Global Solution Strategies for the Network-Constrained Unit Commitment (NCUC) Problem with Nonlinear AC Transmission Models

Jianfeng Liu · Anya Castillo · Jean-Paul Watson · Carl D. Laird

Abstract This paper addresses the globally optimal solution of the network-constrained unit commitment (NCUC) problem incorporating a nonlinear alternating current (AC) model of the transmission network. We formulate the NCUC as a mixed-integer quadratically constrained quadratic programming (MIQCQP) problem. A global optimization algorithm is developed based on a multi-tree approach that iterates between a mixed-integer master problem (relaxation for obtaining lower bounds) and a nonlinear programming subproblem (for obtaining upper bounds). Inspired by the second-order cone (SOC) relaxation originally designed for optimal power flow (OPF) problems, three convex relaxations of the original NCUC problem are presented, which are formulated as either mixed-integer second-order cone programming (MISOCP) or mixed-integer linear programming (MILP) problems. Numerical results on four benchmark problems indicate both good solution quality and computational efficiency of this tailored global solution framework.

Keywords Unit commitment, AC transmission models, multi-tree optimization.

1 Introduction

The unit commitment (UC) problem is a key optimization problem in power systems operations [35]. The objective in UC is to minimize system-wide cost under certain operational constraints and physical limits, to achieve projected (e.g., day-ahead) demand for power. A UC solution specifies on-line/off-line statuses for each generator in the system, and corresponding dispatch levels, for each time period in the planning horizon. UC is typically formulated as a mixed-integer programming problem, where binary variables are introduced to represent generator states over time.
Traditional thermal (as opposed to renewable) generating units require inclusion of various operating constraints in a UC formulation, to capture operational requirements such as minimum uptime/downtime restrictions and ramping limits. When constraints associated with models of the underlying transmission network are additionally taken into consideration, the resulting model is instead a network-constrained unit commitment (NCUC) problem.

Due to the size of real-world power systems, NCUC problems can be extremely large and computationally challenging. Consequently, only approximations of NCUC – typically involving at best linearized (“DC”) network approximations or AC network feasibility checks – are presently solved in practice; real-world systems require solution times in the range of minutes to tens of minutes. However, there is significant interest in obtaining solutions in tractable run-times to higher-fidelity models, as there is evidence that inclusion of AC transmission models into UC can lead to less costly and/or more reliable solutions [4]. Due to the large operating costs associated with power systems, even one or a few percent reductions can translate into hundreds of millions or billions of dollars in annual savings. Further, approximations may ultimately yield generating schedules that are infeasible given the actual AC transmission system, as key nonlinear features (e.g., transmission losses and voltage magnitude limits) are not explicitly considered. To partially remedy this issue, heuristic “repair” methods are often used to determine a feasible commitment through (typically time-consuming) trial-and-error procedures.

To begin to address these disadvantages, we present here (1) an NCUC formulation that includes a nonlinear AC transmission model and (2) an efficient global solution approach for the resulting mixed-integer nonlinear programming (MINLP) problems. To the best of our knowledge, this is the first time that NCUC problems of significant size have been formulated and solved to (within reasonable tolerances of) global optimality. In our approach, the continuous variables (e.g., voltage at each bus, power flow on each transmission line, and the power dispatch of each generator) are optimized simultaneously yet implicitly, via a multi-tree approach – with the binary variables (e.g., unit on-line / off-line statuses). Consequently, critical network nonlinearities are explicitly considered when making commitment decisions, resulting in globally optimal schedules that can be directly implementable.

The remainder of this paper is organized as follows. In Section 2, we briefly review existing base UC skeleton problem formulations, global optimization techniques for mixed-integer nonlinear programming (MINLP) problems, and relaxation strategies for AC transmission network models. Section 3 introduces our NCUC formulation, which includes a nonlinear AC transmission network model. In Section 4 we outline our global solution approach based on iteratively solving a sequence of upper bounding nonlinear programming (NLP) subproblems and lower bounding mixed-integer master problems. To complete the proposed algorithm, we introduce a global solution strategy for the NLP subproblem and three MILP-based relaxations for the master problem. Section 5 provides numerical results on a variety of test systems. We then conclude in Section 6 with a summary of our results, and directions for future work.
2 Background

We now provide a brief literature review concerning the three topics that are central to our contribution: modeling of UC problems, global optimization strategies for MINLP problems, and relaxations of nonlinear AC transmission network models.

2.1 Modeling Strategies for Unit Commitment Problems

Over the last decade, researchers have conducted extensive research on modeling and solving UC problems (without nonlinear AC transmission models), leading to a number of effective formulations [3, 24, 26]. These formulations can be partitioned into two major groups, the single-binary (1BIN) formulation and the three-binary (3BIN) formulation, and differ in the number of binary variables that are used to model generator on-line/off-line statuses. In a 1BIN UC formulation, a single binary variable, $y_{g,t}$, is employed to indicate the on-line/off-line state of generator $g$ at time period $t$. In contrast, a 3BIN formulation introduces two additional binary variables, $u_{g,t}$ and $w_{g,t}$, to respectively represent the startup and shutdown states of a generator $g$ at time period $t$. Mathematically, a 3BIN can be equivalently rewritten as a 1BIN formulation, as both $u_{g,t}$ and $w_{g,t}$ can be directly expressed in terms of $y_{g,t}$ only.

An advantage of the 1BIN formulation is its relatively smaller size, in terms of the number of binary variables; fewer binary variables often (but not always) translate into easier-to-solve MILPs. However, relaxations of the 1BIN formulations tend to be relatively loose when compared with relaxations of 3BIN formulations, allowing modern MILP solvers such as CPLEX and Gurobi to more effectively solve 3BIN formulations – despite the increased number of binary variables and constraints. Moreover, through the introduction of additional valid inequalities, feasible regions of the 3BIN formulation can be made even tighter. Hence, 3BIN formulations have been extensively studied in recent years. Comprehensive surveys of different 3BIN models are available in the literature, e.g., see [26, 27]. Based on these surveys and associated numerical comparisons, we base our work on the tight and compact 3BIN formulation proposed by Morales-Espa˜na et al. [24].

2.2 Global Optimization for Mixed-Integer Nonlinear Programming Problems

As we show subsequently, NCUC problems with nonlinear AC transmission models can be formulated as mixed-integer nonlinear programming (MINLP) problems. MINLP solution techniques can be divided into one of two classes: stochastic methods and deterministic methods. Stochastic methods, which include metaheuristics such as simulated annealing, tabu search, and genetic algorithms, can be easily implemented. However, these algorithms fail to provide any practical guarantee of global optimality, nor provide information concerning optimality gaps.
Deterministic methods, in contrast, are able to rigorously locate globally optimal solutions. Single-tree deterministic algorithms, such as the well-known branch-and-bound (BB) methods [7, 19], seek a global optimum by searching a single tree using a systematic enumeration strategy consisting of three primary steps: branching, bounding, and selecting. BB-based global optimization strategies have been well-studied and specialized, yielding strategies such as Branch-and-Reduce [30], Reduced Space Branch-and-Bound [9], Branch-and-Contract [36], Branch-and-Cut [15], and Branch-and-Sandwich [16]. In general, these approaches are suitable for general, non-convex MINLP problems of small or medium size.

In contrast, multi-tree search methods are based on the strategy of iteratively solving a sequence of related master problems and sub-problems. In this paper, we refer to a mixed-integer problem as convex if the corresponding continuous relaxation is a convex optimization problem. For such “convex” MINLP problems, many multi-tree solution strategies – including generalized benders decomposition (GBD) [11], outer approximation (OA) [8], and exact cutting plane (ECP) methods [33] – are available, and have been applied to a broad range of MINLPs in various application domains. For “non-convex” MINLP problems with special properties (e.g., those that are bilinear, polynomial, linear fractional, or concave separable), extensions of the above mentioned multi-tree methods can be found in literature [28, 29].

Our proposed global optimization algorithm for the NCUC is a multi-tree method, as it relies on iterative solution of a sequence of mixed-integer master problems and nonlinear subproblems. In our approach, convex relaxations of the nonlinear AC transmission model play a crucial role, for two primary reasons. First, a strong relaxation can improve the tightness of the mixed-integer master problem, leading to a tighter lower bound and thus faster convergence. Second, a strong relaxation of the AC transmission network model can be leveraged to efficiently solve the non-convex NLP subproblems to global optimality, which is required to ensure global optimality in the overall MINLP.

2.3 Convex Relaxations of Nonlinear AC Transmission Models

Three promising relaxation strategies for nonlinear AC transmission models have been recently proposed in the literature [2, 17, 22]: semi-definite programming (SDP) [1], quadratic convex (QC) relaxations [13], and second-order cone programming (SOCP) [14]. The SDP-based relaxation has generated significant recent interest and a comparatively extensive literature due to its tightness relative to other approaches [20]. Although SDP relaxations are often empirically exact for a wide range of problems, no sufficient condition ensuring an exact SDP relaxation of a general transmission network has yet been reported. On the contrary, several counterexamples have been highlighted in the literature [18, 21, 23], in which SDP relaxations yield non-zero duality gaps and fail to provide physically meaningful solutions. Additionally, SDP-based relaxations can be computationally prohibitive as the problem size increases.
Finally, SDP-based relaxations are difficult to incorporate within unit commitment models, due to the presence of discrete decision variables.

Recent studies have shown that the QC relaxation can serve as a competitive alternative to SDP due to its tightness as well as computational efficiency [6, 13]. The QC approach is not uniformly better when compared with the SDP relaxation, in that it neither dominates nor is dominated [6]. However, theoretical analysis shows that the strength of the QC relaxation depends on voltage phase angle differences. Numerical results indicate that the QC relaxation has better accuracy than the SOCP relaxation when phase angle difference bounds are tight. For cases with large phase angle differences, on the other hand, the QC relaxation is similar in tightness to the SOCP relaxation. However, the assumption of very small voltage phase angle differences may not be satisfied in real-world power systems.

