Exact algorithms for bi-objective ring tree problems with reliability measures

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Abstract. We introduce bi-objective models for ring tree network design with a focus on network reliability within telecommunication applications. Our approaches generalize the capacitated ring tree problem (CRTP) which asks for a partially reliable topology that connects customers with different security requirements to a depot node by combined ring and tree graphs. While the CRTP aims at optimizing the edge installation costs, we propose four alternative, reliability-oriented objective functions. To that end, we consider network breakdown due to single-edge failures and incorporate the overall number of tree customers and tree edges, the maximal number of subtree customers and the maximal number of tree hops from rings as additional measures. We develop mathematical multi-commodity flow formulations for the corresponding bi-objective problems and identify relationships between the new objectives. For identifying the Pareto fronts, we apply an \( \epsilon \)-constraint method based on integer programming. To increase the computational efficiency, we employ local search heuristics in order to tighten upper bounds and elaborate valid inequalities to strengthen lower bounds in the subproblems. In a computational study we report results, illustrate solution network topologies and extensively analyze the algorithm performance for instances from the literature.

Keywords: capacitated ring tree problem, bi-objective optimization, Steiner tree, reliable network design, mathematical programming, telecommunications

1 Introduction

1.1 Motivation

In telecommunication network design, the minimization of costs for constructing infrastructure is of major importance since it typically represents the predominant cost factor. However, alternative measures such as, e.g., coverage, lead
times or stability are of high practical relevance in order to provide a competitive service level to the customers. Especially the loss of connection for a subset of customers is a serious incident that telecommunication providers seek to avoid by implementing reliable network structures. Due to high infrastructure cost, in particular in wire-based networks, it might not be feasible to install complete redundancy in practice. Thus, a compromise between cost- and reliability-oriented policy is required.

In this work, the failure of a single connection and the implications for the communication are considered within the initial design of the network. In our approach we group customers by their valuation (or need) of a highly stable network service. Customers that require this augmented stability (type 2) will be willing to pay an increased charge and receive special consideration by being connected redundantly. For the remaining customers it is assumed that simple connectivity is sufficient (type 1). A combinatorial optimization model which follows this approach is the NP-hard capacitated ring tree problem (CRTP) \cite{13,11,12}. The solution of a CRTP is a hybrid structure called ring tree, which is a connected graph, consisting of rings that intersect in a single depot node and trees that can be either rooted at this depot or at one of the ring nodes. Type 2 customers have to be located on rings in a solution and thus remain connected in the case of any single edge failure. Type 1 customers may either be tree or ring nodes and non-customer (Steiner) nodes may be used as intermediate nodes anywhere within the ring trees. Moreover, practically relevant capacity restrictions that limit the number of customers in a ring tree as well as the number of ring trees are imposed.

In the CRTP it is assumed that the elevation of the reliability for the type 1 customers can be accomplished by their inclusion in the rings. Since this implies a significant cost increase it is necessary to identify partial reliability improvements for selected customers while measuring the rise of network costs. The following questions arise in this context. It is not obvious how to measure the risk within such a partially reliable network. Hence, which are suitable measures that are of practical relevance? How can we incorporate such an additional objective into a cost-oriented model? Moreover, what solution methods can be used to solve such problems to optimality?

1.2 Contribution

In this work, we introduce four alternative criteria to measure reliability within bi-level ring tree networks and study the relation between these functions. More detailed, we identify the overall number of tree customers and tree edges, the maximal number of subtree customers and the maximal number of hops from a tree customer to a ring as secondary, reliability-oriented objectives. Corresponding bi-objective optimization problems are defined that combine each of these criteria with the overall edge cost objective. We elaborate novel formulations based on multi-commodity flows and develop a generic $\epsilon$-constraint method for determining the Pareto fronts in the fashion of posteriori methods. Our algorithm iteratively solves single-objective variants of the CRTP which incorporate
additional $\epsilon$-constraints. In order to speed up the procedures, we apply upper bounding techniques based on local search and valid inequalities to strengthen the lower bounds. In a computational study we report the performance of our approach for a set of literature instances.

After providing an overview of related work in the literature in the remainder of this section, the single-objective CRTP is formally introduced in Section 2 and is formulated as a mixed-integer program. The reliability oriented objectives, their relationships and the corresponding bi-objective optimization models and formulations are described in Section 3. Our exact $\epsilon$-constraint approach together with accelerating upper and lower bounding techniques is given in Section 4. We close this paper with computational results in Section 5 and some conclusions in Section 6.

1.3 Related work

In the area of network design, multi-objective ideas are applied to various models, an introduction is given by [22] and [17]. In their approaches, alternative objectives consider the network diameter and the maximum degree of nodes in the network. Recent references consider multi-objective network design for application in transport or supply chain networks, see, e.g., [33, 30, 24, 4]. The literature in this field is extensive and we do not aim at giving a comprehensive review here. Rather, as the CRTP joins ring and tree structures, a particular focus is set on multi-objective approaches in ring- or tree-network design. The problem of finding multi-objective minimum spanning trees (MST) is, e.g., considered by [29] and [25]. For each edge, alternative weights are defined, referring to different metrics such as distance or time. Alternative objectives are derived from these weights, see, [10], for a survey on multi-objective MST. An extension of the MST problem is the Steiner tree problem (STP) where only a subset of nodes, the set of terminal nodes, has to be connected. The remaining nodes, the Steiner nodes, may be included into the solution in order to reduce the total costs. The multi-objective version of the STP is obtained analogous to the multi-objective MST problem, see, [23]. However, alternative versions include network capacity [32] or telecommunication-related objectives like total network length or Quality of Service [19]. Recently, [18] give a comparison of five different methods for a bi-objective price-collecting Steiner tree problem.

In ring-network design, multi-objective formulations for the TSP exist. Analog to the multi-objective MST, alternative objectives are generated by defining multiple edge weights, see, e.g., [6, 27]. The prize-collecting TSP and the orienteering problem are variants of the TSP which include profits. At each node, a reward is collected, however it is not requested to visit all nodes. As a bi-objective variant of these two approaches, the TSP with profits considers two objective functions, the minimization of the total travel costs as well as the maximization of the collected profit. For this problem, [8] apply an $\epsilon$-constrained method and [13] presents an multi-objective evolutionary algorithm. An overview over further approaches to the multi-objective TSP is provided by [21].
The vehicle routing problem (VRP) is an extension of the TSP towards multiple vehicles. Alternative objectives for the VRP include the consideration of tour balancing or of time-related aspects. A survey on the multi-objective VRP is given by [16]. Moreover, the multi-objective VRP is extended in various ways, e.g., by including time windows [9, 8]. The traveling purchaser problem (TPP) is related to the TSP and requires the purchase of products that are available at different places for different prices. A bi-objective version is described by [28]. The bi-objective version splits the original objective and defines the minimization of the travel costs and the minimization of the total purchase cost as two separate objectives. A hybrid method combining weighted sum method with the $\epsilon$-constraint method is applied by [28].

