( \tilde{M}'_{-h} \), in the assertion of Lemma 2.2, we may still replace \( M_{-h} \) by the larger set \( \tilde{M}'_{-h} \). The corresponding functional description of a subset of the inner parallel set becomes
\[
T'_{-h} := \{(x, y) \in D_{-h} | g_i(x, y) + L_{\text{Lips}}^i \frac{h}{2} \leq 0, \ i \in I, \ y_i^L \leq y \leq y_i^U\},
\]
and the subsequent improvement of Lemma 2.3 follows.

Proposition 2.4 For any \( h > 0 \) any rounding of any point from \( T'_{-h} \) lies \( M_h \).

In the following it will be important that the assertion of Proposition 2.4 is independent of the actual choice of the relaxed feasible set and its inner parallel set. In fact, we will use the set \( T'_{-h} \) along with the original relaxed feasible set \( \hat{M}_h \).

Proposition 2.4 gives rise to a feasible rounding approach (FRA). To generate “good” points in \( \hat{M}_h \) by this idea, at least the following two methods may be employed.
FRA-ROR (feasible rounding approach by retract-optimize-round):

In this approach we first consider the inner approximation $T^\ell_{-h}$ of the grid relaxation retract, compute an optimal point $(x^\ell_h,y^\ell_h)$ of $f$ over $T^\ell_{-h}$, that is, of the problem

$$P^\ell_h : \min_{(x,y) \in \mathbb{R}^n \times \mathbb{R}^m} f(x,y) \quad \text{s.t.} \quad g_i(x,y) + L_{\text{un}} \frac{h}{2} \leq 0, \ i \in I, \ (x,y) \in D_{-h}, \ y^\ell_h \leq y \leq y^u_h,$$

and then round it to $(\hat{x}^\ell_h,\hat{y}^\ell_h) \in M_h$.

FRA-ROPR (feasible rounding approach by relax-optimize-project-round):

Here, we start with the relaxed problem

$$\overline{\text{MIP}}_h : \min_{(x,y) \in \mathbb{R}^n \times \mathbb{R}^m} f(x,y) \quad \text{s.t.} \quad g_i(x,y) \leq 0, \ i \in I, \ (x,y) \in D, \ y^\ell_h \leq y \leq y^u_h,$$

compute an optimal point $(\tilde{x}^\ell_h,\tilde{y}^\ell_h)$ of $\overline{\text{MIP}}_h$, project it onto $T^\ell_{-h}$, that is, compute an optimal point $(x^\ell_h,y^\ell_h)$ of the projection problem

$$P^\ell_h : \min_{(x,y) \in \mathbb{R}^n \times \mathbb{R}^m} \| (x,y) - (\tilde{x}^\ell_h,\tilde{y}^\ell_h) \| \quad \text{s.t.} \quad g_i(x,y) + L_{\text{un}} \frac{h}{2} \leq 0, \ i \in I,$$

$$(x,y) \in D_{-h}, \ y^\ell_h \leq y \leq y^u_h,$$

and finally round the latter to $(\hat{x}^\ell_h,\hat{y}^\ell_h) \in M_h$.

The problems $P^\ell_h$, $\overline{\text{MIP}}_h$, and $P^\ell_h$ appearing in the two methods may be solved efficiently if, for example, the sets $D$ and, thus, $D_{-h}$ are polyhedral, the functions $f$, $g_i$, $i \in I$, are smooth and convex, and if $\| \cdot \|$ stands for the $\ell_1$-, $\ell_2$-, $\ell_{\infty}$-, or any elliptic norm. If linear constraints describing $D$ are known explicitly, then [30, Lem. 2.3] may be used to compute an explicit description of $D_{-h}$.

Clearly, the computational effort of FRA-ROR is lower than that of FRA-ROPR. However, as we shall see in Section 3, the output of FRA-ROPR is better suited to the formulation of tight error bounds. In any case, one may first run both FRA-ROR and FRA-ROPR and then choose the better of the two outputs.

Example 2.5 For the mixed-integer linear problem $\text{MILP}_h$ from Example 1.1 we obtain

$$T^\ell_{-h} = \{ (x,y) \in D_{-h} | Ax + By - b + \| \hat{\beta} \|_1 \frac{h}{2} \leq 0, \ y^\ell_h \leq y \leq y^u_h \}$$

where, by a slight abuse of notation, $\| \hat{\beta} \|_1$ stands for the vector with entries $\| \beta_i \|_1$, $i = 1, \ldots, p$. The optimization problems $P^\ell_h$ and $P^\ell_h$ thus become a linear optimization problem and a projection problem to a polyhedral set, respectively. Recall that the lattice box constraints cover, in particular, nonnegativity constraints on $y$.

We remark that in the case $D = X \times \mathbb{R}^m$, with some nonempty and closed set $X \subseteq \mathbb{R}^n$, we have $D_{-h} = D$, so that in [30] not only the inclusion from Lemma 2.2 but even the identity $T^\ell_{-h} = \hat{M}_{-h}$ could be shown for the mixed-integer linear case. Lattice box constraints may be included in this result as explained above.

Note that the linearity of $f$ is of minor importance in these observations. For example, a convex function $f$ leads to convex optimization problems $P^\ell_h$ and $\overline{\text{MIP}}_h$ with polyhedral feasible sets. □
Hence, for \( \bar{h} \in H \): max set is positive, and \( g \) in Proposition 2.8] to the presence of lattice box constraints.

**Assumption 2.6** There exist some \( h' > 0 \), bounds \( y^f \in (\mathbb{R} \cup \{-\infty\})^m \) with \( \Delta h \) and \( y^u \in (\mathbb{R} \cup \{+\infty\})^m \) with \( y^u \leq y^u_{h} \) for all \( h \in (0, h'] \).

**Proposition 2.7** Let Assumption 2.6 hold, and let the optimal value \( \bar{h} \) of the problem

\[
H : \max_{(x,y,h) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}} g_i(x,y) + Lg_i \frac{h}{2} \leq 0, \quad i \in I, \quad y^f \leq y \leq y^u, \quad (x,y) \in D_{-h}, \quad 0 \leq h \leq h'
\]

be positive. Then \( T_{-h} \) is nonempty for all \( h \in (0, \bar{h}] \).

