Closed-form solutions for worst-case law invariant risk measures with application to robust portfolio optimization

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Abstract

Worst-case risk measures refer to the calculation of the largest value for risk measures when only partial information of the underlying distribution is available. For the popular risk measures such as Value-at-Risk (VaR) and Conditional Value-at-Risk (CVaR), it is now known that their worst-case counterparts can be evaluated in closed form when only the first two moments are known for the underlying distribution. These results are remarkable since they not only simplify the use of worst-case risk measures but also provide great insight into the connection between the worst-case risk measures and existing risk measures. We show in this paper that somewhat surprisingly similar closed-form solutions also exist for the general class of law invariant coherent risk measures, which consists of spectral risk measures as special cases that are arguably the most important extensions of CVaR. We shed light on the one-to-one correspondence between a worst-case law invariant risk measure and a worst-case CVaR (and a worst-case VaR), which enables one to carry over the development of worst-case VaR in the context of portfolio optimization to the worst-case law invariant risk measures immediately.

1 Introduction

Measuring how risky a random loss is often requires the knowledge of its probability distribution. The industry standard measure of risk, Value-at-Risk (VaR), for example, reports the risk level of a random loss by calculating an extremal quantile of its distribution. Another measure of risk, Conditional Value-at-Risk (CVaR), which has emerged as the most popular alternative to replace VaR as industry standard, calculates the average loss exceeding an extremal quantile to indicate the riskiness of a random loss. The problem however of implementing both of these measures and any other distribution-based risk measure is that in
most practices the exact form of distribution is often lacking and only sample data is available for estimating the distribution, which is inevitably prone to sampling error.

This issue has motivated the development of worst-case risk measures where the goal is to determine the worst-possible risk level over a set of candidate distributions that captures the uncertainty of distribution. Worst-Case Value-at-Risk (WCVaR) was first studied by El Ghaoui et al. (2003) [9], who considered a set of candidate distributions described by the first two moments, and showed how the worst-possible VaR value can be calculated for the set. One of the most noticeable results of El Ghaoui et al. (2003) [9] is perhaps the closed-form solution for WCVaR. The closed-form expression remarkably resembles the risk measure of weighted mean-standard deviation, and hence provides useful insight into how WCVaR can be minimized. El Ghaoui et al. provided also the formulations of semidefinite programs that are equivalent to the closed-form expression, which are useful when an additional layer of uncertainty about the moments needs to be further addressed. It turns out that a closed-form expression also exists for Worst-Case Conditional Value-at-Risk (WCCVaR) when the set of candidate distributions is described by the first two moments (see Chen et al. (2011)[6], Naturajan et al. (2010)[11]), and the expression is identical to the one for WCVaR. Interestingly, this implies that some of the developments in WCVaR such as dealing with moment uncertainty in El Ghaoui et al. (2003) can be directly carried over to the case of WCCVaR. Alternative formulations of worst-case risk measures can also be found in the literature of distributionally robust optimization (DRO) (see for example [5, 7, 11, 21, 22, 23]). Most of these works focus on deriving tractable convex or conic programs for computing the worst-case values (and finding the corresponding robust solutions).

Our work is motivated by the insight gained from the closed-form solutions of the WCVaR and WCCVaR. Given the elegance of the closed form, it is natural to wonder if the closed-form result is just a consequence of the relatively simple structure of VaR and CVaR, or it can be found also for alternative risk measures with more sophisticated structure. On the top of the list of more sophisticated risk measures is the class of spectral risk measures that plays an essential role in both theory and practice. They were first introduced by Acerbi (2002) [1] who attempted to generalize CVaR (and VaR) so that a more realistic description of risk-aversion can be made over a spectrum of CVaRs (VaRs). Later, it became clear that this class of measures is equivalent to the class of distortion risk measures that have applications in insurance [13, 16]. It is also known that spectral risk measures satisfy most, if not all, desirable properties that have been postulated by the modern risk theory ([1, 4, 8, 10]), namely the property of monotonicity, convexity, translation invariance, coherency, and law invariance. A more surprising finding however is that any risk measure that satisfies all these properties, also known as law invariant coherent risk measures, can be represented through spectral risk measures (see [10, 19]). We study in this paper both the case of Worst-Case Spectral Risk Measure (WCSRM) and Worst-Case Law-Invariant Coherent Risk Measure (WCLICRM). Our finding is that despite their seeming complexity, both can be boiled down to a closed-
form expression when only the first two moments are known for the underlying distribution. The closed-form remarkably resembles the measure of weighted mean-standard deviation also, which allows us to shed light on the one-to-one correspondence between any WCLICRM and WCCVaR (and WCVaR). Based on the observation, we demonstrate how the result can be extended and applied in the context of robust portfolio optimization.

This article is organized as follows. In Section 2, we prove the closed-form result for WCSRM and WCLICRM over a set of univariate distributions with fixed first two moments. We show in Section 3 how the result can be applied in the context of robust portfolio optimization.

2 Analytical Results

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space and \(Z\) denote a random variable with its distribution \(F_Z\), i.e. \(Z : (\Omega, \mathcal{F}, \mathbb{P}) \rightarrow \mathbb{R}\) and \(F_Z(t) := \mathbb{P}(Z \leq t)\). The space of random variables is contained in \(L^2(\Omega, \mathcal{F}, \mathbb{P})\). We begin by recalling the following definition of spectral risk measure.