In the related capacitated ring star problem (CRSP) [1], a ring is to be located in a network and all non-ring nodes are to be assigned to the ring via a direct link such that the costs for establishing the ring and for the node assignments are minimized. Differing from the CRTP, non-trivial tree structures are not feasible for assigning non-ring nodes to the ring. The CRSP will appear as a special case of a subproblem that we solve in the method presented in this work. [31] and [20] discuss a bi-objective ring star problem that includes the minimization of both ring costs and customer assignment costs.

Regarding models that address the analysis of network reliability without imposing a specific structure we refer to [2]. Moreover, under the assumption that network breakdowns can be in relationship, e.g., in the case of a geographical event, [20] present measures that include correlation among edge failures.

2 The single-objective capacitated ring tree problem

In this section we provide a formal definition of the cost-oriented CRTP in accordance with its introduction in [13] and present a novel mathematical formulation based on multi-commodity flows.

2.1 The model

A ring tree is a pseudotree, i.e. a connected graph that contains at most one cycle, called a ring. In the CRTP, customers are represented by nodes and connections correspond to edges of a simple undirected graph. We are given disjoint sets of type 1 customers $U_1$ and type 2 customers $U_2$. The set $W$ contains the Steiner nodes which might act as transition points or might facilitate a ring structure. However, nodes in $W$ are not associated to any customer, i.e., are not necessarily included in the ring trees. All potential nodes $V$ are given by $V = U \cup W \cup \{d\}$, where $U$ denotes the set of all customers and node $d$ describes the unique depot. We say that an edge that is part of a ring is a ring edge. All other edges are tree edges. Likewise, a ring customer (ring node) is a customer (node) that is incident to a ring edge and other customers (nodes) are tree customers (nodes). The set of all potential edges between nodes in $V$ is denoted by $E$ and a cost $c_e \geq 0$ is associated to each edge $e \in E$. We are given two capacity bounds $q, m \in \mathbb{N}$. The
CRTP asks for an edge cost minimal set of at most \( m \) edge-disjoint ring trees with at most \( q \) customers each, such that

- they intersect in \( d \) which is a ring node or a leaf if the ring tree is a tree,
- each customer in \( U \) is included in exactly one ring tree,
- each customer in \( U_2 \) is a ring node, and
- each node in \( W \) is used at most once.

Figure 1 illustrates a solution which implements five ring trees and shows two different representations: Euclidean and orthogonal. If there are only type 1 customers (\( U_2 = \emptyset \)), the CRTP reduces to the capacitated STP. If there are only type 2 customers (\( U_1 = \emptyset \)), we obtain a unit-demand VRP.

![Fig. 1. A solution for the CRTP with 5 Steiner nodes, 3 rings and 6 trees (left) and an orthogonal embedding of a directed representation in the plane (right). \( |W| \geq 5, |U_1| = 19, |U_2| = 6, m = 5, q = 6 \)]

### 2.2 A multi-commodity flow formulation

We present a multi-commodity-flow (MCF) formulation \( (F_0) \) which will be proven powerful when incorporating reliability related constraints for our bi-objective approaches in Section 3. This mixed-integer programming (MIP) formulation is related to the formulation presented in [14] for a facility location variant of the CRTP.

First note that a solution for the CRTP can be represented as a directed network as follows. Remove one of the edges incident to \( d \) in each ring, root the remaining tree in \( d \), and re-insert the deleted edges as arcs pointing to \( d \). More precisely, there are \( 2^r \) different representations, due to the two opposed
orientations of each directed cycle (or circuit), where \( r \) is the number of rings. A directed solution trivially induces a unique undirected solution in turn. In the following let \( A = \{(i, j), (j, i) : \{i, j\} \in E\} \) denote the set of these potential arcs and, for each node \( i \in V \), let \( \delta^+(i) [\delta^-(i)] \) be the set of outgoing arcs (incoming arcs). We introduce a continuous flow variable \( f_a^k \) for every arc \( a \in A \) and every customer \( k \in U \) to model the flow of commodity \( k \) on \( a \). Moreover, a binary arc variable \( x_a \) indicates the installation of \( a \) in case of positive flow \( f_a^k > 0 \) for some commodity \( k \), otherwise \( x_a = 0 \) holds. The undirected solution is encoded using binary edge variables \( y_e \) \( \forall e = \{i, j\} \in E \) which take value 1 if arc \((i, j)\) or arc \((j, i)\) is installed, otherwise 0. In addition, a binary ring arc variable \( r_a \) indicates whether an arc \( a \) is used within a directed cycle.

\[
(F_0) \quad \min \quad z_0(y) = \sum_{e \in E} c_e y_e 
\]

subject to

\[
\sum_{a \in \delta^-(i)} f_a^k - \sum_{a \in \delta^+(i)} f_a^k = \begin{cases} 1 & \text{if } i = k \\ -1 & \text{if } i = d \\ 0 & \text{else} \end{cases} \quad \forall i \in V, k \in U, \tag{2}
\]

\[
f_a^k \leq x_a \quad \forall a \in A, k \in U_1, \tag{3}
\]

\[
f_a^k \leq r_a \quad \forall a \in A, k \in U_2, \tag{4}
\]

\[
r_a \leq x_a \quad \forall a \in A, \tag{5}
\]

\[
\sum_{a \in \delta^-(i)} x_a \leq 1 \quad \forall i \in W, \tag{6}
\]

\[
\sum_{a \in \delta^-(i)} x_a = 1 \quad \forall i \in U, \tag{7}
\]

\[
\sum_{k \in U} f_a^k \leq q \quad \forall a \in \delta^+(d), \tag{8}
\]

\[
\sum_{a \in \delta^+(d)} x_a \leq m, \tag{9}
\]

\[
\sum_{a \in \delta^-(i)} r_a = \sum_{a \in \delta^+(i)} r_a \quad \forall i \in V, \tag{10}
\]

\[
x_{ij} + x_{ji} = y_e \quad \forall e = \{i, j\} \in E, \tag{11}
\]

\[
f_a^k \geq 0 \quad \forall a \in A, k \in U, \tag{12}
\]

\[
x_a, r_a \in \{0, 1\} \quad \forall a \in A, \tag{13}
\]
\[
y_e \in \{0, 1\} \quad \forall e \in E.
\] (14)

Inequalities (2) guarantee the flow conservation, the source property of the distributor and the satisfaction of the customer demands. The strong linking inequalities (3), (4) and (5) limit the individual commodity flows on the arcs and link the arc variables to the corresponding arc flow. Ring arc installations are evoked by inequalities (4). The directed ring tree property of in-degree one or at most one for non-distributor nodes is enforced by constraints (6) and (7), respectively. In (8) the commodity flow on an outbound arc incident with the distributor is bounded by the customer per ring tree capacity. The maximum number of ring trees is met due to (9). We forbid the splitting of ring commodity flows by enforcing the directed cycles in (10) and exclude backward arcs by linking the arc variables to the edge variables using inequalities (11).

3 The bi-objective ring tree problems

In this section we present four different ways to measure the risk of service loss related to an eventual single-link failure in a CRTP solution. Each of the developed concepts leads to an additional CRTP-specific reliability-oriented objective. Based on the latter, we introduce different bi-objective models and corresponding mathematical formulations.

