Distributionally robust chance constrained optimal power flow with renewables: A conic reformulation

Weijun Xie* and Shabbir Ahmed†

School of Industrial & Systems Engineering
Georgia Institute of Technology, Atlanta, GA 30332

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Abstract

The uncertainty associated with renewable energy sources introduces significant challenges in optimal power flow (OPF) analysis. A variety of new approaches have been proposed that use chance constraints to limit line or bus overload risk in OPF models. Most existing formulations assume that the probability distributions associated with the uncertainty are known a priori or can be estimated accurately from empirical data, and/or use separate chance constraints for upper and lower line/bus limits. In this paper we propose a data driven distributionally robust chance constrained optimal power flow model (DRCC-OPF), which ensures that the worst-case probability of violating both the upper and lower limit of a line/bus capacity under a wide family of distributions is small. Assuming that we can estimate the first and second moments of the underlying distributions based on empirical data, we propose an exact reformulation of DRCC-OPF as a tractable convex program. The key theoretical result behind this reformulation is a second order cone programming (SOCP) reformulation of a general two-sided distributionally robust chance constrained set by lifting the set to a higher dimensional space. Our numerical study shows that the proposed SOCP formulation can be solved efficiently and that the results of our model are quite robust.

Keywords: Optimal power flow; distributionally robust; chance constraints; conic program; renewable integration.

Nomenclature

Sets

\( \mathcal{V} = \text{set of all buses} \)
\( \mathcal{E} = \text{set of all transmission lines linking two buses} \)
\( \mathcal{G} = \text{subset of buses that house generators} \)
\( \mathcal{W} = \text{subset of buses that holds uncertain power sources (wind farms)} \)

*wxie33@gatech.edu
†sahmed@isye.gatech.edu
Parameters

\( \mu_j \) = average generation at bus \( j \in W \) (\( \mu_j = 0 \), for all \( j \in V \setminus W \))

\( \omega_j \) = random fluctuation of power generation at uncertain power source \( j \in W \)

\( \Sigma \) = known covariance matrix of random vector \( \omega \)

\( d_i \) = demand at bus \( i \in V \)

\( \beta_{ij} \) = the line sustenance between \( (i, j) \in E \) (\( \beta_{ij} = \beta_{ji} \) by symmetry)

\( B \) = weighted Laplacian matrix defined as

\[
B(i, j) = \begin{cases} 
-\beta_{ij}, & \text{if } (i, j) \in E \\
\sum_{k:(k, j) \in E} \beta_{kj}, & \text{if } i = j \\
0, & \text{otherwise} 
\end{cases}
\]

for each \( (i, j) \)

\( \bar{B} \) = submatrix of \( B \) by removing the last row and column

\( \hat{B} \) = pseudo-inverse of \( B \) defined as

\[
\hat{B} = \begin{bmatrix} 
\bar{B}^{-1} & 0 \\
0 & 0 
\end{bmatrix}
\]

\( \bar{B}^W = |V| \times |W| \) submatrix of \( \bar{B} \) where its columns are from \( W \)

\( \bar{B}^W_{i\cdot} \) = \( i \)th row of \( \bar{B}^W \)

\( e \) = all one vector

\( r_i, c_i \) = quadratic cost coefficients of generator \( i \in G \)

\( \tilde{c}_{ij} \) = risk parameter of violating the capacity of transmission line \( (i, j) \in E \)

\( \tilde{e}_i \) = risk parameter of violating the capacity of bus \( i \in G \)

\( f_{ij}^{\max} \) = max capacity of transmission line \( (i, j) \in E \)

\( p_i^{\min} \) = generation lower bound of bus \( i \in G \)

\( p_i^{\max} \) = generation upper bound of bus \( i \in G \)

Decision Variables

\( \theta_j \) = be the phase of bus \( j \in V \)

\( \bar{p}_i \) = regular generation at generator \( i \in G \)

\( \alpha_i \) = \( i \)th assignment of total renewables to generator \( i \in G \)

1 Introduction

Recently with growing interests in environmentally friendly power generation such as wind, solar, geothermal energy \([18, 30, 32]\), optimal power flow (OPF) under uncertainty has attracted much attention from researchers \([1, 4, 31, 34, 36, 37, 38]\). A particular issue caused by renewables is the voltage fluctuations which can lead to severe issues, for example, overloaded transmission lines \([7, 22]\). To mitigate these issues, \([1, 4]\) proposed a chance constrained optimal power flow model (CC-OPF), which controls the overloading within a small probability for each individual line and each bus. This paper will extend this work by enforcing power flow within lower and upper bounds simultaneously and show tractability results of such an approach under data driven distributionally robust setting.
There are many works on solving OPF or unit commitment (UC) problem via stochastic programming [25][33][28], robust optimization [13][14][3], and chance constrained program approaches [4][24][34][36] (see [20] for some discussions). Stochastic programming approaches highly rely on the underlying distribution, which could be unknown in many cases, and the performance of solution algorithms is usually very sensitive to the distribution used [29]. Robust optimization is often too conservative [2], while chance constrained programming is less conservative but is often NP-hard to solve [21][23]. Thus, to overcome the difficulties from these two approaches, here we adopt a distributionally robust chance constrained approach, which allows violation of uncertain constraints with a small probability for a large class of probability distributions and could be reformulated as a tractable convex program [1][6][35][40].

