

# Positive and $\mathbf{Z}$ -operators on closed convex cones

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## Abstract

Let  $K$  be a closed convex cone with dual  $K^*$  in a finite-dimensional real Hilbert space  $V$ . A *positive operator* on  $K$  is a linear operator  $L$  on  $V$  such that  $L(K) \subseteq K$ . Positive operators generalize the nonnegative matrices and are essential to the Perron-Frobenius theory. We say that  $L$  is a  *$\mathbf{Z}$ -operator* on  $K$  if

$$\langle L(x), s \rangle \leq 0 \text{ for all } (x, s) \in K \times K^* \text{ such that } \langle x, s \rangle = 0.$$

The  $\mathbf{Z}$ -operators are generalizations of  $\mathbf{Z}$ -matrices (whose off-diagonal elements are nonpositive) and they arise in dynamical systems, economics, game theory, and elsewhere. We connect the positive and  $\mathbf{Z}$ -operators. This extends the work of Schneider, Vidyasagar, and Tam on proper cones, and reveals some interesting similarities between the two families.

## 1 Introduction

Positive operators arose from the study of integral operators and matrices with nonnegative entries [1]. The latter form the foundation of the Perron-Frobenius theory. Perron showed that a matrix with positive entries has a unique largest eigenvalue whose corresponding eigenvector can be chosen to have positive entries. Frobenius partially extended this result to nonnegative matrices, and the nonnegative matrices are positive operators in that setting.

Suppose that  $V$  is an ordered vector space and that  $x \geq 0$  in  $V$ . In the theory of operators [1],  $x$  is called a positive element of  $V$ . A *positive operator* is a linear operator that sends positive elements of  $V$  to positive elements. Every proper cone  $K$  orders [2] its ambient space by  $x \geq 0 \iff x \in K$ . With respect to this ordering, we denote the set of positive operators by

$$\pi(K) := \{L : V \rightarrow V \mid L \text{ is linear and } L(K) \subseteq K\}.$$

The Perron-Frobenius theorem is thus a statement about positive operators on the cone  $K = \mathbb{R}_+^n$ , the nonnegative orthant in  $\mathbb{R}^n$ . The famous Krein-Rutman theorem extends Perron-Frobenius to a proper cone in a Banach space. This connects positive operators to the theory of dynamical systems [24], to game theory [9], and more. It also inspires the nonlinear Perron-Frobenius theory [18].

A  $\mathbf{Z}$ -matrix is a real square matrix whose off-diagonal entries are nonpositive. Equivalently, a  $\mathbf{Z}$ -matrix has the form  $\lambda I - N$  where  $N$  is a nonnegative matrix (that is, a positive operator on  $\mathbb{R}_+^n$ ). It is therefore not surprising that the two theories are intertwined. From the expression  $\lambda I - N$ , it is clear that the Perron-Frobenius theory influences the invertibility of a  $\mathbf{Z}$ -matrix. Berman and Plemmons [3] cite an astounding number of equivalent conditions for  $\mathbf{Z}$ -matrices, connecting them to many different areas.

Generalizations of  $\mathbf{Z}$ -matrices have started to appear [4, 5]. Our definition of a  $\mathbf{Z}$ -operator is due to Gowda and Tao [12]. If  $K^*$  represents the dual of  $K$ , then  $L$  is a  $\mathbf{Z}$ -operator on  $K$  and we write  $L \in \mathbf{Z}(K)$  if  $\langle L(x), s \rangle \leq 0$  for all  $x \in K$  and  $s \in K^*$  such that  $\langle x, s \rangle = 0$ . This definition reduces to that of a  $\mathbf{Z}$ -matrix when  $K = \mathbb{R}_+^n$ . These  $\mathbf{Z}$ -operators emerge in dynamical systems [12], complementarity problems [12], game theory [10], economics, and everywhere that  $\mathbf{Z}$ -matrices arise [3]. Kuzma et al. [17] recently resolved an open problem that applies  $\mathbf{Z}$ -operators to mathematical finance.

Schneider and Vidyasagar [25] discovered a striking connection between the positive and  $\mathbf{Z}$ -operators on a proper cone  $K$ . We eventually extend this result to any closed convex cone in finite dimensions.

**Theorem.** *If  $K$  is a proper cone in  $\mathbb{R}^n$  and  $A$  is a matrix in  $\mathbb{R}^{n \times n}$ , then  $A \in \mathbf{Z}(K)$  if and only if  $e^{-tA} \in \pi(K)$  for all  $t \geq 0$ .*

The  $\mathbf{Z}$ -operators contain a subspace  $\mathbf{LL}(K) := \mathbf{Z}(K) \cap -\mathbf{Z}(K)$  of *Lyapunov-like* operators. Lyapunov-like operators are important because they can be used to solve the equation  $\langle x, s \rangle = 0$  for  $x \in K$  and  $s \in K^*$  that appears as optimality conditions in convex optimization problems [23]. One motivation for studying the  $\mathbf{Z}$ -operators is their connection to the Lyapunov-like operators. In fact, all three families are related. Schneider and Vidyasagar proved the following.

**Theorem.** *If  $K$  is a proper cone in  $\mathbb{R}^n$ , then  $\mathbf{Z}(K) = \text{cl}(\mathbf{LL}(K) - \pi(K))$ .*

We will also generalize this result. Sometimes the closure is superfluous and  $\mathbf{Z}(K) = \mathbf{LL}(K) - \pi(K)$ —the problem solved by Kuzma et al. was of that type. By studying  $\mathbf{Z}(K)$  and  $\pi(K)$ , we hope to gain insight into similar problems.

## 2 Preliminaries

Throughout this section,  $V$  is a finite-dimensional real Hilbert space.

### 2.1 Standard definitions

Let  $W$  be finite-dimensional real Hilbert space. The set of all linear operators from  $V$  to  $W$  forms a vector space which we denote by  $\mathcal{B}(V, W)$ . We abbreviate  $\mathcal{B}(V, V)$  by  $\mathcal{B}(V)$ . If  $L \in \mathcal{B}(V, W)$  is invertible and preserves inner products, then  $L$  is an *isomorphism*. If  $L \in \mathcal{B}(V)$  is invertible and  $L(X) = X$ , then  $L$  is an automorphism of  $X$  and we write  $L \in \text{Aut}(X)$ .

Any  $L \in \mathcal{B}(V, W)$  has an adjoint  $L^* \in \mathcal{B}(W, V)$  such that  $\langle L(x), y \rangle = \langle x, L^*(y) \rangle$  for all  $x \in V$  and  $y \in W$ . Given two elements  $x$  and  $s$  in  $V$ , we define  $s \otimes x$  to be the operator  $y \mapsto \langle x, y \rangle s$  on  $V$ . For subsets  $S$  and  $X$  of  $V$ , we will write  $S \otimes X := \{s \otimes x \mid s \in S, x \in X\}$ . The adjoint of  $s \otimes x$  is  $x \otimes s$ , and  $s \otimes L^*(x) = (s \otimes x) \circ L$  is the composition of the operators  $s \otimes x$  and  $L \in \mathcal{B}(V)$ . The identity operator on  $V$  is  $\text{id}_V \in \mathcal{B}(V)$ . In  $\mathbb{R}^n$ , the identity matrix of the appropriate size is denoted by  $I$ .

Define the trace operator on  $\mathcal{B}(V)$  to be the sum-of-eigenvalues,  $\text{trace}(L) := \sum_{\lambda \in \sigma(L)} \lambda$ . Then  $\langle L_1, L_2 \rangle := \text{trace}(L_1 \circ L_2^*)$  is our inner product on linear operator spaces. Later we use the fact that  $\text{trace}(s \otimes x) = \text{trace}(x \otimes s) = \langle x, s \rangle$ .

The *convex hull* of a nonempty subset  $X$  of  $V$  is

$$\text{conv}(X) := \left\{ \sum_{i=1}^m \alpha_i x_i \mid x_i \in X, \alpha_i \geq 0, m \in \mathbb{N}, \sum_{i=1}^m \alpha_i = 1 \right\}.$$

Carathéodory's theorem [21] lets us regard any element in the convex hull as a finite convex combination. A generalization of the convex hull is the *affine hull*,

$$\text{aff}(X) := \left\{ \sum_{i=1}^m \alpha_i x_i \mid x_i \in X, m \in \mathbb{N}, \sum_{i=1}^m \alpha_i = 1 \right\}.$$

The topological closure of a subset  $X$  of  $V$  is  $\text{cl}(X)$ . Let  $B_\epsilon(x)$  denote an  $\epsilon$ -ball at  $x$ . If  $X$  is convex, then its *relative boundary* is the set of  $x \in \text{cl}(X)$  such that for any  $\epsilon > 0$ , the set  $B_\epsilon(x) \cap \text{aff}(X)$  intersects both  $X$  and  $V \setminus X$  nontrivially.

If  $W$  is a subspace of  $V$ , then the *orthogonal complement* of  $W$  is another subspace of  $V$  defined by  $W^\perp := \{y \in V \mid \langle x, y \rangle = 0 \text{ for all } x \in W\}$ , and  $V$  has direct sum decomposition  $V = W \oplus W^\perp$ . By  $\text{proj}(W, x)$  we denote the *orthogonal projection* of  $x \in V$  onto the subspace  $W$ . The direct sum of two orthogonal sets  $X$  and  $S$  will be denoted by  $X \oplus S$ ; for example, it is appropriate to write  $V = W \oplus W^\perp$  when  $W$  is a subspace of  $V$ .

The real  $n$ -space  $\mathbb{R}^n$  is equipped with the usual inner product, standard basis  $(e_1, e_2, \dots, e_n)$ , and nonnegative orthant  $\mathbb{R}_+^n := \{x \in \mathbb{R}^n \mid x_i \geq 0 \text{ for all } i\}$ .