3.1 Reliability oriented objectives

The CRTP guarantees partial reliability in the sense of 2-connectivity for the ring nodes. In other words, robustness is guaranteed for the type 2 customers regarding connectivity in the case of a single link failure. In this work we consider such one-edge failures only. More precisely, our focus is on the failure of tree edges since the removal of a ring edge does not result in a service outage for any customer. We assume that edge failures are independent and the probability for such an event is \( p \). Furthermore, we assume that all nodes are perfectly reliable, i.e., no loss of transmission occurs. A reason for perfectly reliable ring edges could be the installation of superior technology. Different edge costs for ring and tree edges are discussed in [13]. Note that our approach is combinatorial since we assume constant probabilities for the edge failures. However, the suggested concepts can be generalized to respect edge dependent risks which is considered for future work. In the following we will use the four ring tree networks (i)-(iv) depicted in Figure 2 to illustrate the reliability criteria.

(A) Risky customers A consistent way to increase the security for type 1 customers is their integration into ring structures. In terms of optimization this corresponds to the maximization of the type 1 ring customers or, equivalently, to the minimization of the tree customers, and can be achieved for \( (F_0) \) by employing the following objective.
Fig. 2. Ring trees that illustrate reliability measures (A)-(D). Objective values \(z_A\), \(z_B\), \(z_C\), \(z_D\): 0, 0, 0, 0 (i); 7, 10, 6, 5 (ii); 7, 7, 1, 1 (iii); 7, 8, 7, 8 (iv).

\[
\min z_A(x) = \sum_{i \in U_1} \sum_{a \in \delta^-(i)} (x_a - r_a) \tag{15}
\]

Certainly, this criterion antagonizes the original cost minimizing objective (1) and takes values in \(\{0, \ldots, |U_1|\}\). An optimal ring-based network is given by (i) \((z_A = 0)\) and suboptimal solutions are given by (ii)-(iv). The corresponding objective values are \(z_A = 7\) for solutions (ii)–(iv). The network in Figure 1 has objective value \(15 \in \{0, \ldots, 19\}\). Objective \(z_A\) represents a conservative viewpoint that aims at protecting as many customers as possible while being indifferent to the particular structure of the trees attached to the rings.

(B) Network vulnerability Since Steiner nodes will never be leaves in a cost-optimal network, the deletion of a tree edge disconnects at least one customer of type 1. As we assume that the failure probabilities of the tree edges are independent and constant, the probability of a service interruption is given by 
\[
1 - (1 - p)^m,
\]
where \(m\) denotes the number of tree edges. Thus, the overall network reliability can be increased by minimizing the number of installed tree edges, which yields the following alternative objective for \((F_0)\).

\[
\min z_B(x) = \sum_{a \in A} (x_a - r_a) \tag{16}
\]

Note that this risk measure is not equivalent to (A) since Steiner nodes can increase the number of tree edges without changing the number of tree customers. Objective \(z_B\) may take values in \(\{0, \ldots, |U_1| + |W|\}\). Optimal solutions of the problem are, as in (A), the purely ring based networks. Network (i) is optimal and solutions (ii)-(iv) are suboptimal. The corresponding objective values are 0 (i), 10 (ii), 7 (iii) and 8 (iv). The objective value for the network in Figure 1 is 17. Objective \(z_B\) focuses on preventing customer service loss in general. The number of affected customers, however, is neglected. Thus, objective \(z_B\) favors solution (iii) over solution (iv), and solution (iv) over solution (ii). Although an identical number of customers lie outside the ring, the probability of a service interruption is the highest in solution (ii).

(C) Degree of damage The failure of a single tree edge disconnects the complete subtree induced by this edge. In the worst case, a breakdown affects a
tree that contains all the type 1 customers, as for instance for (iv). To prevent
from such outages, we are interested in minimizing the maximal number of cus-
tomers on a subtree induced by a single edge. To model this, we incorporate a
non-negative integer variable $u^t$ into ($F_0$) to measure the maximal number of
disconnected customers, which is equivalent to the maximal total commodity
flow on a tree edge, by the following *subtree capacity inequalities*.

$$\sum_{k \in U} f^k_a - |U|r_a \leq u^t \quad \forall a \in A$$

(17)

$$u^t \in \mathbb{N}_0$$

(18)

In other words, $u^t$ corresponds to the the maximal number of customers in a
connected component of the network after removing the rings. The minimization
of the maximal number of potentially isolated customers can be modeled by
substituting objective (1) as follows.

$$\min z_C(u^t) = u^t$$

(19)

Note that $z_C \in \{0, \ldots, \min(|U_1|, q)\}$. An example for an optimal ring-based net-
work for the resulting problem is given by (i), whereas solutions (ii)–(iv) are
suboptimal with objective values 0 (i), 6 (ii), 1 (iii) and 7 (iv). The objective
value for the network in Figure 1 is 5. Objective $z_C$ is based on a pessimistic
point of view since it minimizes the number of customers experiencing a loss
of service in a worst-case scenario. The maximum number of disconnected cus-
tomers equals the maximum number of customers in any subtree and thus, $z_C$
prefers solution (iii) over solution (ii) and solution (ii) over solution (iv). Thus,
this objective aims at preventing very large subtrees which has a balancing effect
on the number of customers in the subtrees.

**D) Individual customer risk** Consider a tree customer $i \in U_1$, if it exists,
and let $P_i$ be the unique path from $i$ to the closest ring node or the depot.
Similar to (B), we say that the edges in $P_i$, called *tree hops*, are the vulnerable
points regarding the disconnection of customer $i$. Note that $P_i$ remains fully
operational with probability $(1-p)^{|P_i|}$. The latter monotonously decreases with
an increasing number of hops. To reduce the worst case individual tree customer
risk in the network, we minimize the length of such a longest path. A related
secondary objective was also considered in [24].

In general, the length of the directed path from the depot to a customer $i \in U_1$
in an integer feasible solution for ($F_0$) can be computed by $\sum_{a \in A} f^t_a$. However,
we need to restrict ourselves to the number of hops from a ring node (or $d$, if $i$ is
part of a pure tree) to $i$. To achieve this, we adapt the path-based hop constraints
for the STP used in [5]. Hence, let $\mathcal{P}(i)$ be the set of directed paths in $G$ with
end node $i \in V$ and $h^t$ the integer *maximum tree hop variable*. Then $h^t$ is an
upper bound on the maximal number of customer tree hops when adding the
following exponential number of linking *tree hop inequalities* to ($F_0$).

$$\sum_{a \in P} (f^k_a - r_a) \leq h^t \quad \forall P \in \mathcal{P}(k), \ k \in U_1$$

(20)
\[ h^t \in \mathbb{N}_0 \]  

The resulting tree customer hop-minimizing objective function can be written as follows.

\[
\min \quad z_D(h^t) = h^t
\]

The number of tree hops for a type 1 customer in a solution, i.e., \( z_D \), may be in \( \{0, \ldots, \min(q, |U_1|) + |W| \} \). An example for an optimal ring-based network for this problem is shown in (i) and suboptimal solutions are depicted in (ii)-(iv). The corresponding objective values are 0 (i), 5 (ii), 1 (iii) and 8 (iv). The objective value for the network in Figure 1 is 4. Objective \( z_D \) aims at preventing the case that specific customers have a very high probability of getting disconnected by minimizing the worst case, similar to \( z_C \).

