On Dantzig figures from lexicographic orders

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December 13, 2016

Abstract

We consider two families of $d$-polytopes defined as convex hulls of initial subsets for the graded lexicographic (grlex) and graded reverse lexicographic (grevlex) orders on $\mathbb{Z}_{\geq 0}^d$. Our considerations are motivated by the nice properties of the lex polytopes which were studied in relation to optimization problems. We show that the grlex and grevlex polytopes are non-simple Dantzig figures which are not combinatorially equivalent but have the same number of vertices, $\mathcal{O}(d^2)$. In fact, we provide an explicit description of the vertices and the facets for both families and show the basic properties of their graphs such as the radius, diameter, existence of Hamiltonian circuits, and chromatic number. The diameter is no more than 3. Moreover, we prove that the graph of the grlex polytope, which has $\mathcal{O}(d^2)$ vertices of degree $d$, has edge expansion 1.

Keywords: grlex, grevlex, polytope, Dantzig figure, diameter

MSC Classification: 52B12, 90C57, 52B05

1 Introduction

A $d$-polytope is a bounded convex polyhedron whose affine dimension is equal to $d$. Equivalently, a $d$-polytope is the convex hull of finitely many points, exactly $d + 1$ of which are affinely independent. A $d$-polytope with $n$ facets is referred to as a $(d, n)$-polytope. When $n = 2d$, we have a $(d, 2d)$-polytope. A $(d, 2d)$-polytope $X$ is said to be a Dantzig figure generated by distinct vertices $u$ and $v$ if $u$ and $v$ do not share a common facet. We denote this by $X = D(X, u, v)$. Since $u$ and $v$ do not lie on the same facet, they are called an antipodal vertex pair. Thus for a Dantzig figure, exactly $d$ distinct facets are incident to each of $u$ and $v$, and every facet contains exactly one of $u$ or $v$. This also means that both $u$ and $v$ have exactly $d$ neighboring vertices. A Dantzig figure is simple if every vertex is defined by exactly $d$ facets, or equivalently, has exactly $d$ neighboring vertices; otherwise it is non-simple.

Dantzig figures were introduced by Dantzig [Dan64] in connection to two famous conjectures, both of which were formulated by Hirsch in 1957 in context of the simplex method for linear programming, on (combinatorial) diameter of polytopes. Denoting $\Delta(d, n)$ as the maximum of the diameters of all $(d, n)$-polytopes, the Hirsch conjecture stated that $\Delta(d, n) \leq n - d$ for all $2 \leq d < n$, whereas the $d$-step conjecture claimed that this bound is tight for the family of $(d, 2d)$-polytopes in the sense that $\Delta(d, 2d) = d$ for every $d \geq 2$. See Klee and Kleinschmidt [KK87] for a survey. Klee and Walkup [KW67] showed that proving the Hirsch conjecture for all $d$ and $n$ is equivalent to proving the $d$-step conjecture for all $d$ and they stoked interest in Dantzig

∗AG is partially supported by ONR grant N00014-16-1-2725.
†SP is partially supported by NSF grant DMS-1312817.
figures by showing that the value $\Delta(d, 2d) = 0$ is attained by some $d$-dimensional simple Dantzig figure $\mathcal{D}(X, u, v)$ and is equal to the length of the shortest path between the antipodes $u$ and $v$. They used this characterization to prove the $d$-step conjecture for $d \leq 5$. In a landmark result, Santos [San12] proved the $d$-step and Hirsch conjectures to be false in general by constructing a $(43, 86)$-polytope whose diameter is at least 44. Santos’ construction makes use of a polytope called $d$-spindle, which is a $d$-polytope with $n \geq 2d$ facets and an antipodal pair $(u, v)$. There also has been considerable work on identifying (see Remark (43a)) of $(d,kd)$-polytopes that satisfy the Hirsch bound [San13] and diameter of special-structured polytopes [BDLF16; BFH15; DLK14; PR74]. Simple Dantzig figures have also been studied with regards to lower bounding the total number of $d$-length paths between the antipodal vertices [HK98].

One may think of Dantzig figures as polytopes with not too many facets, i.e., belonging to the family of $(d, kd)$-polytopes for some small constant $k$. Polytopes with few facets, where few facets generally means $(d, d+k)$-polytopes with $k$ being a small constant (typically $k \leq 6$), have received considerable attention both in terms of understanding their combinatorial properties and enumerating the different classes [BS11; Gri03; Pad16]. On the other hand, identification of different classes of $(d, kd)$-polytopes and/or their explicit construction has received limited attention. It is known that the so-called lexicographic (lex) polytope, which arises from lexicographic ordering of integral points, is a $(d, kd)$-polytope for $k \in \{3, 4\}$. Given $\theta, u \in \mathbb{Z}^d$ with $0 \leq \theta \leq u$, the lex polytope is $P_{\text{lex}} := \text{conv}\{x \in \mathbb{Z}^d : 0 \leq x \leq_{\text{lex}} \theta, x \leq u\}$. Note that the upper bound $x \leq u$ is necessary to obtain a polytope because the lex constraint $x \leq_{\text{lex}} y$ over the reals defines a non-closed convex cone. For $u = 1$, Laurent and Sassano [LS92] were the first to study $0 \setminus 1 P_{\text{lex}}$ in the context of what they called superincreasing knapsack polytopes, and showed that these are $(d, 3d)$-polytopes. Gillmann and Kaibel [GK06] later observed the equivalence of these knapsacks to $0 \setminus 1 P_{\text{lex}}$ and analyzed their graph and extremal properties. Gugte [Gup16] showed that $P_{\text{lex}}$ is a $(d, 3d)$-polytope for general $u$ and also showed that the polytope $\text{conv}\{x \in [0, u] \cap \mathbb{Z}^d : \gamma \leq_{\text{lex}} x \leq_{\text{lex}} \theta, x \leq u\}$ defined by one $\leq_{\text{lex}}$ and $\geq_{\text{lex}}$ order is a $(d, 4d)$-polytope. Lex polytopes, besides their independent interest in identifying families of $(d, kd)$-polytopes, have also been helpful in deriving strong cutting planes for mixed-integer optimization problems [Gup+13].

Due to Klee and Walkup’s results on the Hirsch conjecture, literature on Dantzig figures has focused attention mainly to bounding the diameter of simple Dantzig figures, and to the best of our knowledge, explicit ways of constructing nontrivial Dantzig figures, either simple or non-simple, for arbitrary $d$ are unknown. In this paper, we construct two combinatorial types of non-simple Dantzig figures using two monomial orders related to the lex order. Thus, we not only further the study of polytopes arising from monomial orders but also provide a constructive characterization for some Dantzig figures. Our explicit constructions also add to the vast body of work on specially-structured polytopes. Furthermore, our analysis reveals that there exist non-simple Dantzig figures with very small, and in fact constant, diameter, which is opposite to what we know from the disproval of the Hirsch conjecture that there exist simple Dantzig figures with relatively large diameter.

The polytopes we construct are defined by the graded lex (grlex) order ($\leq_{\text{gr}}$) and the graded reverse lex (grevlex) order ($\leq_{\text{grev}}$). Given $d \geq 3$ and $\theta \in \mathbb{Z}^d$ with $\theta \geq 1$, the grlex and grevlex polytopes are, respectively,

$$P := \text{conv}\{x \in \mathbb{Z}^d : 0 \leq x \leq_{\text{gr}} \theta\}, \quad Q := \text{conv}\{x \in \mathbb{Z}^d : 0 \leq x \leq_{\text{grev}} \theta\}. \quad (1)$$

Our consideration of these lattice polytopes is motivated by lex polytopes being $(d, kd)$-polytopes for $k \in \{3, 4\}$. Also, note that projecting $P$ (or $Q$) gives us a lex polytope over a integral simplex (see Remark 1). We show that $P$ and $Q$ are non-simple Dantzig figures for all $d$ and $\theta$ and although they appear closely related by definition, they are combinatorially not equivalent for $d \geq 3$. Both polytopes have equal number of vertices ($O(d^2)$) and the grevlex polytope has $O(d)$

\footnote{Strictly speaking, lex polytopes are, in general, $(d, kd - \epsilon)$-polytopes for $\epsilon \in \{1, 2\}$.}
times more edges. Although \( P \) is non-simple, it is not highly non-simple in the sense that only \( d - 3 \) of its vertices have more than \( d \) neighbors. We also describe the graphs of these polytopes, \( G(P) \) and \( G(Q) \). These graphs are mostly independent of actual values of \( \theta \), and \( G(P) \) with some \( \theta_k = 1 \) is a minor of \( G(P) \) for some \( \theta > 1 \). So for fixed \( d \), all grlex polytopes with \( \theta > 1 \) are combinatorially equivalent whereas all grevlex polytopes with \( \theta > 1 \) are combinatorially equivalent. We also analyze the radius, diameter, and chromatic number of these graphs. The diameter is a constant small value (\( \leq 3 \)) and therefore satisfies the Hirsch bound. This shows that for fixed \( d \), the smallest diameter over all non-simple Dantzig figures is very small and independent of \( d \), which is opposite of the now known fact that the maximum diameter over all simple Dantzig figures is larger than \( d \). Finally, we also address the expansion properties of these graphs. Bounding the edge expansion of graphs of polytopes is of significant interest due to its importance in studying random walks on such graphs and therefore has received much attention [Kai01]. A graph is said to be an expander if, roughly speaking, it is sparse but has high connectivity, which is quantified in terms of edge expansion of at least 1 (cf. [Sar04]). Noticing that \( G(P) \) is a relatively sparse graph with average vertex degree \( O(d) \), we show that \( G(P) \) is an expander graph by proving that it has edge expansion equal to 1. Thus, our Dantzig figure lies on the threshold for polytopes with good and poor expansion properties. Since \( G(Q) \) has a high average degree of \( O(d^2) \), its expansion is not of much interest.