**Definition 1.** (Spectral risk measure [1]) Given a random variable \(Z\), let \(F_Z^{-1}\) denote its general inverse cdf function, i.e. \(F_Z^{-1}(\alpha) := \inf\{q \mid F_Z(q) \geq \alpha\}\). The function

\[
\rho_\phi(Z) := \int_0^1 \phi(\alpha) F_Z^{-1}(\alpha) d\alpha
\]

is called a spectral risk measure parameterized by \(\phi\), if \(\phi \in L^1[0, 1]\) is a non-decreasing probability density function, i.e. \(\phi \geq 0\) and \(\int_0^1 \phi(\alpha) d\alpha = 1\). The density function \(\phi\) is also called an “admissible” risk spectrum.

Intuitively, a spectral risk measure may be viewed as a weighted sum of Value-at-Risk (VaR), where the admissibility of \(\phi\) enforces that the weight assigned to a larger VaR cannot be less. This characterizes the coherency required for a rational individual who is risk-averse. The most notable example of spectral risk measure is \((1 - \epsilon)\)-Conditional Value-at-Risk ((1 - \epsilon)-CVaR), where the spectrum \(\phi\) takes the form \(\phi(\alpha) := \frac{1}{\epsilon} \mathbb{1}_{[1-\epsilon, 1]}(\alpha)\) and \(\epsilon \in (0, 1]\) stands for a tail probability of \(F_Z\). To understand why spectral risk measures play an central role in the modern theory of risk measures [4, 8, 10], we shall review the following definition about law invariant coherent risk measures.

**Definition.** (Law invariant coherent risk measures) Any risk measure \(\rho : L^2(\Omega, \mathcal{F}, \mathbb{P}) \rightarrow \mathbb{R}\) that satisfies

1) Monotonicity: \(\rho(Z_1) \leq \rho(Z_2)\) for any \(Z_1 \leq Z_2\) almost surely;
2) Convexity : \(\rho((1 - \lambda)Z_1 + \lambda Z_2) \leq (1 - \lambda)\rho(Z_1) + \lambda \rho(Z_2), 0 \leq \lambda \leq 1\);
3) Translation invariance : \(\rho(Z + c) = \rho(Z) + c, c \in \mathbb{R}\);
4) Positive homogeneity : \(\rho(\lambda Z) = \lambda \rho(Z), \lambda \geq 0\);
5) Law invariance : \(\rho(F_{Z_1}) = \rho(F_{Z_2})\) if \(F_{Z_1} \equiv F_{Z_2}\); is said to be a law invariant coherent risk measure.
The importance of the above class of risk measures lies in the fact that it satisfies all the properties that have been postulated by the modern theory of convex risk measures \[4, 8, 10\] about what a reasonable risk measure should satisfy. Interestingly, despite its generality, there is a close link between this general class of risk measures and spectral risk measures, namely that the former can always be represented through the latter via a supremum representation.

**Theorem 1.** Any law invariant coherent risk measure \(\rho : L^2(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}\) has the following representation

\[
\rho(Z) = \sup_{\phi \in \Phi} \rho_\phi(Z),
\]

where \(\Phi \subseteq L^1[0,1]\) denotes a set of admissible spectrums.

**Proof.** It has been discussed in [19] (see Proposition 1) that any law invariant coherent risk measure \(\rho : L^p(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}\) with \(p \in [1, \infty)\) admits the representation of

\[
\rho(Z) = \sup_{\mu \in \mathcal{M}} \int_0^1 AV@R_\gamma(Z) d\mu(\gamma),
\]

where \(AV@R_\gamma\) stands for \(\gamma\)-VaR and \(\mathcal{M}\) denotes some set of probability measures on \([0,1]\). Since \(AV@R_\gamma(Z) = (1-\gamma)^{-1} \int_\gamma^1 F_Z^{-1}(\alpha) d\alpha\), we have

\[
\rho(Z) = \sup_{\mu \in \mathcal{M}} \int_0^1 \int_\gamma^1 (1-\gamma)^{-1} F_Z^{-1}(\alpha) d\alpha d\mu(\gamma)
\]

\[
= \sup_{\phi \in \Phi} \int_0^1 \phi(\alpha) F_Z^{-1}(\alpha) d\alpha = \sup_{\phi \in \Phi} \rho_\phi(Z),
\]

where \(\Phi := \{\varphi \mid \varphi(\kappa) = \int_0^\kappa (1-\gamma)^{-1} d\mu(\gamma), \kappa \in [0,1], \mu \in \mathcal{M}\}\) and every \(\varphi\) is, by definition, a non-decreasing probability density on \([0,1]\), i.e., it is an admissible spectrum.

As mentioned earlier, for both the case of VaR and CVaR, their worst-case counterparts can be evaluated in closed form when only the first two moments are known for the underlying distribution. More specifically, given a pair of mean and standard deviation \((\mu, \sigma)\), the largest \((1-\epsilon)\)-VaR and \((1-\epsilon)\)-CVaR value over the set of distributions having the mean \(\mu\) and standard deviation \(\sigma\) can be calculated by [9, 6]

\[
\rho_{WCVaR}(\mu, \sigma, \epsilon) = \rho_{WCCVaR}(\mu, \sigma, \epsilon) = \mu + \sigma \sqrt{\frac{1-\epsilon}{\epsilon}},
\]

where \(\epsilon \in (0,1]\) is the tail probability.