There is a one-to-one correspondence between the optimal solutions for the single-objective problems in (A)-(D) and the feasible solutions of the 1-VRP obtained after the re-dedication of the type 1 customers as type 2. Furthermore, we point out that the objectives in (B), (C) and (D) are also relevant for related rooted tree-based optimization problems, such as for instance the directed MST and the STP. Approach (A) requires a way to equip customers with additional reliability (e.g., rings) instead.

### 3.2 Objective relationships

In the following we give relationships of the reliability-oriented objectives \( z_A, z_B, z_C \) and \( z_D \). These will be used in Section 4 to derive valid inequalities that accelerate the exact approaches.

Let us assume that every leaf in a solution is a customer. From equations (15) and (16) and for given \( x \) and \( r \), we obtain immediately that \( z_B(x, r) = \sum_{i \in V \setminus U_1} \sum_{a \in S^{-}(i)} (x_a - r_a) \geq 0 \) holds. That is, it holds that \( z_A(x, r) \leq z_B(x, r) \) is satisfied. Similar relations can be identified for the remaining objective functions. As \( z_A \) gives the total number of customers that are located within trees and \( z_C \) gives the maximum number of customers that are located in a single tree, we have \( z_C(u^t) \leq z_A(x, r) \) and consequently \( z_C(u^t) \leq z_B(x, r) \). Moreover, \( z_D \) gives the length of the longest tree path from a ring to a leaf whereas \( z_B \) gives the number of non-ring edges. Thus, \( z_D(h^t) \leq z_B(x, r) \) is satisfied. As the former path may contain Steiner nodes, a comparison of \( z_D \) with objectives \( z_A \) and \( z_C \) (both count customers) is only possible if Steiner nodes are excluded. Let \( \zeta \) be the maximum number of customers contained in such a single path, then \( z_D(h^t) - |W| \leq \zeta \) holds. As each of these paths is contained in some tree, we have that \( \zeta \) is less than or equal to the maximal number of customers located in any tree. That is, \( z_D(h^t) - |W| \leq \zeta \leq z_B(u^t) \) and consequently \( z_D(h^t) - |W| \leq z_A(x, r) \leq z_B(x, r) \) does hold. The described relations are summarized in the following equations

\[
z_D(h^t) - |W| \leq z_C(u^t) \leq z_A(x, r) \leq z_B(x, r)
\]

\[
z_D(h^t) \leq z_B(x, r)
\]
However, (B) is bounded by the number of tree customers (A) plus the number of Steiner nodes, which is expressed by the following inequality.

\[ z_B(x, r) \leq z_A(x, r) + |W|. \] \hfill (25)

If \( z_D(h^i) = 1 \) then it is easy to see that

\[ z_C(u^i) = z_D(h^i). \] \hfill (26)

Similarly, \( z_B(x, r) = 1 \) or \( \min\{z_A(x, r), z_B(x, r), z_C(u^i), z_D(h^i)\} = 0 \) implicates that

\[ z_A(x, r) = z_B(x, r) = z_C(u^i) = z_D(h^i) \] \hfill (27)

and, \( z_A(x, r) = 1 \) implicates that

\[ z_C(u^i) = z_A(x, r). \] \hfill (28)

### 3.3 Bi-objective optimization problems and MIP formulations

We now introduce bi-objective models that minimize the network costs along with one risk measure. Note that each problem has the same solution space as the corresponding single-objective CRTP.

We define the **risky customers-oriented bi-objective CRTP** (RC-2-CRTP) as the optimization problem which minimizes the edge costs and the number of tree customers (objectives (1) and (15)). We can derive the following mathematical formulation for the RC-2-CRTP from \((F_0)\).

\[(F_A) \quad \min \ z_{0,A}(y, x, r) = [z_0(y), z_A(x, r)] \quad \hfill (29)\]

We define the **network vulnerability-oriented bi-objective CRTP** (NV-2-CRTP) as the optimization problem which minimizes the edge costs and the number of tree edges (objectives (1) and (16)). We can derive the following mathematical formulation for the NV-2-CRTP from \((F_0)\).

\[(F_B) \quad \min \ z_{0,B}(y, x, r) = [z_0(y), z_B(x, r)] \quad \hfill (30)\]

Note that the NV-2-CRTP is a special case of another bi-objective CRTP model that minimizes the tree edge costs in the secondary objective with constant tree edge cost 1.

We define the **damage-degree-oriented bi-objective CRTP** (DD-2-CRTP) as the optimization problem which minimizes the edge costs and the maximal number of subtree customers (objectives (1) and (19)). We can derive the following mathematical formulation for the DD-2-CRTP from \((F_0)\).

\[(F_C) \quad \min \ z_{0,C}(y, u^i) = [z_0(y), z_C(u^i)] \quad \hfill (31)\]
We define the individual-risk-oriented bi-objective CRTP (IR-2-CRTP) as the optimization problem which minimizes the edge costs and the number of tree edges (objectives (1) and (2)). We can derive the following mathematical formulation for the IR-2-CRTP from $F_0$.

\[
(F_D) \min z_{0,D}(y, h^t) = [z_0(y), z_D(h^t)]
\]

4 The $\epsilon$-constraint method

We apply an $\epsilon$-constraint method to compute the Pareto sets for the models introduced in the previous section. More precisely, we aim at computing a minimal set of efficient solutions in which equivalent solutions, i.e., solutions with the same objective values, are omitted. The main idea of this posteriori algorithm is to solve each bi-objective problem by solving a series of single-objective optimization problems to optimality in which neglected objectives are bounded to values within a given interval $[\xi, \tau]$ (see, e.g., [7]). These subproblems are solved using the MIP formulations presented in Section 3.3. Advanced bounding techniques will be developed in Sections 4.2 and 4.3 to further enhance the corresponding exact algorithms. In our descriptions we use $z_X$ to represent an arbitrary objective from $z_A$, $z_B$, $z_C$, and $z_D$. We note that the order of a minimal set of efficient solutions for each problem is linear according to the value range of such a secondary objective.

4.1 Search strategy

In our algorithm, a sequence of CRTP-related subproblems is solved to optimality under two types of additional $\epsilon$-constraints. Namely, we either minimize the costs and restrict the reliability objective or vice versa. The first solution within this iterative procedure is obtained by solving the corresponding single-objective CRTP to optimality. Since the resulting network is purely cost-driven, a network of equal cost ($z_0^*$) with an even smaller $z_X$-value might exist. To compute the latter or assert its non-existence we solve the CRTP variant with objective $z_X$ (see Section 3.1) under the $\epsilon$-constraint $z_0 = z_0^* (= \tau_0)$. Let $z_X^*$ be the minimal value for $z_X$ that could be found. Then, we identify a cost minimal solution while requiring $z_X \in [0, z_X^* - 1]$ and optimize with respect to $z_X$ as above. This procedure is repeated until $z_X = 0$. In Algorithm 1 we describe this method using the exact subroutines $solve_0(P, \tau_X)$ and $solve_X(P, \tau_0)$ to solve the single-objective cost-oriented and reliability-oriented CRTP variants of instance $P$ with $z_X \leq \tau_X$ and $z_0 = \tau_0$, respectively. $z_0(S)$ and $z_X(S)$ denote the corresponding objective values of a solution $S$. The strategy is illustrated in Figure 3 for an instance with
5 Pareto-optimal solutions.