**Notation.** The vector of all zeros is \( 0 \) and the vector of all ones is \( 1 \). Let \( \leq_{\text{lex}} \) denote the lexicographic monomial order. For \( x,y \in \mathbb{Z}^d \) we say \( x \leq_{\text{lex}} y \) if either \( x = y \) or there exists some \( i \) with \( x_i < y_i \) and \( x_k = y_k \) for all \( k > i \). \(^2\) The graded lex (grlex) and graded reverse lex (grevlex) monomial orders are denoted as \( \leq_{\text{grlex}} \) and \( \leq_{\text{grevlex}} \), respectively, and defined as follows:

1. \( x \leq_{\text{grlex}} y \) if either \( \sum_{i=1}^d x_i = \sum_{i=1}^d y_i \) or \( \sum_{i=1}^d x_i = \sum_{i=1}^d y_i \) and \( x \leq_{\text{lex}} y \),
2. \( x \leq_{\text{grevlex}} y \) if either \( \sum_{i=1}^d x_i = \sum_{i=1}^d y_i \) or \( \sum_{i=1}^d x_i = \sum_{i=1}^d y_i \) and \( x \geq_{\text{lex}} y \).

Denoting

\[
b := \sum_{i=1}^d \theta_i, \quad \tilde{b}_k := \sum_{i=1}^k \theta_i = b - \sum_{i=k+1}^d \theta_i \quad 1 \leq k \leq d, \quad H_0 := \left\{ x \in \mathbb{R}^d : \sum_{i=1}^d x_i = b \right\}, \tag{2a}
\]

as the total and partial sums of \( \theta \) and the grading hyperplane, it is clear that

\[
\mathcal{P} = \text{conv} \left\{ x \in \mathbb{R}^d_+ : \sum_{i=1}^d x_i \leq b - 1 \right\} \cup \text{conv} \left\{ x \in \mathbb{Z}^d_+ : \sum_{i=1}^d x_i = b, x \leq_{\text{lex}} \theta \right\}, \tag{2b}
\]

\[
\mathcal{Q} = \text{conv} \left\{ x \in \mathbb{R}^d_+ : \sum_{i=1}^d x_i \leq b - 1 \right\} \cup \text{conv} \left\{ x \in \mathbb{Z}^d_+ : \sum_{i=1}^d x_i = b, x \geq_{\text{lex}} \theta \right\}. \tag{2c}
\]

Both \( \mathcal{P} \) and \( \mathcal{Q} \) are \( d \)-polytopes since they contain the standard simplex. It is easy to verify that \( \mathcal{P} \cap H_0 \subseteq H_0 \cap \text{conv} \{ x \in \mathbb{Z}^d : 0 \leq x \leq_{\text{lex}} \theta \} \). Similarly for \( \mathcal{Q} \). Thus \( \mathcal{H} \)-representations of \( \mathcal{P} \) and \( \mathcal{Q} \) are not a trivial implication of the known results for lex polytopes.

Figure 1 illustrates these polytopes for \( d = 3 \) and the same \( \theta \). Both have 7 vertices, 11 edges, and 6 facets, but they are not isomorphic because \( \mathcal{P} \) has one pentagonal, two quadrilateral, and three triangular facets whereas \( \mathcal{Q} \) has two triangular and four quadrilateral facets.

\(^2\)Our right-to-left order of coordinate comparison here is opposite to the left-to-right order generally used in literature, but this is immaterial up to permuting the variables.
Figure 1: The grlex and grevlex polytopes in $d = 3$ defined by the same point $\theta$.

Remark 1. $\mathcal{P}$ (resp. $\mathcal{Q}$) yields the convex hull of all the integral vectors that belong to a standard integral simplex and are lexicographically smaller (resp. greater) than a fixed integer vector. In particular, denoting $\tilde{x} = (x_2, \ldots, x_d)$ and $\tilde{\theta} = (\theta_2, \ldots, \theta_d)$, we have

$$\text{Proj}_{\tilde{x}}(\mathcal{P} \cap H_0) = \text{conv} \left\{ \tilde{x} \in \mathbb{Z}^{d-1}_+ : \sum_{i=2}^d x_i \leq b, \tilde{x} \leq_{\text{lex}} \tilde{\theta} \right\}$$

and

$$\text{Proj}_{\tilde{x}}(\mathcal{Q} \cap H_0) = \text{conv} \left\{ \tilde{x} \in \mathbb{Z}^{d-1}_+ : \sum_{i=2}^d x_i \leq b, \tilde{x} \geq_{\text{lex}} \tilde{\theta} \right\}.$$ 

Outline. We begin by providing in §2 a conic representation of arbitrary polytopes. Our result implies that any Dantzig figure is equal to the intersection of two polyhedral cones, which is obviously the minimum number of cones required to represent any polytope. This suggests a way for obtaining the $\mathcal{H}$-representation of a Dantzig figure. The rest of the paper is divided into two parts. §3 analyzes the grlex polytope $\mathcal{P}$ and §4 analyzes the grevlex polytope $\mathcal{Q}$. For each polytope, we show it is a non-simple Dantzig figure generated by $0$ and $\theta$ and identify all its $O(d^2)$ vertices and $2d$ facet-defining inequalities. Although $\mathcal{P}$ and $\mathcal{Q}$ appear closely related by definition, they are combinatorially not equivalent, as seen in Figure 1. This necessitates separate proofs, especially for showing the Dantzig figure property in Theorems 3.1 and 4.1, but we condense our arguments whenever possible. We describe $G(\mathcal{P})$ and $G(\mathcal{Q})$, the graphs of these polytopes, and their basic properties, including diameter, in §3.3 and §4.3. $G(\mathcal{P})$ has $O(d^3)$ edges whereas $G(\mathcal{Q})$ has $O(d^4)$ edges, confirming our intuition from Figure 1 that $\mathcal{P}$ and $\mathcal{Q}$ define two distinct families of non-simple Dantzig figures. The edge expansion of $G(\mathcal{P})$ is proved in Theorem 3.2.

2 Conic characterization of polytopes

Let $X \subseteq \mathbb{R}^n$ be a $d$-polytope with set of vertices $\text{vert}(X)$. For every $v \in \text{vert}(X)$, $\mathcal{N}_X(v)$ denotes the set of vertices adjacent to $v$. Recall that two vertices of a $d$-polytope are adjacent if and only if there are at least $d - 1$ facets that contain both the vertices. The translated polyhedral
cone at a vertex \( v \) of \( X \) is defined as

\[
C_X(v) := v + \left\{ \sum_{x \in \mathcal{N}_X(v)} \alpha_x (x - v) : \alpha \geq 0 \right\}.
\]

By construction, the dimension of this cone cannot be greater than the dimension of \( X \). Observe that

\[ X \subseteq C_X(v) \quad \forall v \in \text{vert}(X). \tag{4} \]

This can be argued as follows.\(^3\) Let \( X = \{ x \in \mathbb{R}^n : Ax \geq b \} \) be the \( \mathcal{H} \)-representation of \( X \) for some \( A \in \mathbb{R}^{m \times n}, b \in \mathbb{R}^m \). A basis of \( X \) is a \( n \)-subset of \( [m] \) such that the rows of \( A \) indexed by this subset are linearly independent. Consider any \( v \in \text{vert}(X) \). By the equivalence of vertices and basic feasible solutions of a polyhedron, there exists some basis \( B \) such that \( v \) is the unique solution to the linear system \( a_i x = b_i \) for \( i \in B \), where \( a_i \) is the \( i \)th row of \( A \). Let \( B' := \{ i : a_i v = b_i \} \); clearly \( B' \supseteq B \) with the inclusion being strict if and only if \( v \) is a degenerate vertex. It is easy to argue then that the vertex cone at \( v \) can be represented as

\[ C_X(v) = v + \{ y : a_i y \geq 0, i \in B' \}. \tag{5} \]

Now for any \( x \) with \( Ax \geq b \), we have \( x - v \) satisfying \( a_i (x - v) \geq 0 \) for all \( i \in B' \), and therefore \( x \in C_X(v) \).