There are two concerns about existing literatures on CC-OPF formulations. It is known that each transmission line as well as each bus (node) in general has lower and upper bound limits. However, most works [1][4][20] treats the lower- and upper- bound overloading separately, which is an inexact approximation. The reason of such an approximation scheme is simply due to tractability [19]. As far as we are concerned, [19] is the only known work which treated lower and upper bounds simultaneously. However the results in [19] highly depend on the assumption of a Gaussian distribution on the underlying uncertainties. In this paper, we will consider incorporating power flow within lower and upper bounds simultaneously and our results are distribution-free.

Another issue regarding previous CC-OPF studies is that they assumed a particular distribution of renewables’ output. For example, [4][20][27] assumed that the prior distributions of renewables are Gaussian while [17] assumed that it is Weibull. However, these assumptions might not be true in practice [1]. In general the underlying probability distributions of renewables are not known or are hard to estimate from empirical data. Thus, to hedge against the unnecessary assumption on probability distributions, in this paper, we consider a data driven distributionally robust chance constrained optimal power flow model (DRCC-OPF) by considering the overload within the upper and lower bounds jointly with high probability. And the underlying probability distribution comes from a family of distributions (called “ambiguity set”) that share the same mean and covariance matrix estimated from empirical data.

Distributionally robust chance constrained problems with multiple uncertain constraints (joint-DRCCP) are very challenging [12]. There are only few setting under which joint DRCCP can be equivalently reformulated into a convex program. For example, in [11], they assumed right-hand uncertainty with mean dispersion moment ambiguity set, and proposed convex reformulations. Recently, [35] explored several sufficient conditions under which joint DRCCP is convex. However none of known sufficient conditions for convexity can be directly applied to the two-sided DRCC-OPF here. This paper fills gap in [35] by showing that joint DRCCP with two-sided constraint has a convex reformulation.

The remaining of the paper is organized as follows. Section 2 introduces the model formulation and Section 3 shows how to reformulate the model into a convex second order cone program. Section 4 numerically illustrates the strengths of the proposed model. Section 5 concludes the paper.
2 Model Formulation

In this section, we consider an extension of distributionally robust chance constrained optimal flow model (DRCC-OPF) proposed in [4].

In the optimal power flow problem, we suppose that there is a subset \( W \) of the buses holds uncertain power sources (e.g., wind farms), and for each \( j \in W \), the amount of power generated by source \( j \) is \( \mu_j + \omega_j \), where \( \omega_j \) is a random variable. Suppose the random vector \( \omega \) is zero mean and has covariance matrix \( \Sigma \). And for house generator \( i \in G \), we let \( \mu_i = 0 \). We also assume that each bus \( i \in V \) has demand \( d_i \). We also assume the output of bus \( i \in G \) is fluctuated by wind generators. Let \( \alpha_i \) for each \( i \in G \) be the proportional wind power affecting bus \( i \), i.e., the output of bus \( i \in G \) can be denoted as \( \bar{p}_i - (e^\top \omega)\alpha_i \) with nonnegative variables \( \bar{p}_i, \alpha_i \) and \( \sum_{i \in G} \alpha_i = 1 \).

Let \( \theta \) be the phases of all the buses. To approximate nonconvex AC power flow equations, we use DC-approximation. Thus, the power flow between line \((i,j)\) is approximated as \( \beta_{ij}(\bar{\theta}_i - \bar{\theta}_j) \) where \( \beta_{ij} = \beta_{ji} \) denotes the line sustenance.