**Input:** RC-2-CRTP/NV-2-CRTP/DD-2-CRTP/IR-2-CRTP instance $P$;

$$\tau_X \leftarrow \infty;$$

while $\tau_X > 0$ do

$$S \leftarrow solve_0(P, \tau_X - 1);$$

$$S \leftarrow solve_X(P, z_0(S));$$

$$\tau_X \leftarrow z_X(S);$$

end

**Output:** $S$;

**Algorithm 1:** Generic $\epsilon$-constraint method for the bi-objective CRTPs.

The $\epsilon$-constraint CRTP variants (minimize $z_0, z_X \leq \epsilon_X$) that are solved in our method are challenging optimization problems of practical relevance themselves. They are NP-hard since they generalize the CRTP and can be defined as follows. For (A) we face the tree-customer-capacitated ring tree problem, whereas (B) leads to the tree-edge-capacitated ring tree problem. The subtree-capacitated ring tree problem and the tree-hop-constrained CRTP appear as subproblems in (C) and (D), respectively. Note that the capacitated ring star problem [1] is a special case of the latter when the tree-hop bound is set to one.

We use the solutions found by the heuristic multi-start approach for the CRTP developed in [11] to initialize a dynamic solution pool. This pool is used to initialize the cutting plane algorithms. It is extended by integral solutions found during these subroutines. Note that for $\epsilon_D = 1$ the model is equivalent to the capacitated ring star problem (CRSP) [1]. In the following two sections

![Fig. 3. An example for the decremental search strategy in the $\epsilon$-constraint method leading to an efficient set of order 5 after solving 9 single-objective subproblems (left to right).](image-url)

we present techniques to accelerate the algorithm by tightening lower and upper bounds for the subproblems in each iteration.
4.2 Objective space search heuristics

To obtain tighter upper bounds for the MIPs arising in each iteration of our $\epsilon$-constraint algorithm we derive feasible solutions from solutions within the dynamic pool. More precisely, we search for tight upper bounds for the cost-oriented subproblems. Since these might be infeasible with respect to the risk-based $\epsilon$-constraint, we apply a repair technique in order to reduce the corresponding risk objective value while increasing the cost as little as possible. Herewith, we dynamically maintain an upper bound solution set.

The main idea is to reduce the secondary objective $z_X$, $X \in \{A, B, C, D\}$, by turning selected tree nodes into ring nodes. Figure 4 illustrate these model-specific local search techniques that we use to deduce feasible solutions. The particular approaches regarding the objectives defined in (A) – (D) are given next.

(A) To turn a tree customer $k$ into a ring node we re-route its corresponding ring through the tree up to node $k$ by connecting $k$ to a subsequent ring node of the original network afterwards. Let $j \neq d$ be the node in a ring $R = (\ldots, i, j, l, \ldots)$ such that $k$ is a node in a tree attached to $j$. Then $k$ can be integrated into $R$ by either replacing the edge $\{j, l\}$ with edge $\{k, l\}$ or $\{i, j\}$ with $\{i, k\}$. Note that if $k$ is a node in a pure tree (and not adjacent to $d$) then we can still attempt to create a ring by adding the edge $\{k, d\}$.

(B) To reduce the number of tree edges we apply the technique for (A), but extended to any tree node instead of tree customers only.

(C) The search described for (A) can be restricted to trees that exceed the desired maximal number of customers in a tree.

(D) Let $P$ be a path exceeding the tree hop limit and let $j$ be an inner node of $P$ with predecessor $i$ and successor $k$. We can squeeze out $j$ in by removing edge $\{j, k\}$ and inserting edge $\{i, k\}$.

We perform the most cost-efficient reduction until the corresponding objective decreased sufficiently. After each network modification, a local search at-
tempts to reduce the network costs by deleting single Steiner nodes (see [11]). This reduction is applied to each solution in the pool to identify the best upper bound. Figure 5 illustrates the effect of these repair heuristics within the 2-dimensional objective space.

Fig. 5. How repair heuristics on solution pool networks help to tighten upper bounds for the subproblems. (Left: solution pool including repaired networks in the objective search space; right: reduced search spaces of the subproblems.)

4.3 Strengthening the formulations

After strengthening the upper bounds for the single-objective MIPs from Section 4.1 in Section 4.2, we focus on improving the lower bounds. For this purpose, we present valid inequalities for each formulation.

\((F_0)\) The following valid inequality ensures that enough arcs leave the depot with respect to the ring tree capacity and the number of customers.

\[
\sum_{a \in \delta^+(d)} x_a \geq \left\lceil \frac{|U|}{q} \right\rceil 
\] (33)

Formulation \((F_0)\) allows redundant arcs that connect ring tree nodes to the depot without being ring arcs. These will not be installed in an optimal solution (unless of cost zero). Together with \((10)\) the following equalities forbid these arcs.

\[
x_{id} = r_{id} \quad \forall \ i \in V \setminus \{d\} \tag{34}
\]

Note that these constraints only become effective when considering additional capacity constraints in our models in the next section. This also applies to the following inequalities which forbid Steiner leaf nodes.

\[
\sum_{a \in \delta^+(i)} x_a \geq \sum_{a \in \delta^-(i)} x_a \quad \forall \ i \in W \tag{35}
\]
Although inequalities (8) guarantee feasibility in \((F_0)\), the total commodity traversing an arc might exceed \(q\) in the linear relaxation. Therefore, the following strong linking inequalities are valid.

\[
\sum_{k \in U} f_{ij}^k \leq qx_{ij} \quad \forall (i, j) \in A : j \in W \tag{36}
\]

\[
\sum_{k \in U} f_{ij}^k \leq qf_{ij}^j \quad \forall (i, j) \in A : j \in U_1 \tag{37}
\]

\[
\sum_{k \in U \setminus \{j\}} f_{ij}^k + r_{ij} \leq qf_{ij}^j \quad \forall (i, j) \in A : j \in U_2 \tag{38}
\]

We add even stronger versions after exchanging \(q\) by \(q - 1\) in the case that the tail of the arc (node \(i\)) is a customer.