Equation (4) implies two things. First that the affine dimension of \( C_X(v) \) is equal to \( d \). Secondly, it leads to the inclusion \( X \subseteq \cap_{v \in \text{vert}(X)} C_X(v) \). In fact, equality holds, i.e., every polytope is equal to the intersection of its vertex cones. We will use a stronger version of this statement.

**Lemma 2.1.** For any \( \emptyset \neq S \subseteq \text{vert}(X) \), we have \( X = \cap_{v \in S} C_X(v) \) if and only if every facet of \( X \) contains some \( v \in S \).

A special case of the above result arises by considering \( S = \text{vert}(X) \), which leads to

\[ X = \bigcap_{v \in \text{vert}(X)} C_X(v). \tag{6} \]

**Proof.** Suppose that \( X = \cap_{v \in S} C_X(v) \). By (5), \( C_X(v) = \{ x : a_i x \geq a_i v, i \in B'(v) \} \), where \( B'(v) \) is the set of tight inequalities at \( v \). This implies \( X = \{ x : a_i x \geq a_i v, i \in B'(v), v \in S \} \). Note that for every facet of a polyhedron there exists some defining inequality of the polyhedron that represents this facet. Hence if \( F \) is a facet of \( X \), then \( F = \{ x \in X : a_i x = a_i v \} \) for some \( i \in B'(v), v \in S \). Then it is clear that \( v \in F \).

For the reverse direction, we will need the following.

**Claim 2.1.** Let \( H \) be a supporting hyperplane of \( X \). Then \( X \cap H \) is a facet of \( X \) if and only if \( C_X(v) \cap H \) is a facet of \( C_X(v) \) for every \( v \in \text{vert}(X) \cap H \).

**Proof.** We have that \( H \) defines a proper face of \( X \), i.e. the dimension of \( X \cap H \) is at least 0 and at most \( d - 1 \). \( (\Leftarrow) \) Since \( C_X(v) \) is a \( d \)-dimensional polyhedral cone, any \( (d - 1) \)-subset of its generators are linearly independent, meaning that \( v \) and any \( (d - 1) \)-subset of \( \mathcal{N}_X(v) \) are affinely independent. Suppose \( H \) defines a facet of \( C_X(v) \) for every \( v \in \text{vert}(X) \cap H \). Then \( H \) contains \( v \) and at least \( d - 1 \) vertices in \( \mathcal{N}_X(v) \). Therefore \( H \) contains \( d \) affinely independent vertices of \( X \), making \( X \cap H \) a facet of \( X \).

\( (\Rightarrow) \) Suppose \( X \cap H \) is a facet of \( X \). The cone \( C_X(v) \) being \( d \)-dimensional for every \( v \), we need to argue that \( H \) defines a \( (d - 1) \)-dimensional face of \( C_X(v) \) for every \( v \in \text{vert}(X) \cap H \). For every \( v \in \text{vert}(X) \), \( X \subseteq C_X(v) \) tells us that \( C_X(v) \not\subseteq H \) and that the points in \( \mathcal{N}_X(v) \setminus H \)

\(^3\)A different proof is given in Ziegler [Zie95, Lemma 3.6].
are all on one side of $H$. Thus for every $v \in \text{vert}(X) \cap H$, the generators $\{u - v\}_{u \in N_X(v)}$ of $C_X(v)$ belong to one of the halfspaces defined by $H$. Hence $H$ defines a face of $C_X(v)$. Due to $C_X(v) \nsubseteq H$, the dimension of this face is at most $d - 1$. Since $H$ defines a facet of $X$, we have $\text{vert}(X) \cap H = \text{vert}(X) \cap H$ and so $H$ contains $d$ affinely independent vertices of $X$. Now $\text{vert}(X) \cap H \subseteq C_X(v) \cap H$ tells us that the dimension of the face $C_X(v) \cap H$ is at least $d - 1$, thereby implying that $H$ defines a facet of $C_X(v)$ for every $v \in \text{vert}(X) \cap H$.

Now suppose every facet of $X$ contains some $v \in S$. It suffices to prove that $\cap_{v \in S} C_X(v) \subseteq X$ because $X \subseteq \cap_{v \in S} C_X(v)$ is obvious from (4) and $S \subseteq \text{vert}(X)$. For sake of contradiction, let $x \in \cap_{v \in S} C_X(v) \setminus X$. Then $cx > c_0$ for some facet-defining inequality $cx \leq c_0$ of $X$. By assumption, there exists some $\bar{v} \in S$ such that $\bar{v}w = c_0$. By Claim 2.1, we have that $cx \leq c_0$ is a facet-defining inequality of $C_X(\bar{v})$. But then $cx > c_0$ leads to the contradiction $x \notin C_X(\bar{v})$.

Lemma 2.1 poses an interesting question: for a $(d,kd)$-polytope $X$, is there a good lower bound (in terms of $d$ and $k$) on how many polyhedral cones are required to describe $X$? The answer does not seem obvious even for $k = 2$. Even a simpler question does not seem obvious: is there a characterization of $(d,2d)$-polytopes, or $(d,n)$-polytopes, that are equal to the intersection of two vertex cones? Note that any polytope requires at least two vertex cones to describe it. For $(3,6)$-polytopes, which are called hexahedra and have seven distinct combinatorial types as enumerated in [Num16], one can graphically verify the answer to the second question to be yes. For $d = 4$, for the dual of the simplicial 4-polytope with 8 vertices $P_8^4$ in [GS67, pp. 454], it is easy to verify that there does not exist any vertex pair $(u,v)$ such that every facet of $P_8^4$ contains either $u$ or $v$, meaning that $P_8^4$ requires at least three vertex cones for its description. For general $d$, the answer to the second question is clearly yes for Dantzig figures, due to Lemma 2.1.

**Corollary 2.1.** For a Dantzig figure $X = \mathcal{D}(X,u,v)$, we have $X = C_X(u) \cap C_X(v)$.

The converse of Corollary 2.1 is not true — not every $(d,2d)$-polytope that is equal to the intersection of two vertex cones is a Dantzig figure; for example in $\mathbb{R}^3$, a pyramid with a pentagonal base is not a Dantzig figure since every pair of vertices shares a common facet. Therefore, the Dantzig figure property does not seem necessary for a $(d,2d)$-polytope to be described by two cones.

**Corollary 2.1** can be used to derive an explicit $\mathcal{H}$-representation for a Dantzig figure. Since $u$ and $v$ have exactly $d$ neighboring vertices, we may denote $N_X(u) = \{y^1,\ldots,y^d\}$ and $N_X(v) = \{z^1,\ldots,z^d\}$. If we let

$$
M_u := \begin{bmatrix} y^1 - u & y^2 - u & \cdots & y^d - u \end{bmatrix}, \quad M_v := \begin{bmatrix} z^1 - v & z^2 - v & \cdots & z^d - v \end{bmatrix}
$$

 denote the $d \times d$ matrices defined by neighbors of $u$ and $v$, respectively, then the vertex cones $C_X(u)$ and $C_X(v)$ are given by $C_X(u) = u + \{M_u \alpha : \alpha \geq 0\}$ and $C_X(v) = v + \{M_v \alpha : \alpha \geq 0\}$. Since the above matrices are nonsingular, these cones are simplicial and we have $C_X(u) = \{x : M_u^{-1}(x - u) \geq 0\}$ and $C_X(v) = \{x : M_v^{-1}(x - v) \geq 0\}$. This combined with Corollary 2.1 yields the following.

**Proposition 2.1.** For a Dantzig figure $X = \mathcal{D}(X,u,v)$, we have the following minimal inequality representation:

$$
X = \{x : M_u^{-1}(x - u) \geq 0, M_v^{-1}(x - v) \geq 0\}.
$$

We will apply this method of deriving the $\mathcal{H}$-representation to our polytopes $P$ and $Q$.

### 3 The grlex polytope $P$

In this section we will describe the main properties of the polytope $P$. To simplify the notation, throughout, we will use $\preceq$ to denote the grlex order.
3.1 $\mathcal{V}$-polytope

Consider the following integral points:

\begin{align*}
    v := (b - 1)e_d &= (0, 0, \ldots, 0, b - 1) \\
    u^k := ((b_{k-1} + 1)e_{k-1}, \theta_{k-1} - 1, \theta_{k+1}, \ldots, \theta_d) & \quad 3 \leq k \leq d \\
    v^{j,k} := (\tilde{b}_k e_j, 0, \theta_{k+1}, \ldots, \theta_d) & \quad 1 \leq j < k \leq d.
\end{align*}

By construction, we have

\begin{equation*}
    \text{Equation (7)}
\end{equation*}

**Observation 3.1.** $u^k \in H_0$ for all $k$, $v^{j,k} \in H_0$ for all $j, k, \theta \in H_0$, 0, $w \not\in H_0$.

Since $\theta \geq 1$, we have $w, u^k, v^{j,k} \geq 0$. Also, every $u^k$ and $v^{j,k}$ is $\leq_{\text{lex}}$-less than $\theta$. Thus $w, u^k, v^{j,k} \in \mathcal{P}$. Observe that $\theta \geq 1$ implies that $v^{j,k}$ and $u^{k,2}$ coincide if and only if $k_1 = k_2$, $j = \tilde{k}_1 - 1$, and $\theta_{k_1} = 1$.