Along this line of work, we consider the following optimization problem that defines the Worst-Case Law Invariant Coherent Risk Measures (WCLICRM):

\[
\rho_{WCLICRM}(\mu, \sigma, \Phi) := \sup_{Z \in \mathcal{Q}} \rho_\phi(Z)
\]

subject to

\[
E[Z] = \mu
\]

\[
STD[Z] = \sigma,
\]

subject to

\[
E[Z] = \mu
\]

\[
STD[Z] = \sigma,
\]

subject to

\[
E[Z] = \mu
\]

\[
STD[Z] = \sigma,
\]
where $\mathcal{Q}$ denotes the set of all probability distributions on $(-\infty, \infty)$. As a special case of WCLICRM, we define also the Worst-Case Spectral Risk Measures (WCSRM) when a single spectrum $\phi$ is considered

$$\rho_{WCSRM}(\mu, \sigma, \phi) := \rho_{WCLICRM}(\mu, \sigma, \{\phi\}). \quad (3)$$

Before proceeding further, we shall make the following assumption about the risk measure $\rho$ used in defining the problem (2).

**Assumption 1.** For any risk measure $\rho$ employed in the definition of WCLICRM, the set $\Phi$ consists of spectrums in $L^\infty[0, 1]$, i.e. bounded functions only.

As noted in [15], unless all considered random variables are essentially bounded, i.e. $Z \in L^\infty(\Omega, \mathcal{F}, \mathbb{P})$, in general a spectral risk measure $\rho_{\phi}$ with an arbitrary spectrum $\phi \in L^1[0, 1]$ may not be well defined. It is not hard to confirm that with Assumption 1, a spectral risk measure $\rho_{\phi}$ would be finite for any $Z \in L^2(\Omega, \mathcal{F}, \mathbb{P})$ (in fact, for any $Z \in L^1(\Omega, \mathcal{F}, \mathbb{P})$). Moreover, this assumption is not really restrictive since for any general law invariant coherent risk measure $\rho$, there always exists a set $\Phi \subseteq L^\infty[0, 1]$ such that $\rho = \sup_{\phi \in \Phi} \rho_{\phi}$ holds (see Corollary 5, [14]).

It is not clear if the problem (2) is tractable in its full generality or only for special cases like CVaR. The main result of this section is to show that not only can the above problem be tractably solved for the cases where $\Phi := \{\phi\}$, i.e. the case of WCSRM (3), the solution of (2) in general admits an elegant closed form expression. We present the result in two steps. Firstly we focus on the case of spectral risk measures, i.e. (3) and show that in this case the problem (3) reduces to a closed-form. Thereafter, the result of WCLICRM, i.e. (2) in general, can be proved fairly straightforwardly.

Before presenting our main results, we need the following lemma that facilitates our analysis.

**Lemma 1.** Given Assumption 1, any spectral risk measure $\rho_{\phi}(Z)$ can be equivalently formulated as

$$\rho_{\phi}(Z) = \min_{q(\alpha)} E[\phi(0)Z] + \int_0^1 [(1 - \alpha)q(\alpha) + (Z - q(\alpha))^+] d\phi(\alpha),$$

where $q \in L^1(0, 1)$, and there exists a non-decreasing function as the optimal solution.

*Proof. Following Proposition 3.2 in [2] we have

$$\rho(Z) = \min_{q(\alpha)} E[\phi(0)Z] + \int_0^1 [(1 - \alpha)q(\alpha) + E[(Z - q(\alpha))^+]] d\phi(\alpha),$$

and the optimal solution $q^*(\alpha)$ satisfies $q^*(\alpha) \in [F_Z^{-1}(\alpha), F_Z^{-1}(\alpha)^+]$ over $\alpha \in \text{supp}(\phi)$, i.e. the support of the measure defined by $\phi(\alpha)$, and can take arbitrary values otherwise. Hence, one can always construct a non-decreasing function $q^*(\alpha)$ over $(0, 1)$ that attains the optimality.

Applying Fubini’s Theorem, we arrive at the result. \qed

$^1F_Z^{-1}(\alpha)^+ := \inf\{q \mid F_Z(q) > \alpha\}$
Applying Lemma 1, we can equivalently formulated the problem of WCSRM (3) as

\[
\rho_{WCSR}(\mu, \sigma, \phi) = \sup_{F_Z} \min_{q(\alpha)} \int_{-\infty}^{\infty} [\phi(0)z + \int_{\alpha}^{1} [(1 - \alpha)q(\alpha) + (z - q(\alpha))^+]d\phi(\alpha)]dF_Z
\]

subject to

\[
\int_{-\infty}^{\infty} zdF_Z = \mu \\
\int_{-\infty}^{\infty} z^2dF_Z = \mu^2 + \sigma^2 \\
\int_{-\infty}^{\infty} dF_Z = 1, \quad F_Z \in \mathcal{Q}.
\]

For simplicity, from here on the integral \(\int_{-\infty}^{\infty}\) may be written as \(\int\) only. As the main result of this paper, we show in the following theorem that the above problem can be reduced to the form of a weighted sum of mean and standard deviation.

**Theorem 2.** Given Assumption 1, any worst-case spectral risk measure (WCSRM) can be evaluated in closed-form:

\[
\rho_{WCSR}(\mu, \sigma, \phi) = \mu + \sigma \sqrt{\int_{0}^{1} \phi^2(p)dp} - 1.
\]

In the case of \(1-\epsilon\)-CVaR, we have \(\int_{0}^{1} \phi^2(p)dp = \frac{1}{\epsilon}\).

**Proof.** Firstly, given that in (4) for any fixed \(F_Z\), there exists a non-decreasing function as the optimal solution, we can impose without loss of generality the constraint \(q(\alpha) \in Q'\), where \(Q'\) denotes the set of all non-decreasing functions over \((0, 1)\). This will facilitate the rest of the proof. Let also \(g(z; q(\alpha)) := \int_{0}^{1} [(1 - \alpha)q(\alpha) + (z - q(\alpha))^+]d\phi(\alpha)\).