**Proof** Let \( (\bar{x}, \bar{y}, \bar{h}) \) be an optimal point of \( H \) and \( h \in (0, \bar{h}] \). Then, due to

\[
\Delta g_i(\bar{x}, \bar{y}) + Lg_i \frac{h}{2} \leq 0, \quad i \in I, \quad y^f \leq y \leq y^u, \quad (x,y) \in D_{-h}, \quad 0 \leq h \leq h'
\]

the point \( (\bar{x}, \bar{y}) \) lies in \( T_{-h} \).

If, again, the sets \( D \) and, thus, \( D_{-h} \) are polyhedral, and the functions \( g_i, i \in I \), are smooth and convex, then the problem \( H \) in Proposition 2.7] is efficiently solvable. Note that the choice \( h' = +\infty \) may be allowed in Assumption 2.6] and that \( H \) may then be unbounded, leading to an arbitrary choice of \( h > 0 \). Furthermore, if the finite supremum of \( H \) is not attained at an optimal point, then \( T_{-h} \neq \emptyset \) for all \( h \in (0, \bar{h}/2] \) is easily seen. Finally, the subsequent Slater type condition guarantees that the feasible set of \( H \) is nonempty for sufficiently small values \( h > 0 \). Here, \( \text{int} D \) stands for the topological interior of the set \( D \).

**Assumption 2.8** The sets \( \hat{M}_h, \ h > 0 \), satisfy a uniform Slater type condition for small \( h \) in the sense that Assumption 2.6] holds, and that there exist some \( (\bar{x}, \bar{y}) \in \text{int} D \) with \( g_i(\bar{x}, \bar{y}) < 0, i \in I \), and \( y^f \leq \bar{y} \leq y^u \).

**Proposition 2.9** Under Assumption 2.8 there exists some \( \bar{h} > 0 \) such that for all \( h \in (0, \bar{h}] \) the set \( T_{-h} \) as well as the feasible set of problem \( H \) from Proposition 2.7 are nonempty.

**Proof** With the Slater point \( (\bar{x}, \bar{y}) \) and \( h' \) from Assumption 2.8 we have \( (2.2) \) for all \( h \in (0, h'] \). Moreover, the value

\[
h'' := -\max \left\{ \frac{2g_i(\bar{x}, \bar{y})}{Lg_i} \mid i \in I, \ Lg_i > 0 \right\}
\]

is positive, and \( g_i(\bar{x}, \bar{y}) + Lg_i \frac{h''}{2} \leq 0, i \in I, \) holds for any \( h \in (0, h''] \). Finally, as the set \( K \) is bounded, there exists some \( h'' > 0 \) with \( (\bar{x}, \bar{y}) + hK \in D \) for all \( h \in (0, h''] \). Hence, for \( \bar{h} := \min(h', h'', h''') > 0 \) and all \( h \in (0, \bar{h}] \) we have \( (\bar{x}, \bar{y}) \in T_{-h} \) and the first assertion is shown. Furthermore, due to the inner inequalities in \( (2.2) \) the point \( (\bar{x}, \bar{y}, \bar{h}) \) also is feasible for \( H \). This proves the second assertion. \( \square \)
Remark 2.10 As a main result of this section we stress that, under appropriate assumptions, the construction of (even good) feasible points for MIP$_h$ can be carried out by exact and efficient methods, as opposed to heuristics. On the other hand, we emphasize that the particular choice $h = 1$ is not possible in Proposition 2.9 if $\hat{M}_1$ does not contain any translate of the set $K = \{0\} \times B_\infty (0, \frac{1}{2})$. This may happen, for example, if $\hat{M}_1$ is a bounded set with too small diameter or if $\hat{M}_1$ is too flat. Such instances corresponding to empty inner parallel sets give rise to the general NP-hardness of the feasibility problem for MIP$_h$.

\[ \square \]

3 Error bounds

For a two-dimensional purely integer linear problem with two inequality constraints, Figure 3.1 illustrates that, on the one hand, the methods FRA-ROR and FRA-ROPR may easily lead to the same feasible point but that, on the other hand, this point may be far from optimal. Modifying the illustrated situation by forcing the angle between the two constraints to become more acute results in examples which move the constructed points arbitrarily far away from an optimal point.

Fig. 3.1 Possible location of the constructed feasible points for the minimization of $y_1$

To evaluate how “good” any feasible point $(\hat{x}_h, \hat{y}_h) \in M_h$ is, a natural choice is to compare its objective function value $\hat{v}_h := f(\hat{x}_h, \hat{y}_h)$ with the optimal value $v_h$ of MIP$_h$. As the latter is unknown, we rather bound the difference $\hat{v}_h - v_h$ in terms of the optimal value of the relaxed problem by

\[ 0 \leq \hat{v}_h - v_h \leq \bar{v}_h - \hat{v}_h. \tag{3.1} \]

For both, the point $(\bar{x}_h, \bar{y}_h)$ and the point $(\tilde{x}_h, \tilde{y}_h)$, this bound can be computed explicitly. In fact, after the computation of $(\bar{x}_h, \bar{y}_h)$ the bound $\bar{v}_h - \bar{v}_h$ is readily available, whereas for evaluating the bound $\bar{v}_h - \hat{v}_h$ of $(\tilde{x}_h, \tilde{y}_h)$ additionally the relaxed problem $\hat{MIP}_h$ has to be solved.
Unfortunately, such an a-posteriori error bound is not useful for controlling the error in the sense that for a given accuracy \( \epsilon > 0 \) the mesh size \( h \) may be a-priori chosen such that \( \tilde{v}_h - v_h < \epsilon \) holds. Hence, in the following we shall derive a-priori error bounds for \( \tilde{v}_h - v_h \) which do not depend on the solutions of auxiliary optimization problems, but merely on the problem data.

For the next result, recall that the equivalence of norms in finite dimensions yields the existence of some constant \( k > 0 \) such that for all \( (x, y) \in \mathbb{R}^p \times \mathbb{R}^m \) we have \( \| (x, y) \| \leq k \| (x, y) \|_\infty \).