Theoretical studies have shown the SOCP approach – originally introduced by Jabr [14] – has a relatively loose relaxation relative to its SDP and QC counterparts. However, from a practical perspective, SOCP relaxations have proven competitive on many AC transmission problems, especially in situations where no exact SDP relaxations are available. Recently, Kocuk et al. [17] proposed three strong SOCP-based relaxations, which neither dominate nor are dominated by the SDP or QC relaxations. Moreover, these relaxations can be efficiently solved for a wide range of benchmark problems. Inspired by this observed performance, we therefore focus on developing a new SOCP-based relaxation for the nonlinear AC transmission network model.

3 NCUC Problem Formulation

In this section, we introduce our NCUC problem formulation. The core UC model is based on the compact three-binary (3BIN) formulation introduced in Morales-Españo et al. [24]. The widely-used rectangular power-voltage (PQV) model is used to represent the nonlinear AC transmission network. The resulting combination yields a mixed-integer quadratically constrained quadratic programming (MIQCQP) problem, the solution of which is addressed by the algorithm proposed in Section 4.
3.1 Notation

Sets

\( \mathcal{T} \) Set of time periods \{1, ..., T\}
\( \mathcal{B} \) Set of all buses \{1, ..., B\}
\( \mathcal{B}_b \) Set of all buses that are connected to bus \( b \)
\( \mathcal{L} \) Set of all branches (transmission lines) \{1, ..., L\}
\( \mathcal{L}^i_b \) Set of all inbound branches to bus \( b \) (bus \( b \) is at the to end of a branch)
\( \mathcal{L}^{out}_b \) Set of all outbound branches from bus \( b \) (bus \( b \) is at the from end of a branch)
\( \mathcal{G} \) Set of all generators \{1, ..., G\}
\( \mathcal{G}_b \) Set of all generators at bus \( b \)
\( \mathcal{SC} \) Set of all synchronous condensers \{1, ..., SC\}
\( \mathcal{SC}_b \) Set of all synchronous condensers at bus \( b \)
\( \mathcal{S}_g \) Set of startup segments of generator \( g \) \{1, ..., \( S_g \)\}

Indices

\( t \) Time index; \( t \in \mathcal{T} \)
\( b, k \) Bus indices; \( b, k \in \mathcal{B} \)
\( l \) Branch index; \( l \in \mathcal{L}; l = (b, k) \), bus \( b \) is the from end and bus \( k \) is the to end
\( g \) Generator index; \( g \in \mathcal{G} \)
\( sc \) Synchronous condenser index; \( sc \in \mathcal{SC} \)
\( \tau \) Startup segment index; \( \tau \in \mathcal{S}_g \)

Parameters

\( P_{b,t}^D, Q_{b,t}^D \) Real (P) and reactive (Q) power demand (or load) at bus \( b \) at time \( t \)
\( P_t^R \) Global real power reserve requirement at time \( t \)
\( p_g^{G,\text{min}}, p_g^{G,\text{max}} \) Minimum/maximum real power output of generator \( g \)
\( q_g^{G,\text{min}}, q_g^{G,\text{max}} \) Minimum/maximum reactive power output of generator \( g \)
\( q_{\text{SC}, \text{min}}, q_{\text{SC}, \text{max}} \) Minimum/maximum reactive power output of synchronous condenser \( sc \)
\( v_{b,\text{min}}, v_{b,\text{max}} \) Minimum/maximum voltage magnitude at bus \( b \)
\( s_l^{\text{max}} \) Apparent power magnitude limit on branch \( l \)
\( K_{g,\tau}^{\text{su}}, K_{g,\tau}^{\text{sd}} \) Startup / shutdown cost of generator \( g \)
\( T_{\gamma, g}^{\text{su}}, T_{\gamma, g}^{\text{sd}} \) Time segment of startup cost function for generator \( g \)
\( T_g^{u}, T_g^{d} \) Minimum uptime/downtime of generator \( g \)
\( A_{g,0}, A_{g,1}, A_{g,2} \) Coefficients of quadratic production cost function of generator \( g \)
3.2 Unit Commitment Model

We use the term UC skeleton when referring to a unit commitment model consisting only of a cost function, operating constraints, and any associated continuous and binary variables – but not constraints and variables associated with nonlinear AC transmission models. As we discussed previously, 3BIN-based UC formulations can be efficiently solved due to their tightness and compactness. Our approach is based on the 3BIN formulation recently proposed by Morales-Espeña et al. [24]. For the sake of brevity, we only provide a brief introduction to several key components of this model here.

3.2.1 Cost Function

The total cost in UC is the sum of three major components – production costs, startup costs, and shutdown costs – as follows:

\[ f^P + f^{su} + f^{sd}. \]

As is often assumed in UC models, the production cost \( f^P \) is a quadratic function of real power generation. However, this assumption is often relaxed in practice, such that the quadratic is replaced with a piecewise approximation –
which can lead to computational advantages. Computation of $f^p$ in the quadratic case is accomplished by imposing the constraints

$$f^p = \sum_{g \in G} \sum_{t \in T} c_{g,t}$$

(1)

where $A_{g,2}, A_{g,1},$ and $A_{g,0}$ are known coefficients associated with a specific generator $g$.

To formulate the total startup cost, $f^{su}$, we first introduce a new binary variable $\delta_{g,\tau,t}$, which indicates the startup type $\tau$ of generator $g$ at time period $t$. In particular, $\delta_{g,\tau,t}$ takes the value of 1 if the generator $g$ starts up at time $t$ and has been previously offline within $[T_{g,\tau}, T_{g,\tau+1})$ hours. The logical constraints between $w_{g,t}$, $u_{g,t}$, and $\delta_{g,\tau,t}$ are given as

$$\delta_{g,\tau,t} \leq \sum_{t' = t - T_{g,\tau}}^{t+1 - T_{g,\tau+1}} w_{g,t'} \quad \forall g, \tau, t \in [1, S_g)$$

$$u_{g,t} = \sum_{\tau \in S_g} \delta_{g,\tau,t} \quad \forall g, t$$

(2)

where $S_g$ is the number of startup types for generator $g$. Note that $w_{g,t}$ with positive time index $t$ are variables, otherwise $w_{g,t}$ are treated as constants to demonstrate previous system status.

For a thermal unit, the startup cost is assumed to be a monotonically increasing step function with respect to the generator’s previous off-line time. The total startup cost is given by

$$f^{su} = \sum_{g \in G} \sum_{t \in T} \sum_{\tau \in S_g} K^{su}_{g,\tau} \delta_{g,\tau,t}$$

(3)

where $K^{su}_{g,\tau}$ is the cost of startup type $\tau$ for generator $g$. Given logical constraints (2) and the monotonically increasing startup cost function, it can be shown that $\delta_{g,\tau,t}$ will always solve to a binary value. In other words, instead of explicitly defining $\delta_{g,\tau,t}$ as a binary, it can be relaxed as a continuous variable within range $[0, 1]$.

The shutdown cost of generator $g$ is assumed to be independent of its previous on-line states, and the total shutdown cost is:

$$f^{sd} = \sum_{g \in G} \sum_{t \in T} K^{sd}_g w_{g,t}$$

(4)

### 3.2.2 Operating Constraints

According to operating restrictions, a thermal unit must stay in one state (either on-line or off-line) for a certain period of time before its state can be changed again. Such time periods vary between different generator types. To enforce
this requirement, we have to introduce minimum uptime and downtime constraints

\[
\sum_{t'=t-T_g+1}^{t} u_{g,t'} \leq y_{g,t} \quad \forall \; g, \; t
\]

\[
\sum_{t'=t-T_d+1}^{t} w_{g,t'} \leq 1 - y_{g,t} \quad \forall \; g, \; t
\]

(5)

where \(u_{g,t}\) and \(w_{g,t}\) with positive time index \(t\) are unknown variables, otherwise they are treated as constants to indicate previous system status. Additional constraints are required to denote the logical correlation between \(u_{g,t}, w_{g,t}\), and \(y_{g,t}\)

\[
y_{g,t} - y_{g,t-1} = u_{g,t} - w_{g,t} \quad \forall \; g, \; t
\]

(6)

Note that these constraints ensure that a generator cannot start up and shut down within the same time period. Given the fact that \(y_{g,t}\) is a binary variable, imposing constraints (5) and (6) together guarantees that \(u_{g,t}\) and \(w_{g,t}\) take binary values only. Consequently, \(u_{g,t}, w_{g,t}\), and \(\delta_{g,\tau,t}\), though initially defined as binaries, can be relaxed as continuous within \([0,1]\), leaving the \(y_{g,t}\) as the only binary variables in our UC skeleton formulation.

The spinning reserve describes extra generating capacity that is available by increasing the power output of generators that are currently connected to the power system. By doing that, power demand arises from generator failures can be satisfied within a short period of time. Typically, the spinning reserve is a fraction of the current total power demand. The upper and lower bounds of generator power output are dependent on its states. If a generator is on-line, the real and reactive power productions are constrained by \([P_{G,min}^g, P_{G,max}^g]\) and \([Q_{G,min}^g, Q_{G,max}^g]\), respectively. Otherwise, both real and reactive power output are set to 0. The synchronous condenser is always on-line and the reactive power output is constrained by \([Q_{SC,min}^{SC}, Q_{SC,max}^{SC}]\). The spinning reserve constraints and power output bounds are given as

\[
\sum_{b \in B} P_{b,t}^{-D} + P_t^{R} \leq \sum_{g \in G} p_{g,t}^a \quad \forall \; t
\]

\[
P_{G,min}^g y_{g,t} \leq p_{g,t}^G \leq p_{g,t}^a \leq P_{G,max}^g y_{g,t} \quad \forall \; g, \; t
\]

\[
Q_{G,min}^g y_{g,t} \leq q_{g,t}^G \leq Q_{G,max}^g y_{g,t} \quad \forall \; g, \; t
\]

\[
Q_{SC,min}^{SC} \leq q_{sc,t}^{SC} \leq Q_{SC,max}^{SC} \quad \forall \; sc, \; t
\]

(7)

It is worthwhile to point out that proper modeling of other restrictions, such as the ramping up/down limits, can also help to improve the tightness and compactness of the resulting UC skeleton. More details on UC problem constraints can be found in literature [24].
3.3 AC Transmission Network Model

In electric power system analysis, the rectangular PQV model is widely used to represent an AC transmission network. This model imposes balance equations for both real power and reactive power at each bus. A branch (transmission line) is denoted as \( l \equiv (b, k) \), where \( b \) is the index of the bus at the from end of branch \( l \) and \( k \) is the index of the bus at the to end. Real and reactive power injections at either end of a branch are explicitly expressed in terms of complex voltages.