Following [4], a distributionally robust chance constrained optimal power flow problem (DRCC-OPF) is formulated as

\[
v^* = \min_{\bar{p}, \alpha, \bar{\theta}} \sup_{P \in \mathcal{P}} \mathbb{E}_P \left[ \sum_{i \in G} c_i (\bar{p}_i - \alpha_i (e^\top \omega) + r_i)^2 \right]
\]

s.t. \( \sum_{i \in G} \alpha_i = 1 \),

\( \sum_{i \in V} (\bar{p}_i + \mu_i - d_i) = 0 \),

\( B\bar{\theta} = \bar{p} + \mu - d \),

\[
\inf_{P \in \mathcal{P}} \mathbb{P} \{ \omega : |\beta_{ij}(\bar{\theta}_i - \bar{\theta}_j + [\bar{B}^W(\omega - (e^\top \omega)\alpha)]_i - [\bar{B}^W(\omega - (e^\top \omega)\alpha)]_j)| \leq f_{ij}^{\max} \} \geq 1 - \tilde{\epsilon}_{ij},
\]

\( \forall (i,j) \in \mathcal{E} \),

\[
\inf_{P \in \mathcal{P}} \mathbb{P} \{ \omega : p_{i}^{\min} \leq \bar{p}_i - (e^\top \omega)\alpha_i \leq p_{i}^{\max} \}
\]

\( \geq 1 - \tilde{\epsilon}_i, \forall i \in G \),

\( \bar{p} \geq 0, \alpha \geq 0 \).

where (1a) is to optimize cost function where \( c > 0 \) and \( r \in \mathbb{R}^{\vert G \vert} \) are constant, (1b) implies that the total assignment of power from wind is 1, (1c) means on average, the total generation equals to the total demand, (1d) is the DC-approximation equation with

\[
B(i,j) = \begin{cases} 
-\beta_{ij}, & \text{if } (i,j) \in \mathcal{E} \\
\sum_{k:(k,j) \in \mathcal{E}} \beta_{kj}, & \text{if } (i,j) \in \mathcal{E} \\
0, & \text{otherwise}
\end{cases}
\]

for each \((i,j)\), (1e) enforce that the worst case probability that the absolute flow on \((i,j)\) does not exceed the maximum capacity \( f_{ij}^{\max} \) should be no smaller than \( \tilde{\epsilon}_{ij} \) with pseudo-inverse of \( B \)

\[
\bar{B} = \begin{bmatrix} \bar{B}^{-1} & 0 \\ 0 & 0 \end{bmatrix}
\]

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and its submatrix \( \hat{B} \) the submatrix of \( B \) by removing the last row and column; and (1f) ensures that with probability at least \( 1 - \bar{\epsilon}_i \), the generated power at \( i \) satisfied the lower bound \( p^\text{min}_i \) and the upper bound \( p^\text{max}_i \), (1g) defines the boundary of variables. Here we assume the all the risk parameters are within \((0, 1)\).

Similar to \([8, 15, 40]\), let us consider the ambiguity set defined by first and second moments as

\[
\mathcal{P} = \{ \mathbb{P} \in \mathcal{P}_0(\mathbb{R}^{|\mathcal{W}|}) : \mathbb{E}\mathbb{P}[\omega] = 0, \mathbb{E}\mathbb{P}[\omega\omega^\top] = \Sigma \}
\]

where \( \mathcal{P}_0(\mathbb{R}^{|\mathcal{W}|}) \) denotes the set of all of probability measures on \( \mathbb{R}^{|\mathcal{W}|} \) with a sigma algebra \( \mathcal{F} \), and \( \Sigma \in \mathbb{R}^{|\mathcal{W}| \times |\mathcal{W}|} \) is a positive definite matrix (i.e., \( \Sigma \succ 0 \)).

As we know the mean and covariance of \( \omega \), the cost function (1a) is equivalent to

\[
\sup_{\mathbb{P} \in \mathcal{P}} \mathbb{E}\mathbb{P}\left[ \sum_{i \in \mathcal{G}} c_i (\bar{p}_i - \alpha_i e^\top \omega + r_i) \right] = \sum_{i \in \mathcal{G}} \left( c_i (\bar{p}_i + r_i)^2 + c_i \alpha_i^2 e^\top \Sigma e \right).
\]

Thus, apart from the chance constraints (1e) and (1f), the DRCC-OPF formulation (1) is a convex quadratic optimization problem.

### 3 Convex Reformulation of Chance Constraints (1e) and (1f)

In this section, we will develop a deterministic convex formulation of (1) by reformulating the chance constraints (1e) and (1f) into equivalent convex constraints.

To reformulate the chance constraints (1e) and (1f), let us first consider a generic distributionally robust chance constrained set defined as follows:

\[
Z := \left\{ x : \inf_{\mathbb{P} \in \mathcal{P}} \mathbb{P}[|a(x)^\top \omega + b(x)| \leq T] \geq 1 - \epsilon \right\}
\]

where \( a(x), b(x) \) are affine mappings. This set is defined by a distributionally robust two-sided chance constraint.