**Lemma 3.1** For any \( h > 0 \) let \( (\tilde{x}_h, \tilde{y}_h) \) denote any optimal point of \( \widetilde{\text{MIP}}_h \). Then for any point \( (x, y) \in T'_h \) and any of its roundings \( (\tilde{x}_h, \tilde{y}_h) \), the value \( \tilde{v}_h := f(\tilde{x}_h, \tilde{y}_h) \) satisfies

\[
0 \leq \tilde{v}_h - v_h \leq L^f \left( \| (x, y) - (\tilde{x}_h, \tilde{y}_h) \| + \kappa \frac{\delta}{2} \right).
\]

**Proof** As above, the first inequality follows from Proposition 2.4. To see the second inequality, we use (3.1) and note that the Lipschitz continuity of \( f \) on \( D \) and the triangle inequality yield

\[
\tilde{v}_h - v_h = |f(\tilde{x}_h, \tilde{y}_h) - f(\tilde{x}_h, \tilde{y}_h)| \leq L^f \left( \| (\tilde{x}_h, \tilde{y}_h) - (x, y) \| + \| (x, y) - (\tilde{x}_h, \tilde{y}_h) \| \right).
\]

Due to \( \tilde{x}_h = x_h \), the assertion now follows from

\[
\| (\tilde{x}_h, \tilde{y}_h) - (x, y) \| \leq \kappa \| (\tilde{x}_h, \tilde{y}_h) - (x, y) \|_\infty = \kappa \| \tilde{y}_h - y \|_\infty \leq \kappa \frac{\delta}{2}.
\]

\( \Box \)

### 3.1 An error bound for FRA-ROPR

We first study an error bound for the point \( (\tilde{x}_h^*, \tilde{y}_h^*) \) resulting from FRA-ROPR. Since it is the rounding of the projection \( (x_h^*, y_h^*) \) onto \( T'_h \) of an optimal point \( (\tilde{x}_h^*, \tilde{y}_h^*) \) of \( \text{MIP}_h \), the corresponding term \( \| (x_h^*, y_h^*) - (\tilde{x}_h^*, \tilde{y}_h^*) \| \) in Lemma 3.1 coincides with the distance \( \text{dist}( (\tilde{x}_h^*, \tilde{y}_h^*), T'_h) \) of \( (\tilde{x}_h^*, \tilde{y}_h^*) \) to \( T'_h \).

The latter distance may be bounded above in terms of problem data by employing a global error bound for the system of inequalities describing \( T'_h \). To make the description of \( T'_h \) purely functional, in the sequel we will choose \( D = \mathbb{R}^e \times \mathbb{R}^m \).

To state the global error bound, let \( g \) denote the vector of functions \( g_i, i \in I \), and \( L^g_p \) the vector of Lipschitz constants \( L^g_p, i \in I \), both in \( \mathbb{R}^p \). With the componentwise positive-part operator \( a^+ := (\max\{0, a_1\}, \ldots, \max\{0, a_p\})^T \) for vectors \( a \in \mathbb{R}^p \) we may then define the penalty function \( \| (g(x, y) + \frac{1}{2} L^g_p)^+, (y_h^* - y)^+, (y_h^* - y_h^*)^+ \|_\infty \) of the set \( T'_h \). A global error bound relates the geometric distance to the (consistent) set \( T'_h \) with the evaluation of its penalty function by stating the existence of a constant \( \gamma_h > 0 \) such that for all \( (\tilde{x}, \tilde{y}) \in \mathbb{R}^e \times \mathbb{R}^m \) we have

\[
\text{dist}( (\tilde{x}, \tilde{y}), T'_h) \leq \gamma_h \| (g(\tilde{x}, \tilde{y}) + \frac{1}{2} L^g_p)^+, (y_h^* - \tilde{y})^+, (y_h^* - y_h^*)^+ \|_\infty.
\]

As Hoffman showed the existence of such a bound for any linear system of inequalities in his seminal work [15]. \( \gamma_h \) is also called a Hoffman constant, and the error
bound (3.2) is known as a Hoffman error bound. Short proofs of this result for the polyhedral case can be found in [14,17]. For global error bounds of broader problem classes see, for example, [2,11,18,24,29], and [3,4,27] for surveys. These references also contain sufficient conditions for the existence of global error bounds. To cite an early result for the nonlinear case from [29], if for convex functions $g_i, i \in I$, the set $T'_i$ is bounded and satisfies Slater’s condition, then a global error bound holds.

The next result simplifies the error bound for points $(\hat{x}, \hat{y}) \in \hat{M}_h$. It was used analogously in [31, Th. 3.3] and follows from the subadditivity of the max operator, the monotonicity of the $\ell_\infty$-norm, as well as $g^+(\hat{x}, \hat{y}) = 0$ and $(y_h - \hat{y})^+ = (\hat{y} - y_h)^+ = 0$ for any $(\hat{x}, \hat{y}) \in \hat{M}_h$.

**Lemma 3.2** Let $D = \mathbb{R}^n \times \mathbb{R}^m$, let $T'_i \neq \emptyset$, and let the error bound (3.2) hold with some $\gamma_{-h} > 0$. Then all $(\hat{x}, \hat{y}) \in \hat{M}_h$ satisfy

$$\text{dist}(\hat{x}, \hat{y}, T'_i) \leq \gamma_{-h} \|L_{\infty}\|_{\infty} h^2.$$  

The combination of Lemmata 3.1 and 3.2 yields our first main result in this section.

**Theorem 3.3** Let $D = \mathbb{R}^n \times \mathbb{R}^m$, let $T'_i \neq \emptyset$, and let the error bound (3.2) hold with some $\gamma_{-h} > 0$. Then the objective function value $v^*_h$ of any rounding of any projection onto $T'_i$ of any optimal point of $\hat{MIP}_h$ satisfies

$$0 \leq v^*_h - v_h \leq \|L_{\infty}\|_{\infty} (\gamma_{-h} \|L_{\infty}\|_{\infty} + \kappa) h^2.$$  

**Example 3.4** For the mixed-integer linear problem $MILP_h$ from Example 1.1 we obtain

$$\|L_{\infty}\|_{\infty} = \max_{i \in I} \|B_i\|_1 = \|B\|_{\infty},$$  

where $\|B\|_{\infty}$ denotes the maximal absolute row sum of the matrix $B$. Furthermore, from [15] it is known that for polyhedral constraints not only the global error bound always exists, but also that the corresponding Hoffman constant $\gamma$ may be chosen independently of the right-hand side vector and, thus, in our case independently of $h$. The assumption on the existence of an error bound may thus be dropped from Theorem 3.3, and the remaining assumptions yield

$$0 \leq v^*_h - v_h \leq \|(c, d)\|^* (\gamma \|B\|_{\infty} + \kappa) h^2.$$  

Hence, the error $v^*_h - v_h$ tends to zero at least linearly with $h \to 0$. The same holds, of course, for nonlinear objective functions $f$ under our above Lipschitz assumptions, when $\|(c, d)\|_*^*$ is replaced by $L_f$.  