## 2.2 Cone definitions

**Definition 1.** A nonempty subset  $K$  of  $V$  is a *cone* if  $\lambda K = K$  for all  $\lambda \geq 0$ . A *closed convex cone* is a cone that is closed and convex as a subset of  $V$ .

**Definition 2.** The *conic hull* of a nonempty subset  $X$  of  $V$  is

$$\text{cone}(X) := \left\{ \sum_{i=1}^m \alpha_i x_i \mid x_i \in X, \alpha_i \geq 0, m \in \mathbb{N} \right\}.$$

Clearly  $\text{cone}(X)$  is a convex cone. If  $X$  is finite, then  $\text{cone}(X)$  is closed [21].

**Definition 3.** If  $\text{cone}(G) = K$ , then  $G$  *generates*  $K$  and the elements of  $G$  are *generators* of  $K$ . If a finite set generates  $K$ , then  $K$  is *polyhedral*.

**Definition 4.** The dimension of  $K \subseteq V$  is  $\dim(K) := \dim(\text{span}(K))$ , and  $\text{codim}(K) := \dim(V) - \dim(K)$ . A convex cone  $K$  is *solid* if  $\text{span}(K) = V$ .

If  $S$  and  $X$  are subsets of  $V$ , then  $\dim(S \otimes X) = \dim(S) \dim(X)$  [22].

**Definition 5.** The *lineality space* of a convex cone  $K$  is  $\text{linspace}(K) := K \cap -K$ . Its *lineality* is  $\text{lin}(K) := \dim(\text{linspace}(K))$ , and  $K$  is *pointed* if  $\text{lin}(K) = 0$ .

**Definition 6.** A pointed, solid, and closed convex cone is *proper*.

There is a duality between pointed and solid cones.

**Definition 7.** If  $K$  is a subset of  $V$ , then the *dual cone*  $K^*$  of  $K$  is

$$K^* := \{y \in V \mid \forall x \in K, \langle x, y \rangle \geq 0\}.$$

The dual  $K^*$  is always a closed convex cone. If  $K$  is a closed convex cone, then the duality is faithful and  $(K^*)^* = K$ . The next two results are well-known and are given as Rockafellar's [21] Corollary 14.6.1 and Corollary 16.4.2.

**Proposition 1.** *If  $K$  is a closed convex cone in  $V$ , then  $K$  is pointed if and only if  $K^*$  is solid. Moreover  $\text{linspace}(K) = \text{span}(K^*)^\perp$  and  $\text{lin}(K) = \text{codim}(K^*)$ .*

**Proposition 2.** *If  $K_1$  and  $K_2$  are convex cones, then  $(K_1 + K_2)^* = K_1^* \cap K_2^*$  and  $(\text{cl}(K_1) \cap \text{cl}(K_2))^* = \text{cl}(K_1^* + K_2^*)$ . If  $K_1$  and  $K_2$  are closed, then it follows that the dual of  $\text{cl}(K_1 + K_2)$  is  $K_1 \cap K_2$ .*

Often we will need to find pairs  $(x, s) \in K \times K^*$  having  $\langle x, s \rangle > 0$ . When  $K$  is not proper, they may not exist; the following is a consequence of Proposition 1.

**Corollary 1.** *Let  $K$  be a closed convex cone in  $V$ . If  $x \in K$  but  $x \notin \text{linspace}(K)$  then there exists an  $s \in K^*$  such that  $\langle x, s \rangle > 0$ .*

## 2.3 Classes of linear operators

Our main results concern the following classes of linear operators.

**Definition 8.** The operator  $L \in \mathcal{B}(V)$  is a *positive operator* on  $K$  if  $L(K) \subseteq K$ . The set of all such operators is denoted by  $\pi(K)$ .

The prototypical positive operators are nonnegative matrices [3] on  $K = \mathbb{R}_+^n$ . If  $K$  is a closed convex cone, then we have an alternative characterization:

$$L \in \pi(K) \iff \langle L(x), s \rangle \geq 0 \text{ for all } (x, s) \in K \times K^*.$$

The requisite property of a **Z**-operator is similar, but it need only hold on pairs of orthogonal vectors in  $K \times K^*$ .

**Definition 9.** The *complementarity set* of  $K$  is

$$C(K) := \{(x, s) \in K \times K^* \mid \langle x, s \rangle = 0\}.$$

**Definition 10.** The operator  $L \in \mathcal{B}(V)$  is a  $\mathbf{Z}$ -operator on  $K$  if

$$\langle L(x), s \rangle \leq 0 \text{ for all } (x, s) \in C(K).$$

By  $\mathbf{Z}(K)$  we denote the set of all  $\mathbf{Z}$ -operators on  $K$ .

When  $K = \mathbb{R}_+^n$ , the complementarity set  $C(\mathbb{R}_+^n)$  consists of pairs of standard basis vectors. The requirement on  $\mathbf{Z}(\mathbb{R}_+^n)$  gives rise to matrices whose off-diagonal elements are nonpositive—the  $\mathbf{Z}$ -matrices. The set  $\mathbf{Z}(K)$  is a closed convex cone and it contains the subspace of Lyapunov-like operators.

**Definition 11.** The operator  $L \in \mathcal{B}(V)$  is *Lyapunov-like* on  $K$  if

$$\langle L(x), s \rangle = 0 \text{ for all } (x, s) \in C(K).$$

By  $\mathbf{LL}(K)$  we denote the set of all Lyapunov-like operators on  $K$ . The *Lyapunov rank* of  $K$  is  $\beta(K) := \dim(\mathbf{LL}(K))$ .

The set  $\mathbf{LL}(K)$  is a vector space and  $\mathbf{LL}(K) = \text{linspace}(\mathbf{Z}(K))$ . Finding Lyapunov-like operators is an interesting problem. The search began with Rudolf et al. [23] and has been continued by others [11, 13, 14, 19, 20].

### 3 Positive operators

Observe that the positive operators on a closed convex cone  $K$  themselves form a closed convex cone. The three criteria—that  $\pi(K)$  is closed, convex, and a cone—are easy to verify and depend on the same properties of  $K$ .

**Proposition 3.** *If  $K$  is a closed convex cone, then so is  $\pi(K)$ .*

If  $K$  is proper, then both  $\pi(K)$  and its dual are proper [25]. To determine if some linear operator belongs to  $\pi(K)$ , it suffices to check positivity on a generating set of  $K$ . This can be seen by expanding any element of  $K$  in terms of its generators and using the linearity of the operator.

**Proposition 4.** *If  $K = \text{cone}(G)$  in a finite-dimensional real Hilbert space  $V$  and if  $L \in \mathcal{B}(V)$ , then  $L \in \pi(K)$  if and only if  $L(G) \subseteq K$ .*

Tam [30] found a simple expression for the generators of the dual of  $\pi(K)$  when  $K$  is proper. He uses the fact that  $\text{cone}(K^* \otimes K)$  is closed to prove that

$$\pi(K)^* = \text{cone}(K^* \otimes K) \text{ if } K \text{ is proper.}$$

These generators will work when  $K$  is merely closed and convex, but the proof requires some modification. When  $K$  is a closed convex cone, we will usually define  $\mathcal{T} := \text{cone}(K^* \otimes K)$  to be the cone of Tam’s generators. Note that any element  $\phi \in \mathcal{T}$  can be written  $\phi = \sum_{i=1}^m s_i \otimes x_i$  without the scalar factors, since they can be absorbed into  $s_i \otimes x_i$ . Our first step will be to decompose  $\mathcal{T}$  into two components, one of which is pointed [21, 27].

**Proposition 5.** *If  $K$  is a convex cone in a finite-dimensional real Hilbert space, then  $K$  has an orthogonal direct sum decomposition into two convex cones,*

$$K = K \cap \text{linspace}(K)^\perp \oplus \text{linspace}(K).$$

*Its first factor  $K \cap \text{linspace}(K)^\perp = \text{proj}(\text{linspace}(K)^\perp, K)$  is pointed.*

**Corollary 2.** *If  $K$  is a closed convex cone in a finite-dimensional real Hilbert space, then  $\text{proj}(\text{linspace}(K)^\perp, K) = K \cap \text{linspace}(K)^\perp$  is closed.*

We will ultimately apply this decomposition to  $\mathcal{T} := \text{cone}(K^* \otimes K)$  which is itself a convex cone. The crux of our argument is that  $\mathcal{T}$  is closed even if  $K$  is not proper. This next result reduces the burden of proof to the pointed component of  $\mathcal{T}$  obtained from Proposition 5.

**Proposition 6.** *If  $K$  is a convex cone in a finite-dimensional real Hilbert space and if  $\text{proj}(\text{linspace}(K)^\perp, K)$  is closed, then  $K$  is closed.*

*Proof.* Using Proposition 5, any convergent sequence  $(v + w)_n$  in  $K$  decomposes into orthogonal parts  $v_n + w_n$  where  $(v)_n$  converges in  $\text{proj}(\text{linspace}(K)^\perp, K)$  and  $(w)_n$  converges in  $\text{linspace}(K)$ . Thus  $(v + w)_n$  converges in  $K$ .  $\square$

To exploit Proposition 5, we'll need the generators of its pointed component. The next proposition shows what those generators are, albeit not explicitly.