Our first result shows that the points defined in (7), along with 0 and $\theta$, provide a vertex characterization of $\mathcal{P}$.

**Proposition 3.1.** The vertices of $\mathcal{P}$ are

\begin{equation*}
    \text{vert}(\mathcal{P}) = \{0, \theta, w\} \cup \{u^k : 3 \leq k \leq d\} \cup \{v^{j,k} : 1 \leq j < k \leq d\}.
\end{equation*}

**Proof.** It is clear from the definition of $\mathcal{P}$ that 0 and $w$ cannot be written as a nontrivial convex combination of integral points in $\mathcal{P}$. Suppose $\theta = \sum_{i=1}^s \lambda_i x^i$ is a nontrivial convex combination of some $x^i \in \mathcal{P} \cap \mathbb{Z}^d$. Since $\theta \in \mathcal{P} \cap H_0$ and $H_0$ defines a face of $\mathcal{P}$, we have $x^i \in \mathcal{P} \cap H_0$ for all $i$. Let

\begin{equation*}
    m = \max\{j : x^i_j \neq \theta_j \text{ for some } i\}.
\end{equation*}

Then $x^i \leq_{\text{lex}} \theta$ implies $x^i_m \leq \theta_m$, leading to the contradiction $\sum_{i=1}^s \lambda_i x^i_m < \theta_m$. Next, suppose $u^k = \sum_{i=1}^s \lambda_i x^i$ is a nontrivial convex combination of some $x^i \in \mathcal{P} \cap \mathbb{Z}^d$. Since $u^k \in H_0$, $x^i \in H_0$ as well. By the same reasoning as for $\theta$ we get $x^i_j = \theta_j$ for all $i$ and $j > k$. Also, $x^i_j = 0$ for all $i$ and $j < k - 1$. Now, if $x^i_k = \theta_k$ for some $i$ then $x^i_{k-1} = \tilde{b}_{k-1}$ which contradicts $x^i_{k-1} \leq \theta$ because $\theta \geq 1$ and $k \geq 3$. So, the only possibility is $x^i = u^k$ for all $i$. Similarly, one can conclude that all $v^{j,k}$ have to be vertices of $\mathcal{P}$ as well.

Now we argue that if $v \in \text{vert}(\mathcal{P}) \setminus \{0, w, \theta\}$, then $v$ must be equal to some $u^k$ or $v^{j,k}$. Equation (2b) gives us

\begin{equation*}
    \text{vert}(\mathcal{P}) \subseteq \text{vert}\{x \in \mathbb{R}^d : \sum_{i=1}^d x_i \leq b - 1\} \cup \text{vert}(\mathcal{P} \cap H_0).
\end{equation*}

The vertices of the simplex \{ $x \in \mathbb{R}^d : \sum_{i=1}^d x_i \leq b - 1$ \} are 0 and $(b - 1)e_i$ for $i = 1, \ldots, d$. The point $(b - 1)e_i$ is a convex combination of 0 and $be_i$, and for $i \leq d - 1$, $\theta_d \geq 1$ implies $be_i \leq_{\text{lex}} \theta$, and hence $be_i \in \mathcal{P} \cap H_0$. Therefore $(b - 1)e_i \notin \text{vert}(\mathcal{P})$ for $i \leq d - 1$ and we have

\begin{equation*}
    \text{vert}(\mathcal{P}) \subseteq \{0, w\} \cup \text{vert}(\mathcal{P} \cap H_0).
\end{equation*}

Since $H_0$ defines a face of $\mathcal{P}$, we have $\text{vert}(\mathcal{P} \cap H_0) = \text{vert}(\mathcal{P}) \cap H_0 = \text{vert}(\mathcal{P}) \cap H_0$ and then 0, $w \in \text{vert}(\mathcal{P})$ leads to the equality

\begin{equation*}
    \text{vert}(\mathcal{P}) = \{0, w\} \cup \text{vert}(\mathcal{P} \cap H_0).
\end{equation*}

We argued in the first paragraph that $\theta \in \text{vert}(\mathcal{P}) \cap H_0$ and because $\text{vert}(\mathcal{P}) \cap H_0 = \text{vert}(\mathcal{P} \cap H_0)$, we have $\theta \in \text{vert}(\mathcal{P} \cap H_0)$. Let $\bar{x} \neq \theta$ be an arbitrary vertex of $\mathcal{P} \cap H_0$ and define $k := \max\{i : \bar{x}_i \neq \theta_i\}$. Since $\text{vert}(\mathcal{P}) \subseteq \mathbb{Z}^d$, we have $\bar{x} \in \mathbb{Z}^d$ and $\bar{x}_k \in \{0, 1, \ldots, \theta_k - 1\}$. Suppose $1 \leq \bar{x}_k \leq \theta_k - 2$. Then $\bar{x} \in H_0$ implies there exists a $i < k$ such that $1 \leq \bar{x}_i \leq b - 1$. For $x', x'' \in H_0 \cap \mathbb{Z}^d$
defined as \( x' = \bar{x} + e_i - e_k \) and \( x'' = \bar{x} - e_i + e_k \), note that \( 1 \leq \bar{x}_k \leq \theta_k - 2 \) implies that \( 0 \leq x', x'' \leq_{\text{lex}} \theta \). Hence \( x', x'' \in \mathcal{P} \cap H_0 \) and since \( \bar{x} = (x' + x'')/2 \), we have a contradiction to \( \bar{x} \in \text{vert}(\mathcal{P} \cap H_0) \), thereby implying that \( \bar{x}_k \in \{0, \theta_k - 1\} \). The assumption \( \theta \geq 1 \) allows us to make similar arguments when \( \bar{x}_k \in \{0, \theta_k - 1\} \) and \( \bar{x}_i, \bar{x}_j \geq 1 \) for distinct \( i, j, k \leq 1 \). Thus if \( \bar{x}_k = 0 \), then \( \bar{x} \in \text{vert}(\mathcal{P} \cap H_0) \) only if \( \bar{x} = v_{i,k}^1 \) for some \( j \leq 1 \). Finally let \( \bar{x}_k = \theta_k - 1 \), \( \bar{x}_i = \bar{b}_{i-1} + 1 \) for some \( i \leq k - 2 \) and \( \bar{x}_t = 0 \) for \( t \neq i, k \). In this case, \( \bar{x} = \frac{\theta_k - 1}{\theta_k} \sigma^1 + \frac{1}{\theta_k} \sigma^2 \), where

\[
\sigma^1 := (\bar{b}_{k-1} e_i, \theta_k, \theta_{k+1}, \ldots, \theta_d), \quad \sigma^2 := (\bar{b}_k e_i, 0, \theta_{k+1}, \theta_{k+2}, \ldots, \theta_d),
\]

and \( \sigma^1, \sigma^2 \in \mathcal{P} \cap H_0 \) due to \( \theta \geq 1, k \geq 3, i \leq k - 2 \). Hence \( \bar{x} \) must be equal to \( w^k \) when \( \bar{x}_k = \theta_k - 1 \).

**Observation 3.2.** The \( d \) coordinate planes are facets of \( \mathcal{P} \), which we call trivial facets.

**Proof.** We know \( 0 \in \mathcal{P} \). The assumption \( \theta \geq 1 \) implies that \( e_i \not\in \theta \) for \( 1 \leq i \leq d \).

The remaining facets are defined by supporting hyperplanes that have a monotone coefficient property, which we prove next.

**Lemma 3.1.** Suppose \( \mathcal{P} \subseteq \{x: cx \leq c_0\} \) and let \( F = \mathcal{P} \cap \{x: cx = c_0\} \) be a face of \( \mathcal{P} \). If \( F \) is not contained in \( x_{i+1} = 0 \) for some \( i \in \{1, \ldots, d-1\} \) then \( c_{i+1} \geq \max\{c_i, 0\} \). Consequently, if \( F \) is a nontrivial facet then \( 0 \leq c_i \leq c_{i+1} \) for all \( 1 \leq i \leq d - 1 \).

**Proof.** Let \( \bar{x} \) be a vertex on \( F \) with \( \bar{x}_{i+1} = 1 \). Consider first \( x' = \bar{x} - e_{i+1} \). Since \( 0 \leq x' \leq \bar{x} \), we have \( cx' \leq c_0 = c\bar{x} \), which implies \( c_{i+1} \geq 0 \). Similarly, the point \( x'' = \bar{x} + e_{i} - e_{i+1} \) also has the property \( 0 \leq x'' \leq \bar{x} \). Therefore, we have \( cx'' \leq c_0 = c\bar{x} \), which yields \( c_i \leq c_{i+1} \).