Since the set \(Q'\) is convex, and the objective function in (4) is convex in \(q(\alpha)\) for any fixed \(F_Z\) and linear in \(F_Z\) for any fixed \(q(\alpha)\), we can apply Sion’s minmax theorem [20] to switch the sup and min and arrive at the following equivalent problem

\[
\min_{q(\alpha) \in Q'} \{\sup_{F_Z \in \mathcal{Q}} \int [\phi(0)z + g(z; q(\alpha))]dF_Z \mid \int zdF_Z = \mu, \int z^2dF_Z = \mu^2 + \sigma^2\}.
\]

Applying duality theory of conic linear problems (Shapiro 2001 [18]), we can replace the inner maximization problem by its dual, which leads to

\[
\min_{q(\alpha) \in Q'} \min_{\lambda_0, \lambda_1, \lambda_2} \lambda_0 + \mu \lambda_1 + (\mu^2 + \sigma^2) \lambda_2
\]

subject to

\[
\lambda_0 + z \lambda_1 + z^2 \lambda_2 \geq \phi(0)z + g(z; q(\alpha)), \quad \forall z.
\]
Strong duality holds when $\sigma > 0$. It is easy to verify that when $\sigma = 0$, we have $\rho_{WCSRM}(\mu, 0, \phi) = \mu$ and therefore only the case $\sigma > 0$ requires further investigation.

We claim that given any fixed $q(\alpha) \in Q^+$, the function $g(z; q(\alpha))$ is equivalent to the following function

$$ h(z; q(\alpha)) := \sup_{\beta \in [0, 1]} \int_0^1 [(1 - \alpha)q(\alpha) + 1_{(0, \beta]}(\alpha)(z - q(\alpha))]d\phi(\alpha). $$

We verify this by considering the following two cases for any $z$: based on the given $q(\alpha)$, either there exists $\alpha \in (0, 1)$ such that $q(\alpha) \leq z$ or otherwise. For the first case, let $\beta(z) := \arg \max_{\alpha \in (0, 1)} \{\alpha : q(\alpha) \leq z\}$. Since $q(\alpha)$ is non-decreasing, we have $g(\alpha) \leq z, \forall \alpha \in (0, \beta(z))$. Thus, we can equivalently re-write $g(z; q(\alpha))$

$$ g(z; q(\alpha)) = \int_0^1 [(1 - \alpha)q(\alpha) + 1_{(0, \beta(z)]}(\alpha)(z - q(\alpha))]d\phi(\alpha). $$

By definition, $h(z; q(\alpha)) \geq g(z; q(\alpha))$ follows. To show the other direction, let $\beta^*(z)$ denote the optimal solution for the problem in $h(z; q(\alpha))$. There are two possible cases: either $\beta^*(z) \leq \beta(z)$ or otherwise. If $\beta^*(z) \leq \beta(z)$, we have

$$ h(z; q(\alpha)) = \int_0^1 [(1 - \alpha)q(\alpha) + 1_{(0, \beta^*(z)]}(\alpha)(z - q(\alpha))]d\phi(\alpha) $$

$$ \leq \int_0^1 [(1 - \alpha)q(\alpha) + 1_{(0, \beta(z)]}(\alpha)(z - q(\alpha)) + 1_{(\beta^*(z), \beta(z)]}(\alpha)(z - q(\alpha))]d\phi(\alpha) $$

$$ = g(z; q(\alpha)), $$

whereas for the case $\beta^*(z) > \beta(z)$ we have

$$ h(z; q(\alpha)) = \int_0^1 [(1 - \alpha)q(\alpha) + 1_{(0, \beta(z)]}(\alpha)(z - q(\alpha)) + 1_{(\beta(z), \beta^*(z)]}(\alpha)(z - q(\alpha))]d\phi(\alpha) $$

$$ \leq g(z; q(\alpha)), $$

where the last inequality is due to the definition of $\beta(z)$.

Now, for the case that there exists no $\alpha \in (0, 1)$ such that $q(\alpha) \leq z$, we immediately have

$$ h(z; q(\alpha)) = \int_0^1 (1 - \alpha)q(\alpha)d\phi(\alpha) = g(z; q(\alpha)), $$

where the first equality is due to that the optimal $\beta^*(z)$ in $h(z; q(\alpha))$ must be zero, and the second one follows the definition of $g(z; q(\alpha))$.

Hence, we can replace $g(z; q(\alpha))$ by $h(z, q(\alpha))$ in the constraint (6), which leads to

$$ \lambda_0 + z\lambda_1 + z^2\lambda_2 \geq \phi(0)z + \int_0^\beta [(1 - \alpha)q(\alpha) + (z - q(\alpha))]d\phi(\alpha) + \int_0^1 [(1 - \alpha)q(\alpha)]d\phi(\alpha), \forall z, \forall \beta \in [0, 1], $$
where the second constraint can be recast as a second order cone constraint

\[
\min \{ \lambda_0 - \int_0^1 [(1 - \alpha) - 1_{(0, \beta]}(\alpha)]q(\alpha)d\phi(\alpha) + (\lambda_1 - (\phi(0) + \int_0^\beta d\phi(\alpha)))z + \lambda_2 z^2 \} \geq 0, \ \forall \beta \in [0, 1].
\]

By definition, \( \phi(0) + \int_0^\beta d\phi(\alpha) = \phi(\beta) \). For any fixed \( \beta \), the left-hand-side of the above inequality is an elementary minimization problem of a univariate quadratic function. It is bounded below if any only if \( \lambda_2 \geq 0 \). By replacing the optimization problem by the formula known for its optima value, we have the following equivalent formulation:

\[
(\lambda_0 - \varphi(\beta)) - \frac{(\lambda_1 - \phi(\beta))^2}{4\lambda_2} \geq 0, \ \forall \beta \in [0, 1],
\]

where \( \lambda_2 \geq 0 \) and \( \varphi(\beta) := \int_0^1 [(1 - \alpha) - 1_{(0, \beta]}(\alpha)]q(\alpha)d\phi(\alpha) \).