In the general nonlinear setting Theorem 3.3 shows that the dependence of the error bound on the mesh size $h$ is intimately related to the dependence of the Hoffman constant $\gamma_{-h}$ on $h$. Then the linear decrease of these errors with $h \to 0$, as in the situation of Example 3.3, is possible if Hoffman constants remain bounded under small perturbations of the underlying inequality system. This was shown in [29] for
convex problems under mild assumptions and, under Assumption [2,8] for sufficiently small $h > 0$ it may be applied to the inequalities describing the set

$$T''_{-h} := \{(x,y) \in \mathbb{R}^n \times \mathbb{R}^m | g_i(x,y) + L_{f,\frac{1}{2}} h \leq 0, i \in I, y^i \leq y^h \}$$

along the lines of the proof of [31, Cor. 3.6]. As the inclusion $T''_{-h} \subseteq T'_{-h}$ entails $\text{dist}((\tilde{x},\tilde{y}), T'_{-h}) \leq \text{dist}((\tilde{x},\tilde{y}), T''_{-h})$ for any $(\tilde{x},\tilde{y}) \in \mathbb{R}^n \times \mathbb{R}^m$, this shows the following assertion.

**Corollary 3.5** In addition to the above Lipschitz assumptions on $f$ and $g_i$, $i \in I$, let $D = \mathbb{R}^n \times \mathbb{R}^m$, let the functions $g_i$, $i \in I$, be real-valued convex, let Assumption [2,8] hold, and let $M_h$ be bounded for all sufficiently small $h > 0$. Then for $h \to 0$ the decreasing rate of the error $\tilde{v}_h^f - v_h$ is at least linear.

3.2 An error bound for FRA-ROR

We conclude this section with an analogous error bound analysis for the point $(\tilde{x}_h^f, \tilde{y}_h^f)$ generated by FRA-ROR, which is easier to compute than the point $(\hat{x}_h^f, \hat{y}_h^f)$ from FRA-ROP. To this end, let $L_{\infty} \geq 0$ denote a Lipschitz constant with respect to $y$ uniformly in $x$ for $f$ on $D$ with respect to the $\ell_\infty$-norm (e.g., $L_{\infty} := \kappa L_1$). Then for any point $(x,y) \in T_{-h}'$ and its rounding $(\tilde{x}_h, \tilde{y}_h) \in D$ the identity $x = \tilde{x}_h$ implies

$$|f(x,y) - f(\tilde{x}_h, \tilde{y}_h)| \leq L_{\infty} \|(x,y) - (\tilde{x}_h, \tilde{y}_h)\|_\infty \leq L_{\infty} h^2. \tag{3.3}$$

**Lemma 3.6** For $T_{-h}' \neq \emptyset$ let $(\tilde{x}_h^f, \tilde{y}_h^f)$ denote any rounding of any optimal point $(x_h^f, y_h^f)$ of $f$ over $T_{-h}'$. Then the value $\tilde{v}_h^f = f(\tilde{x}_h^f, \tilde{y}_h^f)$ satisfies

$$0 \leq \tilde{v}_h^f - v_h \leq (\tilde{v}_h^f - v_h) + L_{\infty} h.$$

**Proof** As before, the first inequality stems from Proposition [2,4]. For the proof of the second inequality we write

$$\tilde{v}_h^f - v_h = (\tilde{v}_h^f - \tilde{v}_h^f) + (\tilde{v}_h^f - v_h)$$

and apply (3.3) twice to obtain

$$\tilde{v}_h^f - \tilde{v}_h^f = (f(\tilde{x}_h^f, \tilde{y}_h^f) - f(x_h^f, y_h^f)) + (f(x_h^f, y_h^f) - f(x_h^f, y_h^f)) \leq L_{\infty} h + f(x_h^f, y_h^f) - f(x_h^f, y_h^f).$$

Finally, since $(x_h^f, y_h^f)$ is a minimal point of $f$ over $T_{-h}'$, and $(x_h^f, y_h^f)$ is some point in $T_{-h}'$, we arrive at

$$f(x_h^f, y_h^f) - f(x_h^f, y_h^f) \leq 0,$$

and the assertion is shown.

Examples show that the upper bound given in Lemma 3.6 is tight.

The combination of Theorem 3.3 with Lemma 3.6 immediately yields our next main result.
Theorem 3.7 Let $D = \mathbb{R}^n \times \mathbb{R}^m$, let $T_{-h} \neq \emptyset$, and let the error bound $(3.2)$ hold with some $\gamma > 0$. Then the objective function value $\bar{v}_h^f$ of any rounding of any optimal point of $f$ over $T_{-h}$ satisfies

$$0 \leq \bar{v}_h^f - v_h \leq L^f (\gamma \|L\|_\infty + \kappa) \frac{h}{2} + L^f h.$$ 

Example 3.8 For the mixed-integer linear problem MILP, from Example 1.1 we may choose $L^h = \|d\|_1$. As in Example 3.4 the Hoffman constant $\gamma$ may be chosen independently of $h$, the assumption on the existence of an error bound may be dropped from Theorem 3.7 and the remaining assumptions yield

$$0 \leq \bar{v}_h^f - v_h \leq \|(c, d)\|^* (\gamma \|B\|_\infty + \kappa) \frac{h}{2} + \|d\|_1 h.$$ 

Again, the error tends to zero at least linearly with $h \to 0$, as it would also do for nonlinear objective functions $f$ under our above Lipschitz assumptions, when $\|(c, d)\|^*$ is replaced by $L^f$. \hfill \square

For the nonlinear case under appropriate convexity and regularity assumptions, of course Corollary 3.5 holds analogously, that is, for $h \to 0$ also the decreasing rate of the error $\bar{v}_h^f - v_h$ is at least linear.

4 A post processing step for FRA-ROR and FRA-ROPR

In this section we add a post processing step to FRA-ROR and FRA-ROPR, respectively, which may improve the generated feasible point even further with respect to its objective function value. We discuss this technique only for FRA-ROR, since the modification to FRA-ROPR is straightforward.