**Proposition 7.** *If  $K = \text{cone}(G)$  in a finite-dimensional real Hilbert space  $V$  and if  $W$  is a subspace of  $V$ , then  $\text{proj}(W, G)$  generates  $\text{proj}(W, K)$ .*

*Proof.* Since  $\text{proj}(W, K) = \{\text{proj}(W, \sum_{i=1}^m \alpha_i g_i) \mid \alpha_i \geq 0, g_i \in G, m \in \mathbb{N}\}$ , the linearity of the projection gives  $\text{proj}(W, K) = \text{cone}(\text{proj}(W, G))$ .  $\square$

So, the generators of the pointed component in Proposition 5 are orthogonal projections of the cone's generators. But we want to apply this result to the cone of Tam's generators, so we are missing two pieces of information. First, we lack a description of  $\text{linspace}(\mathcal{T})^\perp$ . And if we had one, we would still need to determine the form of the projected generators. We can compute the lineality space of  $\mathcal{T}$  explicitly, but we need a trivial proposition before we can do that.

**Proposition 8.** *If  $K$  is a closed convex cone in a finite-dimensional real Hilbert space and if  $\mathcal{T} = \text{cone}(K^* \otimes K)$ , then  $\mathcal{T} \subseteq \pi(K^*)$ .*

*Proof.* If  $L = \sum s_i \otimes x_i \in \mathcal{T}$  and  $t \in K^*$ , then  $L(t) = \sum \langle x_i, t \rangle s_i \in K^*$ .  $\square$

Now we'll find  $\text{linspace}(\mathcal{T})$  because we need  $\text{linspace}(\mathcal{T})^\perp$  for Proposition 6.

**Proposition 9.** *If  $K$  is a closed convex cone in a finite-dimensional real Hilbert space and if  $\mathcal{T} = \text{cone}(K^* \otimes K)$ , then  $\text{linspace}(\mathcal{T}) = U_1 + U_2$  where*

$$U_1 := \text{span}(K^* \otimes \text{linspace}(K)); \quad U_2 := \text{span}(\text{linspace}(K^*) \otimes K).$$

*Proof.* First we show that  $U_1 + U_2 \subseteq \text{linspace}(\mathcal{T})$ . For any  $u_1 \in U_1$ ,

$$u_1 = \sum_{i=1}^m \alpha_i (s_i \otimes x_i) \text{ where } \alpha_i \in \mathbb{R}, x_i \in \text{linspace}(K), \text{ and } s_i \in K^*.$$

Write  $\alpha_i (s_i \otimes x_i) = s_i \otimes (\alpha_i x_i)$  where  $\alpha_i x_i \in K$  since  $x_i \in \text{linspace}(K)$ . Each  $\alpha_i (s_i \otimes x_i)$  is thus an element of  $\mathcal{T}$ , and so  $u_1 \in \mathcal{T}$ . Repeat the argument to show  $-u_1 \in \mathcal{T}$  and thus  $u_1 \in \text{linspace}(\mathcal{T})$ . Similarly,  $u_2 \in U_2$  implies  $u_2 \in \text{linspace}(\mathcal{T})$ . It follows that  $U_1 + U_2 \subseteq \text{linspace}(\mathcal{T})$ .

For the other inclusion, let  $h \in \text{linspace}(\mathcal{T})$ . Then  $\pm h \in \mathcal{T}$ , so,

$$-h = \sum_{i=1}^m -(s_i \otimes x_i) \text{ where } (x_i, s_i) \in K \times K^*.$$

By adding elements of  $\mathcal{T}$  to both sides, we can show that  $h$  is a sum of  $s_j \otimes x_j \in \text{linspace}(\mathcal{T})$ . For example, when  $j = 1$ ,

$$-h + \sum_{i=2}^m (s_i \otimes x_i) = -(s_1 \otimes x_1) \in \mathcal{T} \implies s_1 \otimes x_1 \in \text{linspace}(\mathcal{T}).$$

If every such  $s_j \otimes x_j$  is an element of  $U_1 + U_2$ , then  $h \in U_1 + U_2$  and we are done. Consider any particular  $s_j \otimes x_j$ :

**Case 1:**  $x_j \in \text{linspace}(K)$ .

Take  $u_1 := s_j \otimes x_j \in U_1$ .

**Case 2:**  $s_j \in \text{linspace}(K^*)$ .

Take  $u_2 := s_j \otimes x_j \in U_2$ .

**Case 3:**  $x_j \notin \text{linspace}(K)$  and  $s_j \notin \text{linspace}(K^*)$ .

By Corollary 1, there exists a  $t \in K^*$  such that  $\langle x_j, t \rangle > 0$ . Apply  $\pm (s_j \otimes x_j) \in \mathcal{T}$  to  $t$ , and use Proposition 8 to show that  $\pm s_j \in K^*$ , contradicting  $s_j \notin \text{linspace}(K^*)$ .  $\square$

The lineality of a cone is the dimension of its lineality space, so we are now in a position to compute  $\text{lin}(\mathcal{T})$ . This will be used to find  $\text{linspace}(\mathcal{T})^\perp$ .

**Proposition 10.** *If  $K$  is a closed convex cone in a finite-dimensional real Hilbert space  $V$  and if  $\mathcal{T} = \text{cone}(K^* \otimes K)$ , then*

$$\text{lin}(\mathcal{T}) = \text{lin}(K) \dim(K^*) + \dim(K) \text{lin}(K^*) - \text{lin}(K) \text{lin}(K^*).$$

*Proof.* Use Proposition 9 and apply the dimension formula to  $\text{linspace}(\mathcal{T})$ ,

$$\dim(\text{linspace}(\mathcal{T})) = \dim(U_1) + \dim(U_2) - \dim(U_1 \cap U_2).$$

Clearly  $\dim(U_1) = \text{lin}(K) \dim(K^*)$  and  $\dim(U_2) = \dim(K) \text{lin}(K^*)$ . Since  $\text{linspace}(K^*) \otimes \text{linspace}(K) \subseteq U_1 \cap U_2$ , we have  $\dim(U_1 \cap U_2) \geq \text{lin}(K) \text{lin}(K^*)$ . Yet any  $u_1 \in U_1$  and  $u_2 \in U_2$  satisfy

$$u_1 \left( \text{linspace}(K)^\perp \right) = \{0\} \text{ and } u_2(V) \subseteq \text{linspace}(K^*),$$

so  $\dim(U_1 \cap U_2) \leq \dim(\mathcal{B}(\text{linspace}(K), \text{linspace}(K^*))) = \text{lin}(K) \text{lin}(K^*)$  for equality. Substitute into the dimension formula to obtain the result.  $\square$

Combined with our knowledge of  $\text{linspace}(\mathcal{T})$  from Proposition 9 and its dimension from Proposition 10, this next fact will allow us to find  $\text{linspace}(\mathcal{T})^\perp$ .

**Proposition 11.** *Suppose that  $V$  is a finite-dimensional real Hilbert space and that  $\{x, s, t, y\} \subseteq V$ . If  $\langle x, y \rangle = 0$  or  $\langle t, s \rangle = 0$ , then  $\langle s \otimes x, t \otimes y \rangle = 0$  in  $\mathcal{B}(V)$ .*

*Proof.* We defined  $\langle s \otimes x, t \otimes y \rangle := \text{trace}((s \otimes x) \circ (y \otimes t))$ . If  $(s \otimes x) \circ (y \otimes t)$  has an eigenvalue  $\lambda$ , then it has a nonzero eigenvector  $v \in V$  with

$$[(s \otimes x) \circ (y \otimes t)](v) = \langle t, v \rangle \langle x, y \rangle s = \lambda v.$$

If  $\langle x, y \rangle = 0$ , then clearly  $\lambda = 0$ . If  $\langle t, s \rangle = 0$ , then  $v \in \text{span}(\{s\})$  implies that  $\langle t, v \rangle = 0$  and again that  $\lambda = 0$ . All eigenvalues of  $(s \otimes x) \circ (y \otimes t)$  are zero.  $\square$

**Proposition 12.** *If  $K$  is a closed convex cone in a finite-dimensional real Hilbert space  $V$  and if  $\mathcal{T} = \text{cone}(K^* \otimes K)$ , then*

$$\begin{aligned} \text{linspace}(\mathcal{T})^\perp &= \text{span} \left( (K^*)^\perp \otimes \text{linspace}(K) \right) \\ &\oplus \text{span} \left( \text{linspace}(K^*)^\perp \otimes \text{linspace}(K)^\perp \right) \\ &\oplus \text{span} \left( \text{linspace}(K^*) \otimes K^\perp \right). \end{aligned} \quad (1)$$

*Proof.* Proposition 11 shows that all subspaces involved are mutually orthogonal to each other and to  $\text{linspace}(\mathcal{T})$ . The dimension of (1) and  $\text{lin}(\mathcal{T})$  sum to  $\dim(\mathcal{B}(V))$  by Proposition 10. Therefore (1) must be  $\text{linspace}(\mathcal{T})^\perp$ .  $\square$

**Corollary 3.** *If  $K$  is a closed convex cone in a finite-dimensional real Hilbert space and if  $\mathcal{T} = \text{cone}(K^* \otimes K)$ , then*

$$\text{proj} \left( \text{linspace}(\mathcal{T})^\perp, \mathcal{T} \right) \subseteq \text{span} \left( \text{linspace}(K^*)^\perp \otimes \text{linspace}(K)^\perp \right).$$

*Proof.* Using Proposition 11, it is easy to see that  $\text{span}(\mathcal{T})$  is orthogonal to two of the three subspaces that constitute  $\text{linspace}(\mathcal{T})^\perp$  in Proposition 12. Therefore the projection of  $\mathcal{T}$  onto  $\text{linspace}(\mathcal{T})^\perp$  lies entirely in the third subspace.  $\square$

Corollary 3 shows that without loss of generality, the projection of  $\mathcal{T}$  onto  $\text{linspace}(\mathcal{T})^\perp$  is onto  $\text{span} \left( \text{linspace}(K^*)^\perp \otimes \text{linspace}(K)^\perp \right)$  instead. This allows us to bring the following proposition and its corollary to bear, providing a description of the generators that appear when Proposition 7 is applied to  $\mathcal{T}$ .