We also note that \( H_0 \) defines a facet of \( \mathcal{P} \).

**Observation 3.3.** \( \sum_{i=1}^d x_i \leq b \) is a facet-defining inequality for \( \mathcal{P} \).

**Proof.** The face \( \mathcal{P} \cap H_0 \) contains \( \theta \) and the \( d - 1 \) coordinate vectors \( v_1^1, v_2^2, \ldots, v_{d-1}^{d-1} \), and these vertices are affinely independent because of \( \theta_d \geq 1 \).

**Proposition 3.2.** The neighbors of \( \theta \) and \( 0 \) are

\[
\mathcal{N}_{\mathcal{P}}(\theta) = \{w, v_1^1, v_2^2\} \cup \{u^k: 1 \leq k \leq d\}
\]

\[
\mathcal{N}_{\mathcal{P}}(0) = \{w\} \cup \{v_j^j: 1 \leq j \leq d - 1\}.
\]

**Proof.** Since \( \theta \) has at least \( d \) neighbors, it suffices to show that the other vertices are not neighbors of \( \theta \). Now, suppose \( \mathcal{P} \subseteq \{x: cx \leq c_0\} \) and \( F = \mathcal{P} \cap \{x: cx = c_0\} \) is an edge through \( \theta \) and \( v_j^j \) for some \( k \geq 3 \) and \( j < k \). Lemma 3.1 implies

\[
(c_{k-1} - c_j)(\theta_1 + \cdots + \theta_{k-1} + 1) + (c_k - c_j)(\theta_k - 1) \geq 0
\]

which, in turn, implies \( cu_k \geq cv_j^j = c_0 \). Therefore \( u^k \in F \), which contradicts the assumption that \( F \) is an edge.

Since the coordinate planes define trivial facets of \( \mathcal{P} \) and two vertices of a \( d \)-polytope are neighbors if and only if there are at least \( d - 1 \) facets that contain both the vertices, it follows that \( w \) and \( be_j \), for \( j \leq d - 1 \), are neighbors of \( 0 \). These neighbors are coordinate vectors and their conic hull is \( \mathbb{R}^d_+ \). Hence \( \mathcal{C}_{\mathcal{P}}(0) \), the vertex cone at \( 0 \), contains \( \mathbb{R}^d_+ \) but since \( \mathcal{P} \subseteq \mathbb{R}^d_+ \), we in fact have \( \mathcal{C}_{\mathcal{P}}(0) = \mathbb{R}^d_+ \). Therefore there does not exist another vertex of \( \mathcal{P} \) which is a neighbor of \( 0 \).

**Theorem 3.1.** \( \mathcal{P} = \mathcal{D}(\mathcal{P}, 0, \theta) \) and \( (0, \theta) \) is the only pair of antipodal vertices that generates this Dantzig figure.
Proof. Proposition 3.2 gives us that each of \( 0 \) and \( \theta \) has exactly \( d \) neighboring vertices. For any vertex \( v \) of a \( d \)-polytope, every facet containing \( v \) also contains at least \( d - 1 \) neighbors of \( v \). Hence each of \( 0 \) and \( \theta \) lies on exactly \( d \) facets. The nonnegativity of the coefficients of the supporting hyperplanes from Lemma 3.1 and the assumption \( \theta \geq 1 \) imply that there is no nontrivial face containing both \( 0 \) and \( \theta \).

Now we need to prove that every facet of \( \mathcal{P} \) contains either \( 0 \) or \( \theta \). Suppose \( F \) is a facet of \( \mathcal{P} \) given by \( cx \leq c_0 \). If \( F \) doesn’t contain \( \theta \) nor any of the vertices \( v_{1,k} \), then it is contained in the subspace \( x_1 = 0 \) and hence be equal to the facet defined by \( x_1 \geq 1 \) and therefore contain \( 0 \). So, suppose \( F \) contains a vertex \( v_{1,k} \) for some \( k \) and suppose \( c \theta < c_0 \) and \( c \theta < c_0 \). Then \( v_{1,k} \in F \) implies

\[
c \theta < c_0 = c_1 \sum_{i=1}^{k} \theta_i + \sum_{i=k+1}^{d} c_i \theta_i
\]

and using Lemma 3.1 we get

\[
0 \leq \sum_{i=2}^{k} (c_i - c_1) \theta_i < 0
\]

which is a contradiction.

The vertex \( w \) is a neighbor of both \( 0 \) and \( \theta \) and the vertices \( \theta, u_k \) and \( v_{1,k} \) all belong to the facet defined by \( H_0 \). Hence any other antipodal pair of vertices must be of the form \((0, v)\) or \((w, v)\), where \( v = u_k \) for some \( 3 \leq k \leq d \), or \( v = v_{j,k} \) for some \( 1 \leq j < k \leq d \). However each of these pairs has a common facet defined by some coordinate plane: each of the two pairs \((0, u_k)\) and \((w, u_k)\) shares the facet \( x_1 = 0 \); \((0, v_{j,k})\) share the facet \( x_k = 0 \); \((w, v_{j,k})\) share the facet \( x_k = 0 \) if \( k < d \) and otherwise, \( x_i = 0 \) for some \( i \leq d - 1, i \neq j \).

\[\Box\]

### 3.2 \( \mathcal{H} \)-polytope

Proposition 3.2 tells us that the vertex cone at \( 0 \) is \( \mathcal{C}_P(0) = \mathbb{R}^d_+ \). Then Proposition 2.1 gives us \( \mathcal{P} = \{ x \geq 0 : M^{-1}(x - \theta) \geq 0 \} \), where

\[
M := \begin{bmatrix}
v^{1,2} - \theta & u^3 - \theta & u^4 - \theta & \cdots & u^d - \theta & w - \theta \\
\theta_2 & -\theta_1 & -\theta_1 & \cdots & -\theta_1 & -\theta_1 \\
-\theta_2 & b_1 + 1 & -\theta_2 & \cdots & -\theta_2 & -\theta_2 \\
0 & -1 & b_2 + 1 & \cdots & -\theta_3 & -\theta_3 \\
\vdots & \vdots & 0 & -1 & \vdots & \vdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & \cdots & \cdots & 0 & -1 & b_{d-2} + 1 \\
\cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 & b_d - 1 - 1 \\
\end{bmatrix}
\]

To describe the inverse of \( M \), denote

\[
p_{ij}^j := \begin{cases} 
b_i \prod_{k=i+1}^j (b_k + 1) & j > i \\
b_i & j = i \\
1 & j < i. \\
\end{cases}
\]
Proposition 3.3. \( \mathcal{P} = \{ x \geq 0 : Nx \geq N\theta \} \) where \( N = M^{-1} \) with

\[
\begin{align*}
N_{d,i} &= -1, \quad 1 \leq i \leq d, \quad N_{i,d} = -p_i^{d-1}, \quad 2 \leq i \leq d, \quad N_{1,d} = \frac{-p_1^{d-1}}{\theta_2}, \\
N_{i,j} &= \begin{cases} 
N_{i,j+1} + \frac{p_{j-1}}{\theta_2} & i = 1, \ 1 \leq j \leq d - 1 \\
N_{i,j+1} + p_{i-1} & 2 \leq i \leq d - 1 \\
N_{i,j+1} & 1 \leq j < i \leq d - 1.
\end{cases}
\end{align*}
\]

Proof. We need to consider several cases when computing \( N_i.M_j \). First note that

\[
N_{1,M_1} = \theta_2(N_{1,1} - N_{1,2}) = \theta_2/\theta_2 = 1
\]

and for \( i \geq 2, \)

\[
N_i.M_i = -\sum_{l=1}^{i-1} N_{i,l} \theta_l + (\tilde{b}_{i-1} + 1)N_{i,i} - N_{i,i+1}
\]

\[
= -N_{i,i} \sum_{l=1}^{i-1} \theta_l + (\tilde{b}_{i-1} + 1)N_{i,i} - N_{i,i+1}
\]

\[
= N_{i,i} - N_{i,i+1}
\]

\[
= 1.
\]

Consider now the case \( 1 \leq j < i \leq d \). It is readily seen that

\[
N_{d,M_j} = -\sum_{l=1}^{d} M_{l,j} = 0
\]

and for \( i \leq d - 1, \)

\[
N_i.M_j = -N_{i,i} \sum_{l=1}^{j-1} \theta_l + (\tilde{b}_{j-1} + 1)N_{i,j} - N_{i,j+1}
\]

\[
= -N_{i,i} \sum_{l=1}^{j-1} \theta_l + (\tilde{b}_{j-1} + 1)N_{i,i} - N_{i,i}
\]

\[
= 0.
\]