In fact, so far we have used that, on the one hand, it is unlikely that some rounding of the computed optimal point $(\hat{x}_h^*, \hat{y}_h^*)$ of the relaxed problem $MIP_h$ is feasible for the original problem $MIP_0$, and that, on the other hand, definitely any rounding $(\hat{x}_h^*, \hat{y}_h^*)$ of $(x_h^*, y_h^*)$ lies in $M_0$. However, Figure 4.1 illustrates for the case $n = m = 1$ that roundings of points from their connecting line segment $[(\hat{x}_h^*, \hat{y}_h^*), (x_h^*, y_h^*)]$ may well also be feasible. Since the inequality $f(\hat{x}_h^*, \hat{y}_h^*) \leq f(x_h^*, y_h^*)$ is strict under mild assumptions, roundings of points from $[(\hat{x}_h^*, \hat{y}_h^*), (x_h^*, y_h^*)]$ may even generate points in $M_0$ with a better objective function value than $(x_h^*, y_h^*)$.

Our proposition for a post processing step systematically checks if this is the case. In particular we will see that, as in Figure 4.1, also in general the set of roundings is a disconnected manifold composed of line segments which are constant with respect to the integer variables $y$, and we will determine the corresponding switching points. In the subsequent analysis, again one may think of $h$ to be fixed, for example to $h = 1$.

With the direction vectors

$$\xi := x_h^* - \hat{x}_h^* \quad \text{and} \quad \eta := y_h^* - \hat{y}_h^*$$

let us parametrize the line segment as

$$[(\hat{x}_h^*, \hat{y}_h^*), (x_h^*, y_h^*)] = \left\{ (\hat{x}_h^*, \hat{y}_h^*) + t(\xi, \eta) \mid t \in [0, 1] \right\}.$$
Then, for any $t \in [0, 1]$, $(\tilde{x}(t), \tilde{y}(t)) \in \mathbb{R}^n \times h\mathbb{Z}^m$ is a rounding of the point

$$(x(t), y(t)) = (\tilde{x}_h, \tilde{y}_h) + t(\xi, \eta) \in [(\tilde{x}_h, \tilde{y}_h), (x^t_h, y^t_h)]$$

if and only if we have

$$\tilde{x}(t) = x(t) \quad \text{and} \quad |y_j(t) - \tilde{y}_j(t)| \leq \frac{h}{2}, \ j = 1, \ldots, m.$$ 

For a more explicit representation of roundings, let us introduce Gaussian brackets with respect to the mesh $h\mathbb{Z}$, that is

$$[a]_h := \max\{y \in h\mathbb{Z} | y \leq a\} \quad \text{and} \quad [a]_h := \min\{y \in h\mathbb{Z} | y \geq a\}$$

for $a \in \mathbb{R}$. Then for any $t \in [0, 1]$ each value $y_j(t), \ j = 1, \ldots, m,$ lies in the interval $[\lfloor y_j(t) \rfloor_h, \lceil y_j(t) \rceil_h]$ and $\tilde{y}(t)$ is a rounding of $y(t)$ if and only if for each $j \in \{1, \ldots, m\}$ we have

$$\tilde{y}_j(t) = \begin{cases} [y_j(t)]_h, & y_j(t) - [y_j(t)]_h \leq \frac{h}{2} \\ \lceil y_j(t) \rceil_h, & y_j(t) - [y_j(t)]_h > \frac{h}{2} \\ \in \lfloor [y_j(t)]_h, \lceil y_j(t) \rceil_h \rfloor, & y_j(t) - [y_j(t)]_h = \frac{h}{2} \end{cases} \quad (4.1)$$

The roundings of $y(t)$ thus constitute the set

$$\text{Rd}(y(t)) := \{\tilde{y}(t) \in h\mathbb{Z}^m | (4.1) \text{ holds for all } j \in \{1, \ldots, m\}\}. \quad (4.2)$$

In particular, the set of roundings may contain up to $2^m$ elements. Since the third case in $(4.1)$ gives rise to such ambiguous roundings, let us define the active set

$$J(t) = \{j \in \{1, \ldots, m\} | y_j(t) - [y_j(t)]_h = \frac{h}{2}\}$$

for any $t \in [0, 1]$.

For $\bar{t} \in [0, 1]$ with $J(\bar{t}) = \emptyset$ not only the rounding $(x(\bar{t}), \tilde{y}(\bar{t}))$ of $(x(\bar{t}), y(\bar{t}))$ is unique but, due to the continuity of $y(\cdot)$, also for all $t$ from some neighborhood of $\bar{t}$ (relative to $[0, 1]$) the rounding of $(x(t), y(t))$ is unique, and it coincides with $(x(t), \tilde{y}(t))$. If this neighborhood possesses endpoints $t_-$ and $t_+$ within the interval $[0, 1]$, then at these we must necessarily have $J(t_\pm) \neq \emptyset$. This shows that the set of rounded points possesses a piecewise constant structure with respect to $y$. 

---

Fig. 4.1 Idea of the post processing step for some MILP, with objective function $x + y$
Lemma 4.1

a) The parameter value \( \bar{t} \in [0, 1] \) is a switching point if and only if there exists some \( j \in J(\bar{t}) \) with \( \eta_j \neq 0 \).

b) When passing a switching point \( \bar{t} \) for increasing values of \( t \), the set of roundings changes from

\[
\text{Rd}(y(\bar{t})) \setminus \{ \{ |y_j(\bar{t})|_h \mid j \in J(\bar{t}), \eta_j > 0 \} \cup \{ |y_j(\bar{t})|_h \mid j \in J(\bar{t}), \eta_j < 0 \} \}
\]

to

\[
\text{Rd}(y(t)) \setminus \{ \{ |y_j(t)|_h \mid j \in J(t), \eta_j > 0 \} \cup \{ |y_j(t)|_h \mid j \in J(t), \eta_j < 0 \} \}.
\]

Proof The cases in which either \( J(\bar{t}) = \emptyset \) or \( J(\bar{t}) \neq \emptyset \) with \( \eta_j = 0, j \in J(\bar{t}) \), holds, rule out that \( \bar{t} \) is a switching point, as discussed above. Conversely, consider some \( j \in J(\bar{t}) \) with \( \eta_j \neq 0 \). Then we have

\[
\frac{h}{2} = y_j(\bar{t}) - |y_j(\bar{t})|_h = (\hat{y}_j^* + \bar{\eta}_j - |y_j(\bar{t})|_h).
\]