When integrated in our UC skeleton, the rectangular PQV model is given by

\[
\begin{align*}
\sum_{l \in L_{b}^a} p^{l}_{t,t} + \sum_{l \in L_{b}^a} p^{f}_{l,t} + G^{sh}_{b} v^{2}_{b,t} + P^{D}_{b,t} - \sum_{g \in G_{b}} q^{G}_{g,t} &= 0 \quad \forall b, t \\
\sum_{l \in L_{b}^a} q^{l}_{t,t} + \sum_{l \in L_{b}^a} q^{f}_{l,t} - B^{sh}_{b} v^{2}_{b,t} + Q^{D}_{b,t} - \sum_{g \in G_{b}} q^{G}_{g,t} - \sum_{s \in SC_{b}} q^{SC}_{s,t} &= 0 \quad \forall b, t \\
p^{l}_{t,t} &= G^{ff}_{l} v^{2}_{b,t} + G^{fl}_{l} (v^{r}_{b,t} v^{r*}_{k,t} + v^{i}_{b,t} v^{i*}_{k,t}) - B^{ff}_{l} (v^{r}_{b,t} v^{i}_{k,t} - v^{i}_{b,t} v^{r*}_{k,t}) \quad \forall l, t \\
q^{l}_{t,t} &= -B^{ff}_{l} v^{2}_{b,t} - B^{fl}_{l} (v^{r}_{b,t} v^{r*}_{k,t} + v^{i}_{b,t} v^{i*}_{k,t}) - G^{ff}_{l} (v^{r}_{b,t} v^{i}_{k,t} - v^{i}_{b,t} v^{r*}_{k,t}) \quad \forall l, t \\
p^{f}_{l,t} &= G^{ff}_{l} v^{2}_{k,t} + G^{fl}_{l} (v^{r}_{k,t} v^{r*}_{b,t} + v^{i}_{k,t} v^{i*}_{b,t}) - B^{ff}_{l} (v^{r}_{k,t} v^{i}_{b,t} - v^{i}_{k,t} v^{r*}_{b,t}) \quad \forall l, t \\
q^{f}_{l,t} &= -B^{ff}_{l} v^{2}_{k,t} - B^{fl}_{l} (v^{r}_{k,t} v^{r*}_{b,t} + v^{i}_{k,t} v^{i*}_{b,t}) - G^{ff}_{l} (v^{r}_{k,t} v^{i}_{b,t} - v^{i}_{k,t} v^{r*}_{b,t}) \quad \forall l, t \\
(V^{min}_{b})^2 \leq v^{2}_{b,t} &= (v^{r}_{b,t})^2 + (v^{i}_{b,t})^2 \leq (V^{max}_{b})^2 \quad \forall b, t
\end{align*}
\]

where \( G^{sh}_{b} \) and \( B^{sh}_{b} \) are the shunt conductance and susceptance at bus \( b \), respectively.

\[
G_{l} = \begin{bmatrix} G^{ff}_{l}, G^{fl}_{l} \\ G^{ff}_{l}, G^{tt}_{l} \end{bmatrix}, B_{l} = \begin{bmatrix} B^{ff}_{l}, B^{fl}_{l} \\ B^{ff}_{l}, B^{tt}_{l} \end{bmatrix}
\]

are the real and imaginary parts of the admittance matrix \( Y_{l} \) of branch \( l \), respectively. Matrix \( Y_{l} \) can be expressed in terms of branch parameters, such as resistance \( (r_{l}) \), reactance \( (x_{l}) \), susceptance \( (b_{l}) \), transformer off-nominal turns ratio \( (\tau_{l}) \), and phase shift angle \( (\theta_{l}) \):.

\[
Y_{l} = \begin{bmatrix} \frac{(y_{l} + j \frac{b_{l}}{2})}{\tau_{l} e^{-\frac{\theta_{l}}{2}}}, -y_{l} \frac{1}{\tau_{l} e^{-\frac{\theta_{l}}{2}}} \\ -y_{l} \frac{1}{\tau_{l} e^{-\frac{\theta_{l}}{2}}}, y_{l} + j \frac{b_{l}}{2} \end{bmatrix}
\]

where \( y_{l} = 1/(r_{l} + j x_{l}) \). We use (M-PQV) to highlight that it is a multiple-period rectangular PQV model since multiple time periods are considered.

The thermal limits of a branch can be expressed in various ways. In our model, without loss of generality, thermal limit is measured by apparent power flow injections at both ends of branch \( l \):

\[
(p^{l}_{t,t})^2 + (q^{l}_{t,t})^2 \leq (S^{max}_{l,t})^2 \quad \forall l, t
\]

where \( S^{max}_{l,t} \) is the thermal limit of branch \( l \) at time \( t \).
For detailed information on rectangular \textit{PQV} models, please refer to the \textit{MATPOWER} manual, a commonly-used \textit{MATLAB}-based power system analysis tool [37].

3.4 NCUC Problem Formulation

Combining the UC skeleton with the multiple-period rectangular \textit{PQV} model gives a NCUC formulation:

$$\begin{align*}
\min & \quad f^p + f^{su} + f^{sd} \\
\text{s.t.} & \quad (1-7), (M-PQV), (8)
\end{align*}$$

(NCUC)

Note that the optimization model given above is a mixed-integer quadratically constrained quadratic programming (MIQCQP) problem due to the quadratic cost functions from the UC skeleton and the bilinear and quadratic terms from the multiple-period \textit{PQV} model.

4 Global Solution Framework for NCUC

We now propose a global solution strategy to solve NCUC problems. Our global solution framework is described in Section 4.1, and is based on solving a sequence of mixed-integer master problems and nonlinear programming (NLP) subproblems. To complete our algorithm, we introduce a sub-algorithm in Section 4.2 to solve the NLP subproblems associated with AC transmission models to global optimality. Finally, we propose in Section 4.3 three formulations for the mixed-integer master problem, based on second-order cone relaxations originally used for solving the AC optimal power flow (OPF) problem.

4.1 Global Solution Algorithm

Our proposed global solution framework, regarded as a multi-tree method, is an iterative algorithm that solves a sequence of mixed-integer master problems and nonlinear subproblems. The master problem, formulated as a mixed-integer programming problem, is a convex (in the sense of the term introduced in Section 2) relaxation of the original NCUC problem. Thus, if the mixed-integer master problem is infeasible, the original NCUC problem is also infeasible and the algorithm terminates. Otherwise, the master problem provides a valid lower bound and a candidate set of values for discrete decision variables for use when forming the nonlinear subproblems. We observe that the master problems can be refined by adding \textit{integer cuts}, which eliminate all integer solutions that have been previously visited from consideration. This enhancement ensures that different solutions are obtained during each major iteration of our algorithm, where a major iteration is defined as a pair of master / subproblem solves. A corresponding subproblem is then obtained by fixing all of the binary variables present in the original NCUC problem to the values specified in
the solution to the master problem. The resulting subproblem is a multi-period AC OPF, which is an NLP problem. If a feasible solution is found, the subproblem provides a valid upper bound for the NCUC problem. A globally optimal solution to subproblems is sought to ensure tightness of the upper bound as the global algorithm progresses. The algorithm proceeds through a series of major iterations, cycling between the solution of a mixed-integer master problem and a nonlinear subproblem – yielding a sequence of lower and upper bounds. The algorithm terminates when the relative optimality gap is below a given tolerance.

More formally, our global optimization algorithm for the NCUC is given as follows:

G.0 Initialization The initial mixed-integer master problem (without integer cuts) is formulated and solved. If the problem is infeasible, then the original NCUC problem is also infeasible and the algorithm terminates. Otherwise, go to Step G.2 Solve the NLP Subproblem.

G.1 Solve the Mixed-Integer Master Problem The mixed-integer master problem is formulated and solved; any of the three different master problem formulations proposed in Section 4.3 can be used here. If the master problem is feasible, update the lower bound and proceed to the Step G.2 Solve the NLP Subproblem. Otherwise, the algorithm terminates and the current upper bound is a globally optimal solution.

G.2 Solve the NLP Subproblem Given the integer solution obtained in the previous step, the nonlinear subproblem is solved to global optimality; the sub-algorithm used is described in Section 4.2. If the subproblem is feasible and the objective function value is better than that associated with the best known solution, then update the upper bound. If the optimality gap between the current upper and lower bound is below the tolerance, the algorithm terminates and the global solution is obtained. Otherwise proceed to Step G.3 Refine the Mixed-Integer Master Problem.

G.4 Refine the Mixed-Integer Master Problem The current mixed-integer master problem is tightened by adding integer cuts. Proceed to Step G.1 Solve the Mixed-Integer Master Problem.

The convergence of our proposed global solution framework strongly depends on two important components. First, it requires the global solution of NLP subproblems, which unfortunately is a non-convex optimization problem. Therefore, a sub-algorithm is necessary for their solution. Second, the tightness of the mixed-integer master problem may impact convergence speed. Thus, there is a need for a strong but computationally efficient relaxation for the master problem. In the remainder of this section, we focus on addressing both issues: a sub-algorithm for the NLP subproblem is proposed and tested in Section 4.2 and various master problem formulations are introduced in Section 4.3.

4.2 Sub-Algorithm for Non-Convex Subproblems

The subproblem, which is obtained by fixing all integer variables in the original NCUC problem, can be regarded as an multiple-period AC OPF problem. In this section, we will describe the global optimization strategy for this non-
convex, nonlinear subproblem. Relaxation techniques initially designed for general AC OPF problems are well suited here. Particularly, we will use relaxations based on second-order cone programming (SOCP).