Note that (1e) and (1f) are special cases of (4), where in (1e), we let \( a(x) = \beta_{ij} (\hat{B}^{\mathcal{W}}_i - \alpha_i e - \hat{B}^{\mathcal{W}}_j + \alpha_j e), b(x) = \beta_{ij} (\bar{\theta}_i - \bar{\theta}_j), T = f^\text{max}_{ij} \) and in (1f), we let \( a(x) = -\alpha_i e, b(x) = \frac{p^\text{max}_i + p^\text{min}_i}{2} - p^\text{min}_i, T = \frac{p^\text{max}_i - p^\text{min}_i}{2} \).

#### 3.1 Approximation by Two Single-sided Chance Constraints

Recently, including \([1, 4, 36, 37]\), many studies tried to approximate the two-side chance constrained set (4) by two single-sided chance constraints. In particular, let

\[
Z_A(\alpha) = \left\{ x : \begin{array}{l}
\inf_{\mathbb{P} \in \mathcal{P}} \mathbb{P}[a(x)^\top \omega + b(x) \leq T] \geq 1 - \alpha, \\
\inf_{\mathbb{P} \in \mathcal{P}} \mathbb{P}[a(x)^\top \omega + b(x) \geq -T] \geq 1 - \alpha.
\end{array} \right\}
\]

By choosing \( \alpha \sim \epsilon \) existing works use \( Z_A(\alpha) \) to approximate \( Z \). It turns out that sets \( Z_A(\epsilon) \) and \( Z_A(\epsilon/2) \) are outer and inner approximations of set \( Z \) and can be formulated as second order cone programs (SOCP).
Theorem 1. Suppose the ambiguity set \( P \) is defined as in (2), then set \( Z_A \) is equivalent to the following SOCP

\[
Z_A(\alpha) = \begin{cases} 
  x : & b(x) + \sqrt{\frac{1 - \alpha}{\alpha}} a(x)^\top \Sigma a(x) \leq T, \\
  -b(x) + \sqrt{\frac{1 - \alpha}{\alpha}} a(x)^\top \Sigma a(x) \leq T.
\end{cases}
\]

and \( Z_A(\varepsilon/2) \subseteq Z \subseteq Z_A(\varepsilon). \)

Proof. For \( x \in Z \), clearly, we have

\[
\inf_{P \in P} \mathbb{P}[|a(x)^\top \omega + b(x)| \leq T] \geq \inf_{P \in P} \mathbb{P}[a(x)^\top \omega + b(x) \leq T]
\]

and

\[
\inf_{P \in P} \mathbb{P}[|a(x)^\top \omega + b(x)| \leq T] \geq \inf_{P \in P} \mathbb{P}[a(x)^\top \omega + b(x) \geq -T].
\]

Clearly, \( Z \subseteq Z_A(\varepsilon). \)

The result that \( Z_A(\varepsilon/2) \subseteq Z \) follows by Bonferroni approximation of joint chance constrained set (c.f. [23]). The equivalent reformulation of \( Z_A(\alpha) \) follows by Theorem 3.1 in [6]. \( \square \)

As discussed in the sequel the approximations offered by \( Z_A(\alpha) \) could very crude, especially when the risk parameter \( \varepsilon \) is modest. In the next subsection, we will explore an exact convex reformulation of the set \( Z \).

3.2 Exact Reformulation

Our main result is the following theorem which provides a convex reformulation of the two-sided chance constrained set (4) as an SOCP.

Theorem 2. Suppose the ambiguity set \( P \) is defined in (2), then set \( Z \) is equivalent to the following convex SOCP (involving two additional variables):

\[
Z = \begin{cases} 
  y^2 + a(x)^\top \Sigma a(x) \leq \varepsilon (T - \pi)^2, \\
  x : |b(x)| \leq y + \pi, \\
  T \geq \pi \geq 0, y \geq 0.
\end{cases}
\]

Proof. First of all, we observe that \( |b(x)| \leq T \) for each \( x \in Z \), this is because by choosing \( \omega_1 = 0 \) with probability \( 1 - \varepsilon \), \( \omega_{i+1} = \sqrt{\frac{2|\mathcal{W}|}{\varepsilon}} \tilde{\lambda}_i 0.5 u_i, \omega_{i+1+|\mathcal{W}|+1} = -\sqrt{\frac{2|\mathcal{W}|}{\varepsilon}} \tilde{\lambda}_i 0.5 u_i \) with probability \( \frac{\varepsilon}{2|\mathcal{W}|} \) for each \( i \in \mathcal{W} \), where \( \tilde{\lambda}_i \) and \( u_i \) are \( i \)th eigenvalue and eigenvector of \( \Sigma \). Then, under this particular construction, we have \( |b(x)| \leq T \).

We separate our proof into two steps.