**Proposition 13.** *Let  $V$  be a real Hilbert space with basis  $\mathbf{v} := \{v_1, v_2, \dots, v_n\}$ . If  $W_1$  and  $W_2$  are subspaces of  $V$ , then the linear operator defined by*

$$\text{proj}(\text{span}(W_1 \otimes W_2), v_i \otimes v_j) := \text{proj}(W_1, v_i) \otimes \text{proj}(W_2, v_j) \quad (2)$$

*is the orthogonal projection of  $\mathcal{B}(V)$  onto  $\text{span}(W_1 \otimes W_2)$ .*

*Proof.* The existence and uniqueness of such a projection is guaranteed, since  $\text{span}(W_1 \otimes W_2)$  is a subspace of  $\text{span}(V \otimes V) = \mathcal{B}(V)$ . A dimension argument shows that  $\mathbf{v} \otimes \mathbf{v}$  is a basis for  $\mathcal{B}(V)$ , so (2) is well-defined, and linear by definition.

From (2) it is clear that  $\text{proj}(\text{span}(W_1 \otimes W_2), \mathcal{B}(V)) \subseteq \text{span}(W_1 \otimes W_2)$ . It is equally clear that  $\text{proj}(\text{span}(W_1 \otimes W_2), \phi) = \phi$  for any  $\phi \in \text{span}(W_1 \otimes W_2)$ . If instead we have  $\phi \in \text{span}(W_1 \otimes W_2)^\perp = \text{span}(W_1^\perp \otimes V) + \text{span}(V \otimes W_2^\perp)$ , then  $\text{proj}(\text{span}(W_1 \otimes W_2), \phi) = 0$  by Proposition 11. The only such operator is the orthogonal projection onto the subspace  $\text{span}(W_1 \otimes W_2)$ .  $\square$

**Corollary 4.** *Let  $V$  be a finite-dimensional real Hilbert space and let  $W_1$  and  $W_2$  be subspaces of  $V$ . If  $P \subseteq V \times V$ , then*

$$\text{proj}(\text{span}(W_1 \otimes W_2), \text{cone}(\{s \otimes x \mid (x, s) \in P\}))$$

*is equal to*

$$\text{cone}(\{\text{proj}(W_1, s) \otimes \text{proj}(W_2, x) \mid (x, s) \in P\}).$$

*Proof.* Use Proposition 7 and then Proposition 13.  $\square$

It will be convenient to work with bounded generating sets of cones like  $\text{cone}(\{s \otimes x \mid (x, s) \in P\})$  from Corollary 4. We construct such a set explicitly.

**Proposition 14.** *Let  $V$  be a finite-dimensional real Hilbert space. If  $P \subseteq V \times V$  is nonempty and such that  $(x, s) \in P$  implies that both  $(\lambda x, s) \in P$  and  $(x, \lambda s) \in P$  for all  $\lambda \geq 0$ , then*

$$\text{cone}(\{s \otimes x \mid (x, s) \in P\}) = \text{cone}(\{s \otimes x \mid (x, s) \in P, \|x\| = \|s\| = 1\}).$$

*Proof.* One set is clearly a subset of the other, so suppose that we are given a  $\phi \in \text{cone}(\{s \otimes x \mid (x, s) \in P\})$ . Pull out nonnegative factors so that

$$\phi = \sum_{i=1}^m \alpha_i (s_i \otimes x_i) = \sum_{i=1}^m \alpha_i \|s_i\| \|x_i\| \left( \frac{s_i}{\|s_i\|} \otimes \frac{x_i}{\|x_i\|} \right).$$

The latter belongs to  $\text{cone}(\{s \otimes x \mid (x, s) \in P, \|x\| = \|s\| = 1\})$ .  $\square$

The form of the set  $P$  in Proposition 14 is a generalization of the two sets  $K \times K^*$  and  $C(K)$  to which we will apply the proposition. We finally have all of the tools that we need to prove that  $\mathcal{T}$  is closed.

**Lemma 1.** *If  $K$  is a closed convex cone in a finite-dimensional real Hilbert space and if  $\mathcal{T} = \text{cone}(K^* \otimes K)$ , then  $\mathcal{T}$  is closed.*

*Proof.* It suffices (Proposition 6) to show that  $\text{proj} \left( \text{linspace}(\mathcal{T})^\perp, \mathcal{T} \right)$  is closed. We will construct a generating set  $G$  of that projection such that  $\text{cone}(G)$  is closed. First, use Corollary 3 to show that  $\text{proj} \left( \text{linspace}(\mathcal{T})^\perp, \mathcal{T} \right)$  is equal to

$$\text{proj} \left( \text{span} \left( \text{linspace}(K^*)^\perp \otimes \text{linspace}(K)^\perp \right), \mathcal{T} \right). \quad (3)$$

Then, from Corollary 4 and the fact that  $\mathcal{T} = \text{cone}(K^* \otimes K)$ , the set (3) is

$$\text{cone} \left( \text{proj} \left( \text{linspace}(K^*)^\perp, K^* \right) \otimes \text{proj} \left( \text{linspace}(K)^\perp, K \right) \right). \quad (4)$$

If either projection in (4) is  $\{0\}$ , then the entire cone is the closed set  $\{0\}$  and the proof is complete. We can therefore assume that both projections contain nonzero elements. And since they are both closed convex cones by Corollary 2, they contain elements of unit norm. Thus we have two nonempty compact sets,

$$\begin{aligned} J_1 &:= \left\{ x \in \text{proj} \left( \text{linspace}(K)^\perp, K \right) \mid \|x\| = 1 \right\} \\ J_2 &:= \left\{ s \in \text{proj} \left( \text{linspace}(K^*)^\perp, K^* \right) \mid \|s\| = 1 \right\}. \end{aligned}$$

Use Proposition 14 to see that  $\text{cone}(J_2 \otimes J_1)$  is equal to (4), and let  $G := J_2 \otimes J_1 \neq \emptyset$ . The set  $G$  is also compact, since it is the image of the continuous mapping  $(x, s) \mapsto s \otimes x$  on the compact set  $J_1 \times J_2$ . It follows that  $\text{conv}(G)$  is compact [1]. For  $\text{cone}(G)$  to be closed, it suffices [21, 30] to show that  $0 \notin \text{conv}(G)$ . Suppose on the contrary that

$$0 = \lambda_1 g_1 + \lambda_2 g_2 + \cdots + \lambda_m g_m, \text{ where } g_i \in G, \lambda_i > 0, \text{ and } \sum_{i=1}^m \lambda_i = 1.$$

This can be rearranged to  $-\lambda_1 g_1 = \lambda_2 g_2 + \cdots + \lambda_m g_m \in \mathcal{T}$ , contradicting the fact that  $g_1 \in \text{linspace}(\mathcal{T})^\perp$  unless  $g_1 = 0$ . The elements of  $G$  were constructed to have nonzero norm, so that cannot happen.  $\square$

**Theorem 1.** *If  $K = \text{cone}(G_1)$  is closed in a finite-dimensional real Hilbert space and if  $K^* = \text{cone}(G_2)$ , then  $\pi(K)^* = \text{cone}(G_2 \otimes G_1)$ .*

*Proof.* Expand the elements of  $K$  and  $K^*$  in terms of  $G_1$  and  $G_2$  to verify that  $\text{cone}(G_2 \otimes G_1) = \text{cone}(K^* \otimes K)$ . From Definition 7 we have  $L \in \pi(K)$  if and only if  $\langle L(x), s \rangle \geq 0$  for all  $(x, s) \in K \times K^*$ . And  $\langle L(x), s \rangle = \langle L, s \otimes x \rangle$  by properties of the trace. But,

$$\langle L, s \otimes x \rangle \geq 0 \text{ for all } (x, s) \in K \times K^* \iff L \in \text{cone}(G_2 \otimes G_1)^*.$$

Therefore,  $\pi(K) = \text{cone}(G_2 \otimes G_1)^*$  and Lemma 1 shows that  $\text{cone}(G_2 \otimes G_1)$  is closed. Take duals on both sides.  $\square$

This result was known for proper cones, so we look elsewhere for examples.

**Example 1.** If  $K = \{0\}$  in  $V$ , then there are no nonzero generators of  $K$ , so  $\pi(K)^* = \{0\}$ , and thus  $\pi(K) = \mathcal{B}(V)$ . Likewise if  $K = V$  and  $K^* = \{0\}$ .

**Example 2.** If  $K = \text{cone}(\{e_1, \pm e_2\})$  is the right half-space in  $V = \mathbb{R}^2$ , then  $K^* = \text{cone}(\{e_1\})$  and Theorem 1 gives

$$\pi(K)^* = \text{cone}(\{e_1 e_1^T, \pm e_1 e_2^T\}); \quad \pi(K) = \text{cone}(\{e_1 e_1^T, \pm e_2 e_1^T, \pm e_2 e_2^T\}).$$

This result is verified using Proposition 4.

Using their extremal vectors, Tam [30] proved the following for proper cones. The non-uniqueness of generating sets complicates things but is not fatal.