To ensure that enough arcs emanate from a type 1 customer or a Steiner node based on the total commodity in-flow, we add the following inequalities.

\[
\sum_{a \in \delta^+(j)} x_a \geq \frac{1}{q} \sum_{k \in U \setminus \{j\}} \sum_{a \in \delta^-(j)} f_a^k \quad \forall j \in U_1 \tag{39}
\]

\[
\sum_{a \in \delta^+(j)} x_a \geq \frac{1}{q+1} \sum_{k \in U \setminus \{j\}} \sum_{a \in \delta^-(j)} f_a^k \quad \forall j \in W \tag{40}
\]

Inequalities (2) allow that more than one unit of a commodity leaves the depot which is prevented by adding

\[
\sum_{a \in \delta^+(d)} f_a^i = 1 \quad \forall i \in U. \tag{41}
\]

Finally, we break ties by dynamically separating the following valid inequalities which are closely related to the corresponding inequalities introduced in [13]. The capacitated connectivity inequalities ensure that sufficient ring tree structures are present to satisfy the ring tree customer capacity for each node subset \(S \subseteq V \setminus \{d\}\).

\[
\sum_{a \in \delta^-(S)} x_a \geq \left\lceil \frac{|U \cap S|}{q} \right\rceil \quad \forall S \subseteq V \setminus \{d\} \tag{42}
\]

The capacitated ring tree multi-star inequalities follow the idea above but take into account additional capacity needed by customer outside of \(S\) that are connected to a node in \(S\). Here we assume that from a customer node \(i\), at most \(q\) arcs can enter \(S\) if considering its tree arc value \(x_{ij} - r_{ij}\), whereas at most one corresponding ring arc may exists since \(\sum_{a \in \delta^+(i)} r_a \leq 1\). Moreover, we use that an arc’s commodity flow is a lower bound for its arc value to strengthen the left hand side of the inequality. We denote by \(\delta(X : Y)\) the set of arcs from \(X\) to \(Y\).

\[
\sum_{a \in \delta^-(S \setminus U)} x_a + \sum_{ij \in \delta^-(S \setminus U)} f_{ij}^j \geq \frac{1}{q} \left( |U \cap S| + \sum_{a \in \delta(U \setminus S \setminus U_2 \cap S)} x_a + \sum_{a \in \delta(U \setminus S \setminus U_2 \setminus S)} [r_a + \frac{x_a - r_a}{q}] \right) \quad \forall S \subseteq V \setminus \{d\} \tag{43}
\]
We add an even stronger version of (43) in which we lift the right hand side by $1/q \sum_{a \in \delta(S:U)} x_a$ (not separated, only added).

\[
\sum_{a \in \delta(S:V \setminus U_2)} r_a + \sum_{ij \in \delta(S:U_2)} f_{ij} \geq \frac{1}{q} \left( |U_2 \cap S| + \sum_{i \in U_1 \cap S, a \in \delta^-(i)} r_a + \sum_{a \in \delta(U_2 \cap S:U \setminus S)} r_a + \sum_{a \in \delta(U_2 \cap S:U \setminus S)} x_a \right) \quad \forall S \subseteq V \setminus \{d\} \tag{44}
\]

We use an even stronger version of (44) in which we add $1/q \sum_{a \in \delta(U:S \cap U_2)} x_a$ and $1/q \sum_{a \in \delta(U:S \setminus U_2)} r_a$ to the right hand side (not separated, only added).

Connectivity regarding the ring structure is strengthened in fractional solutions by ring node connectivity constraints.

\[
\sum_{a \in \delta^-(S)} r_a \geq \sum_{a \in \delta^-(i)} r_a \quad \forall S \subseteq V \setminus \{d\} : i \in S, i \in V \setminus \{d\} \tag{45}
\]

Similarly, the following ring closure inequalities are valid, since they force the closure of rings and eliminate fractional sub-rings.

\[
\sum_{a \in \delta^+(S)} r_a \geq \sum_{a \in \delta^-(i)} r_a \quad \forall S \subseteq V \setminus \{d\} : i \in S, i \in V \setminus \{d\} \tag{46}
\]

The separation procedures for inequalities (42), (43), (44), (45) and (46) are closely related to the ones described in [13] to which we refer for the details.

**$(F_A)$** Bounding objective function (15) by a constant $\tau_A$ may enforce the installation of additional rings, even if no customers are of type 2, for $\tau_A < |W| + |U_1|$. The used ring arc variables indicating a directed cycle should then be at least equal to a lower bound for the number of rings, e.g., $\lceil (|U_1| - \tau_A)/q \rceil$.

\[
\sum_{a \in \delta^+(d)} r_a \geq \left\lceil \frac{|U_1| - \tau_A}{q} \right\rceil \tag{47}
\]

With type 2 nodes present we obtain the following inequality.

\[
\sum_{a \in \delta^+(d)} r_a \geq \left\lceil \frac{|U_2| + |U_1| - \tau_A}{q} \right\rceil \tag{48}
\]

Inequality (48) remains valid after exchanging the upper tree customer bound $\tau_A$ with the actual number of tree customers. Even though the resulting modified inequality is non-linear in $x$ and $r$, we can add the variant below which omits to round the right hand side.

\[
\sum_{a \in \delta^+(d)} r_a \geq \frac{|U| - z_A(x, r)}{q} \tag{49}
\]
(FB) As in inequality (48), we can enforce a lower bound on the number of installed rings using the tree edge limit $\bar{r}_B$. To this end, we take into account that each Steiner tree node (equivalently, each Steiner tree edge) consumes capacity regarding $\epsilon_B$.

$$\sum \limits_{a \in \delta^+(d)} r_a \geq \frac{1}{q} \left[ |U| + \sum \limits_{i \in W} \sum \limits_{a \in \delta^-(i)} (x_a - r_a) - \bar{r}_B \right]$$ (50)

(F_C) Knowing that the number of customers in a tree induced by an edge incident to a ring node is bounded by $\epsilon_C$ we can limit the tree node out-degree to $\epsilon_C$ for Steiner nodes and to $\epsilon_C - 1$ for type 1 customer nodes. Here we used the fact that a branch in a tree contains at least one type 1 customer. This leads to the following tree out-degree-bounding inequalities in which the summation term on the right hand side is needed to relax the requirement in case that $i$ is a ring node.

$$\sum \limits_{a \in \delta^+(i)} x_a \leq z_C(u_t) + (q + 1) \sum \limits_{a \in \delta^-(i)} r_a \quad \forall \ i \in W$$ (51)

$$\sum \limits_{a \in \delta^+(i)} x_a \leq (z_C(u_t) - 1) + q \sum \limits_{a \in \delta^-(i)} r_a \quad \forall \ i \in U_1$$ (52)

(F_D) Inequality (20) can be strengthened by taking into account all the tree paths leading to a type 1 customers instead of only the ones used by its commodity:

$$\sum \limits_{a \in P} (x_a - r_a) \leq h^t \quad \forall \ P \in \mathcal{P}(k), \ k \in U_1$$ (53)

Let $x^*_a$ and $r^*_a$ be the arc value and the ring arc value for the arc $a \in A$ in an optimal solution of the linear relaxation of the formulation described above. The separation of inequality (20) for customer $i$ can be done on a directed auxiliary graph $G'$ with node set $V' = V$ and arc set $A' = \{(j,k) : (k,j) \in A \land x^*_k,j - r^*_k,j > 0\}$. Then we assign the weight $x^*_k,j - r^*_k,j$ to each arc $(j,k) \in A'$ and a path maximally violating inequality (20) is given by a longest directed path in $G'$. This longest directed path problem is well known to be NP-hard on general graphs. We solve this problem heuristically for each type 1 customer $i$ by searching for a longest path from $i$ in a greedy fashion. We identify the $\eta$ violated shortest paths (15 in our experiments). Furthermore, the following cuts are very important for the algorithm’s performance.

$$z_D(h^t) \geq \sum \limits_{a \in \mathcal{A}} (f^k_a - r_a) \quad \forall \ k \in U_1$$ (54)

We add the four cuts corresponding to the relationships given in (23) and (24). The first one can be lifted as follows using the argument that a longest tree path
contains at most \( \tau_C \) customers plus the current number of Steiner nodes used in
trees.

\[
z_D(h^t) - \sum_{i \in W} \sum_{a \in \delta^-(i)} (x_a - r_a) \leq z_C(u^t) \tag{55}
\]

Similarly, The following equality can be derived from (25) stating that the number
of tree arcs is equal to the number of tree customers plus the number of tree
Steiner nodes.

\[
z_B(x, r) - \sum_{i \in W} \sum_{a \in \delta^-(i)} (x_a - r_a) = z_A(x, r) \tag{56}
\]

5 Computational study

In the following computational study we identify both, the performance of our
exact approaches on a set of hard test instances and the structure of the Pareto
fronts for these problems. The computations were carried out on an Intel i7-3667U 2.00 GHz machine with 3.5 GB effective RAM. The CPLEX 12.6 branch
and cut framework was used for the implementation of the cutting plane algorithms. We fully allowed the solver-internal cutting planes which was proven
useful in our experiments.