On the other hand, any \( t \) from a sufficiently small neighborhood of \( \bar{t} \) satisfies

\[
y_j(t) = (\hat{y}_j^* + \bar{\eta}_j + t \eta_j) = y_j(\bar{t}),
\]

but still

\[
|y_j(t)|_h = |(\hat{y}_j^* + \bar{\eta}_j + t \eta_j)|_h = |y_j(\bar{t})|_h.
\]

This results in \( y_j(t) = |y_j(\bar{t})|_h \neq y_j(\bar{t}) - |y_j(\bar{t})|_h = \frac{h}{2} \) so that, according to (4.1), \( \bar{t} \) is a switching point, and part a) is shown. The assertion of part b) immediately follows from the above considerations by using (4.3) for \( j \in J(\bar{t}) \) with the sign of \( \eta_j \) taken into account.

To determine the set \( S = \{ s \in [0, 1] \mid (\hat{y}_j^* + \bar{\eta}_j + t \eta_j) = y_j(\bar{t}) \} \) we may compute the set of switches \( S_j \) stemming from this coordinate direction separately, and put \( S = \bigcup_{j=1}^m S_j \). By Lemma 4.1b), for any \( j \) with \( \eta_j = 0 \) we find \( S_j = \emptyset \), and for \( \eta_j \neq 0 \)

\[
S_j = \{ t \in [0, 1] \mid (\hat{y}_j^* + \bar{\eta}_j + t \eta_j - |(\hat{y}_j^* + \bar{\eta}_j)|_h = \frac{h}{2} \}.
\]

The switching points in \( S_j \) thus are the solutions

\[
t = \frac{\frac{h}{2} - (\hat{y}_j^* + \bar{\eta}_j + \ell)}{\eta_j}
\]

of the equations

\[
(\hat{y}_j^* + \bar{\eta}_j - \ell) = \frac{h}{2}
\]

with \( \ell \in \mathbb{Z} \) chosen such that \( 0 \leq t \leq 1 \) holds. To account for the different signs of \( \eta_j \) in a closed formula, we may formulate the corresponding result using the positive and negative parts \( a^+ := \max(0, a) \) and \( a^- := \min(0, a) \) of a real number \( a \in \mathbb{R} \). Note that the subsequent formula also covers the case \( \eta_j = 0 \).
Then the set of roundings of points from the line segment
\[
\mathcal{S} = \bigcup_{j=1}^{m} \left\{ \frac{\frac{h}{2} - (\hat{y}_j^*)_j + \ell}{\eta_j} \in h\mathbb{Z}, \quad \left(\hat{y}_j^*\right)_j + \eta_j^- \leq \ell \leq \left(\hat{y}_j^*\right)_j + \eta_j^+ \right\}.
\]

The next result immediately follows from Lemma 4.1).

**Proposition 4.2** The set of switching points is given by
\[
\mathcal{S} = \{ t_k \mid 1 \leq k \leq r \} \text{ with } r \in \mathbb{N}_0 \text{ as well as } t_0 = 0 \text{ and } t_{r+1} = 1. \quad \text{Moreover, for any point } \hat{y}_0^* \in \text{Rd}(\hat{y}_h^*) \text{ define}
\]
\[
\hat{y}_{j+1}^* = \begin{cases} \hat{y}_j^* + \text{sign}(\eta_j)h, & j \in J(t_{k+1}), \\ \hat{y}_j^*, & j \neq J(t_{k+1}). \end{cases}
\]

Then the set of roundings of points from the line segment \([(\hat{y}_h^*,\hat{y}_h^+), (x_h^*,y_h^+)]\) coincides with
\[
\bigcup_{y^* \in \text{Rd}(\hat{y}_h^*)} \bigcup_{r=0}^{r} \left\{ (\hat{y}_h^* + t\hat{\xi}, \hat{y}_h^*) \mid t \in [t_k, t_{k+1}] \right\}.
\]

Observe that, by definition of \(\mathcal{S}\), all intervals \([t_k, t_{k+1}]\) with \(1 \leq k \leq r-1\) possess positive length. In particular, switching points which simultaneously result from discontinuities with respect to different components \(j\) are listed in \(\mathcal{S}\) only once. On the other hand, if \(t_0 = 0\) or \(t_{r+1} = 1\) happen to be switching points, the intervals \([t_0, t_1]\) and \([t_r, t_{r+1}]\), respectively, have zero length. We emphasize that Proposition 4.3 also covers these special cases. Furthermore, note that by Proposition 4.4, the set of roundings is unique on the interiors of the intervals \([t_k, t_{k+1}]\), \(k = 0, \ldots, r\), if the rounding of \(\hat{y}_h^*\) is unique.

It remains to check whether some connected component of the set of roundings from Proposition 4.3 contains a point from \(\hat{M}_h\) and, thus, from \(M_h\). To even find \textit{good} feasible points, this may be carried out by restricting the relaxed problem \(\hat{MIP}_h\) to the single line segments and solve
\[
P^k : \min_{t \in \mathbb{R}} f(\hat{y}_h^* + t\hat{\xi}, \hat{y}_h^*) \quad \text{s.t.} \quad (\hat{y}_h^* + t\hat{\xi}, \hat{y}_h^*) \in \hat{M}_h, \quad t_k \leq t \leq t_{k+1}
\]
for all \(\hat{y}_0^* \in \text{Rd}(\hat{y}_h^*)\) and \(k = 0, \ldots, r\). If no feasible points exist on a line segment, the above problem is, of course, inconsistent. Again, the treatment of these problems is efficiently possible under the above convexity or linearity assumptions. If a monotonicity property of \(f\) along the line segments is guaranteed (as for linear objective functions), for given \(\hat{y}_0^* \in \text{Rd}(\hat{y}_h^*)\) it is sufficient to search feasible points for increasing values of \(t\) until the first one is found, as this will also be the best one. This monotonicity may often be expected. Moreover, the point \(\hat{y}_0^* \in \text{Rd}(\hat{y}_h^*)\) may be expected to be uniquely determined. If it is not, it seems plausible to choose some rounding such that \((\hat{y}_h^*, \hat{\xi})\) minimizes a penalty function of \(\hat{M}_h\). For this reason, subsequently we shall again assume \(D = \mathbb{R}^n \times \mathbb{R}^m\), and use the penalty function
\[
\rho(\hat{x}, \hat{y}) = \|g^+(\hat{x}, \hat{y}), (\hat{y}_h^*, \hat{\xi})^+, (\hat{y}^* - \hat{y})^+\|_\infty
\]
of $\tilde{M}_h$. Our resulting proposition for the post processing step is formulated in Algorithm 4.1.