To reduce the notation burden, we only consider a general ‘single-period’ AC OPF problem based on the rectangular power-voltage (PQV) model:

\[
\begin{equation}
\begin{aligned}
\min \quad & f^p = \sum_{g \in \mathcal{G}} c_g^p \\
s.t. \quad & A_g^2(p_g^G)^2 + A_g^1 p_g^G + A_g^0 \leq c_g^p \quad \forall g \\
& \sum_{l \in \mathcal{L}_b^m} p_l^l + \sum_{l \in \mathcal{L}_b^{out}} p_l^l + G^{sh}_b v_b^2 + P_b^D - \sum_{g \in \mathcal{G}_b} v_g^G = 0 \quad \forall b \\
& \sum_{l \in \mathcal{L}_b^m} q_l^l + \sum_{l \in \mathcal{L}_b^{out}} q_l^l - B^{sh}_b v_b^2 + Q_b^D - \sum_{g \in \mathcal{G}_b} q_g^G - \sum_{sc \in \mathcal{S}_b} q_{sc}^G = 0 \quad \forall b \\
& p_l^l = G^{ff}_l v_b^2 + G^{f}_l (v_b v_k + v^*_b v^*_k) - B^{ff}_l (v_b v_k - v^*_b v^*_k) \quad \forall l \\
& q_l^l = -B^{ff}_l v_b^2 - B^{ff}_l (v_b v_k + v^*_b v^*_k) - G^{ff}_l (v_b v_k - v^*_b v^*_k) \quad \forall l \\
& (p_l^l)^2 + (q_l^l)^2 \leq (S_{l_{\max}}^i)^2, \quad (p_l^l)^2 + (q_l^l)^2 \leq (S_{l_{\max}}^q)^2 \\
& (v_b^{\min})^2 \leq v_b^2 = (v_b^*)^2 \leq (v_b^{\max})^2 \\
& P_g^{G_{\min}} \leq P_g^{G_{\max}}, \quad Q_g^{G_{\min}} \leq Q_g^{G_{\max}} \quad \forall g \\
& Q_{sc_{\min}}^{S_{\min}} \leq Q_{sc_{\max}}^{S_{\max}} \quad \forall sc
\end{aligned}
\end{equation}
\]

Note that (AC-OPF) is a non-convex problem due to a large number of quadratic and bilinear terms in equality constraints. To obtain a strict convex relaxation, our approach is based on the idea of the second-order cone relaxation method which was first proposed by Jabr [14]. A set of new variables are first defined as

\[
\begin{aligned}
c_{b,k} &= (v_b^*)^2 + (v_b^*)^2 = v_b^2 \\
c_{b,k} &= v_b^* v_k^* + v_b^* v_k^* = |v_b||v_k| \cos \theta_{b,k} \\
s_{b,k} &= v_b^* v_k^* - v_b^* v_k^* = -|v_b||v_k| \sin \theta_{b,k}
\end{aligned}
\]
The quadratic and bilinear terms can be replaced by new variables, $c_{b,k}$ and $s_{b,k}$, leading to the relaxation:

$$\min \ f^p = \sum_{g \in \mathcal{G}} c_g^p$$

s.t. \hspace{1cm} \begin{align*}
A_g^2 (p_g^G)^2 + A_g^1 p_g^G + A_g^0 \leq c_g^p & \hspace{1cm} \forall g \\
\sum_{l \in \mathcal{L}_{in}^b} p_{l}^f + \sum_{l \in \mathcal{L}_{out}^b} p_{l}^f + G_{b,k}^{lh} c_{b,k} + P_b^D - \sum_{g \in \mathcal{G}} p_g^G = 0 & \hspace{1cm} \forall b \\
\sum_{l \in \mathcal{L}_{in}^b} q_{l}^f - \sum_{l \in \mathcal{L}_{out}^b} q_{l}^f - B_b^{lh} c_{b,k} + Q_b^D - \sum_{g \in \mathcal{G}} q_g^G - \sum_{sc \in \mathcal{S}_{bc}} q_{sc}^G = 0 & \hspace{1cm} \forall b \\
p_{l}^f = G_{l}^{ff} c_{b,k} + G_{l}^{ft} c_{k,b} - B_{l}^{ft} s_{b,k} & \hspace{1cm} \forall l \\
q_{l}^f = -B_{l}^{ff} c_{b,k} - B_{l}^{ft} c_{k,b} - G_{l}^{ft} s_{b,k} & \hspace{1cm} \forall l \\
p_{l}^f = G_{l}^{ft} c_{k,b} + G_{l}^{ff} c_{b,k} - B_{l}^{ft} s_{b,k} & \hspace{1cm} \forall l \\
q_{l}^f = -B_{l}^{ff} c_{k,b} - B_{l}^{ft} c_{b,k} - G_{l}^{ft} s_{b,k} & \hspace{1cm} \forall l \\
(p_{l}^f)^2 + (q_{l}^f)^2 \leq (S_l^{max})^2, (p_{l}^f)^2 + (q_{l}^f)^2 \leq (S_l^{max})^2 & \hspace{1cm} \forall l \\
(V_b^{min})^2 \leq c_{b,k} \leq (V_b^{max})^2 & \hspace{1cm} \forall b \\
P_g^{G, min} \leq p_g^G \leq P_g^{G, max}, Q_g^{SC, min} \leq q_g^G \leq Q_g^{SC, max} & \hspace{1cm} \forall g \\
Q_{sc}^{SC, min} \leq q_{sc}^{SC} \leq Q_{sc}^{SC, max} & \hspace{1cm} \forall sc \\
c_{b,k} = c_{k,b}, s_{b,k} = -s_{k,b} & \hspace{1cm} \forall l = (b, k)
\end{align*}$$

(AC-OPF-R)

where the power balance and branch injection constraints are all linear. However, we need to introduce additional constraints

$$c_{b,k}^2 + s_{b,k}^2 \leq c_{b,k} c_{k,b} \hspace{1cm} \forall l = (b, k)$$

(9)

which are known as the second-order cone inequalities. Though quadratic and bilinear terms are included there, second-order cone constraints are convex since the corresponding feasible region includes both the surface and the inner space of a second-order cone in $\mathbb{R}^4$. The original SOCP relaxation proposed by Jabr [14], which includes (AC-OPF-R) and (9), gives a strict convex relaxation of the original problem (AC-OPF).

To improve the tightness of the original SOCP relaxation, Kocuk et al. [17] proposed three strong relaxations. Relaxation SOCPA is obtained by introducing “cycle constraints” and arctangent envelopes to the original SOCP relaxation. SOCPA is further strengthened by imposing 3- and 4- cycle decompositions, leading to a tighter relaxation S34A. The relaxation SSDP, on the other hand, is obtained by introducing a semi-definite programming (SDP) separation routine to the original SOCP relaxation. Numerical results have shown that all these relaxations are tighter than the classic SOCP formulation.
In our global solution sub-algorithm, particularly, we focus on strong SOCP relaxation with cycle constraints. The cycle constraints ensure that the sum of angle differences along each cycle (or loop) equals zero

\[ \sum_{l \in L_c} \theta_l = 0 \quad \forall \ L_c \]

(10)

\[ \theta_l \equiv \theta_{b,k} = -\arctan \left( \frac{s_{b,k}}{c_{b,k}} \right) \quad \forall \ l = (b, k) \]

where \( L_c \) denotes a simple cycle in the network. Previous work has shown that cycle constraints can be neglected for some systems with certain special properties. For a general transmission network, unfortunately, these constraints have to be explicitly considered to obtain a tighter relaxation.

To attain the convexity of the relaxation problem, we need to replace the nonlinear \( \arctan \) terms in cycle constraints (10). Particularly, we are interested in constructing linear outer approximations including over- and under-estimators. Note that for each \( \arctan \) term, the feasible region of \((c, s)\) is a rectangle constrained by \([c^\min, c^\max]\) and \([s^\min, s^\max]\).

Thus, a simple idea is to identify the “tightest” hyperplane which passes a vertex of the rectangle and lies strictly above or below the arctangent function over the entire feasible region. This way, we introduce four over-estimators and four under-estimators for each \( \arctan \) term. Each estimator can be obtained by solving a linear programming (LP) problem. To further refine the tightness of the outer approximations, it is necessary to reduce the bounds of \( c \) and \( s \). In our NLP sub-algorithm, the optimality-based bound tightening (OBBT) method is used to dynamically tighten bounds of these variables.

Previous work has shown that we only need to impose cycle constraints for a set of simple cycles, which is called a cycle basis. For instance, in a connected graph with \( n \) buses and \( m \) branches, the cycle basis contains a number of \( m - n + 1 \) simple cycles. In a real-world sized network, however, \( m - n + 1 \) may still be a large number and the resulting relaxation may become very challenging to solve. Therefore, instead of imposing all cycle constraints at once, we will progressively add them as necessary. We first solve a relaxation problem with a few cycle constraints and identify simple cycles with large angle summation mismatches. Additional constraints corresponding to these cycles are then added to the current relaxation. The goal is to reduce the optimality gap significantly while considering as few cycle constraints as possible.
The resulting SOCP, shown below, is a strict convex relaxation of the original AC OPF problem.

\[
\begin{align*}
\min & \quad f^p = \sum_{g \in G} c^p_g \\
\text{s.t.} & \quad A^2_g (p^G_g)^2 + A^1_g p^G_g + A^0_g \leq c^p_g \quad \forall \ g \\
& \quad \sum_{l \in L_b^{in}} p^f_{il} + \sum_{l \in L_b^{out}} p^f_{il} + G^t_{b} c_{b,k} + P^D_b - \sum_{g \in G_b} p^G_g = 0 \quad \forall \ b \\
& \quad \sum_{l \in L_b^{in}} q^f_{il} + \sum_{l \in L_b^{out}} q^f_{il} - B^h_{b} c_{b,k} + Q^D_b - \sum_{g \in G_b} q^G_g - \sum_{sc \in SC_b} q^SC_g = 0 \quad \forall \ b \\
& \quad p^f_l = G^f_{l} c_{b,k} + G^f_{l} c_{b,k} - B^f_{l} s_{b,k} \quad \forall \ l \\
& \quad q^f_l = -B^f_{l} c_{b,k} - B^f_{l} c_{b,k} - G^f_{l} s_{b,k} \quad \forall \ l \\
& \quad p^f_l = G^f_{l} c_{b,k} + G^f_{l} c_{b,k} - B^f_{l} s_{b,k} \quad \forall \ l \\
& \quad q^f_l = -B^f_{l} c_{b,k} - B^f_{l} c_{b,k} - G^f_{l} s_{b,k} \quad \forall \ l \\
& \quad (p^f_{il})^2 + (q^f_{il})^2 \leq (S^f_{max})^2 \quad \forall \ l \\
& \quad (V^b_{min})^2 \leq c_{b,k} \leq (V^b_{max})^2 \quad \forall \ b \\
& \quad p^G_{g, min} \leq p^G_g \leq p^G_{g, max} \quad \forall \ g \\
& \quad Q^SC_{g, min} \leq q^SC_g \leq Q^SC_{g, max} \quad \forall \ sc \\
& \quad c_{b,k} = c_{k,b}, \ s_{b,k} = -s_{k,b} \quad \forall \ l = (b, k) \\
& \quad c_{b,k}^2 + s_{b,k}^2 \leq c_{b,k} c_{k,b} \quad \forall \ l = (b, k) \\
& \quad \sum_{l \in L_c} \theta_l = 0 \quad \forall \ L_c \\
& \quad \theta_l \leq \arctan \left( \frac{s_{b,k}}{c_{b,k}} \right) \quad \forall \ l = (b, k) \\
& \quad \theta_l \geq \arctan \left( \frac{s_{b,k}}{c_{b,k}} \right) \quad \forall \ l = (b, k)
\end{align*}
\]

where \( \arctan(\cdot) \) and \( \arctan(\cdot) \) are linear over- and under-estimators of \( \arctan \) function, respectively.