5.1 Algorithm implementation

The strategy of the \( \epsilon \)-constraint method presented in Section 4.1 foresees the
solution of single-objective subproblems using the procedure \( \text{solve}_X(P, \epsilon_0) \). A
good upper bound on \( z_X \) is \( z_X(S) \), where \( S \) is the optimal solution for the
preceding cost-minimizing subproblem. Note that the incumbent solution \( S \) will
only be replaced if a network with reduced \( z_X \) can be found. Therefore, we
experimented with solving the subproblem with an additional constraint \( z_X \leq z_X(S) - 1 \). This turned out to increase the run time significantly since it suffers
from the missing start solution (or upper bound).

In each step of the algorithm, we omitted to add a constraint \( \epsilon_0 \geq z_0(S) \) that
bounds the cost objective from below. It is known that these constraints lead to
degenerate problems which we also observed in our experiments.

5.2 Instances

We use a subset of the TSPlib-based CRTP instances\(^3\) described in [13]. The
customer types in these problems are assigned uniformly according to the type
1 customer rates 0.25, 0.5, 0.75 and 1. Note that we neglect purely ring based
instances in our study since they are trivial regarding the reliability oriented
objectives. Moreover, we limit ourselves to the first 12 instances with 26 nodes.

5.3 Results

The computational results are given in the following Table 1, followed by illustrations of efficient solution sets for each model introduced in Section 3. The columns list instance properties such as the base problem from [13] \((P)\), type 1 customer rate \((r_1)\), number of nodes \((|V|)\), number of type 2 customers \((|U_2|)\), number of type 1 customers \((|U_1|)\), number of Steiner nodes \((|W|)\), ring tree limit \((m)\), and ring tree customer limit \((q)\). Column \(z^*_0\) contains the lowest costs of an efficient solution. Note that \(z^*_0\) equals the optimal solution value of the corresponding single-objective CRTP. In column \(z^*_0\) we provide the highest costs of an efficient solution, which is obtained for \(z_X = 0, X \in \{A, B, C, D\}\). We recall that \(z^*_0\) corresponds to the optimal costs for the CRTP in which all customers are of type 2 and \(z_A = z_B = z_C = z_D = 0\) holds for each corresponding network.

For each bi-objective model, and the reliability criterion \(X\), we give the largest secondary objective value \((z_X^*)\), number of efficient solutions \((\#)\) and the overall run time of our algorithms. Consequently, the objective values describing the ideal point are given by \([z^*_0, 0 = z_X^*]\).

Table 1. Results for the bi-objective models on CRTP instances up to 26 nodes.

<table>
<thead>
<tr>
<th>Instance</th>
<th>Cost (z^*_0)</th>
<th>(z^*_0) (t(\text{min}))</th>
<th>(z^*_A) (# t(\text{min}))</th>
<th>(z^*_B) (# t(\text{min}))</th>
<th>(z^*_C) (# t(\text{min}))</th>
<th>(z^*_D) (# t(\text{min}))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Q-1</td>
<td>157 242</td>
<td>12 12 213.9</td>
<td>12 12 238.9</td>
<td>5 5 142.4</td>
<td>5 5 443.3</td>
<td>Q-2 3 9 12 13 3 5 210 242 7 4 51.2 7 4 49.1 5 4 202.8 3 3 80.2</td>
</tr>
<tr>
<td>Q-3</td>
<td>227 242</td>
<td>2 2 24.7</td>
<td>2 2 21.6</td>
<td>1 1 23.2</td>
<td>1 1 14.4</td>
<td>Q-4 9 3 13 3 5 236 242 1 1 16.5 1 1 19.2 1 1 19.5 1 1 17.5</td>
</tr>
<tr>
<td>Q-6</td>
<td>163 242</td>
<td>12 12 464.7</td>
<td>12 12 491.4</td>
<td>4 4 28.4</td>
<td>4 4 46.9</td>
<td>Q-7 3 9 13 3 4 4 207 251 8 7 96.4 8 7 95.4 3 3 19.1 3 3 54.3</td>
</tr>
<tr>
<td>Q-8</td>
<td>240 251</td>
<td>3 3 232.4</td>
<td>3 3 197.5</td>
<td>2 2 284.1</td>
<td>2 2 618.2</td>
<td>Q-9 9 3 13 4 4 249 251 1 1 76.4 1 1 34.4 1 1 51.4 1 1 57.6</td>
</tr>
<tr>
<td>Q-11</td>
<td>170 279</td>
<td>12 12 42.4</td>
<td>13 13 50.9</td>
<td>3 3 15.1</td>
<td>3 3 22.3</td>
<td>Q-12 3 9 13 3 5 3 242 279 6 5 25.6 6 5 24.3 3 3 34.9 3 3 158.4</td>
</tr>
<tr>
<td>Q-13</td>
<td>251 279</td>
<td>4 4 56.2</td>
<td>4 4 48.9</td>
<td>3 3 77.7</td>
<td>3 3 83.2</td>
<td>Q-14 9 3 13 5 3 279 279 0 0 1.2 0 0 1.1 0 0 1.1 0 0 2.1</td>
</tr>
</tbody>
</table>

Even though our algorithm is not as efficient for the CRTP as the branch and cut method described in [13], we recall that the presented techniques allow an implementation based on compact mathematical formulations (except for (D)). Therefore, small instances of these extremely hard combinatorial problems can be solved without the elaboration of sophisticated cut separation procedures and corresponding cut insertion procedures.

5.4 Pareto front illustrations

In order to provide an even more intuitive presentation of our approach, we continue with graphical representations of obtained solutions. Therefore, we illustrate all the networks of Pareto fronts for selected problem instances for the four different models. We omit an illustration for NV-2-CRTP (B) since due to a low utilization of Steiner nodes the results are very similar to the results for RC-2-CRTP (A). For details about the Steiner node impact we refer to Section 5.5.
**RC-2-CRTP (A)** Figure 6 depicts the efficient solutions for the RC-2-CRTP and instance Q-1. Since no type 2 customers present, the CRTP induced by Q-1 is a capacitated Steiner tree problem. Optional rings need to be installed to reduce the tree customers once $z_A < |U_1|$. Note that a Steiner node is utilized to establish a single-customer ring in the second and the penultimate network. The

![Diagram](image1)

**Fig. 6.** Efficient RC-2-CRTP solutions for instance Q-1 ($|W| = |U_1| = 12, |U_2| = 0, m = 3, q = 5$) with $(z_0, z_A) \in \{(157, 12), (172, 11), (173, 10), (174, 9), (184, 8), (186, 7), (195, 6), (201, 5), (203, 4), (213, 3), (222, 2), (235, 1), (242, 0)\}$.