(i) First, we can show that set \( Z \) is equivalent to \( Z_1 \cup Z_2 \cup Z_3 \), where

\[
Z_1 = \left\{ x : \sqrt{\frac{1}{\varepsilon}} \sqrt{a(x)^\top \Sigma a(x) + b(x)^2} \leq T \right\}
\]

\[
|b(x)| \leq \varepsilon T,
\]
\[ Z_2 = \left\{ x : b(x) + \sqrt{\frac{1 - \epsilon}{\epsilon}} \sqrt{a(x)\Sigma a(x)} \leq T, \right\} \]

and

\[ Z_3 = \left\{ x : -b(x) + \sqrt{\frac{1 - \epsilon}{\epsilon}} \sqrt{a(x)\Sigma a(x)} \leq T, \right\} \]

This can be proven by three steps.

(1) Suppose \( x \in Z \), then by the standard random variable changing (c.f. [9]) and Theorem 1 in [26] (see also in [16, 35]), set \( Z \) is equivalent to

\[ Z := \left\{ x : \inf_{P \in \mathcal{P}_1} \mathbb{P}[|\xi| \leq T] \geq 1 - \epsilon \right\} \]

where

\[ \mathcal{P}_1 = \left\{ P \in \mathcal{P}_0(\mathbb{R}) : \mathbb{E}_P[\xi] = b(x), \mathbb{E}_P[\xi^2] = a(x)\Sigma a(x) + b(x)^2, \right\} \]

Theorem 5.99 in [5] implies that for any \( x \in Z \), \( \inf_{P \in \mathcal{P}_1} \mathbb{P}[|\xi| \leq T] \) is equivalent to

\[
\begin{align*}
\max_{\lambda, \gamma, \beta} & \quad \lambda + b(x) \gamma + (a(x)\Sigma a(x) + b(x)^2)\beta \\
\text{s.t.} & \quad \lambda + \xi \gamma + \xi^2 \beta \leq 1, \forall \xi \in \mathbb{R} \\
& \quad \lambda + \xi \gamma + \xi^2 \beta \leq 0, \forall \xi : \xi > T \\
& \quad \lambda + \xi \gamma + \xi^2 \beta \leq 0, \forall \xi : \xi < -T.
\end{align*}
\]

Note that in (11), we must have \( \beta \leq 0 \). And we also have \( \frac{\gamma^2}{2\beta} \leq T \), otherwise, sup \( \xi \lambda + \xi \gamma + \xi^2 \beta \leq 0 \). This implies that for any probability measure \( P \in \mathcal{P}_1 \), we have

\[ \lambda + b(x)\gamma + (a(x)\Sigma a(x) + b(x)^2)\beta = \lambda + \mathbb{E}_P[\gamma \xi + \xi^2 \beta] \leq 0 \]

contradiction that \( x \in Z \).

Since \( \beta < 0 \) and \( \frac{\gamma^2}{2\beta} \leq T \), (11) is equal to

\[
\begin{align*}
\max_{\lambda, \gamma, \beta} & \quad \lambda + b(x) \gamma + (a(x)\Sigma a(x) + b(x)^2)\beta \\
\text{s.t.} & \quad \lambda - \frac{\gamma^2}{4\beta} \leq 1, \\
& \quad \lambda + T|\gamma| + T^2 \beta \leq 0, \\
& \quad \beta < 0.
\end{align*}
\]

Note that in (12), the optimal \( \gamma \) must have the same sign as \( b(x) \) so as to maximize the objective function. Thus, set \( Z \) is equivalent to
(2) Now in (13), let $\pi := -\frac{1}{\beta}, \gamma := -|\gamma|/\beta, \lambda := -\lambda/\beta$ and define a new set below

\[
\bar{Z} = \left\{ x : \lambda - \frac{\gamma^2}{4\beta} \leq 1, \lambda + T|\gamma| + T^2\beta \leq 0, \beta < 0 \right\}
\]

(13)

Now we claim that

**Claim 1:** $\bar{Z} = Z$.

**Proof.** $\bar{Z} \subseteq Z$ Given $x \in \bar{Z}$, there exists $(\lambda, \gamma, \pi)$ such that $(\lambda, \gamma, \pi, x)$ satisfy (14). First of all, by letting (14b) minus (14a), we have

\[
\left( \frac{\gamma}{2} - |b(x)| \right)^2 + a(x)^\top \Sigma a(x) \leq \epsilon \pi
\]

thus, $\pi \geq 0$.

There are two cases:

Case 1. if $\pi = 0$, then by (14b) and (14a), we have $\lambda = \gamma = 0$ and $a(x)^\top \Sigma a(x) = 0, b(x)^2 = 0$. Since $\Sigma > 0$, we must have $a(x) = 0, b(x) = 0$. Hence, $z \in Z$.

Case 2. if $\pi > 0$, now define $(\lambda, \gamma) = (\lambda, \gamma)/\pi$ and $\beta = -1/\pi < 0$. Clearly, $(\lambda, \gamma, \beta, x)$ satisfy (13). Thus, $x \in Z$.