To solve the non-convex, nonlinear AC OPF problem (or the subproblem in Section 4.1) to global optimality, the sub-algorithm proposed here depends on iteratively solving a sequence of original NLP problems and strict convex relaxation problems.

A description of the sub-algorithm is given below:

N.0 Initialization An initial relaxation (AC-OPF-SOCP) without cycle constraints is generated.
N.1 Solving the SOCP Relaxation Problem

The relaxation problem (AC-OPF-SOCP) is solved to global optimality. If the relaxation problem is infeasible, the original AC OPF problem is infeasible and the algorithm terminates. Otherwise, update the current lower bound and proceed to Step N.2 Solving the Non-Convex AC OPF Problem.

N.2 Solving the Non-Convex AC OPF Problem

The original non-convex problem (AC-OPF) is initialized by the optimal solution obtained from the previous step and then solved to local optimality. If the problem is feasible and the objective is lower than the best known feasible solution, update the upper bound. If the optimality gap between the current upper and lower bounds is below the given tolerance, the algorithm terminates and the global solution is obtained. Otherwise, proceed to Step N.3 Refining the SOCP Relaxation Problem.

N.3 Refining the SOCP Relaxation Problem

Simple cycles which violate the cycle constraints most strongly are identified. Bounds of variables associated with these cycles are tightened using OBBT. New cycle constraints along with linear outer approximations of arctangent terms are added to the current relaxation (AC-OPF-SOCP). Repeat the process from Step N.1 Solving the SOCP Relaxation Problem.

Note this sub-algorithm is executed in Step G.2 Solving the NLP Subproblem of the global optimization framework proposed in Section 4.1. By doing this, the subproblem, which is equivalent to a multiple-period AC OPF problem, can be solved to global optimality.

We test our sub-algorithm on 14 benchmark AC OPF problems: 7 standard IEEE instances from MATPOWER [37] and 7 instances from the NESTA test case archive [5]. The interior point nonlinear optimization code Ipopt is used to solve the upper and lower bounding problems, with default parameter settings. The tolerance for the relative optimality gap is set to 0.01%. Computational results are reported in Table 1.

The results demonstrate that our sub-algorithm can successfully solve nonlinear AC OPF benchmark problems to within a very small tolerance of global optimality. As expected, the computational burden is strongly correlated with problem size. It is worthwhile to point out that the CPU time reported here is the total time to solve AC OPF problems and the corresponding SOCP-based relaxations. The computational time associated with handling data, generating models, and bound tightening, is not included. For small-scale problems, e.g., Case6ww and Case14, global optimality can be easily achieved. However, medium-size problems, e.g., Case118 and Case300, require many more iterations and thus higher total runtime. We do not consider larger instances here, as UC models with AC transmission components are not presently available at that scale. The major cause of larger numbers of iterations for medium-scale problems is that many more cycle constraints are required to strengthen the relaxations; we observe that only one cycle constraint is added at each iteration. To mitigate this issue, a heuristic could be implemented to select and add an appropriate number of cycle constraints at each iteration; the specific number could be determined by considering factors such as problem size, optimality gap, and angle difference summation mismatch. The goal with such a strategy would be to identify a number of “key” cycle constraints within a small number of iterations.
Table 1: Problem Size and Performance Results

<table>
<thead>
<tr>
<th>Case Name</th>
<th>Optimal Solution</th>
<th>Optimality Gap (%)</th>
<th>CPU Time (s)</th>
<th>Iterations</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case6ww</td>
<td>3126.36</td>
<td>$8 \times 10^{-3}$</td>
<td>0.26</td>
<td>4</td>
</tr>
<tr>
<td>Case14</td>
<td>8081.52</td>
<td>$3 \times 10^{-3}$</td>
<td>0.43</td>
<td>3</td>
</tr>
<tr>
<td>Case30</td>
<td>574.52</td>
<td>0.0</td>
<td>0.95</td>
<td>6</td>
</tr>
<tr>
<td>Case39</td>
<td>41864.18</td>
<td>$5 \times 10^{-3}$</td>
<td>0.81</td>
<td>3</td>
</tr>
<tr>
<td>Case57</td>
<td>41737.79</td>
<td>$6 \times 10^{-3}$</td>
<td>9.03</td>
<td>13</td>
</tr>
<tr>
<td>Case118</td>
<td>129660.69</td>
<td>$6 \times 10^{-3}$</td>
<td>32.4</td>
<td>26</td>
</tr>
<tr>
<td>Case300</td>
<td>719725.10</td>
<td>$9 \times 10^{-3}$</td>
<td>253.1</td>
<td>36</td>
</tr>
<tr>
<td>NESTA Case6ww</td>
<td>3143.97</td>
<td>0.0</td>
<td>0.74</td>
<td>7</td>
</tr>
<tr>
<td>NESTA Case14</td>
<td>244.05</td>
<td>$3 \times 10^{-3}$</td>
<td>0.22</td>
<td>3</td>
</tr>
<tr>
<td>NESTA Case30</td>
<td>204.97</td>
<td>0.0</td>
<td>2.61</td>
<td>11</td>
</tr>
<tr>
<td>NESTA Case39</td>
<td>96505.52</td>
<td>$9 \times 10^{-3}$</td>
<td>4.10</td>
<td>9</td>
</tr>
<tr>
<td>NESTA Case57</td>
<td>1143.27</td>
<td>$6 \times 10^{-3}$</td>
<td>9.62</td>
<td>20</td>
</tr>
<tr>
<td>NESTA Case118</td>
<td>3718.64</td>
<td>0.0</td>
<td>60.0</td>
<td>41</td>
</tr>
<tr>
<td>NESTA Case300</td>
<td>16891.28</td>
<td>0.0</td>
<td>134.0</td>
<td>45</td>
</tr>
</tbody>
</table>

As indicated above, this sub-algorithm is based on relaxations proposed by Kocuk et al. [17]. To validate our implementation and its computational performance, we compare the solution times with those given in Kocuk et al. [17] for the original relaxation, denoted \textit{SOCP}, and the relaxation strengthened by arctangent envelopes, denoted \textit{SOCPA}. Kocuk et al. [17] report the sum of the time to solve the convex relaxation and the first instance of the AC OPF subproblem (providing enough information to report the gap). As such, their reported times correspond to a single iteration of our sub-algorithm (which solves the NLP completely to global optimality within a specified tolerance). Therefore, we report their results along with the average time per iteration for our approach, shown in Table 2.

As expected, these results indicate that our approach requires more computational time than the original relaxation \textit{SOCP}, as our relaxation with select cycle constraints and corresponding arctangent envelopes is more challenging to solve. In contrast, our approach is not as difficult as the relaxation \textit{SOCPA}, which considers all cycle constraints and arctangent approximations. We observe that other strong relaxations, such as the \textit{S34A} and \textit{SSDP} relaxations proposed by Kocuk et al. [17], can also be implemented in our sub-algorithm. The computational time for the \textit{SOCP} and \textit{SOCPA} relaxations are taken directly from [17]; the results are not directly comparable, due to differing hardware and programming languages, but rather provide validation that the performance of our implementation on individual relaxations is consistent with expectations.

4.3 Mixed-Integer Master Problem

The tightness of the master problem is important to the convergence of the global framework proposed in Section 4.1. Particularly, the lower bounding master problem is a “convex” relaxation of the original NCUC problem. To reduce the nonlinearity arising from the AC transmission network model, we can use the same idea of the second-order cone...
Table 2: Solution Time of Different SOCP Relaxations