The optimization problem (5) with the above reformulated constraints can be further reformulated into

\[
\min_{q(\alpha) \in Q^-, \lambda_1, \lambda_2 \geq 0, \beta \in [0, 1]} \sup \{ \frac{(\lambda_1 - \phi(\beta))^2}{4\lambda_2} + \varphi(\beta) + \mu \lambda_1 + (\mu^2 + \sigma^2)\lambda_2 \}
\]

\[
\Rightarrow \min_{q(\alpha) \in Q^-, \lambda_1, \lambda_2 \geq 0, \beta \in [0, 1]} \sup \{ \frac{\lambda_2^2}{4\lambda_2} - \frac{\lambda_1 \phi(\beta)}{2\lambda_2} + \frac{\phi(\beta)^2}{4\lambda_2} + \varphi(\beta) + \mu \lambda_1 + (\mu^2 + \sigma^2)\lambda_2 \}
\]

\[
\Rightarrow \min_{q(\alpha) \in Q^-, q, r, \beta \geq 0, \beta \in [0, 1]} \sup \{ \phi(\beta)^2r + \phi(\beta)q + \varphi(\beta) \} + \frac{q^2}{4r} + \mu(\frac{q}{2r}) + (\mu^2 + \sigma^2) \frac{1}{4r}
\]

\[
\Rightarrow \min_{q(\alpha) \in Q^-, q, r, \beta \geq 0, \beta \in [0, 1]} \sup \{ \phi(\beta)^2r + \phi(\beta)q + \varphi(\beta) \} + \frac{(q - \mu)^2 + \sigma^2}{4r}
\]

where \( r = \frac{1}{4\lambda_2} \) and \( q = \frac{-\lambda_1}{2\lambda_2} \) is applied in the third line.

By introducing dummy variables \( s, t \in \mathbb{R} \), we have the following equivalent formulation

\[
\min_{q(\alpha) \in Q^-, q, r, s, t} s + \int_0^1 (1 - \alpha)q(\alpha)d\phi(\alpha) + t
\]

\[
\int_0^1 1_{(0, \beta]}(\alpha)q(\alpha)d\phi(\alpha) - \phi(\beta)^2r - \phi(\beta)q + s \geq 0, \ \forall \beta \in [0, 1] \quad (7)
\]

\[
4rt \geq (q - \mu)^2 + \sigma^2
\]

\[
r \geq 0,
\]

where the second constraint can be recast as a second order cone constraint

\[
\begin{pmatrix}
q - \mu \\
\sigma \\
r - t \\
r + t
\end{pmatrix} \in Q_4,
\]

(8)

where \( Q_4 := \{(u, t) \in \mathbb{R}^4 | ||u|| \leq t\} \) (see, e.g. [3]). To further reduce the problem, we relax first the constraint \( q(\alpha) \in Q^- \) and will verify later that the
relaxation is tight. We apply again the theory of conic linear program [18] and derive the dual of the relaxed problem.

We can define the dual variable for (7) by \( y(\beta) \in \mathcal{Y}[0, 1] \), where \( \mathcal{Y}[0, 1] \) denotes the set of right continuous functions of bounded variation on \([0, 1]\) that corresponds to the space of all finite signed Borel measures on \([0, 1]\). The integral over \( y(\beta) \) follows Lebesgue-Stieltjes integral. In addition, let \( y \in \mathbb{R}^4 \) denote the dual variables corresponding to the second order cone constraint (8). We can write the Lagrange function as follows, where \( x := (q(\alpha), q, r, s, t) \),

\[
L(x, y(\beta), y) = \int_0^1 \left( (1 - \alpha)q(\alpha)d\phi(\alpha) + s + t - \int_0^\beta q(\alpha)d\phi(\alpha) - \phi(\beta)^2r - \phi(\beta)q + s \right)dy(\beta) \\
- y_0(q - \mu) - y_1(\gamma) - y_2(r - t) - y_3(r + t) \\
= \int_0^1 \left( (1 - \alpha)q(\alpha)d\phi(\alpha) - \int_0^\beta q(\alpha)d\phi(\alpha) \right) \\
+ s \left( 1 - \int_0^\beta dy(\beta) \right) + t(1 + y_2 - y_3) + r \left( \int_0^1 \phi(\beta)^2dy(\beta) - y_2 - y_3 \right) \\
+ q \left( \int_0^1 \phi(\beta)dy(\beta) - y_0 \right) + y_0\mu - y_1\gamma,
\]

where in the second line of equality the second term is obtained by interchanging the order of integration. The dual problem \( \max_{y(\beta), y} \min_x L(x, y(\beta), y) \) reduces to the following problem

\[
\max_{y(\beta), y} \mu y_0 - \sigma y_1 \\
\text{subject to } (1 - \alpha) - \int_0^1 dy(\beta) = 0, \forall \alpha \in \text{supp}(\phi) \quad (9) \\
\int_0^1 dy(\beta) = 1 \quad (10) \\
y \succeq_+ 0 \quad (11) \\
y_2 - y_3 = -1 \quad (12) \\
y_2 + y_3 \leq \int_0^1 \phi(\beta)^2dy(\beta) \quad (13) \\
y_0 = \int_0^1 \phi(\beta)dy(\beta) \quad (14) \\
\sqrt{y_0^2 + y_1^2 + y_2^2} \leq y_3, \quad (15)
\]

where \( y \succeq_+ 0 \) refers to that \( y(\beta) \) is non-decreasing on \([0, 1]\).