Pareto front objective values for edge costs $z_0$ and number of tree customers $z_A$ for instance Q-1 are illustrated in the diagram in Figure 7.

![Diagram](image2)

**Fig. 7.** Pareto-fronts for instance Q-1.

**DD-2-CRTP (C)** Figure 8 illustrates the Pareto-optimal solutions for instance Q-1, whereas the front can be found in Figure 7.

**IR-2-CRTP (D)** Figure 9 illustrates the Pareto front for an instance for "cost vs (D)". Note that rings are only installed in solution seven which results in a notable cost increase.
Fig. 8. Efficient DD-2-CRTP (C) solutions for instance Q-1 ($|U_1| = |W| = 12, |U_2| = 0, m = 3, q = 5$) with $(z_0, z_C) \in \{(157, 5), (166, 4), (192, 3), (194, 2), (218, 1), (242, 0)\}$.

Fig. 9. Efficient IR-2-CRTP (D) solutions for instance Q-6 ($|U_1| = 12, |W| = 13, |U_2| = 0, m = 4, q = 4$) with $(z_0, z_D) \in \{(163, 4), (164, 3), (188, 2), (226, 1), (251, 0)\}$.

Fig. 10. Pareto-fronts for instance Q6.

5.5 Algorithm performance

Search effort In the following we analyze the algorithmic search effort based on the run times. We differentiate between the subproblems that were solved during the horizontal and the vertical search. Figure 11 shows the run times for all four models. The average run time spent on the horizontal search was 2946s (47%) and the vertical search required 3298s (53%), respectively. It can be seen that instances Q-1 and Q-8 were challenging regarding all models. IR-2-CRTP (D) seems to be the hardest for our algorithm whereas DD-2-CRTP (C) was solved within the smallest overall run time.

Cutting plane impact We now show the impact of the addition of the valid inequalities presented in Section 4.3. Therefore, we differentiate between three following classes of cuts.

- $C_+$: cuts that are added to our formulations before starting the branch and bound (constraints (33), (34), (35), (36), (37), (38), (39), (40), (41), (48), (49), (50), (51), (52), (54), (23), (24), (25), (26), (27), (28), (55), and (56)).
- $C_{dyn}$: contains the cuts that are dynamically separated during the branch and bound (inequalities (42), (43), (44), (45), and (46)). We consider this
class since their implementation requires notably increased effort due to the separation procedures (see [13]).

- \( C_{\text{cplex}} \): generic MIP cuts that are implemented within the CPLEX solver\(^4\) using an automatically chosen application intensity.

Omitting any of the cut classes above results in a significant increase of run time when computing the complete Pareto fronts, such that we do not report the corresponding impact on the \( \epsilon \)-constraint method. However, to give an indication of the effect of the cutting planes we compare the root node lower bounds for the single-cost-objective CRTP with and without their incorporation. We note that this problem is solved in the first iteration of our method independent from the bi-objective model. We recall that the used multi-commodity flow formulation does not outperform the cut-set formulation presented in [13] but is compact and advantageous for our \( \epsilon \)-constraint method, i.e., it can incorporate all the side constraints needed for computing the Pareto fronts in a straightforward way, except for the tree hop constraints used in \((F_D)\) (see Section 3.3). Figure 12 shows the relative optimality gap improvement in the root node for the used instances for the different cut classes. The average relative root gap could be improved significantly by the addition of cuts \( C_+ \); i.e., from 19.4% to 3.9%. The further

\(^4\) IBM ILOG CPLEX Optimization Studio V12.6 documentation
Fig. 12. Relative CRTP root node optimality gaps and averages for the different cut classes $C_{\text{cplex}}, C_{\text{dyn}}, C_{+}$ and the pure model ($F_0$).

dynamic separation of cuts and the usage of the cplex cutting techniques reduced the average relative root gap to 3.0% and 2.5%, respectively. As concluded in [13] for the cut set formulation, the instances with a balanced amount of type 1 and type 2 customers are the most challenging for our MCF formulation in terms of optimality gap tightness.

**Heuristic impact** The dynamic solution pool was initialized by 70 networks on average and contained 164 solutions when the algorithm terminated. Figure 13

In 162 cases (69%) out of 234, the presented objective space heuristics from Section 4.2 were able to provide a cost optimal solution to the subsequent search step.

Fig. 13. Solution pool dynamics for the RC-2-CRTP (A).
Steiner node impact To study the impact of the Steiner nodes on the solution structures we applied the following transformation. We reduced the cost of an edge between two Steiner nodes by 50% and the cost of an edge connecting a Steiner node and a type 1 customer by 30%. The resulting edge costs were rounded up to the next integer. This modification favors the usage of Steiner nodes and aims at emphasizing the difference between measures (A)-(D). Figure 14 illustrates the Pareto front for the modified instance Q-3 for NV-2-CRTP (B). The corresponding objective values for all models are depicted in Figure 15.

Fig. 14. Efficient NV-2-CRTP (B) solutions for instance Q-3 with modified edge costs to increase the number of Steiner nodes in solutions ($|W| = 13$, $|U_1| = |U_2| = 6$, $m = 3$, $q = 5$) with $(z_0, z_B) \in \{(213, 4), (214, 3), (219, 2), (224, 1), (228, 0)\}$.

Fig. 15. Pareto-fronts for Q-3 with modified Steiner edge costs.

6 Conclusion

In this work we introduce four novel bi-objective optimization models that generalize the NP-hard capacitated ring tree problem. They all share a primary network cost objective but differ in a reliability-oriented secondary objective.

5 The modified instances for bi-objective models used in this paper can be obtained from the corresponding author.
We are concerned about network reliability in terms of customer connectivity in case of a single tree edge failure. The presented concepts are also relevant for related network design models such as, e.g., capacitated Steiner tree problems.

Each optimization problem is formulated as a mixed-integer program and a generic algorithmic framework is developed based on the $\epsilon$-method to compute the corresponding Pareto sets. Exact cutting plane algorithms are elaborated to solve novel variants of the capacitated ring tree problem covering tree-customer, tree-edge, maximal edge-induced subtree, and maximal tree-hop constraints. We present valid inequalities to increase the performance of these subroutines. Furthermore, we integrate efficient objective space search heuristics to accelerate the overall algorithms.

We study our models and approaches on a hard problem test set and illustrate some efficient frontiers. The provided detailed computational results show the positive impact of the valid inequalities and that sets of efficient solutions can be obtained within reasonable time. A statistical analysis of the computed solutions shows that the results can be used for manifold strategic decision support.

Regarding future work, we currently develop an approach measuring reliability in the case of node failures. Moreover, it will be meaningful to model an edge- (or node-) dependent failure risk to consider, e.g., varying conditions of infrastructure or environment. Finally, in order to address instances of larger scale, the development of heuristics for detecting an approximate Pareto set is required.

References