$\bar{Z} \supseteq Z$ Given $x \in \bar{Z}$, there exists $(\lambda, \gamma, \beta)$ such that $(\lambda, \gamma, \beta, x)$ satisfy (13). Now let $(\lambda, \gamma) = -\lambda/\beta \geq \gamma/\beta$ and $\pi = -1/\beta$, then $(\lambda, \gamma, \beta, x)$ satisfy (14). Hence, $x \in \bar{Z}$.

(3) Next, eliminating variables $\lambda, \pi$ by Fourier-Motzkin procedure, (14) yields

\[
Z = \left\{ x : (|b(x)| - \gamma/2)^2 + a(x)^\top \Sigma a(x) \leq \epsilon(T - \gamma/2)^2, \gamma \geq 0 \right\}
\]

(15a)

Optimizing $\gamma$ in (15) by discussing that $\epsilon T \geq |b(x)| (\gamma^* = 0), b(x) \leq -\epsilon T (\gamma^* = \frac{2}{1-\epsilon}(|b(x)| - \epsilon T))$ and $b(x) \geq \epsilon T (\gamma^* = \frac{2}{1-\epsilon}(|b(x)| - \epsilon T))$, set $Z$ can be reformulated as a disjunction of three sets $Z_1, Z_2$ and $Z_3$.

This complete the first part of the proof.
(ii) In (15), let \( \pi := \frac{1}{2} \). By the discussion above, we observe that the best \( \lambda \) must be no larger than \( 2|b(x)| \); otherwise, we will arrive at a smaller set. Thus, (15) is equivalent to

\[
Z = \left\{ x : (|b(x)| - \pi)^2 + a(x)^\top \Sigma a(x) \leq \epsilon (T - \pi)^2, \right. \\
\left. 0 \leq \pi \leq |b(x)| \right\} 
\]

(16a)

(16b)

Now let set \( \tilde{Z} \) denote the right-hand side in (7). We would like to prove the following claim:

**Claim 2:** \( \tilde{Z} = Z \).

*Proof.* \( \tilde{Z} \supseteq Z \) Given \( x \in Z \), there exists a \( \pi \) such that \((\pi, x)\) satisfy (16). Now let \( y = |b(x)| - \pi \), then \((y, \pi, x)\) satisfy (7). Hence, \( x \in \tilde{Z} \).

\( \tilde{Z} \subseteq Z \) Given \( x \in \tilde{Z} \), there exists \((y, \pi)\) such that \((y, \pi, x)\) satisfy (7). There are two cases:

Case 1. if \(|b(x)| \leq \pi \leq T\), then by (7), we have

\[
a(x)^\top \Sigma a(x) \leq y^2 + a(x)^\top \Sigma a(x) \leq (T - \pi)^2 \leq \epsilon (T - |b(x)|)^2
\]

where the first inequality is due to \( y \geq 0 \) and the second inequality is because of \(|b(x)| \leq \pi \leq T\).

Now we discuss that whether \(|b(x)| \leq \epsilon T\) or not.

a) if \(|b(x)| \leq \epsilon T\), then by (7), we have

\[
a(x)^\top \Sigma a(x) \leq (T - \pi)^2 \leq \epsilon (T - |b(x)|)^2
\]

where the first inequality is due to \( \sqrt{1 - \frac{\epsilon}{\epsilon}} \leq \sqrt{1 - \frac{1}{\epsilon}} \)

b) if \(|b(x)| \geq \epsilon T\), then (17) implies that

\[
|b(x)| + \sqrt{\frac{1}{\epsilon} |a(x)^\top \Sigma a(x)|} \leq T.
\]

Since \( \sqrt{1 - \frac{1}{\epsilon}} \leq \sqrt{\frac{1}{\epsilon}} \), we must have \( x \in Z_2 \cup Z_3 \subseteq Z \).

Case 2. if \( 0 \leq \pi \leq |b(x)| \), then \( y \geq |b(x)| - \pi \geq 0 \). Hence, (7) implies that

\[
(|b(x)| - \pi)^2 + a(x)^\top \Sigma a(x) \\
\leq y^2 + a(x)^\top \Sigma a(x) \\
\leq (T - \pi)^2
\]

where the first inequality is due to \( y \geq |b(x)| - \pi \geq 0 \) and thus \((\pi, x)\) satisfies (16), i.e., \( x \in Z \).
This completes the proof.