Following Shapiro (2001)[18], strong duality holds if there exists a feasible \((q^*(\alpha), q^*, r^*, s^*, t^*)\) such that (generalized) slater condition can be satisfied. This is the case for (7), since given any feasible solution that does not satisfy the slater condition, we can always find alternative feasible \( s \) and \( t \) so that the condition can be satisfied.
Observe that from (15) we have
\[ y_1^2 \leq y_3^2 - y_2^2 - y_0^2 \]
\[ \Rightarrow y_1 \geq -\sqrt{y_3^2 - y_2^2 - y_0^2} \]
\[ \Rightarrow y_1 \geq -\sqrt{(y_3 - y_2)(y_3 + y_2) - y_0^2} \]
\[ \Rightarrow -\sigma y_1 \leq \sigma \sqrt{\int_0^1 \phi(\beta)^2 dy(\beta) - (\int_0^1 \phi(\beta) dy(\beta))^2} \quad \text{(due to (12), (13), (14))} \]
\[ (16) \]

Since \( y_1 \) is only constrained by the above inequality, the equality must hold for the optimal solution.

Observe also that by applying integration by parts, we can write (9) as
\[ y(1) - y(\alpha) = 1 - \alpha, \forall \alpha \in \text{supp}(\phi). \]  
(17)

Without loss of generality, we can assume \( y(\beta) \) is normalized by \( y(1) = 1 \). Together with (14) and (16), the objective function can now be reformulated into

\[ \mu(\int_{\text{supp}(\phi)} \phi(\beta) dy(\beta)) + \sigma \sqrt{\int_0^1 \phi(\beta)^2 dy(\beta) - (\int_0^1 \phi(\beta) dy(\beta))^2} \]
\[ = \mu(\int_{\text{supp}(\phi)} \phi(\beta) dy(\beta) + \int_{[0,1] \setminus \text{supp}(\phi)} \phi(\beta) dy(\beta)) + \]
\[ \sigma \sqrt{\left( \int_{\text{supp}(\phi)} \phi(\beta)^2 dy(\beta) + \int_{[0,1] \setminus \text{supp}(\phi)} \phi(\beta)^2 dy(\beta) \right) - (\int_{\text{supp}(\phi)} \phi(\beta) dy(\beta) + \int_{[0,1] \setminus \text{supp}(\phi)} \phi(\beta) dy(\beta))^2}. \]
\[ (18) \]

\[ (19) \]

\[ (20) \]

Observe that the integrals \( \int_{[0,1] \setminus \text{supp}(\phi)} \) can be carried out independently from the exact shape of \( y(\beta) \) over \([0,1] \setminus \text{supp}(\phi)\). Hence, we can always find an optimal \( y(\beta) \) by setting \( y(\beta) = \beta \) that satisfies (10), (11), (17), i.e. it corresponds to a uniform measure over \([0,1]\). The objective function (20) can thus be reduced to

\[ \mu \int_0^1 \phi(\beta) d\beta + \sigma \sqrt{\int_0^1 \phi(\beta)^2 d\beta - (\int_0^1 \phi(\beta) d\beta)^2} \]
\[ = \mu + \sigma \sqrt{\int_0^1 \phi(\beta)^2 d\beta - 1}, \]

since \( \int_0^1 \phi(\beta) d\beta = 1 \).

We are left to show that the problem (7) remains tight after relaxing the constraint \( q(\alpha) \in Q' \). By Shapiro (2001) [18], given that strong duality holds,
we have the following complementary condition hold for the primal optimal solution \((q^*(\alpha), q^*, r^*, s^*, t^*)\) and dual optimal solution \((y^*(\beta), y^*)\)

\[
\int_0^1 \left[ \int_0^\beta q^*(\alpha) \phi(\alpha) - \phi(\beta)^2 r^* - \phi(\beta) q^* + s^* \right] dy^*(\beta) = 0
\]

\[\Rightarrow \int_0^\beta q^*(\alpha) d\phi(\alpha) = \phi(\beta)^2 r^* + \phi(\beta) q^* - s^*, \forall \beta \in [0, 1] \text{ (since } y^*(\beta) \text{ is uniform over } [0, 1])
\]

\[\Rightarrow \int_0^\beta q^*(\alpha) d\phi(\alpha) = \int_0^\beta (2\phi(\alpha)r^* + q^*) d\phi(\alpha) + \phi(0)^2 r^* + \phi(0) q^* - s^*, \forall \beta \in [0, 1]
\]

\[\Rightarrow \int_0^\beta (q^*(\alpha) - 2\phi(\alpha)r^* - q^*) d\phi(\alpha) = \phi(0)^2 r^* + \phi(0) q^* - s^*, \forall \beta \in [0, 1],
\]

\[\Rightarrow \int_{\beta_1}^{\beta_2} (q^*(\alpha) - 2\phi(\alpha)r^* - q^*) d\phi(\alpha) = 0, \forall \beta_1, \beta_2 \in [0, 1].
\]

where the second line can also see [17], and the third is because of applying integration by parts to the right-hand-side. Hence, for any \(\alpha \in \text{supp}(\phi)\), we must have \(q^*(\alpha) = 2\phi(\alpha)r^* + q^*\), which is non-decreasing given that \(\phi\) is so (note that \(r^* \geq 0\)). Since for any \(\alpha \in [0, 1] \setminus \text{supp}(\phi)\), the change of \(q^*(\alpha)\) does not make a difference in (7), we thus confirmed that there exists also a non-decreasing function that is optimal in the relaxed problem. This completes the proof.