Observe that the while loop in Algorithm 4.1 always terminates with a consistent set, since for $t_{r+1} = 1$ all roundings are feasible. Under the mentioned monotonicity property of $f_i$, the output of Algorithm 4.1 will satisfy $f(\bar{x}, \bar{y}) \leq f(\tilde{x}_i^r, \tilde{y}_i^r)$ for some rounding $(\tilde{x}_i^r, \tilde{y}_i^r)$ of $(\tilde{x}_i^r, \tilde{y}_i^r)$. If unexpectedly this inequality is violated, of course one may invest computing time into checking further problems $P^r$ for good feasible points. As Proposition 4.3 identifies all connected components of the set of roundings, with the corresponding computational effort even all of them may be checked for feasible points.

Example 4.4 For the mixed-integer linear optimization problem MILP$_h$ with $D = \mathbb{R}^n \times \mathbb{R}^m$ we obtain

$$\tilde{M}_h = \{(x, y) \in \mathbb{R}^n \times \mathbb{R}^m | Ax + By \leq b, y_h^l \leq y \leq y_h^u\}$$
and thus
\[
\{ t \in [t_k, t_{k+1}] \mid (\hat{x}_h^t + t\xi, \hat{y}_h^t) \in \hat{M}_h \}
\]
\[
= \{ t \in [t_k, t_{k+1}] \mid tA^T \xi \leq b - A\hat{x}_h^t - B\hat{y}_h^t \},
\]
\[
y_h^t \leq \hat{y}_h^t \}
\]
\[
= \left\{ \begin{array}{ll}
\max \left\{ t_k, \max_{i \in I} \frac{b_i - a_i^T \hat{x}_h^t - \beta_i^T \hat{y}_h^t}{a_i^T \xi} \right\}, & \min \left\{ t_{k+1}, \min_{i \in I^c} \frac{b_i - a_i^T \hat{x}_h^t - \beta_i^T \hat{y}_h^t}{a_i^T \xi} \right\} , \\
\text{if} & b_i - a_i^T \hat{x}_h^t - \beta_i^T \hat{y}_h^t \geq 0, \ i \in I_0, \\
\emptyset, & \text{else},
\end{array} \right.
\]
where we put
\[
I_{-/+} = \{ i \in I \mid a_i^T \xi < / = / > 0 \}.
\]
Consequently, the while loop in Algorithm 4.1 terminates iff the conditions
\[
\max \left\{ t_k, \max_{i \in I} \frac{b_i - a_i^T \hat{x}_h^t - \beta_i^T \hat{y}_h^t}{a_i^T \xi} \right\} \leq \min \left\{ t_{k+1}, \min_{i \in I^c} \frac{b_i - a_i^T \hat{x}_h^t - \beta_i^T \hat{y}_h^t}{a_i^T \xi} \right\} ,
\]
\[
b_i - a_i^T \hat{x}_h^t - \beta_i^T \hat{y}_h^t \geq 0, \ i \in I_0,
\]
\[
y_h^t \leq \hat{y}_h^t \]
simultaneously hold. Furthermore, due to
\[
f(\hat{x}_h^t + t\xi, \hat{y}_h^t) = c^T \hat{x}_h^t + tc^T \xi + d^T \hat{y}_h^t,
\]
an optimal point in line 13 of Algorithm 4.1 is
\[
t^* = \left\{ \begin{array}{ll}
\max \left\{ t_k, \max_{i \in I} \frac{b_i - a_i^T \hat{x}_h^t - \beta_i^T \hat{y}_h^t}{a_i^T \xi} \right\}, & \text{if} c^T \xi \geq 0 , \\
\min \left\{ t_{k+1}, \min_{i \in I^c} \frac{b_i - a_i^T \hat{x}_h^t - \beta_i^T \hat{y}_h^t}{a_i^T \xi} \right\} , & \text{if} c^T \xi < 0 .
\end{array} \right.
\]
Note that the inequality
\[
0 \leq f(x_h^t, y_h^t) - f(\hat{x}_h^t, \hat{y}_h^t) = c^T \xi + d^T \eta
\]
is too weak to exclude the second case in the above formula. In fact, appropriate modifications of the optimization problem sketched in Figure 5.1 show that $c^T \xi$ may possess an arbitrary sign. \hfill \Box

5 Numerical results for knapsack problems

In order to demonstrate the effectiveness of the feasible rounding approach we consider the celebrated knapsack problem which was introduced in [10] and is an NP-hard optimization problem (cf. [15, pp. 483-491]). In its original formulation, which is also called the 0-1 knapsack problem, all decision variables are binary. The bounded knapsack problem (BKP) is a generalization of the 0-1 knapsack problem where it is possible to pick more than one piece per item, that is, the integer decision variables may be not binary. A possible numerical approach to bounded knapsack problems is
to transform them into equivalent 0-1 knapsack problems for which efficient solution techniques exist. In contrast to this approach we exploit the finer granularity of the BKP and obtain very good feasible points by applying FRA-ROR to test instances of the bounded knapsack problem. Furthermore, the post processing technique in Algorithm 4.1 improves the results of FRA-ROR considerably.