**Theorem 2.** *If  $K$  is a closed convex cone in a finite-dimensional real Hilbert space, then  $\pi(K)$  is polyhedral if and only if  $K$  is polyhedral.*

*Proof.* Necessity is trivial, so suppose that  $\pi(K)$  is polyhedral. Then  $\pi(K)^*$  is polyhedral as well, and from Theorem 1 we have  $\pi(K)^* = \text{cone}(G_2 \otimes G_1)$  for generating sets  $G_1$  and  $G_2$  of  $K$  and  $K^*$  respectively. Since  $\pi(K)^*$  is polyhedral, it is generated by a finite set  $P$ . Each  $p \in P$  belongs to  $\pi(K)^*$ , so by Carathéodory's theorem, it is a finite conic combination of elements of  $G_2 \otimes G_1$ . Collect these elements—a finite number for each  $p$  in the finite set  $P$ —to construct a finite subset  $F_1 \times F_2 \subseteq G_1 \times G_2$  such that  $\pi(K)^* = \text{cone}(F_2 \otimes F_1)$ . We show that  $\text{cone}(F_2) = K^*$  and it follows that both  $K^*$  and  $K$  are polyhedral.

If  $K = \text{linspace}(K)$ , then  $K$  is polyhedral, so assume otherwise. Choose a nonzero  $x \in K \cap \text{linspace}(K)^\perp$  and any  $s \in K^*$ . Then  $s \otimes x \in \pi(K)^*$ , so

$$s \otimes x = \sum_{i=1}^m \alpha_i (s_i \otimes x_i) \quad \text{where } \alpha_i \geq 0 \text{ and } (x_i, s_i) \in F_1 \times F_2.$$

Since  $x \notin \text{linspace}(K)$ , there exists a  $t \in K^*$  such that  $\langle x, t \rangle > 0$  by Corollary 1. Apply  $s \otimes x$  to  $t$  to obtain  $s = \frac{1}{\langle x, t \rangle} \sum_{i=1}^m \alpha_i \langle x_i, t \rangle s_i \in \text{cone}(F_2)$ .  $\square$

Proposition 9 and Proposition 10 now have interpretations in terms of  $\pi(K)$ .

**Corollary 5.** *If  $K$  is a closed convex cone in a finite-dimensional real Hilbert space, then  $\text{linspace}(\pi(K)^*) = U_1 + U_2$  where*

$$U_1 := \text{span}(K^* \otimes \text{linspace}(K)); \quad U_2 := \text{span}(\text{linspace}(K^*) \otimes K).$$

*Proof.* Combine Proposition 9 with Theorem 1.  $\square$

**Corollary 6.** *Let  $K$  be a closed convex cone in a finite-dimensional real Hilbert space  $V$ . If  $n = \dim(V)$ ,  $m = \dim(K)$ , and  $\ell = \text{lin}(K)$ , then  $\text{lin}(\pi(K)^*) = \ell(m - \ell) + m(n - m)$ .*

*Proof.* One form of  $\text{lin}(\pi(K)^*)$  is given by Theorem 1 and Proposition 10. Use Proposition 1 to achieve the desired form.  $\square$

**Lemma 2.** *Let  $K$  be a closed convex cone in a finite-dimensional real Hilbert space  $V$ , and let  $n = \dim(V)$ ,  $m = \dim(K)$ , and  $\ell = \text{lin}(K)$ . Then,*

$$\dim(\pi(K)) = n^2 - \ell(m - \ell) - m(n - m).$$

*Proof.* Apply Corollary 6 to  $\text{codim}(\pi(K)) = \text{lin}(\pi(K)^*)$ .  $\square$

**Example 3.** If  $K = \{0\}$  in  $V$ , then  $m = \ell = 0$ , and  $\dim(\pi(K)) = n^2$  which agrees with the obvious fact that  $\pi(K) = \mathcal{B}(V)$ .

**Example 4.** If  $K$  is proper, then in Lemma 2, we have  $m = n$  and  $\ell = 0$ . Thus  $\dim(\pi(K)) = n^2$  and  $\pi(K)$  is solid.

**Example 5.** Example 2 has  $n = m = 2$  and  $\ell = 1$  giving  $\dim(\pi(K)) = 3$ .

**Lemma 3.** *If  $K$  is a closed convex cone in a finite-dimensional real Hilbert space  $V$ , then*

$$\text{lin}(\pi(K)) = \dim(V)^2 - \dim(K) \dim(K^*).$$

*Proof.* By Proposition 1 we have  $\text{lin}(\pi(K)) = \dim(V)^2 - \dim(\pi(K)^*)$ , and from Theorem 1 it follows that  $\dim(\pi(K)^*) = \dim(K) \dim(K^*)$ .  $\square$

**Example 6.** If  $K = \{0\}$  in  $V$ , then  $\dim(K) = 0$ , and  $\text{lin}(\pi(K)) = \dim(V)^2$  in agreement with the fact that  $\pi(K) = \mathcal{B}(V)$ .

**Example 7.** If  $K$  is proper, then  $\dim(K) = \dim(K^*) = \dim(V)$ . Lemma 3 gives  $\text{lin}(\pi(K)) = 0$ , showing that  $\pi(K)$  is pointed.

**Example 8.** In Example 2, we have  $\text{lin}(\pi(K)) = 4 - 2 \cdot 1 = 2$ .

These corollaries and examples reaffirm that if  $K$  is proper, then  $\pi(K)$  is proper [24]. Lemma 3 allows us to prove the converse.

**Theorem 3.** *If  $K$  is a closed convex cone in a finite-dimensional real Hilbert space, then  $\pi(K)$  is proper if and only if  $K$  is proper.*

When  $K$  is polyhedral, Theorem 1 allows us to compute a generating set of  $\pi(K)$ . Algorithms to compute the dual generators of a polyhedral cone are known, and the inverse operations  $\text{vec}()$  and  $\text{mat}()$  are isomorphisms.

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**Algorithm 1** Compute generators of  $\pi(K)$

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**Input:** A closed convex cone  $K$

**Output:** A generating set of  $\pi(K)$

**function** PI( $K$ )

$G_1 \leftarrow$  a finite set of generators for  $K$

$G_2 \leftarrow \text{dual}(G_1)$

$\triangleright$  a finite set of generators for  $K^*$

$G \leftarrow G_2 \otimes G_1$

**return**  $\text{mat}(\text{dual}(\text{vec}(G)))$

**end function**

---

Though limited to polyhedral cones, Algorithm 3 can be used to generate conjectures and counterexamples. Recall the space  $\mathbf{LL}(K)$  of all Lyapunov-like operators on  $K$  and its Lyapunov rank  $\beta(K)$  from Definition 11. The Lyapunov rank of a cone is important because the complementarity condition  $\langle x, s \rangle = 0$  can be decomposed into a system of  $\beta(K)$  equations which may be easier to solve [23]. Characterizing the Lyapunov rank of  $\pi(K)$  appears to be a difficult problem, but empirically we observe the following.

**Conjecture 1.** *If  $K$  is a proper polyhedral cone in a real Hilbert space  $V$ , then  $\pi(K)$  is a proper polyhedral cone in  $\mathcal{B}(V)$  and  $\beta(\pi(K)) = \beta(K)^2$ .*

## 4 $\mathbf{Z}$ -operators

We now move on to the  $\mathbf{Z}$ -operators of Definition 10. Every  $\mathbf{Z}$ -operator is the negation of some *cross-positive* operator—the class originally introduced by Schneider and Vidyasagar [25]. Tam answered some early open questions about cross-positive operators [28]. More work was done later by Gritzmann, Klee and Tam [15, 29]. Recently, Kuzma et al. [17] used cross-positive operators in the context of mathematical finance to answer an open question posed by Damm [6].

Many of the results hereafter would appear more natural (that is, without a minus sign) if stated in terms of cross-positive operators. However, the connection between cross-positive operators and  $\mathbf{Z}$ -matrices does not seem to have been made until Gowda and Tao defined  $\mathbf{Z}$ -operators as the negations of cross-positive operators [12], and showed that the  $\mathbf{Z}$ -operators on the cone  $\mathbb{R}_+^n$  are simply  $\mathbf{Z}$ -matrices. The  $\mathbf{Z}$ -matrices have historically received more attention, so we will stress that connection and present our results in terms of  $\mathbf{Z}$ -operators.

The work that we have done for the positive operators will pay off in this section. The cone of  $\mathbf{Z}$ -operators (when it is nontrivial) is never proper. By considering an improper cone  $K$ , we were forced to generalize the proofs for  $\pi(K)$  to such a case. Having done so, those techniques now apply to  $\mathbf{Z}(K)$  as well. As before, we begin by pointing out that the set of all  $\mathbf{Z}$ -operators on  $K$  forms a closed convex cone. Verification of the three criteria is straightforward, but in this case, none of them depend on properties of  $K$ .

**Proposition 15.** *If  $K$  is a closed convex cone, then so is  $\mathbf{Z}(K)$ .*

If the ambient space is nontrivial, then  $\mathbf{Z}(K)$  contains the nontrivial subspace  $\mathbf{LL}(K)$  and is neither pointed nor proper in contrast with Theorem 3. It does however suffice to verify the  $\mathbf{Z}$ -operator property on generating sets.