The above result not only provides a unified perspective on generating WCSRM in closed form for different choice of spectrum \(\phi\), i.e. modifying the scale factor for standard deviation accordingly, it also enables one to re-interpret the earlier result of WCCVaR. While the scale factor for standard deviation in WCCVaR has often been expressed as \(\sqrt{\frac{1 - \epsilon}{\epsilon}}\), which appears to be the square root of the ratio between the probability of non-tail part and tail part, the above result explains that the ratio can also be interpreted as how much more “skewed” the \(\phi\) is, i.e. \(\int_0^1 \phi^2(p) dp\), compared to the case where \(\phi\) is uniform over \([0, 1]\), in which case \(\int_0^1 \phi(p)^2 dp = 1\). We have \(\rho_{WCSR}(\mu, \sigma, \phi^*) = \mu\) when \(\phi^*\) is uniform. Hence, the closed-form might be roughly read as “the risk neutral value where \(\phi\) is uniform plus standard deviation multiplied by how much more skewed a given \(\phi\) is compared to the case of uniform measure, i.e. \(\sqrt{\int \phi^2(p) dp - 1}\)”. Despite its elegance, the result of Theorem 2 is in fact not obvious and can be quite counter-intuitive if one takes the perspective from the nominal risk measures, i.e. for a fixed distribution. To see why the result might be surprising, let us highlight the following implication from the result.

**Corollary 1.** Given any \((\mu, \sigma)\), the worst-case spectral risk measure with spectrum \(\phi\), i.e. \(\rho_{WCSR}(\mu, \sigma, \phi)\) is equivalent to the worst-case \((1 - \epsilon')\)-value-at-risk and \((1 - \epsilon')\)-conditional value-at-risk respectively with \(\epsilon = \frac{1}{\int_0^1 \phi(p)^2 dp}\), i.e. \(\rho_{WCVaR}(\mu, \sigma, \frac{1}{\int_0^1 \phi(p)^2 dp})\) and \(\rho_{WCCVaR}(\mu, \sigma, \frac{1}{\int_0^1 \phi(p)^2 dp})\) respectively.
Obviously, the above statement might not be true for the case of nominal risk measures, since in order to match the value of a spectral risk measure, the corresponding tail probability of a \((1 - \epsilon')\)-CVaR could depend on the shape of the given distribution, i.e. \(\epsilon' := \epsilon(F_Z)\). In the above corollary however, the equivalency between WCSRM and WCCVaR can be established independently from the structure of the distribution, i.e. the mean and standard deviation.

We are now ready to present the result for the general case of worst-case law invariant coherent risk measures, which can be straightforwardly obtained from the result of Theorem 2.

**Theorem 3.** Given Assumption 1, any worst-case law invariant coherent risk measure defined based on \(\rho_\Phi = \sup_{\phi \in \Phi} \rho_\phi\) can be evaluated in closed-form

\[
\rho_{WCLICRM}(\mu, \sigma, \Phi) = \mu + \sigma \sqrt{\sup_{\phi \in \Phi} \int_0^1 \phi^2(p) dp - 1},
\]

and is equivalent to the worst-case \((1 - \epsilon')\)-VaR and \((1 - \epsilon')\)-CVaR by setting \(\epsilon' = \frac{1}{\sup_{\phi \in \Phi} \phi^2(p) dp}\).

**Proof.** For simplicity, we write \(F_Z \sim (\mu, \sigma)\) to denote any distribution with mean \(\mu\) and standard deviation \(\sigma\). We have

\[
\rho_{WCLICRM}(\mu, \sigma, \Phi) := \sup_{F_Z \sim (\mu, \sigma)} \sup_{\phi \in \Phi} \rho_\phi(Z)
\]

\[
= \sup_{\phi \in \Phi} \sup_{F_Z \sim (\mu, \sigma)} \min_{q(\alpha)} \int_0^1 \left[ \phi(0)z + \int_0^1 (1 - \alpha)q(\alpha) + (z - q(\alpha))^+ \right] d\phi(\alpha) dF_Z
\]

\[
= \sup_{\phi \in \Phi} \mu + \sigma \sqrt{\int_0^1 \phi^2(p) dp - 1}
\]

\[
= \mu + \sigma \sqrt{\sup_{\phi \in \Phi} \int_0^1 \phi^2(p) dp - 1},
\]

where the last equality is simply due to the fact that \(\sigma\sqrt{\cdot}\) is an increasing function.

We end this section by concluding that the closed-form insight from WCVaR and WWCVaR can be well carried over to many risk measures that are considered sensible in the modern risk theory.

### 3 Robust Portfolio Optimization

The observation made in Corollary 1 (or Theorem 3) can be found particularly useful in the context of robust portfolio optimization. We provide necessary details in this section to draw the connection between robust portfolio optimization for VaR and general law invariant risk measures. The problem of robust portfolio optimization seeks a portfolio that minimizes the worst-case
risk while satisfying a number of constraints such as no short-sale requirement. It can be generally formulated as the following minmax problem when a law invariant coherent risk measure is employed

$$\min_{x \in \mathcal{X}} \sup_{F_R} \rho_{\phi}(-(R)^T x)$$

subject to

$$E[R] = \mu$$
$$\text{COV}[R] = \Sigma,$$

where $\mathcal{X} \subseteq \mathbb{R}^n$ denotes a set of admissible portfolio allocation vectors over $n$ different assets, and $R : (\Omega, \mathcal{F}, \mathbb{P}) \rightarrow \mathbb{R}^n$ stands for the vector of random returns of the $n$ assets with its distribution $F_R$. The set $\mathcal{X}$ is assumed to be a bounded polytope that does not contain 0. In the above formulation, we assume that only the mean $\mu \in \mathbb{R}^n$ and covariance $\Sigma \in \mathbb{R}^{n \times n}$ of the joint distribution of returns are known, and a portfolio $x \in \mathcal{X}$ is sought that minimizes the worst-case risk over the set of multivariate distributions $F_R$ having $\mu$ and $\Sigma$ as mean and covariance. The above problem appears to be difficult due to its minmax form of objective function and the high dimensionality of the random returns. Fortunately, we can apply the following result first to simplify the robust problem.