<table>
<thead>
<tr>
<th>Case Name</th>
<th>SOCP CPU Time (s)</th>
<th>SOCPA CPU Time (s)</th>
<th>Our Approach CPU Time (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Case6ww</td>
<td>0.13</td>
<td>0.48</td>
<td>0.07</td>
</tr>
<tr>
<td>Case14</td>
<td>0.05</td>
<td>0.44</td>
<td>0.14</td>
</tr>
<tr>
<td>Case30</td>
<td>0.12</td>
<td>1.01</td>
<td>0.16</td>
</tr>
<tr>
<td>Case39</td>
<td>0.10</td>
<td>0.96</td>
<td>0.27</td>
</tr>
<tr>
<td>Case57</td>
<td>0.11</td>
<td>1.50</td>
<td>0.69</td>
</tr>
<tr>
<td>Case118</td>
<td>0.27</td>
<td>3.86</td>
<td>1.25</td>
</tr>
<tr>
<td>Case300</td>
<td>0.62</td>
<td>8.04</td>
<td>7.03</td>
</tr>
<tr>
<td>NESTA Case6ww</td>
<td>0.02</td>
<td>0.22</td>
<td>0.11</td>
</tr>
<tr>
<td>NESTA Case14</td>
<td>0.02</td>
<td>0.45</td>
<td>0.07</td>
</tr>
<tr>
<td>NESTA Case30</td>
<td>0.04</td>
<td>0.88</td>
<td>0.24</td>
</tr>
<tr>
<td>NESTA Case39</td>
<td>0.04</td>
<td>0.89</td>
<td>0.46</td>
</tr>
<tr>
<td>NESTA Case57</td>
<td>0.08</td>
<td>2.04</td>
<td>0.48</td>
</tr>
<tr>
<td>NESTA Case118</td>
<td>0.25</td>
<td>4.98</td>
<td>1.46</td>
</tr>
<tr>
<td>NESTA Case300</td>
<td>0.47</td>
<td>9.93</td>
<td>2.98</td>
</tr>
</tbody>
</table>

relaxation introduced above. The relaxation of a multiple-period AC transmission model (M-PQV) can be rewritten as

\[
\begin{align*}
\sum_{l \in L_{in}^n} p_{l,t}^f + \sum_{l \in L_{out}^n} p_{l,t}^f + G_{b}^{sh} c_{b,b,t} + P_{b,t}^D - \sum_{g \in G_b} p_{g,t}^G = 0 & \quad \forall b, t \\
\sum_{l \in L_{in}^n} q_{l,t}^f + \sum_{l \in L_{out}^n} q_{l,t}^f - B_{b}^{sh} c_{b,b,t} + Q_{b,t}^D - \sum_{g \in G_b} q_{g,t}^G - \sum_{sc \in SC_b} q_{sc,t}^{SC} = 0 & \quad \forall b, t \\
p_{l,t}^f = G_{1}^{ff} c_{b,b,t} + G_{1}^{tt} c_{k,k,t} - B_{1}^{ft} s_{b,k,t} & \quad \forall l, t \\
q_{l,t}^f = -B_{1}^{ff} c_{b,b,t} - B_{1}^{tt} c_{k,k,t} - G_{1}^{tt} s_{b,k,t} & \quad \forall l, t \\
p_{l,t}^t = G_{1}^{tt} c_{k,k,t} + G_{1}^{ff} c_{b,k,t} + B_{1}^{tt} s_{b,k,t} & \quad \forall l, t \\
q_{l,t}^t = -B_{1}^{tt} c_{k,k,t} - B_{1}^{ff} c_{b,k,t} + G_{1}^{tt} s_{b,k,t} & \quad \forall l, t \\
(V_b^{min})^2 \leq c_{b,b,t} \leq (V_b^{max})^2 & \quad \forall b, t \\
c_{b,k,t} = c_{b,k,t}, s_{b,k,t} = -s_{b,k,t} & \quad \forall l, t
\end{align*}
\]

and the corresponding second-order cone inequalities are given by

\[
c_{b,k,t}^2 + s_{b,k,t}^2 \leq c_{b,b,t} c_{k,k,t} \quad \forall l = (b, k), t
\]

Note that the cycle constraints are not included here to reduce the computational complexity. Integrating the UC skeleton with the relaxed multiple-period AC model gives our first formulation of the master problem

\[
\begin{align*}
\min & \quad f^p + f^{su} + f^{sd} \\
\text{s.t.} & \quad (1-8), (M-PQV-R), (11)
\end{align*}
\]

(19)
The label (NCUC-RQ) is used to emphasize the quadratic terms in production costs and thermal limit constraints. Note this problem is formulated as a mixed-integer second-order cone programming (MISOCP) problem and it is a strict convex relaxation of the original problem (NCUC).

To further reduce nonlinearities in formulation (NCUC-RQ), we eliminate quadratic terms in cost functions and thermal limit constraints. First, let us consider the total production costs calculated from (1). In this minimization problem, the convex quadratic cost function can be replaced by a set of linear under-estimators

$$f^P = \sum_{g \in G} \sum_{t \in T} c^P_{g,t}$$

where the feasible region $[P^G_{g,t}^{min}, P^G_{g,t}^{max}]$ is divided into $N$ segments, $P^*_{g,t,n}$ denote the segment points, and the parameters are given by $B^*_{g,t,n} = 2A_{g,2}^*P^*_{g,t,n} + A_{g,1}$ and $C^*_{g,t,n} = -A_{g,2}(P^*_{g,t,n})^2$.

The thermal limit restrictions (8) can be replaced by linear outer approximations. Due to the fact that these thermal limits are rarely violated, however, we exclude these constraints when solving the NCUC problem. A post-verification is necessary to check whether the current optimal solution causes any overloads on branches. If there exists any overload, outer approximations of the thermal limit constraints corresponding to that branch will be added to the current problem. Otherwise, the optimal solution is feasible and there is no need to add any new thermal constraints. By doing this, we can better control the problem size and reduce the computational burden.

The formulation (NCUC-RQ) with additional relaxation described above is given by

$$\min \quad f^P + f^{su} + f^{sd}$$

s.t. \quad (2-7), (M-PQV-R), (11-12)

(NCUC-R)

Note that this relaxation is still an MISOCP problem due to second-order cone constraints (11).

Based on the idea of outer approximations, our third relaxation is derived from formulation (NCUC-R) with all second-order cone constraints (11) replaced by linear cutting planes:

$$2c^*_{b,k,t,m}c_{b,k,t} + 2s^*_{b,k,t,n}s_{b,k,t} \leq c^*_{b,b,t,p}c_{k,k,t} + c^*_{k,k,t,q}c_{b,b,t} \quad \forall \ l, t, m, n, p$$

(13)

where $m, n, p$, and $q$ are indices of segment points of $c_{b,k,t}, s_{b,k,t}, c_{b,b,t}$, and $c_{k,k,t}$, respectively. Note that only three segment points, e.g., $c^*_{b,k,t,m}, s^*_{b,k,t,n}$, and $c^*_{b,b,t,p}$, are independent, while the fourth one, say $c^*_{k,k,t,q}$, must be calculated from the following equation:

$$(c^*_{b,k,t,m})^2 + (s^*_{b,k,t,n})^2 = c^*_{b,b,t,p}c^*_{k,k,t,q} \quad \forall \ l, t, m, n, p$$
This way, the point \( [c_{b,k,t,m}^*, s_{b,k,t,n}^*, c_{b,b,t,p}^*, c_{k,k,t,q}^*] \) is guaranteed to lie on the surface of the 4-D second-order cone.

The resulting relaxation, which can now be formulated as a mixed-integer programming (MILP) problem, is given by

\[
\begin{align*}
\text{min} & \quad f^P + f^{su} + f^{sd} \\
\text{s.t.} & \quad (2-7), (M-PQV-R), (12-13)
\end{align*}
\]

Compared with two MISOPC relaxations, (NCUC-RQ) and (NCUC-R), the fully linear relaxation (NCUC-RL) is weaker in tightness. However, it can be solved with existing MILP algorithms. In the following comparison, we use different number of segment points in (NCUC-RL). In relaxation (NCUC-RL)-5, the feasible domains of \( c_{b,k,t,m}^* \), \( s_{b,k,t,n}^* \), and \( c_{b,b,t,p}^* \) are separated into 5 segments, leading to 125 linear outer approximations for a single second-order cone constraint. In relaxation (NCUC-RL)-10, feasible regions are divided into 10 segments and 1000 linear outer approximations are used to replace a single second-order cone constraint.

5 Numerical Results

We now test our global solution algorithm for NCUC on four benchmark problems: a 6-bus test system [10], two 24-bus test systems including RTS-79 [31] and RTS-96 [34], and a modified IEEE 118-bus test system [10]. The sole difference between the two 24-bus cases is that the RTS-96 case does not specify a positive lower bound on the real power output of each generator. We first compare the computational performance of our algorithm considering each of the three proposed master problem relaxations. We then discuss the schedules and costs associated with globally (near-) optimal solutions to the NCUC problem, and compare our results with those previously reported in the literature.

5.1 Performance Comparison of Mixed-Integer Relaxations

In our global solution framework, three formulations – (NCUC-RQ), (NCUC-R), and (NCUC-RL) – can be used for the mixed-integer master problem relaxation, as they all provide valid lower bounds on the original NCUC problem. We specifically consider four problem formulations: (NCUC-RQ), (NCUC-R), and two variants of (NCUC-RL) with different numbers of segment points – which we denote (NCUC-RL)-5 and (NCUC-RL)-10. The performance of these relaxations is tested and compared in terms of solution quality and computational efficiency.

Our global solution framework is implemented in Pysomo, a Python-based optimization modeling language [12]. The mixed-integer master problem and the non-convex, nonlinear subproblem are solved to an optimality gap below 0.1%. The relative optimality gap of the global algorithm is set to be 0.5%. The total computational time limit is 10 hours and the outer iteration number limit is set to 30. All master problems are solved with Gurobi [25] and all continuous problems (NLP subproblems and SOCP-based relaxations) are solved with IPOPT [32]. Computational
results are shown in Table 3, which includes break-outs of several key components of our global solution framework proposed in Section 4.1. The third column reports the best upper bound (corresponding to the best feasible solution) and the fourth column reports the best lower bound. The relative optimality gap is shown in the fifth column, followed by the total computational time and the number of major iterations. If no feasible solution was obtained, the upper bound and the corresponding optimal gap are indicated by “−.”

First, we consider the simplest unit commitment problem with only 6 buses, where the same globally optimal solution (upper bound) is reported for all master relaxations. The most computationally attractive relaxation is formulation (NCUC-RQ). With tighter lower bounds, it only requires 3 major iterations to converge. The MISOCP relaxation (NCUC-R) with linear under-estimators of cost function is also computationally efficient. Finally, the performance of the MILP relaxation (NCUC-RL) strongly depends on the tightness of linear outer approximation of the second-order cone constraint. Thus, the MISOCP relaxations (NCUC-RQ) and (NCUC-R) are empirically more attractive when solving small-scale NCUC problems.

For the two 24-bus test problems RTS-79 and RTS-96, the MISOCP relaxations significantly outperform the MILP relaxations. Both MILP relaxations fail to report any feasible solution within 10 hours or 30 iterations. Both MISOCP relaxations, on the other hand, obtain feasible solutions within small optimality gaps. For problem RTS-79, the performance of relaxation (NCUC-RQ) is slightly better. Interestingly, for test problem RTS-96, the relatively weaker relaxation (NCUC-R) is more computationally efficient. After one iteration, which takes 83 seconds,
a feasible solution is returned within 0.1% of global optimality. Formulation (NCUC-RQ) can also lead to globally (near-) optimal solutions after one iteration; however, it is more challenging to solve.