**Proposition 16.** *If  $K = \text{cone}(G_1)$  is closed in a finite-dimensional real Hilbert space and if  $K^* = \text{cone}(G_2)$ , then  $L \in \mathbf{Z}(K)$  if and only if*

$$\langle L(x), s \rangle \leq 0 \text{ for all } (x, s) \in C(K) \cap (G_1 \times G_2). \quad (5)$$

*Proof.* Clearly, if  $L \in \mathbf{Z}(K)$ , then  $L$  satisfies (5). So suppose that  $L$  satisfies (5) and let  $(x, s) \in C(K)$ . Since  $G_1$  generates  $K$  and  $G_2$  generates  $K^*$ , we can write

$x = \sum_{i=1}^{\ell} \alpha_i x_i$  and  $s = \sum_{j=1}^m \gamma_j s_j$ . By expanding  $\langle x, s \rangle = 0$  and noting that  $\langle x_i, s_j \rangle \geq 0$ , we see that each  $(x_i, s_j) \in C(K)$ . Linearity gives  $\langle L(x), s \rangle \leq 0$ .  $\square$

As with  $\pi(K)^*$ , we will eventually want to find a generating set of  $\mathbf{Z}(K)^*$  and use that to prove some results about  $\mathbf{Z}(K)$ . Recall the cone of Tam's generators  $\mathcal{T} := \text{cone}(K^* \otimes K)$  from Section 3. There we showed that  $\mathcal{T} = \pi(K)^*$ , and a similar set will generate  $\mathbf{Z}(K)^*$ . When  $K$  is a closed convex cone, we will now usually define the cone of complementary Tam generators,

$$\mathcal{T}_C := \text{cone}(\{s \otimes x \mid (x, s) \in C(K)\}).$$

The lineality space of  $\mathcal{T}$  was crucial to our proof of Theorem 1, and as we will see, the lineality space of  $\mathcal{T}_C$  is identical.

**Proposition 17.** *Let  $K$  be a closed convex cone in a finite-dimensional real Hilbert space. If  $\mathcal{T} = \text{cone}(K^* \otimes K)$  and  $\mathcal{T}_C = \text{cone}(\{s \otimes x \mid (x, s) \in C(K)\})$ , then  $\text{linspace}(\mathcal{T}_C) = \text{linspace}(\mathcal{T})$ .*

*Proof.* Since  $\mathcal{T}_C \subseteq \mathcal{T}$ , we obviously have  $\text{linspace}(\mathcal{T}_C) \subseteq \text{linspace}(\mathcal{T})$ . But notice that every generator  $s \otimes x$  of  $\text{linspace}(\mathcal{T})$  in Proposition 9 has  $\langle x, s \rangle = 0$  by Proposition 1. Therefore any such  $s \otimes x$  belongs to  $\mathcal{T}_C$ , and since  $s \otimes x \in \text{linspace}(\mathcal{T})$ , we can say the same about  $-s \otimes x$ . As a result we have  $s \otimes x \in \text{linspace}(\mathcal{T}_C)$  implying that  $\text{linspace}(\mathcal{T}) \subseteq \text{linspace}(\mathcal{T}_C)$ .  $\square$

**Lemma 4.** *If  $K$  is a closed convex cone in a finite-dimensional real Hilbert space and if  $\mathcal{T}_C = \text{cone}(\{s \otimes x \mid (x, s) \in C(K)\})$ , then  $\mathcal{T}_C$  is closed.*

*Proof.* It suffices by Proposition 6 to show that  $\text{proj}(\text{linspace}(\mathcal{T}_C)^\perp, \mathcal{T}_C)$  is closed. We mimic the proof of Lemma 1 and construct a generating set  $G$  of that projection such that  $\text{cone}(G)$  is closed. First, use Corollary 3 to show that  $\text{proj}(\text{linspace}(\mathcal{T}_C)^\perp, \mathcal{T}_C)$  is equal to

$$\text{proj}\left(\text{span}\left(\text{linspace}(K^*)^\perp \otimes \text{linspace}(K)^\perp\right), \mathcal{T}_C\right). \quad (6)$$

Define the set of “projected complementary pairs” to be,

$$C_p(K) := \left\{ (x_p, s_p) \left| \begin{array}{l} (x, s) \in C(K) \\ x_p = \text{proj}\left(\text{linspace}(K)^\perp, x\right) \\ s_p = \text{proj}\left(\text{linspace}(K^*)^\perp, s\right) \end{array} \right. \right\}.$$

Then, from Corollary 4 and the definition of  $\mathcal{T}_C$ , the set (6) is

$$\text{cone}(\{s \otimes x \mid (x, s) \in C_p(K)\}). \quad (7)$$

We finally define the generating set,

$$G := \{s \otimes x \mid (x, s) \in C_p(K), \|x\| = \|s\| = 1\}.$$

If  $G$  is nonempty, then Proposition 14 shows that  $\text{cone}(G)$  is equal to the set in equation (7), and thus to  $\text{proj}\left(\text{linspace}(\mathcal{T}_C)^\perp, \mathcal{T}_C\right)$ . If  $G$  were empty, then every pair  $(x, s) \in C_p(K)$  would have either  $x = 0$  or  $s = 0$ ; otherwise, we could scale them by their norms to find a pair in  $G$ . But if every such pair has one of its components zero, then the set in equation (7) is  $\{0\}$  (which is closed) and the proof is complete. So we proceed under the assumption that  $G$  is nonempty.

Our next goal is to show that  $C_p(K)$  is topologically closed so that we may claim that  $G$  is compact. Define the pointed components of  $K$  and  $K^*$ ,

$$K_1 := \text{proj}\left(\text{linspace}(K)^\perp, K\right); \quad K_2 := \text{proj}\left(\text{linspace}(K^*)^\perp, K^*\right).$$

Take any  $(x_p, s_p) \in C_p(K)$  so that  $x_p = \text{proj}\left(\text{linspace}(K)^\perp, x\right)$  and  $s_p = \text{proj}\left(\text{linspace}(K^*)^\perp, s\right)$  for appropriate  $(x, s) \in C(K)$ . By Proposition 5, we can write  $x = x_p + x_\ell$  with  $x_\ell \in \text{linspace}(K)$  and  $s = s_p + s_\ell$  with  $s_\ell \in \text{linspace}(K^*)$ . Now  $\langle x, s \rangle = 0$  since  $(x, s) \in C(K)$ , but

$$\langle x, s \rangle = \langle x_p, s_p \rangle + \langle x_p, s_\ell \rangle + \langle x_\ell, s_p \rangle + \langle x_\ell, s_\ell \rangle = \langle x_p, s_p \rangle$$

because  $x_\ell \in \text{linspace}(K)$  and  $s_\ell \in \text{linspace}(K^*)$ . Thus  $\langle x_p, s_p \rangle = 0$ . On the other hand, if  $x_p \in K_1$  and  $s_p \in K_2$  with  $\langle x_p, s_p \rangle = 0$ , then  $(x_p, s_p) \in C_p(K)$ . We have shown that,

$$C_p(K) = \{(x, s) \in K_1 \times K_2 \mid \langle x, s \rangle = 0\}.$$

It is easy to see that this representation is topologically closed. For example, the set  $K_1 \times K_2$  is closed in  $V \times V$ . If we define  $\phi : K_1 \times K_2 \rightarrow \mathbb{R}$  by  $\phi(x, s) = \langle x, s \rangle$ , then  $C_p(K)$  is the preimage of the closed set  $\{0\}$  under the continuous function  $\phi$ . It follows that  $C_p(K)$  is closed not only in  $K_1 \times K_2$ , but in  $V \times V$  as well.

With  $C_p(K)$  closed, both  $G$  and  $\text{conv}(G)$  are compact. The same reasoning as in Lemma 1 shows that the latter set does not contain zero, and that therefore  $\text{cone}(G)$  is closed.  $\square$

**Theorem 4.** *If  $K = \text{cone}(G_1)$  is closed in a finite-dimensional real Hilbert space and if  $K^* = \text{cone}(G_2)$ , then  $\mathbf{Z}(K)^* = \text{cone}(G)$  where  $G$  is defined to be  $G := \{-s \otimes x \mid (x, s) \in C(K) \cap (G_1 \times G_2)\}$ .*

*Proof.* Follow Theorem 1, let  $\mathcal{T}_C = \text{cone}(\{s \otimes x \mid (x, s) \in C(K)\})$ ; we will show that  $\text{cone}(G) = -\mathcal{T}_C$  which by Lemma 4 is closed. Obviously,  $\text{cone}(G) \subseteq -\mathcal{T}_C$ , so let  $h \in -\mathcal{T}_C$  be given with  $h = \sum_{i=1}^\ell -s_i \otimes x_i$  where each  $(x_i, s_i) \in C(K)$ . Take any term  $-t \otimes y$  in this sum, and write  $t$  and  $y$  in terms of  $G_2$  and  $G_1$ ,

$$-t \otimes y = -\left(\sum_{k=1}^n \beta_k h_k\right) \otimes \left(\sum_{j=1}^m \alpha_j g_j\right) = \sum_{k=1}^n \sum_{j=1}^m \alpha_j \beta_k (-h_k \otimes g_j).$$

Expand  $\langle y, t \rangle = 0$  and note that each  $\langle g_j, h_k \rangle \geq 0$  implying  $(g_j, h_k) \in C(K)$ . Thus each  $-t \otimes y \in \text{cone}(G)$ , and so  $h \in \text{cone}(G)$ . Conclude that  $\text{cone}(G) =$

$-\mathcal{T}_C$ . By definition,  $L \in \mathbf{Z}(K)$  if and only if  $\langle -L(x), s \rangle \geq 0$  for all  $(x, s) \in C(K)$ . And  $\langle -L(x), s \rangle = \langle L, -s \otimes x \rangle$  by properties of the trace, so

$$L \in \mathbf{Z}(K) \iff \langle L, -s \otimes x \rangle \geq 0 \text{ for all } (x, s) \in C(K) \iff L \in \text{cone}(G)^*.$$

Take duals on both sides of  $\mathbf{Z}(K) = \text{cone}(G)^*$ .  $\square$

There is no corresponding result for proper cones, so we include the classical case along with some improper cones as examples of Theorem 4.

**Example 9.** If  $K = \mathbb{R}_+^n$  in  $V = \mathbb{R}^n$ , then  $C(K) = \{(e_i, e_j) \mid i \neq j\}$ . Form  $G := \{-e_j e_i^T \mid i \neq j\}$  to find that  $\mathbf{Z}(K)^* = \text{cone}(G)$  is the set of matrices whose diagonal entries are zero and whose off-diagonal entries are nonpositive. Its dual is the cone of  $\mathbf{Z}$ -matrices.