**Lemma 2.** ([6]) Let $A := \{a^T \xi \mid E[\xi] = \mu, \text{COV}[\xi] = \Sigma\}$, $B := \{\eta \mid E[\eta] = a^T \mu, \forall \mathbb{R}[\eta] = a^T \Sigma a\}$. For any $a \neq 0 \in \mathbb{R}^n$, it holds that $A = B$.

In other words, we can equivalently reformulated the above problem as

$$\min_{x \in \mathcal{X}} \sup_{F_Z} \rho_{\phi}(-Z)$$

subject to

$$E[Z] = \mu^T x$$
$$\text{STD}[Z] = \sqrt{x^T \Sigma x}$$

where for any fixed $x$, the random variable $Z$ is simply a random variable with a univariate distribution $F_Z$. The inner maximization problem can now be reformulated using the result of Theorem 3 and the whole problem can be reduced to the following minimization problem

$$\min_{x \in \mathcal{X}} -\mu^T x + \sqrt{x^T \Sigma x} \sqrt{\sup_{\phi \in \Phi} \int_0^1 \phi(p)^2 dp - 1.}$$

Provided that the term $\sup_{\phi \in \Phi} \int_0^1 \phi(p)^2 dp$ can be solved offline, this final problem can be solved easily by a SOCP solver [3]. Moreover, it is identical to the robust portfolio optimization for VaR ([9]) except the scale factor, which confirms the following fact aligned with the observation made in Corollary 1.

**Corollary 2.** Given Assumption 1, solving robust portfolio optimization problem with law invariant coherent risk measure is equivalent to solving the robust problem with $(1 - \epsilon') - \text{VaR}$ (or $(1 - \epsilon') - \text{CVaR}$) with $\epsilon' = \frac{1}{\sup_{\phi \in \Phi} \int_0^1 \phi^2(p) dp}$. 

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The above fact immediately implies that one can easily extend the above robust portfolio problem (21) to the case where the first two moments are uncertain, which has been well addressed in the literature of robust VaR and CVaR optimization [9, 6, 11]. To demonstrate the idea, we provide below a few possible extensions to deal with moment uncertainty based on the work of El Ghaoui et al. (2003) [9], where the resulting formulations can often be recast as conic programs [12]. We skip the proofs since they can be found in El Ghaoui et al. (2003) once Corollary 2 is applied.

**Corollary 3.** (c.f.[9] Section 2.2-2.4, 3.1) Given Assumption 1, if the mean $\mu$ and covariance $\Sigma$ of the distribution $F_R$ of random returns $R$ are only known to belong to a convex set $C \subseteq \mathbb{R}^n \times \mathbb{R}^{n \times n}$, the robust portfolio optimization problem (21) with a law invariant coherent risk measure $\rho_\phi := \sup_{\phi \in \Phi} \rho_\phi$ can be solved by the following minmax problem

$$\min_{x \in X} \max_{r, \mu, \Sigma} - r^T x$$

subject to

$$\begin{bmatrix} \Sigma \\ (r - \mu)^T \end{bmatrix} \sup_{\phi \in \Phi} \int_0^1 \phi(p)^2 dp - 1 \geq 0,$$

$$(\mu, \Sigma) \in C$$

where $\geq 0$ stands for that the left-hand-side matrix is positive semi-definite. The problem further reduces to conic programs [12] for the following special cases

1) (Polytopic uncertainty) $C := \text{Co}\{(\mu_k, \Sigma_k)\}_{k=1, \ldots, K}$, where $\text{Co}$ is the convex hull operator

2) (Componentwise bounds) $C := \{ (\mu, \Sigma) \mid \mu_L \leq \mu \leq \mu_U, \Sigma_L \leq \Sigma \leq \Sigma_U \}$,

3) (Uncertainty in factor’s model) $C := \{ (\mu, \Sigma) \mid \exists (\mu_f, S) \mu = A \mu_f, \Sigma = D + ASA^T, \mu_f_L \leq \mu_f \leq \mu_f_U, S_L \leq S \leq S_U \}$, where a factor model $R = A f + u$ is assumed for the random returns and $u$ are residuals with diagonal covariance matrix $D$.

Finally, it is worth pointing out also that based on the result of [6], the robust portfolio optimization problem (21) can also be solved in closed form when the feasible set $X$ is described by a simple budget constraint, i.e. $1^T x = 1$. Interested readers are referred to Theorem 2.9 in [6].

## 4 Concluding Remarks

In this paper, we showed that closed-form solutions also exist for a general class of worst-case risk measures defined based on law invariant coherent risk measures. The result generalizes to a great extent the existing closed-form result of worst-case Value-at-Risk and worst-case Conditional Value-at-Risk, which have received a considerable amount of attention in the past decade. The closed-form solutions for the general class of measures are remarkably similar to that of VaR and CVaR, and thus are immediately applicable in many settings where worst-case VaR and CVaR have been implemented.

\[2\]and robust mean-variance optimization.
References