Finally, we consider the 118 bus case. Both linear relaxations, again, lead to large-scale problems that cannot be solved within 10 hours. The MISOPC relaxation (NCUC-RQ) with the quadratic cost function also fails to report a feasible solution within the time limit. Among all possible formulations, only relaxation (NCUC-R) yields globally (near-) optimal solution in less than 2 hours of run time.

In summary, our analysis indicates that relaxation (NCUC-R) with second-order inequalities and linear under-estimators of quadratic cost functions is the most promising formulation in terms of both solution quality and computational efficiency. Relaxation (NCUC-RQ), on the other hand, is suitable for solving small-scale NCUC problems. The MILP approach (NCUC-RL) is empirically the least attractive.

5.2 Unit Commitment Results

The optimal schedules for our test UC problems are shown in Table 4, 5, and 6. Recent work of Castillo et al. [4] used a specialized sequential linear approach to solve the nonlinear NCUC problem. We compare our optimal solutions with their results on three UC problems: Case6, RTS-79, and Case118. For Case6, the same optimal solution is found by both approaches. For case RTS-79, however, our objective value is 895,149, which is slightly lower than their value of 895,776. For Case118, our global solution algorithm yields an optimal objective value of 835,776, compared with 843,591 using the sequential linear approximation approach. Note that the sequential linear approach proposed by Castillo et al. [4] is an approximation of the NCUC problem, and cannot guarantee global optimality or quality bounds. However, the sequential linear approach can be solved efficiently, and the obtained solutions are close to globally optimal solutions, especially for small problems.

We note that multiple globally optimal solutions have been reported in both 24-bus and 118-bus cases. Analysis has implied that the symmetric feature of these networks may be the root cause. For instance, in both 24-bus systems, there exist multiple identical generating units located at a single bus, which can not be automatically distinguished. As a result, different combinations of identical generators may lead to the same objective function value. For the 118-bus case, on the other hand, the symmetry may arise from other problem structure, such as identical branches. To partially remedy this problem, symmetric-breaking constraints can be imposed on our NCUC formulation and associated relaxations in order to break network symmetry.

6 Conclusions

Solution of the network-constrained unit commitment (NCUC) problem plays an important role in determining optimal and physically feasible schedules for thermal generating units to satisfy forecasted electricity demand in a power
system. Previously proposed NCUC problem formulations suffer from poor solution quality due to the fact that they fail to consider some critical physics present in AC transmission network models. To address this problem, we present an NCUC formulation coupled with a nonlinear AC transmission model. The resulting optimization problem is a mixed-integer quadratically constrained quadratic programming (MIQCQP) problem. A multi-tree global solution...
Table 6: Commitments for the 118-Bus System

<table>
<thead>
<tr>
<th>Generator</th>
<th>Commitment (h)</th>
<th>Generator</th>
<th>Commitment (h)</th>
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<tr>
<td>G1</td>
<td>Ø</td>
<td>G28</td>
<td>Ø</td>
</tr>
<tr>
<td>G2</td>
<td>8-24</td>
<td>G29</td>
<td>1-24</td>
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<tr>
<td>G3</td>
<td>Ø</td>
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<td>G4</td>
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<td>G31</td>
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<td>1-24</td>
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</table>

strategy is proposed to solve the resulting NCUC problems. In our approach, three convex relaxations – formulated as mixed-integer second-order cone programming (MISOCP) and mixed-integer linear programming (MILP) problems – are implemented for the lower-bounding master problem. The non-convex, nonlinear subproblem is solved to global optimality by a sub-algorithm based also on an SOCP relaxation. This sub-algorithm is also well-suited to solve general AC optimal power flow (OPF) problems. Numerical results on four benchmark problems have shown excellent performance of our tailored algorithm in terms of both solution quality and computational efficiency.

For purposes of simplicity, only the basic version of our global solution framework is described and implemented here. For instance, a relatively loose set of variable bounds is considered without using any bound tightening strategy (e.g., optimality-based bound tightening (OBBT)) in the initialization step. During each major iteration of our algorithm, the current lower bounding master problem is tightened only by adding a single integer cut based on the discrete solution from the last iteration, while no cycle constraints or arctangent envelopes are imposed. Other possible enhancements include adding multiple, stronger integer cuts simultaneously, combining or excluding redundant constraints based on special properties of the NCUC problem, and heuristically searching around the current feasible
solution. By taking all of these complexities into consideration, we expect to obtain even better solutions in significantly less computational time.

Finally, although the run times associated with our algorithm are still longer than that required for operations, we have demonstrated the utility of our approach to analyzing NCUC solutions. Specifically, our algorithm can be used to obtain ”off-line” provably (near-) globally optimal solutions, which can be used to test and validate other algorithmic approaches – including heuristics – that may be developed, e.g., that of [4].

Acknowledgments

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References


A Problem Formulation

A.1 NCUC Problem Formulation (NCUC)

\[
\begin{align*}
\text{min} & \quad f_p + f_{su} + f_{sd} \\
\text{s.t.} & \quad A_{g,t}(y_{g,t})^2 + A_{g,t}p_{g,t} + A_{g,t}0y_{g,t} \leq v_{g,t}^p \\
& \quad f^p = \sum_{g \in G} \sum_{t \in T} c^g_{g,t} \\
& \quad \delta_{g,t} \leq \sum_{t'=t-T_y^{g,t}+1} w_{g,t'} \\
& \quad u_{g,t} = \sum_{r \in S_g} d_{g,r,t} \\
& \quad f_{su} = \sum_{g \in G} \sum_{t \in T} \sum_{r \in S_g} K_{su}^{g,r} \delta_{g,r,t} \\
& \quad f_{sd} = \sum_{g \in G} \sum_{t \in T} \sum_{r \in S_g} K_{sd}^{g,r} w_{g,t} \\
& \quad \sum_{t'=t-T_y^{g,t}+1} u_{g,t'} \leq y_{g,t} \\
& \quad \sum_{t'=t-T_y^{g,t}+1} w_{g,t'} \leq 1 - y_{g,t} \\
& \quad y_{g,t} - y_{g,t-1} = u_{g,t} - w_{g,t} \\
& \quad y_{g,t} \in \{0, 1\} \\
& \quad \sum_{b \in B} P_{b,t}^{G} + P_{t}^{R} \leq \sum_{g \in G} p_{g,t}^a \\
& \quad P_{g,t}^{G_{min}} y_{g,t} \leq p_{g,t}^a \leq P_{g,t}^a \leq P_{g,t}^{G_{max}} y_{g,t} \\
& \quad Q_{g,t}^{G_{min}} y_{g,t} \leq q_{g,t}^a \leq Q_{g,t}^{G_{max}} y_{g,t} \\
& \quad Q_{sc,t}^{SC_{min}} \leq q_{sc,t}^{SC} \leq Q_{sc,t}^{SC_{max}} \\
& \quad \sum_{l \in L^o_b} P_{l,t}^d + \sum_{l \in L^o_b} P_{l,t}^{G} + P_{b,t}^D - \sum_{g \in \mathcal{G}} p_{g,t}^D = 0 \\
& \quad \sum_{l \in L^a_b} q_{l,t}^d + \sum_{l \in L^a_b} q_{l,t}^{G} - B_{b,t}^{G} v_{b,t}^2 + Q_{b,t}^D - \sum_{g \in \mathcal{G}} q_{g,t}^D - \sum_{s \in \mathcal{S}_b} q_{sc,t}^{SC} = 0 \\
& \quad p_{l,t}^d = G_{l,t}(v_{b,t}^2 + G_{l,t}(v_{b,t}^2 - v_{b,t}^2)) - B_{l,t}(v_{b,t}^2 - v_{b,t}^2) \\
& \quad q_{l,t}^d = -B_{l,t}(v_{b,t}^2 - v_{b,t}^2) \\
& \quad p_{l,t} = G_{l,t}(v_{b,t}^2 - G_{l,t}(v_{b,t}^2 - v_{b,t}^2)) - B_{l,t}(v_{b,t}^2 - v_{b,t}^2) \\
& \quad q_{l,t} = -B_{l,t}(v_{b,t}^2 - v_{b,t}^2) \\
& \quad (v_{b,t}^{min})^2 \leq v_{b,t}^2 \leq (v_{b,t}^{max})^2 \quad \forall b, t \\
& \quad (p_{l,t}^d)^2 + (q_{l,t}^d)^2 \leq (S_{l,t}^{max})^2 \quad \forall l, t \\
& \quad (p_{l,t}^d)^2 + (q_{l,t}^d)^2 \leq (S_{l,t}^{max})^2 \quad \forall l, t 
\end{align*}
\]
\[
\min \quad f^p + f^{su} + f^{sd} \\
\text{s.t.} \quad A_{y,t}(p_y^G)^2 + A_{y,t}p_{y,t} + A_{y,0}y_{g,t} \leq c_{y,t}^p \quad \forall g, t \\
f^p = \sum_{g \in G} \sum_{t \in T} p_{g,t}^{p} \\
\delta_{g,\tau,t} \leq \sum_{t' = t - T_y^g}^{t + 1} w_{g,t'} \quad \forall g, \tau, t \\
u_{g,t} = \sum_{\tau \in S_g} \delta_{g,\tau,t} \quad \forall g, t \\
f^{su} = \sum_{g \in G} \sum_{t \in T} \sum_{\tau \in S_g} K_u^{su} \delta_{g,\tau,t} \\
f^{sd} = \sum_{g \in G} \sum_{t \in T} K_g^{sd} u_{g,t} \\
\sum_{t' = t - T_y^g}^{t + 1} u_{g,t'} \leq y_{g,t} \quad \forall g, t \\
\sum_{t' = t - T_y^g}^{t + 1} w_{g,t'} \leq 1 - y_{g,t} \quad \forall g, t \\
y_{g,t} - y_{g,t-1} = u_{g,t} - w_{g,t} \quad \forall g, t \\
y_{g,t} \in \{0, 1\} \quad \forall g, t \\
\sum_{b \in B} p_{b,t}^D + P_t^P \leq \sum_{g \in G} p_{g,t}^a \quad \forall t \\
P_g^{G,\text{min}} y_{g,t} \leq p_{g,t}^G \leq p_{g,t}^a \leq P_g^{G,\text{max}} y_{g,t} \quad \forall g, t \\
Q_g^{G,\text{min}} y_{g,t} \leq q_{g,t}^G \leq Q_g^{G,\text{max}} y_{g,t} \quad \forall g, t \\
Q_{sc}^{\text{min}} \leq q_{sc,t}^{SC} \leq Q_{sc}^{\text{max}} \quad \forall sc, t \\
\sum_{l \in C_b^l} p_{l,t}^f + \sum_{l \in C_b^l, t} p_{l,t}^f + G_{b}^{ab} c_{b,b,t} + P_{b,t}^D - \sum_{g \in G_b} p_{g,t}^G = 0 \quad \forall b, t \\
\sum_{l \in C_b^l} q_{l,t}^f + \sum_{l \in C_b^l, t} q_{l,t}^f - P_b^{ab} c_{b,b,t} + Q_{b,t}^D - \sum_{g \in G_b} q_{g,t}^G - \sum_{sc \in SC_b} q_{sc,t}^{SC} = 0 \quad \forall b, t \\
p_{l,t}^f = G_{l,b} f c_{b,b,t} + G_{l,b} f c_{b,k,t} - B_{l,b} f s_{b,k,t} \quad \forall l, t \\
q_{l,t}^f = -B_{l,b} f c_{b,b,t} - B_{l,b} f c_{b,k,t} - G_{l,b} f s_{b,k,t} \quad \forall l, t \\
p_{l,t}^f = G_{l,k} f c_{k,k,t} + G_{l,b} f c_{b,k,t} + B_{l,b} f s_{b,k,t} \quad \forall l, t \\
q_{l,t}^f = -B_{l,k} f c_{k,k,t} - B_{l,b} f c_{b,k,t} + G_{l,b} f s_{b,k,t} \quad \forall l, t \\
(V_{b}^{min})^2 \leq c_{b,b,t} \leq (V_{b}^{max})^2 \quad \forall b, t \\
(p_{l,t}^f)^2 + (q_{l,t}^f)^2 \leq (S_{l,t}^{max})^2 \quad \forall l, t \\
(p_{l,t}^f)^2 + (q_{l,t}^f)^2 \leq (S_{l,t}^{max})^2 \quad \forall l, t \\
c_{b,k,t} = c_{k,b,t}, s_{b,k,t} = -s_{k,b,t} \quad \forall l, t \\
c_{b,k,t}^2 + s_{b,k,t}^2 \leq c_{b,k,t} c_{k,k,t} \quad \forall l = (b,k), t
A.3 NCUC Relaxation (NCUC-R)