**Example 10.** If  $K = \{0\}$  or  $K = V$ , then  $\mathbf{Z}(K)^* = \{0\}$  and  $\mathbf{Z}(K) = \mathcal{B}(V)$ .

**Example 11.** If  $K$  is the half-space from Example 2, then Theorem 4 gives  $\mathbf{Z}(K)^* = \mathbf{Z}(K)^\perp = \text{span}(\{e_1 e_2^T\})$ . This result is verified by Proposition 16.

Now that we know that  $\mathbf{Z}(K)^* = \mathcal{T}_C$ , we can find the dimension of  $\mathbf{Z}(K)$ .

**Theorem 5.** *If  $K$  is a closed convex cone in a finite-dimensional real Hilbert space, then  $\dim(\mathbf{Z}(K)) = \dim(\pi(K))$ .*

*Proof.* Theorem 1, Proposition 17, and Theorem 4 give  $\text{linspace}(\mathbf{Z}(K)^*) = \text{linspace}(\pi(K)^*)$ . Proposition 1 gives  $\text{codim}(\mathbf{Z}(K)) = \text{codim}(\pi(K))$ .  $\square$

The trivial cone, half-space, full space, and nonnegative orthant (from our previous examples) all corroborate Theorem 5. Our next result is an analogue of Theorem 2 and its proof is similar.

**Theorem 6.** *If  $K$  is a closed convex cone in a finite-dimensional real Hilbert space, then  $\mathbf{Z}(K)$  is polyhedral if and only if  $K$  is polyhedral.*

*Proof.* Refer to the proof of Theorem 2. Necessity is trivial, so suppose that  $\mathbf{Z}(K)$  is polyhedral. Without loss of generality,  $\mathbf{Z}(K)^* = \text{cone}(-F_2 \otimes F_1)$  for finite sets  $F_1 \subseteq K$  and  $F_2 \subseteq K^*$ . If  $\dim(K^*) = 1$ , or if  $K^*$  is a (half) subspace, then  $K$  is polyhedral. We can therefore assume by Fenchel's [8] Theorem 13 that  $K^*$  is generated by its relative boundary rays.

Let  $s$  be any nonzero relative boundary ray of  $K^*$ . Define the closed convex cone  $J := K^* + \text{linspace}(K)$  which by Proposition 1 equals  $K^* \oplus \text{span}(K^*)^\perp$ . Then  $s$  lies on the boundary of  $J$ , and by Fenchel's Corollary 2, there exists a nonzero  $x \in J^*$  such that  $\langle x, s \rangle = 0$ . Proposition 2 gives  $J^* = K \cap \text{linspace}(K)^\perp$ , so this pair satisfies  $(x, s) \in C(K)$  and thus  $-s \otimes x \in \mathbf{Z}(K)^*$ . The argument used in the proof of Theorem 2 can be repeated to show that  $s \in \text{cone}(F_2)$ .  $\square$

This result is confirmed by the polyhedral cones we have examined: the trivial cone, full space, nonnegative orthant, and half-space—all of which have polyhedral cones of  $\mathbf{Z}$ -operators.

**Corollary 7.** *If  $K$  is a closed convex cone in a finite-dimensional real Hilbert space, then  $\mathbf{Z}(K)$  is polyhedral if and only if  $\pi(K)$  is polyhedral.*

There are no simple characterizations of  $\mathbf{Z}(K)$  for nonpolyhedral  $K$ . One sees an example in the work of Stern and Wolkowicz [26] who characterize the  $\mathbf{Z}$ -operators on the Lorentz “ice cream” cone. We close this section with an algorithm, based on Theorem 4, to compute  $\mathbf{Z}(K)$  for polyhedral  $K$ .

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**Algorithm 2** Compute generators of  $\mathbf{Z}(K)$

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**Input:** A closed convex cone  $K$

**Output:** A generating set of  $\mathbf{Z}(K)$

**function**  $\mathbf{Z}(K)$

$G_1 \leftarrow$  a finite set of generators for  $K$

$G_2 \leftarrow$  dual( $G_1$ )  $\triangleright$  a finite set of generators for  $K^*$

$G \leftarrow \{-s \otimes x \mid x \in G_1, s \in G_2, \langle x, s \rangle = 0\}$

**return** mat(dual(vec( $G$ )))

**end function**

---

## 5 The exponential connection

Finally we exhibit an explicit connection between positive and  $\mathbf{Z}$ -operators. As discovered by Schneider and Vidyasagar [25], it applies to proper cones. We restate their theorem in slightly more general language.

**Theorem 7.** *If  $K$  is a proper cone in a finite-dimensional real Hilbert space  $V$  and  $L \in \mathcal{B}(V)$ , then  $L \in \mathbf{Z}(K)$  if and only if  $e^{-tL} \in \pi(K)$  for all  $t \geq 0$ .*

This theorem has been used effectively. Elsner [7] equates exponentially-positive, resolvent-positive, essentially-positive, cross-positive, and quasimonotone operators. Damm [6] shows that Lyapunov-like operators on the positive-semidefinite cone are the familiar Lyapunov transformations from dynamical systems. Gowda and Tao [13] characterize the Lie algebra of the automorphism group of a cone. We prove the general version in two lemmata. The same technique [19], described in more detail, was used to demonstrate a connection between  $\mathbf{LL}(K)$  and  $\text{Aut}(K)$ .

**Lemma 5.** *If  $K$  is a subset of a finite-dimensional real Hilbert space  $V$  and  $L \in \mathcal{B}(V)$ , then  $e^{-tL} \in \pi(K)$  for all  $t \geq 0$  implies that  $L \in \mathbf{Z}(K)$ .*

*Proof.* Let  $e^{-tL} \in \pi(K)$  for all  $t \geq 0$ , and take any  $(x, s) \in C(K)$ . We show that  $\langle L(x), s \rangle \leq 0$  and it follows that  $L \in \mathbf{Z}(K)$ . Since  $e^{-tL}(x) \in K$ ,

$$\frac{1}{t} \langle [e^{-tL} - \text{id}_V](x), s \rangle = \frac{1}{t} \langle e^{-tL}(x), s \rangle \geq 0 \text{ for all } t \geq 0.$$

Take the limit as  $t \rightarrow 0$  to find  $\langle L(x), s \rangle \leq 0$ . □

To prove the other inclusion, we will ultimately rely on Theorem 7 for proper cones. To do that, we use a proper subcone of the closed convex cone  $K$ . Define the cone  $K_S$  to be the set  $K$  living in the ambient space  $S := \text{span}(K)$ . There is an obvious isomorphism  $\psi : V \rightarrow S \times \{0\}$  such that  $\psi(K) = K_S \times \{0\}$ . Here, the trivial cone  $\{0\}$  lives in the ambient space  $S^\perp$ . The cone  $K_S$  is considered solid, since it is full-dimensional in its ambient space (by construction). We take these definitions for granted in the rest of the argument.

The cone  $K_S \times \{0\}$  is obtained from  $K$  by isomorphism. The following propositions relate its positive and  $\mathbf{Z}$ -operators to those of  $K$ .

**Proposition 18.** *If  $K$  is a closed convex cone and  $\psi$  is an isomorphism, then  $\mathbf{Z}(\psi(K)) = \psi\mathbf{Z}(K)\psi^{-1}$  and  $\pi(\psi(K)) = \psi\pi(K)\psi^{-1}$ .*

**Proposition 19.** *If  $K$  is a closed convex cone, then  $L \in \mathbf{Z}(K)$  if and only if  $L^* \in \mathbf{Z}(K^*)$ , and  $L \in \pi(K)$  if and only if  $L^* \in \pi(K^*)$ .*

Both  $\pi(K_S \times \{0\})$  and  $\mathbf{Z}(K_S \times \{0\})$  are easy to describe. Computation of  $\pi(K_S \times \{0\})$  is trivial, and the form of  $\mathbf{Z}(K_S \times \{0\})$  follows from the fact that

$$(x, s) \in C(K_S) \iff \left( (x, 0)^T, (s, t)^T \right) \in C(K_S \times \{0\}) \text{ for all } t \in S^\perp.$$

**Proposition 20.** *If  $K$  is a closed convex cone in a finite-dimensional real Hilbert space, then*

$$\begin{aligned} \pi(K_S \times \{0\}) &= \left\{ \begin{bmatrix} A & B \\ 0 & D \end{bmatrix} \mid A \in \pi(K_S) \right\} \\ \mathbf{Z}(K_S \times \{0\}) &= \left\{ \begin{bmatrix} A & B \\ 0 & D \end{bmatrix} \mid A \in \mathbf{Z}(K_S) \right\}. \end{aligned}$$

The next few results constitute the converse of Lemma 5. The first demonstrates the converse for the pointed cone  $K_S \times \{0\}$ . We then do away with the isomorphism and show that the converse holds for any pointed closed convex cone. Using duality, we show the same for solid closed convex cones, and the general case follows as a consequence.