For \( 1 \leq i < j \leq d - 1, \)

\[
N_i.M_j - N_i.M_{j-1} = N_i(M_j - M_{j-1})
\]

\[
= N_{i,j-1}(\theta_{j-1} - \tilde{b}_{j-2} - 1) + N_{j}(\tilde{b}_{j-1} + 2) + N_{i,j+1}(-1)
\]

\[
= -N_{i,j-1}(b_{j-1} + 1) + N_{i,j}(b_{j-1} + 2) - N_{i,j+1}
\]

\[
= (b_{j-1} + 1)(N_{i,j} - N_{i,j-1}) + (N_{i,j} - N_{i,j+1})
\]

\[
= \begin{cases} 
(b_{j-1} + 1)(-p_{j-2}^{i-2}) + p_{j-1}^{i-1}, & 1 \leq i \leq j \leq d - 1 \\
(b_{j-1} + 1)(-p_i^{j-2}) + p_i^{j-1}, & 1 < i \leq j \leq d - 1
\end{cases}
\]

\[
= \begin{cases} 
0, & i + 1 = j \leq d - 1 \\
-1, & i + 1 < j \leq d - 1.
\end{cases}
\]
Therefore, \( N_i, M_j = 0 \) for \( 1 \leq i < j \leq d - 1 \).

\[
N_i M_d - N_i M_{d-1} = N_i(M_d - M_{d-1})
\]
\[
= N_i, d-1(-b_{d-2} - 1 - \theta_{d-1}) + N_i, d(1 + \tilde{b}_{d-1} - 1)
\]
\[
= \begin{cases} 
    \left( N_i, d + \frac{p_i^{d-2}}{b_2} \right)(\tilde{b}_{d-1} + 1) - \tilde{b}_{d-1} N_i, d, & i = 1 \\
    (N_i, d + \frac{p_i^{d-2}}{b_2})(\tilde{b}_{d-1} + 1) - \tilde{b}_{d-1} N_i, d, & 1 < i \leq d - 1 \\
    N_i, d + \frac{p_i^{d-2}}{b_2} - \tilde{b}_{d-1} N_i, d, & 1 < i \leq d - 1 \\
    0, & 1 \leq i \leq d - 2 \\
    -1, & i = d - 1.
\end{cases}
\]

Therefore, \( N_i, M_d = 0 \) for \( 1 \leq i \leq d - 1 \).

### 3.3 Graph of the polytope

Let \( G(\mathcal{P}) \) denote the graph of \( \mathcal{P} \). Based on Proposition 3.1, it is clear that \( G(\mathcal{P}) \) has \( \frac{d^2 + d + 2}{2} \) many vertices. We next characterize the vertex-facet incidence for \( \mathcal{P} \). This leads us to finding all the edges of \( G(\mathcal{P}) \) since \( \mathcal{P} \) is a \( d \)-polytope and so for any \( v, v' \in \text{vert}(\mathcal{P}) \), \((v, v')\) is an edge in \( G(\mathcal{P}) \) if and only if there are at least \( d - 1 \) facets incident to both \( v \) and \( v' \). As can be seen from Corollary 3.1, \( G(\mathcal{P}) \) depends only on which entries of \( \theta \) are 1, and for a fixed \( d \), these graphs are isomorphic for all \( \theta > 1 \) (Figure 2). We then derive some of the basic properties of \( G(\mathcal{P}) \) such as radius, diameter, coloring number, etc. We also show that when \( \theta > 1 \), this graph is an expander [Sar04] in the sense that the number of vertices is \( n \sim d^2/2 \), the average degree \( \delta_d \sim 4d/3 \), as \( d \to \infty \), and the edge expansion is equal to one.

Figure 2: The graph \( G(\mathcal{P}) \) for a vertex \( \theta > 1 \) in \( \mathbb{R}^d \). The circled vertices form cliques. The dashed edges represent connections between a vertex and a clique.

Since \( \mathcal{P} = \mathcal{D}(\mathcal{P}, 0, \theta) \) as per Theorem 3.1, the vertex cones \( C_\theta(0) \) and \( C_\theta(\theta) \) are simplicial. Thus the \( d \) facet-defining hyperplanes incident to \( \theta \) (resp. 0) are in a one-to-one correspondence with the \( d \) neighbors of \( \theta \) (resp. 0). Furthermore, we know that 0 and \( \theta \) do not belong to a
common facet. Hence we denote the facet-defining hyperplanes incident to $\theta$ as:

$$H_w = \{x: N_d(x - \theta) = 0\} = H_0, \quad H_{v_{1,2}} = \{x: N_1(x - \theta) = 0\},$$
$$H_{v_k} = \{x: N_{(k-1)}(x - \theta) = 0\} \quad 3 \leq k \leq d,$$

where $N$ is the inverse of $M$ from Proposition 3.3 and $H_v$, for $v \in \mathcal{N}_P(\theta)$, signifies the only facet-defining hyperplane that contains $\theta$ but not $v$. Let $\mathcal{H}_P^\theta$ denote the collection of these hyperplanes, i.e.,

$$\mathcal{H}_P^\theta := \{H_{v_{1,2}}, H_{v_3}, \ldots, H_{v_d}, H_w\}. \quad (8a)$$

The $d$ facet-defining hyperplanes incident to $0$ are the coordinate planes $H_i := \{x: x_i = 0\}$ for $1 \leq i \leq d$, and we denote this collection by

$$\mathcal{H}_P^0 := \{H_1, H_2, \ldots, H_d\}, \quad \mathcal{H}_P^{0,i} := \{H_1, H_2, \ldots, H_i\} \quad 1 \leq i \leq d. \quad (8b)$$

The vertex-facet incidence for $P$ is stated in the following result. For $v \in \text{vert}(\mathcal{P})$, let $\psi_P(v)$ denote the subset of facet-defining hyperplanes of $\mathcal{P}$ that contain $v$.

**Proposition 3.4.** We have

$$\psi_P(0) = \mathcal{H}_P^0, \quad \psi_P(\theta) = \mathcal{H}_P^\theta,$$
$$\psi_P(w) = (\mathcal{H}_P^\theta \setminus \{H_w\}) \cup \mathcal{H}_P^{0,d-1},$$
$$\psi_P(v^k) = (\mathcal{H}_P^\theta \setminus \{H_{v^{k}}\}) \cup \mathcal{H}_P^{0,k-2} \cup \begin{cases} H_k & \text{if } \theta_k = 1 \\ \emptyset & \text{if } \theta_k \geq 2 \end{cases} \quad 3 \leq k \leq d$$
$$\psi_P(v^{j,k}) = (\mathcal{H}_P^\theta \setminus \{H_{v^{j,2}}, H_{v^3}, \ldots, H_{v^{k}}\}) \cup (\mathcal{H}_P^{0,k} \setminus \{H_j\}), \quad 1 \leq j < k \leq d, (j,k) \neq (2,3)$$
$$\psi_P(v^{2,3}) = \{H_1, H_3\} \cup \begin{cases} (\mathcal{H}_P^\theta \setminus \{H_{v^2}\}) & \text{if } \theta_3 = 1 \\ (\mathcal{H}_P^\theta \setminus \{H_{v^{2,2}}, H_{v^3}\}) & \text{if } \theta_3 \geq 2. \end{cases}$$

**Proof.** The expressions for $\psi_P(0)$ and $\psi_P(\theta)$ are obvious. For the other vertices, because it is trivial to check containment in a coordinate plane using our assumption $\theta \geq 1$, we only argue the incidence of the elements of $\mathcal{H}_P^\theta$. Since $w \in \mathcal{N}_P(\theta)$, we have $\psi_P(w) \supset \mathcal{H}_P^\theta \setminus \{H_w\}$ by construction of $\mathcal{H}_P^\theta$. The value of $\psi_P(v^k)$ follows by a similar reasoning. Now fix $k \geq 2$ and $1 \leq j \leq k - 1$. Define $\xi := M^{-1}(v^{j,k} - \theta)$. Proposition 3.3 gives us $\xi \geq 0$. The rows of $M^{-1}$ correspond to the hyperplanes $H_{v^{j,1}}, H_{v^2}, \ldots, H_{v^d}, H_w$, respectively. Then to prove the claimed expression for $\psi_P(v^{j,k})$, we need to show that $\xi_k = \xi_{k-1} = \cdots = \xi_d = 0$ and $\xi_i > 0$ for $1 \leq i \leq k - 1$, except that $\xi_1 = 0$ when $j = 2, k = 3, \theta_3 = 1$.