In the bounded knapsack problem we have $m \in \mathbb{N}$ item types and denote the value and weight of item $j \in \{1, \ldots, m\}$ by $v_j$ and $w_j$, respectively. Furthermore, there are at most $b_j \in \mathbb{N}$ units of item $j$ available and the capacity of the knapsack is given by $c > 0$. By maximizing the total value of all items in the knapsack we arrive at the purely integer optimization problem

$$\text{BKP}: \max_{y \in \mathbb{Z}^m} \sum_{j=1}^{m} v_j y_j \quad \text{s.t.} \quad \sum_{j=1}^{m} w_j y_j \leq c, \quad 0 \leq y_j \leq b_j, \quad j = 1, \ldots, m.$$ 

In order to obtain hard test examples of the BKP we create so-called strongly correlated instances (cf. [28] for an analogous treatment in the context of 0-1 knapsack problems), that is, the weights $w_j$ are uniformly distributed in the interval $[1, 10000]$ and we have $v_j = w_j + 1000$. Furthermore, $b_j, j \in \{1, \ldots, m\}$, is uniformly distributed within the set $\{1, \ldots, U\}$ for an integer upper bound $U \in \mathbb{N}$ and in order to avoid trivial solutions we set $c = \sigma \sum_{j=1}^{m} w_j b_j$ for some $\sigma \in (0, 1)$. Note that the granularity of the BKP is controlled by the upper bounds $b_j, j \in \{1, \ldots, m\}$ and $\sigma$, and the problem actually is stated in the transformed version $(M)ILP'_h$ from Remark [1,2]. The inner parallel set of the BKP is given by

$$T' := \{y \in \mathbb{R}^m \mid \sum_{j=1}^{m} w_j y_j \leq c - \frac{1}{2} \sum_{j=1}^{m} w_j, \quad 0 \leq y_j \leq b_j, \quad j = 1, \ldots, m\},$$

and we see that $T'$ is nonempty if and only if $c - \frac{1}{2} \sum_{j=1}^{m} w_j \geq 0$ holds. For our specific choice of $c$ the latter is equivalent to

$$\sum_{j=1}^{m} w_j (\sigma b_j - \frac{1}{2}) \geq 0.$$ 

In particular, $T'$ may be empty for small values of $\sigma$ and $b_j, j \in \{1, \ldots, m\}$. In the remainder of this section we set $\sigma = \frac{1}{2}$.

In our numerical analysis, FRA-ROR is implemented in MATLAB R2016a and all arising optimization problems are solved with GUROBI 6.5. The inner parallel set $T'$ is nonempty in all created test instances which is due to the fact that we choose $U \geq 5$. All tests are run on a personal computer with four cores à 3.2 GHz and 12 GB RAM.

In Table [5,1] we consider the relative optimality gap

$$\frac{\hat{v} - \hat{v}}{\hat{v}}$$

of FRA-ROR applied to different instances of the BKP without post processing. The results seem to indicate that the optimality gap is independent of the problem size $m$. However, we see a strong dependency of the optimality gap on the upper bound $U$. 


Table 5.1 Relative optimality gap of FRA-ROR without post processing for different choices of $U$ and $m$

<table>
<thead>
<tr>
<th>$m$</th>
<th>5</th>
<th>10</th>
<th>100</th>
<th>1000</th>
<th>10000</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>4.35e-01</td>
<td>2.27e-01</td>
<td>2.43e-02</td>
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<td>2.45e-04</td>
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<td>4.62e-01</td>
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<td>2.70e-02</td>
<td>2.72e-03</td>
<td>2.72e-04</td>
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<tr>
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<tr>
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<td>2.41e-01</td>
<td>2.60e-02</td>
<td>2.62e-03</td>
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</tr>
<tr>
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<td>4.47e-01</td>
<td>2.40e-01</td>
<td>2.59e-02</td>
<td>2.61e-03</td>
<td>2.61e-04</td>
</tr>
</tbody>
</table>

This is caused by the fact that $U$ implicitly controls the granularity of our optimization problem which plays a crucial role in the error bound obtained for FRA-ROR. Note that the error bound given in Example 3.8 actually bounds the absolute relaxation error, and that this bound decreases linearly with refined granularity. Thus, for the current setting this result predicts a hyperbolic decrease of the relative relaxation error with increasing values of $U$. This is confirmed by Figure 5.1.

The relative optimality gap of FRA-ROR without post processing is poor for small values of $U$. However, using the post processing procedure from Algorithm 4.1 it is possible to obtain feasible points with a very small optimality gap as we see in Table 5.2. This fact is also illustrated in Figure 5.1.

Table 5.2 Relative optimality gap of FRA-ROR with post processing for different choices of $U$ and $m$

<table>
<thead>
<tr>
<th>$m$</th>
<th>5</th>
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<th>1000</th>
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<td>2.67e-10</td>
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Fig. 5.1 Optimality gap for $m = 1000$ and different choices of $U$ with and without post processing
A feasible rounding approach for mixed-integer nonlinear optimization problems

As mentioned above, solving the BKP to optimality is an NP-hard optimization problem. Instead, for nonempty inner parallel sets the main effort of our feasible rounding approach consists of solving a continuous linear optimization problem which can be done in polynomial time. This fact is demonstrated in Table 5.3 and Figure 5.2 where we see that especially for the larger test instances FRA-ROR is able to find very good feasible points in reasonable time. FRA-ROR with post processing yields feasible points for the largest test instances with relative optimality gaps between $3.32 \cdot 10^{-7}$ and $2.67 \cdot 10^{-10}$ (cf. Table 5.2), that is, the additional time that GUROBI needs to find the global optimal point only yields a marginal benefit.

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<table>
<thead>
<tr>
<th>$U$</th>
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<th>100</th>
<th>1000</th>
<th>10000</th>
</tr>
</thead>
<tbody>
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<td>0.016</td>
<td>0.003</td>
<td>0.031</td>
<td>0.001</td>
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<tr>
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<td>0.006</td>
<td>0.016</td>
<td>0.005</td>
<td>0.016</td>
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<td>2.063</td>
<td>0.755</td>
<td>2.109</td>
<td>0.721</td>
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</table>

Table 5.3 Computing time in seconds for FRA-ROR (left) and GUROBI (right) with post processing for different choices of $U$ and $m$

6 Conclusions

In this article, two feasible rounding approaches are introduced which are able to compute good feasible points for mixed-integer nonlinear optimization problems. Both techniques make use of purely continuous optimization problems over the inner parallel set of the relaxed feasible set which possesses the crucial property that rounding of any of its points sustains feasibility for the original problem. In particular, no
mixed-integer auxiliary problems have to be treated. For assessing the actual quality of the generated feasible points, their optimality gap is estimated by error bounds for both methods. Furthermore, a post processing step is presented which may improve the obtained feasible points considerably. As a numerical illustration, we show that a feasible rounding approach computes feasible points of the bounded knapsack problem with a very small optimality gap. Possible adaptations of the presented feasible rounding approaches to optimization problems with void inner parallel sets are left for future research.

References

A feasible rounding approach for mixed-integer nonlinear optimization problems