Using the above result we can now provide an exact SOCP formulation of DRCC-OPF [1] as follows:

\[
v^* = \min_{\bar{p}, \alpha, \theta} \sum_{i \in G} \left( c_i (\bar{p}_i + r_i)^2 + c_i \alpha_i^2 e^\top \Sigma e \right)
\]

s.t. (1b) - (1d), (1g)

\[
\hat{y}_{ij}^2 + \beta_{ij}^2 \left( \hat{B}_{ij}^W - \alpha_i e - \hat{B}_{ij}^W + \alpha_j e \right)^\top \left( \hat{B}_{ij}^W - \alpha_i e - \hat{B}_{ij}^W + \alpha_j e \right) \leq \hat{e}_{ij} (\hat{f}_{ij}^{\max} - \hat{\pi}_{ij})^2,
\]

\[
\forall (i, j) \in \mathcal{E}, \beta_{ij} (\hat{\theta}_i - \hat{\theta}_j) \leq \hat{y}_{ij} + \hat{\pi}_{ij}, \forall (i, j) \in \mathcal{E},
\]

\[
\beta_{ij} (\hat{\theta}_j - \hat{\theta}_i) \leq \hat{y}_{ij} + \hat{\pi}_{ij}, \forall (i, j) \in \mathcal{E},
\]

\[
\hat{y}_{ij} \geq 0, 0 \leq \hat{\pi}_{ij} \leq \hat{f}_{ij}^{\max}, \forall (i, j) \in \mathcal{E},
\]

\[
\hat{y}_{ij}^2 + \alpha_i^2 e^\top \Sigma e \leq \hat{e}_i \left( \frac{p_{i}^{\max} - p_{i}^{\min}}{2} - \hat{\pi}_i \right)^2, \forall i \in \mathcal{G},
\]

\[
\bar{p}_i - \frac{p_{i}^{\max} + p_{i}^{\min}}{2} \leq \bar{y}_i + \hat{\pi}_i, \forall i \in \mathcal{G},
\]

\[
- \bar{p}_i + \frac{p_{i}^{\max} + p_{i}^{\min}}{2} \leq \bar{y}_i + \hat{\pi}_i, \forall i \in \mathcal{G},
\]

\[
0 \leq \bar{y}_i, 0 \leq \hat{\pi}_i \leq \frac{p_{i}^{\max} - p_{i}^{\min}}{2}, \forall i \in \mathcal{G},
\]

where \( \hat{\pi}, \bar{y}, \hat{\pi}, \bar{y} \) are auxiliary nonnegative variables.

### 3.3 Quality of approximation of \( Z \) by \( Z_A(\epsilon/2), Z_A(\epsilon) \)

We know that from Theorem [1] we have \( Z_A(\epsilon/2) \subseteq Z \subseteq Z_A(\epsilon) \) and usually the inclusion is strict. The following example shows that the distances between set \( Z \) and \( Z_A(\epsilon) \) and between set \( Z \) and \( Z_A(\epsilon/2) \) can be large.

**Example 1.** Let \( b(x) = 0, \Sigma = I \), then

\[
Z_A(\epsilon/2) = \left\{ x : \|a(x)\|_2^2 \leq \frac{\epsilon}{2 - \epsilon} T^2 \right\}
\]

\[
Z_A(\epsilon) = \left\{ x : \|a(x)\|_2^2 \leq \frac{\epsilon}{1 - \epsilon} T^2 \right\}
\]

\[
Z = \left\{ x : \|a(x)\|_2^2 \leq \epsilon T^2 \right\}
\]

Clearly, when \( \epsilon \to 1, Z_A(\epsilon) \to \mathbb{R}^n \) but \( Z \) is close to a ball \( \{ x : \|a(x)\|_2^2 \leq T^2 \} \). Hence the distance between \( Z \) and \( Z_A(\epsilon) \) tends to infinity.

On the other hand, we know that \( \frac{\epsilon}{2 - \epsilon} \approx \frac{\epsilon}{2} \) when \( \epsilon \) is small. Thus, in this case, the radius of ball \( Z \) could be almost \( \sqrt{2} \) larger than \( Z_A(\epsilon/2) \). This inner approximation could easily lead the feasible region of a DRCC-OPF to be infeasible. For example, if there is an additional constraint \( S = \{ x : a(x) \geq T \sqrt{\frac{2 \epsilon}{3m}} \} \) where \( m \) is the dimension of \( a(x) \), then clearly \( S \cap Z_A(\epsilon/2) = \emptyset \) when \( \epsilon < 0.5 \), but \( S \cap Z \) even has a nonempty interior.
4 Numerical Illustration

We test the DRCC-OPF model (18) with an example used in [4]: case39 of MATPOWER data originally from [39]. The case is available at [http://www.pserc.cornell.edu/matpower/](http://www.pserc.cornell.edu/matpower/). In this data set, there are 39 buses (set \(V\)), 46 lines (set \(E\)) and 10 generators (set \(G\)). We assume that renewable power can be generated from buses 1 to 4 (set \(W\)) with mean \(\mu_i = 40\) (MW) for each \(i \in W\) and its covariance matrix \(\Sigma\) is diagonal with \(\Sigma(i,i) = 400\) for each \(i \in W\). All of the instances are solved by CVX [10].