The last row in $M^{-1}$, denoted by $M_d^{-1}$, is a vector of $-1$’s, meaning that the facet-defining inequality $M_d^{-1} x \geq M_d^{-1} \theta$ corresponds to the hyperplane $H_0$, which we know contains $v^{j,k}$. Thus

$$\xi_d = M_d^{-1}(v^{j,k} - \theta) = 0. \quad (9a)$$

Now consider the linear system $M\xi = v^{j,k} - \theta$. The upper Hessenberg structure of $M$ gives us the following recursion:

$$\xi_{i-1} = (b_{i-1} + 1)\xi_i - \xi_i \sum_{t=i+1}^d \xi_t - v^{j,k}_i + \theta_i, \quad 3 \leq i \leq d \quad (9b)$$
$$\xi_1 = \frac{b_1 + 1}{\theta_2} \xi_2 - \sum_{t=3}^d \xi_t - \frac{v^{j,k}_2}{\theta_2} + 1 = \frac{\theta_1}{\theta_2} \sum_{t=2}^d \xi_t + \frac{v^{j,k}_2}{\theta_2} - \frac{\theta_1}{\theta_2}. \quad (9c)$$

12
Note that
\[
v_i^{j,k} - \theta_i = \begin{cases} 
0 & \text{if } 1 \leq i \leq d  \\
\theta_i & \text{otherwise.} 
\end{cases}
\] (10)

First let us apply (9b) with \( i = d \). Invoking \( \xi_d = 0 \) from (9a) gives us \( \xi_{d-1} = \theta_d - v_d^{1,k} \), which is equal to \( \theta_d \geq 1 \) if \( k = d \), otherwise it is zero. Equation (10) and a backward induction on \( i \) in (9b) then lead us to
\[
\xi_k = \xi_{k+1} = \cdots = \xi_d = 0.
\]
The expressions for the remaining \( \xi \)'s can be obtained from (9b) and (9c) as
\[
\xi_i = (\tilde{b}_{i-1} + 1)\xi_i - \theta_i \sum_{t=i+1}^{k-1} \xi_t - v_i^{j,k} + \theta_i, \quad 3 \leq i \leq k
\] (11a)
\[
\xi_1 = \frac{\tilde{b}_1}{\theta_2} \xi_2 - \sum_{t=3}^{k-1} \xi_t - v_1^{j,k} + 1 = \frac{\theta_1}{\theta_2} \sum_{i=2}^{k-1} \xi_t + \frac{v_1^{j,k} - \theta_1}{\theta_2}.
\] (11b)

For \( \xi_2, \ldots, \xi_{k-1} \), we claim the following:
\[
\xi_i = \tilde{b}_i \left( \sum_{t=i+1}^{k-1} \xi_t - 1 \right) + \sum_{t=i+2}^{j-1} \theta_t \quad i = 2, \ldots, j - 1
\] (12a)
\[
\xi_i = \tilde{b}_i \sum_{t=i+1}^{k-1} \xi_t + \sum_{t=i+1}^{k} \theta_t \quad i = j, \ldots, k - 1.
\] (12b)

Proving this claim implies \( \xi_2 > \xi_3 > \cdots > \xi_{k-1} \geq 1 \) since \( \theta \geq 1 \). Equation (11b) gives us \( \xi_1 = (\theta_1/\theta_2)(\sum_{t=2}^{k-1} \xi_t - 1) \) if \( j > 1 \) or \( \xi_1 = (\theta_1/\theta_2) \sum_{t=2}^{k-1} \xi_t + (\sum_{t=2}^{k} \theta_t)/\theta_2 \) if \( j = 1 \). For \( (j, k) \neq (2, 3) \), we then have \( \xi_1 > 0 \). For \( (j, k) = (2, 3) \), \( \xi_1 = \theta_1(\theta_3 - 1)/\theta_2 \), which is equal to zero if and only if \( \theta_3 = 1 \). Thus, proving equations (12) finishes our proof for \( \psi_P(v^{i,j,k}) \).

We prove (12a) and (12b) separately by backward induction on \( i \). For (12b), the base case \( \xi_{k-1} = \theta_k \) follows by using \( \xi_k = 0 \) and (10) in (11a). Assume (12b) to be true for \( j < l \leq k - 1 \) and consider \( \xi_{l-1} \). Applying (11a) and using the induction hypothesis yields
\[
\xi_{l-1} = \tilde{b}_{l-1} \xi_l + \xi_l - \theta_l \sum_{t=l+1}^{k-1} \xi_t + \theta_l
\]
\[
= \tilde{b}_{l-1} \xi_l + \tilde{b}_l \sum_{t=l+1}^{k-1} \xi_t + \sum_{t=l+1}^{k} \theta_t - \theta_l \sum_{t=l+1}^{k-1} \xi_t + \theta_l
\]
\[
= \tilde{b}_{l-1} \xi_l + (\tilde{b}_l - \theta_l) \sum_{t=l+1}^{k-1} \xi_t + \sum_{t=l}^{k} \theta_t
\]
\[
= \tilde{b}_{l-1} \sum_{t=l}^{k-1} \xi_t + \sum_{t=l}^{k} \theta_t.
\]
For (12a), the base case formula for $\xi_{j-1}$ can be obtained as follows: from (11a) we have

$$
\xi_{j-1} = \tilde{b}_{j-1} \xi_j + \xi_j - \theta_j \sum_{t=j+1}^{k-1} \xi_t - \tilde{b}_k + \theta_j
$$

$$
= \tilde{b}_{j-1} \xi_j + \tilde{b}_j \sum_{t=j+1}^{k-1} \xi_t + \sum_{t=j+1}^{k} \theta_t - \theta_j \sum_{t=j}^{k-1} \xi_t - \tilde{b}_k + \theta_j
$$

$$
= \tilde{b}_{j-1} \xi_j + (\tilde{b}_j - \theta_j) \sum_{t=j+1}^{k-1} \xi_t + \sum_{t=j}^{k} \theta_t - \tilde{b}_k
$$

$$
= \tilde{b}_{j-1} \xi_j + \tilde{b}_{j-1} \sum_{t=j+1}^{k-1} \xi_t - \tilde{b}_{j-1}
$$

$$
= \tilde{b}_{j-1} \left( \sum_{t=j}^{k-1} \xi_t - 1 \right).
$$

The inductive step is similar to that for (12b). \qed

\textbf{Corollary 3.1.} If $\theta > 1$, $G(P)$ has $\frac{1}{2}(d^2 + d + 2)$ vertices, the edges between which are as follows:

1. $(\theta, w), (\theta, v^{1,2})$, and $(\theta, u^k)$ for $k \geq 3$,
2. $(0, w)$ and $(0, v^{j,d})$ for $1 \leq j \leq d - 1$,
3. $(w, v^{2,3})$ if $d \geq 4$,
4. $(w, v)$ for $v \in \text{vert}(P) \setminus \{v^{2,3}, v^{1,d}, \ldots, v^{d-1,d}\}$,
5. $(u^{k_1}, u^{k_2})$ for $k_1, k_2 \geq 3$,
6. $(v^{k-1,k}, u^k)$ for $k \geq 3$,
7. $(v^{j,k_1}, u^{k_2})$ for $2 \leq k_1 \leq k_2 - 2$,
8. $(v^{j,1,k}, v^{j,2,k})$ for $1 \leq j_1, j_2 \leq k - 1$,
9. $(v^{j,k}, v^{j,k+1})$ for $1 \leq j < k \leq d - 1$.

If $\theta_k = 1$, then $u^k = v^{k-1,k}$, and $G(P)$ is a minor of the above-described graph obtained by contracting the edges $\{(u^k, v^{k-1,k}): \theta_k = 1\}$.

\textbf{Proof.} The neighbors of 0 and $\theta$ are from Proposition 3.2. Let us first argue neighbors of $w$. For $d \geq 4$, $w$ and $v^{2,3}$ share $H_1$ and $H_3$ and at least $d - 3$ planes from $H^\theta_P$, implying an edge between the two. For $d = 3$, the only coordinate plane they share is $H_1$ and so an edge exists if and only if they share $d - 2$ planes from $H^\theta_P$, which happens only when $\theta_3 = 1$. Since $H^\theta_P \setminus \{H_{w^{2,3}}, H_{w^{1,3}}, \ldots, H_{w^d}\} = H_w$, the coordinate vertex $v^{j,d}$ does not share any plane from $H^\theta_P$ with $w$ and shares all the coordinate planes except $H_j$ and $H_d$. Now $j \leq d - 1$ implies that $|\psi_P(w) \cap \psi_P(v^{j,d})| \leq d - 2$ and so there cannot be an edge between $w$ and $v^{j,d}$. For $k \leq d - 1$, edge $(w, v^{j,k})$ exists because $|\psi_P(w) \cap \psi_P(v^{j,k})| \geq d - 1$ due to $\psi_P(w) \cap \psi_P(v^{j,k}) \supseteq \{H_{u^{k+1}}, \ldots, H_{u^d}\} \cup (H^\theta_P \setminus \{H_j\})$. Arguments for the edge $(w, u^k)$ are similar. The $u^k$’s form a clique since for any $3 \leq k_1 < k_2 \leq d$, $\psi_P(u^{k_1}) \cap \psi_P(u^{k_2}) \supseteq \{H_1\} \cup (H^\theta_P \setminus \{H_{u^{k_1}}, H_{u^{k_2}}\})$.