\[
\begin{align*}
\min & \quad f^p + f^{su} + f^{sd} \\
\text{s.t.} & \quad B_{g,t,n}^p p_{g,t} + C_{g,t,n}^p + \Delta y_{g,t} \leq c^p_{g,t} & \forall g, t, n \\
& \quad f^p = \sum_{g \in G} \sum_{t \in T} c^p_{g,t} \\
& \quad \delta_{g,\tau,t} \leq \sum_{t'=\tau-T_{g,t}}^{t-1} w_{g,t'} & \forall g, t, \tau \in [1, S_g) \\
& \quad u_{g,t} = \sum_{\tau \in S_g} \delta_{g,\tau,t} & \forall g, t \\
& \quad f^{su} = \sum_{g \in G} \sum_{t \in T} \sum_{\tau \in S_g} K_{g,\tau,t}^s \delta_{g,\tau,t} \\
& \quad f^{sd} = \sum_{g \in G} \sum_{t \in T} K_{g,t}^d u_{g,t} \\
& \quad \sum_{t'=t-T_{g,t}+1}^{t} u_{g,t'} \leq y_{g,t} & \forall g, t \\
& \quad \sum_{t'=t-T_{g,t}+1}^{t} w_{g,t'} \leq 1 - y_{g,t} & \forall g, t \\
& \quad y_{g,t} - y_{g,t-1} = u_{g,t} - w_{g,t} & \forall g, t \\
& \quad y_{g,t} \in \{0, 1\} & \forall g, t \\
& \quad \sum_{b \in B} P_{b,t}^R + P_{b,t}^R \leq \sum_{g \in G} p_{g,t} & \forall t \\
& \quad P_{b,t}^G_{\min} \leq P_{b,t}^G \leq P_{b,t}^G_{\max} & \forall b, t \\
& \quad Q_{g,t}^y_{\min} \leq Q_{g,t}^y \leq Q_{g,t}^y_{\max} & \forall g, t \\
& \quad Q_{sc,t}^y_{\min} \leq Q_{sc,t}^y \leq Q_{sc,t}^y_{\max} & \forall sc, t \\
& \quad \sum_{l \in l_c} p_{l,t} + \sum_{l \in l_c} P_{b,t}^D + P_{b,t}^D + P_{b,t}^D + \sum_{g \in G} p_{g,t} = 0 & \forall b, t \\
& \quad \sum_{l \in l_c} q_{l,t} + \sum_{l \in l_c} P_{b,t}^D + P_{b,t}^D + P_{b,t}^D + \sum_{g \in G} q_{g,t} - \sum_{s \in s_c} q_{s,c,t} = 0 & \forall b, t \\
& \quad p_{l,t} = G_{l}^t c_{b,b,t} + G_{l}^t c_{b,b,t} - B_{l}^t s_{b,b,t} & \forall l, t \\
& \quad q_{l,t} = -B_{l}^t c_{b,b,t} + G_{l}^t c_{b,b,t} + G_{l}^t c_{b,b,t} - B_{l}^t s_{b,b,t} & \forall l, t \\
& \quad c_{b,b,t} = c_{b,b,t}, s_{b,b,t} = -s_{b,b,t} & \forall l, t \\
& \quad (V_{b}^y)_{\min} \leq c_{b,b,t} \leq (V_{b}^y)_{\max} & \forall b, t \\
& \quad c_{b,b,t} + s_{b,b,t} \leq c_{b,b,t} s_{b,b,t} & \forall l = (b,k), t 
\end{align*}
\]
A.4 NCUC Relaxation (NCUC-RL)

\[
\begin{align*}
\min & \quad f^p + f^{su} + f^{sd} \\
\text{s.t.} & \quad B_{g,t}^* p_{g,t} + C_{g,t}^* + A^0_y y_{g,t} \leq y_{g,t} && \forall \ g, \ t, \ n \\
& \quad f^p = \sum_{g \in G} \sum_{t \in T} p_{g,t} \\
& \quad \delta_{g,t} \leq \sum_{t' = t - T_g + 1}^{t + 1 - T_g + 1} w_{g,t'} && \forall \ g, \ t, \ \tau \in [1, S_g] \\
& \quad u_{g,t} = \sum_{\tau \in S_g} \delta_{g,t} && \forall \ g, \ t \\
& \quad f^{su} = \sum_{g \in G} \sum_{t \in T} \sum_{\tau \in S_g} K_{g,t}^{su} \delta_{g,t} && \\
& \quad f^{sd} = \sum_{g \in G} \sum_{t \in T} K_{g}^{sd} w_{g,t} \\
& \quad \sum_{t' = t - T_g + 1}^t u_{g,t'} \leq y_{g,t} && \forall \ g, \ t \\
& \quad \sum_{t' = t - T_g + 1}^t w_{g,t'} \leq 1 - y_{g,t} && \forall \ g, \ t \\
& \quad y_{g,t} - y_{g,t-1} = u_{g,t} - w_{g,t} && \forall \ g, \ t \\
& \quad y_{g,t} \in \{0, 1\} && \forall \ g, \ t \\
& \quad \sum_{b \in B} p_{b,t}^D + p_{b,t}^R \leq \sum_{g \in G} \sum_{t \in T} p_{g,t}^u \\
& \quad P_{g}^{G,min} y_{g,t} \leq p_{g,t}^u \leq p_{g,t}^u \leq p_{g,t}^{G,max} && \forall \ g, \ t \\
& \quad Q_{g}^{G,min} y_{g,t} \leq q_{g,t}^G \leq Q_{g}^{G,max} y_{g,t} && \forall \ g, \ t \\
& \quad Q_{sc}^{\min} \leq q_{sc}^{SC} \leq Q_{sc}^{\max} && \forall sc, \ t \\
& \quad \sum_{l \in \mathcal{L}_b^u} p_{l,t}^f + \sum_{l \in \mathcal{L}_b^u} q_{l,t}^f - B_{cb,b,t}^G c_{b} + B_{b,t}^D - \sum_{g \in G_b} p_{g,t}^G = 0 && \forall b, \ t \\
& \quad \sum_{l \in \mathcal{L}_b^u} q_{l,t}^f - B_{cb,b,t}^G c_{b} + B_{b,t}^D - \sum_{sc \in SC_b} q_{sc}^{SC} = 0 && \forall b, \ t \\
& \quad \sum_{l \in \mathcal{L}_b^u} p_{l,t}^f = G_{cb,b,t}^f + G_{cb,k,t}^f c_{b} - B_{cb,b,t}^f s_{b,k,t} && \forall l, \ t \\
& \quad \sum_{l \in \mathcal{L}_b^u} q_{l,t}^f = -B_{cb,b,t}^f c_{b} + G_{cb,k,t}^f c_{b} + B_{cb,b,t}^f s_{b,k,t} && \forall l, \ t \\
& \quad \sum_{l \in \mathcal{L}_b^u} p_{l,t}^f = G_{ck,k,t}^f c_{k} + G_{cb,k,t}^f c_{b} + B_{cb,k,t}^f s_{b,k,t} && \forall l, \ t \\
& \quad \sum_{l \in \mathcal{L}_b^u} q_{l,t}^f = -B_{cb,b,t}^f c_{b} + G_{cb,k,t}^f c_{b} + G_{cb,k,t}^f s_{b,k,t} && \forall l, \ t \\
& \quad (V_b^{min})^2 \leq c_{b,t} \leq (V_b^{max})^2 && \forall b, \ t \\
& \quad c_{b,t} = c_{b,t} - s_{b,k,t} && \forall l, \ m, \ n, \ p
\end{align*}
\]