**Proposition 21.** *If  $K$  is a pointed closed convex cone in a finite-dimensional real Hilbert space  $V$ , then  $L \in \mathbf{Z}(K_S \times \{0\})$  implies  $e^{-tL} \in \pi(K_S \times \{0\})$  for all  $t \geq 0$ .*

*Proof.* Use Proposition 20 and expand  $e^{-tL} = \sum_{n=0}^{\infty} (-tL)^n / n!$ . Note that  $K_S$  is pointed and thus proper and then apply Theorem 7 to the result.  $\square$

**Proposition 22.** *If  $K$  is a pointed closed convex cone in a finite-dimensional real Hilbert space  $V$ , then  $L \in \mathbf{Z}(K)$  implies  $e^{-tL} \in \pi(K)$  for all  $t \geq 0$ .*

*Proof.* Write  $\phi(K) = K_S \times \{0\}$  where  $\phi$  is an isomorphism and  $K_S$  is proper. Take any  $L \in \mathbf{Z}(K)$ . Then  $\phi L \phi^{-1} \in \mathbf{Z}(K_S \times \{0\})$  by Proposition 18, and  $e^{-t\phi L \phi^{-1}} = \phi e^{-tL} \phi^{-1} \in \pi(K_S \times \{0\})$  for all  $t \geq 0$  by Proposition 21. Use Proposition 18 again to obtain  $e^{-tL} \in \pi(K)$  for all  $t \geq 0$ .  $\square$

**Proposition 23.** *If  $K$  is a solid closed convex cone in a finite-dimensional real Hilbert space  $V$ , then  $L \in \mathbf{Z}(K)$  implies that  $e^{-tL} \in \pi(K)$  for all  $t \geq 0$ .*

*Proof.* The closed convex cone  $K^*$  is pointed by Proposition 1. Apply Proposition 22 to  $K^*$  to obtain  $L^* \in \mathbf{Z}(K^*)$  if and only if  $e^{-t(L^*)} \in \pi(K^*)$  for all  $t \geq 0$ . Now apply Proposition 19 to both expressions.  $\square$

**Lemma 6.** *If  $K$  is a closed convex cone in a finite-dimensional real Hilbert space  $V$ , then  $L \in \mathbf{Z}(K)$  implies that  $e^{-tL} \in \pi(K)$  for all  $t \geq 0$ .*

*Proof.* The proof of Proposition 21 relies on the fact that  $K_S$  is proper, or that  $K$  is pointed. However, using Proposition 23, we can prove Proposition 21 and Proposition 22 without the assumption that  $K$  and (thus)  $K_S$  are pointed.  $\square$

**Theorem 8.** *If  $K$  is a closed convex cone in a finite-dimensional real Hilbert space  $V$  and  $L \in \mathcal{B}(V)$ , then  $L \in \mathbf{Z}(K)$  if and only if  $e^{-tL} \in \pi(K)$  for all  $t \geq 0$ .*

*Proof.* Combine Lemma 5 and Lemma 6.  $\square$

A similar result appears in Hilgert, Hofmann, and Lawson [16]. The first two items of their Theorem III.1.9 state that  $L \in \mathbf{Z}(K)$  if and only if  $e^{-tL} \in \pi(K)$  for all  $t \geq 0$ . However, the remaining items suggest hidden assumptions, and its proof relies on another Theorem I.5.27 where the cone is solid. Nevertheless, their Theorem I.5.17 seems to provide the machinery needed to prove the result.

All of our previous examples all corroborate Theorem 8. The next provides an application to dynamical systems.

**Example 12.** The system  $x'(t) = -L(x(t))$  has solution  $x(t) = e^{-tL}(x(0))$ . If  $L \in \mathbf{Z}(K)$  for some closed convex cone  $K$ , then Theorem 8 shows that  $e^{-tL} \in \pi(K)$  for all  $t \geq 0$ . Therefore  $x(t)$  remains in  $K$  for  $t \geq 0$  if  $x(0) \in K$ .

This next result now follows from  $\text{linspace}(\mathbf{Z}(K)) = \mathbf{LL}(K)$  as a corollary.

**Corollary 8** (Orlitzky [19], Theorem 4). *If  $K$  is a closed convex cone in a finite-dimensional real Hilbert space, then  $\mathbf{LL}(K)$  is the Lie algebra of  $\text{Aut}(K)$ .*

*Proof.* Apply Theorem 8 to  $\pm L \in \mathbf{Z}(K)$ .  $\square$

When  $K = V = \mathbb{R}^n$ , this witnesses the well-known fact that the  $n \times n$  real matrices are the Lie algebra of the general linear group of degree  $n$  over  $\mathbb{R}$ .

## 6 Decomposing $\mathbf{Z}$ -operators

Any  $L \in \mathbf{Z}(\mathbb{R}_+^n)$  is of the form  $L = \lambda I - N$  where  $\lambda \in \mathbb{R}$  and  $N \in \pi(\mathbb{R}_+^n)$  is a nonnegative matrix [3]. Schneider and Vidyasagar [25] show that a similar decomposition exists for any proper polyhedral cone: if  $K$  is proper and polyhedral in  $V$ , then  $\mathbf{Z}(K) = \text{span}(\{\text{id}_V\}) - \pi(K)$ . The authors leave open the question of when such a decomposition exists. The answer is “almost never” [15], but we do always have  $\mathbf{Z}(K) = \text{cl}(\text{span}(\{\text{id}_V\}) - \pi(K))$  if we take the closure [25].

From Definition 11 it should be obvious that  $\text{span}(\{\text{id}_V\}) \subseteq \mathbf{LL}(K) \subseteq \mathbf{Z}(K)$  for any closed convex cone  $K$ . It therefore makes sense to investigate when  $\mathbf{LL}(K) - \pi(K) = \mathbf{Z}(K)$ . Damm [6] asks if this is true for the cone of symmetric or Hermitian positive-semidefinite matrices (either real or complex). Kuzma et al. [17] provide an answer, constructing a counterexample when the matrices are larger than  $2 \times 2$ . With Theorem 8 at our disposal, we can prove an analogue of the result obtained by Schneider and Vidyasagar.

**Theorem 9.** *If  $K$  is a closed convex cone in a finite-dimensional real Hilbert space  $V$ , then  $\mathbf{Z}(K) = \text{cl}(\text{span}(\{\text{id}_V\}) - \pi(K))$ .*

*Proof.* If  $\lambda \in \mathbb{R}$  and  $P \in \pi(K)$ , then  $\lambda \text{id}_V - P \in \mathbf{Z}(K)$  from the definitions. Thus  $\text{span}(\{\text{id}_V\}) - \pi(K)$  is contained in  $\mathbf{Z}(K)$ . But  $\mathbf{Z}(K)$  is closed by Proposition 15, so we may take closures.

For the other inclusion, suppose that  $L \in \mathbf{Z}(K)$ . Then  $e^{-tL} \in \pi(K)$  for all  $t \geq 0$  by Theorem 8. Define  $f(t) := (\text{id}_V - e^{-tL})/t$ . It is clear that  $f(t)$  converges to  $L$  as  $t > 0$  approaches zero, and that  $f(t) \in \text{span}(\{\text{id}_V\}) - \pi(K)$  for all  $t \geq 0$ . Therefore  $L \in \text{cl}(\text{span}(\{\text{id}_V\}) - \pi(K))$ .  $\square$

To demonstrate the power of Theorem 9, we will use it to construct new proofs of Theorem 4, Theorem 5, and half of Theorem 6.

**Corollary 9.** *If  $K$  is a closed convex cone in a finite-dimensional real Hilbert space  $V$ , then  $\mathbf{Z}(K)^* = \text{cone}(G)$  where  $G := \{-s \otimes x \mid (x, s) \in C(K)\}$ .*

*Proof.* Take duals in Theorem 9 and apply Proposition 2 to find  $\mathbf{Z}(K)^* = \text{span}(\{\text{id}_V\})^\perp \cap (-\pi(K)^*)$ . Any  $L \in \text{span}(\{\text{id}_V\})^\perp$  must have a trace of zero. But if  $-L \in \pi(K)^*$  as well, then  $-L = \sum_{i=1}^m s_i \otimes x_i$  where  $(x_i, s_i) \in K \times K^*$ . The linearity of the trace and the fact that  $s_i \otimes x_i$  has a single eigenvalue  $\langle x_i, s_i \rangle \geq 0$  together imply that  $\langle x_i, s_i \rangle = 0$  for  $1 \leq i \leq m$ . Thus  $\mathbf{Z}(K)^* \subseteq \text{cone}(G)$ .

Suppose now that  $L \in \mathbf{Z}(K)$  and let  $\sum_{i=1}^m -s_i \otimes x_i \in \text{cone}(G)$ . Then  $\langle L, \sum_{i=1}^m -s_i \otimes x_i \rangle$  expands and can be rearranged to  $\sum_{i=1}^m -\langle L(x_i), s_i \rangle \geq 0$ . Since our choices were arbitrary, we have  $\text{cone}(G) \subseteq \mathbf{Z}(K)^*$ .  $\square$

**Corollary 10.** *If  $K$  is a closed convex cone in a finite-dimensional real Hilbert space  $V$ , then  $\dim(\mathbf{Z}(K)) = \dim(\pi(K))$ .*

*Proof.* Use Theorem 9 to find  $\dim(\mathbf{Z}(K)) = \dim(\text{span}(\{\text{id}_V\}) - \pi(K))$ , which is defined to be  $\dim(\text{span}(\text{span}(\{\text{id}_V\}) - \pi(K)))$ . But  $\text{id}_V \in \pi(K)$ , so

$$\dim(\mathbf{Z}(K)) = \dim(\text{span}(\pi(K))) := \dim(\pi(K)). \quad \square$$

**Corollary 11.** *If  $K$  is a polyhedral convex cone in a real Hilbert space, then  $\mathbf{Z}(K)$  is polyhedral.*

*Proof.* If  $K$  is polyhedral, then  $\pi(K)$  is polyhedral, and the closure in  $\mathbf{Z}(K) = \text{cl}(\text{span}(\{\text{id}_V\}) - \pi(K))$  is superfluous. Thus  $\mathbf{Z}(K)$  is generated by the finite set of generators of  $-\pi(K)$  and/or  $\pm \text{id}_V$ .  $\square$

The converse of Corollary 11 seems more elusive.

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