In our first test, we let \(\hat{\epsilon}_{ij} = \bar{\epsilon}_i = 0.2\). We compare our method with a “risk free” model by assuming there is no uncertainty in (1e) and (1f), i.e., reformulating these constraints as

\[
|\beta_{ij}(\bar{\theta}_i - \bar{\theta}_j)| \leq f_{ij}^{\max}, \forall (i,j) \in E, \tag{1e'}
\]
\[
p_i^{\min} \leq \bar{p}_i - \bar{p}_i^{\max}, \forall i \in G; \tag{1f'}
\]

and to the model in [4] (we call it “BCH model”) where they assume the underlying distribution is Gaussian. All three models can be solved within a second, with total costs 36059.1, 36448.6, 37885.3 for the risk free model, BCH model, and our model (18), respectively. Thus, there is no significant difference (within 5\%) of total costs among all the three models.

We also test the reliability of models by simulating different distributions of renewables’ output, i.e., Gaussian, student, Laplace, Logistic and uniform distributions. We generate 100,000 samples from each distribution and check the violation of line flow capacity and bus capacity for each transmission line and bus. In Table 1, we compute the maximum probability of violations across all the lines and buses under each distribution. It can be seen that even when the risk parameters are all equal to 0.2, our model is quite robust and the chance that a line or bus capacity will be violated is close to zero for most of distributions. However, in the risk free model, there is a 50\% chance that a line or bus capacity is violated almost for each distribution. The BCH model does slightly better, but still under some distributions (e.g., Logistic), the probability of failure is relatively high (31\%).

Table 1: Maximum probability of violations and total costs among model (18), risk free model and BCH model

<table>
<thead>
<tr>
<th>Distribution</th>
<th>Model (18)</th>
<th>Risk Free</th>
<th>BCH</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total Cost</td>
<td>37885.3</td>
<td>36059.1</td>
<td>36448.6</td>
</tr>
<tr>
<td>Gaussian</td>
<td>0.02279</td>
<td>0.50149</td>
<td>0.2022</td>
</tr>
<tr>
<td>Student</td>
<td>1.00E-05</td>
<td>0.50128</td>
<td>5.00E-05</td>
</tr>
<tr>
<td>Laplace</td>
<td>0.0274</td>
<td>0.50366</td>
<td>0.18197</td>
</tr>
<tr>
<td>Logistic</td>
<td>0.12856</td>
<td>0.5022</td>
<td>0.31184</td>
</tr>
<tr>
<td>Uniform</td>
<td>0.0211</td>
<td>0.5016</td>
<td>0.21614</td>
</tr>
</tbody>
</table>

In the second test, we let the risk parameters \(\hat{\epsilon}_{ij}\) and \(\bar{\epsilon}_i\) range from 0.15 to 0.5 and observe how this affects the solutions. We compare our results on maximum probability of violations with the ones of BCH model through generating 100,000 samples from Logistic distribution. In Figure 1(a), we see that the results from BCH model are quite sensitive to the risk parameters, i.e., the probability of violating line capacity or bus capacity increases almost linearly as the risk parameters grows. Since the probability of violation curve is always above the neutral line which
tells whether the probability of violations is larger than the prespecified risk parameter \( \epsilon \) or not, hence the solution of BCH is not robust at all. Therefore, in the BCH model, one might need to stick to small risk parameters. Our model (18) turns out to be quite robust with the risk parameters. Even when all of the risk parameters are equal to 0.5, the chance of capacity violation is still quite small (around 28%). We also observe that in Figure 1(b) cost difference between two models reduces when the risk parameter increases. Another observation is that the total cost of our model (18) is the most costly due to its conservativeness, but the difference between ours and risk-free model is small (at most 6%). This could be because in the objective function (1a), there is only production cost of regular generators but no cost on renewables. Hence influence of renewables to the total costs is small but to the system reliability is dramatic.

\[\text{Figure 1: Comparison between model (18) and BCH model}\]

5 Conclusion

This paper studies a distributionally robust chance constrained optimal power flow problem (DRCC-OPF) with known first and second moments. We propose an exact second order cone program reformulation of DRCC-OPF. Our numerical study shows the proposed model can be solved efficiently and the results are quite robust even with larger risk parameters. We note that the same uncertain parameters appear on multiple chance constraints from the lines and buses, therefore, it is interesting as a future study to explore the tractability of the model by considering the uncertain constraints of lines and buses jointly (i.e., joint chance constraint) instead of individual ones.

References