Consider $v^{j,k_1}$ and $u^{k_2}$ for $k_2 \geq 3, 1 \leq j < k_1 \leq d$. We use the following cases.
$2 \leq k_1 \leq k_2 - 1$: Here

$$\psi_P(v^{j,k_1}) \cap \psi_P(u^{k_2}) \cap \mathcal{H}_P^0 = \begin{cases} \mathcal{H}_P^0 \setminus \{H_{u^3, H_{u^2_2}\} & \text{if } k_1 = 3, \theta_3 = 1 \\
\mathcal{H}_P^0 \setminus \{H_{v^{i,2}, H_{u^3}, \ldots, H_{u^{k_1}}, H_{u^{k_2}}} \} & k_1 = 3, \theta_3 \geq 2 \text{ or } k_1 \neq 3. \end{cases}$$

The cardinality of this set is $d - 2$ for $k_1 = 3$ or $d - k_1$ otherwise. Also

$$\psi_P(v^{j,k_1}) \cap \psi_P(u^{k_2}) \cap \mathcal{H}_P^0 = \mathcal{H}_P^{0,k_1-1} \setminus \{H_j\} \cup \begin{cases} \{H_{k_1}\} & \text{if } k_1 \leq k_2 - 2 \\
\emptyset & \text{if } k_1 \leq k_2 - 1. \end{cases}$$

Therefore $|\psi_P(v^{j,k_1}) \cap \psi_P(u^{k_2})| \geq d - 1$ if and only if $3 \neq k_1 \leq k_2 - 2$ or $k_1 = 3, \theta_3 = 1, k_2 \geq 4$ or $k_1 = 3, \theta_3 \geq 2, k_2 \geq 5$.

$3 \leq k_2 \leq k_1 - 1$: Here $k_1 \geq 4$. We have $\psi_P(v^{j,k_1}) \cap \psi_P(u^{k_2}) \cap \mathcal{H}_P^0 = \mathcal{H}_P^0 \setminus \{H_{v^{i,2}, H_{u^3}, \ldots, H_{u^{k_1}}\} \}$, which are exactly $d - (k_1 - 1)$ common planes from $\mathcal{H}_P^0$. So an edge exists if and only if there are at least $k_1 - 2$ common coordinate planes. Note that $\psi_P(v^{j,k_1}) \cap \psi_P(u^{k_2}) \cap \mathcal{H}_P^0 \subseteq \{H_1, \ldots, H_{k_2-2}, H_{k_2}\}$ and so we can have at most $k_2 - 1$ common coordinate planes. Hence, for $k_2 \leq k_1 - 2$, there is no edge, and for $k_1 = k_2 + 1$, an edge exists if and only if $j = k_2 - 1$ and $\theta_k = 1$.

$3 \leq k_1 = k_2 = k$: We have $\psi_P(v^{j,k_1}) \cap \psi_P(u^k) \cap \mathcal{H}_P^0 = \psi_P(v^{j,k}) \cap \mathcal{H}_P^0$ and so the number of common planes from $\mathcal{H}_P^0$ is $d - 1$ when $k = 3, \theta_3 = 1$ or $d - (k - 1)$ otherwise. For $j = k - 1$, the first $k - 2$ coordinate planes are common, giving us an edge between $v^{j-1,k}$ and $u^k$ for all $k$. For $1 \leq j \leq k - 2$, we get $k - 2$ common coordinate planes if and only if $\theta_k = 1$.

Finally, we argue edges between the $v$ vertices. Each $v^{j,k}$ component is a clique because $v^{i,j,k}$ and $v^{j,k} \in \mathcal{H}_P^0 \setminus \{H_{j_1}, H_{j_2}\}$. Now consider $v^{i,j,k_1}$ and $v^{j_2,j,k_2}$ with $k_1 < k_2$. At most $k_1 - 1$ coordinate planes are shared and exactly $d - (k_2 - 1)$ planes from $\mathcal{H}_P^0$ are shared, making the total number at most $d + k_1 - k_2$. This upper bound is less than $d - 1$ if $k_2 \geq k_1 + 2$, meaning that in this case, no edges exist between the cliques $v^{j,k_1}$ and $v^{j,k_2}$. If $k_2 = k_1 + 1$, then the upper bound is equal to $d - 1$ and is attained if and only if the first $k_1 - 1$ coordinate planes are shared, which happens only when $j_1 = j_2$. \hfill \square

**Corollary 3.2.** For $\theta > 1$, the degrees of the vertices of $G(P)$ are

$$\deg(\theta) = \deg(0) = d, \quad \deg(w) = \frac{d^2 - d + 2}{2}, \quad \deg(u^k) = d + \frac{(k - 2)(k - 3)}{2}, \quad k \geq 3.$$ $\deg(v^{j,k}) = d, \quad 2 \leq k \leq d, 1 \leq j \leq k - 1.$

The total number of edges is $\frac{1}{3}(d^3 + 2d)$ and the average degree is $\frac{2}{3}(d - 1 + \frac{d^2 + 2}{d^2 + d - 2}).$

**Proof.** The degree of each vertex follows from the list of edges in Corollary 3.1. The number of
edges is half the sum of all the degrees, making it equal to

$$\frac{1}{2} \left[ 2d + \frac{d^2 - d + 2}{2} + d(d - 2) + \sum_{k=3}^{d} \frac{(k-2)(k-3)}{2} + d \sum_{k=2}^{d} (k-1) \right]$$

$$= \frac{1}{2} \left[ 2d + \frac{d^2 - d + 2}{2} + d^2 - 2d + \frac{1}{2} \sum_{k=1}^{d-3} k(k + 1) + \frac{d^2(d - 1)}{2} \right]$$

$$= \frac{1}{2} \left[ \frac{d^3 + 2d^2 - d + 2}{2} + \frac{1}{2} \sum_{k=1}^{d-3} k(k + 1) \right]$$

$$= \frac{1}{2} \left[ \frac{d^3 + 2d^2 - d + 2}{2} + \frac{1}{2} \left( \frac{(d - 3)(d - 2)(2d - 5)}{6} + \frac{(d - 3)(d - 2)}{2} \right) \right]$$

$$= \frac{1}{2} \left[ \frac{d^3 + 2d^2 - d + 2}{2} + \frac{(d - 3)(d - 2)(d - 1)}{6} \right]$$

$$= \frac{1}{12} \left[ 3d^3 + 6d^2 - 3d + 6 + d^3 - 6d^2 + 11d - 6 \right]$$

$$= \frac{d^3 + 2d}{3}.$$  

The average degree is obtained by dividing twice the above number with the number of vertices $(d^2 + d + 2)/2$.

\[ \square \]

**Corollary 3.3.** The graph of $\mathcal{P}$ has the following properties.

(a) The radius of $G(\mathcal{P})$ is $r(G(\mathcal{P})) = 2$.

(b) The diameter of $G(\mathcal{P})$ is

$$d(G(\mathcal{P})) = \begin{cases} 3 & d \geq 4 \\ 2 & d \in \{2,3\} \end{cases}$$

(c) $G(\mathcal{P})$ is Hamiltonian.

(d) If $\theta > 1$, the chromatic number of $G(\mathcal{P})$ is $\chi(G(\mathcal{P})) = d$.

**Proof.** (a) Since the only common neighbor of $\theta$ and $0$ is $w$, $r(G(\mathcal{P})) \geq 2$. The equality follows from the fact that $w$ can be chosen as a center: for every non-neighbor of $w$ we have $d(w, v^{j,d}) = 2$, because there is a path $w - v^{j,d-1} - v^{j,d}$.

(b) The distance between the non-neighbors of $w$ is $d(v^{j_1,d}, v^{j_2,d}) = 1$. Therefore, $d(G(\mathcal{P})) \leq 3$. The equality follows from the fact that $v^{1,d}$ and $\theta$ have no common neighbors.

(c) Suppose first that $\theta > 1$. For each $k, 3 \leq k \leq d - 1$, let $p_k$ be a Hamiltonian path in the clique $\{v^{j,k}: 1 \leq j < k\}$ between $v^{k-2,k}$ and $v^{k-1,k}$. Then

$$\{0\} - v^{1,d} - v^{2,d} - \ldots - v^{d-1,d} - u^d - u^{d-1} - \ldots - u^3 - \theta - v^{1,2} - p_2 - p_3 - \ldots - p_{d-1} - w - 0$$

is a Hamiltonian cycle in $G(\mathcal{P})$. If $\theta_k = 1$, the same construction gives a Hamiltonian cycle if we take $p_0$ be a Hamiltonian path in the clique $\{v^{j,k}: 1 \leq j < k\}$.

(d) Since $\{v^{j,d}: 1 \leq j \leq d - 1\} \cup \{0\}$ is a $d$-clique, $\chi(G(\mathcal{P})) \geq d$. On the other hand, one can readily check that $\varphi : V \to \{1, 2, \ldots, d\}$, given by $\varphi(0) = \varphi(\theta) = 1, \varphi(w) = 2, \varphi(u^k) = k, \varphi(v^{j,k}) = k - j + 1$, is a proper coloring of $G(\mathcal{P})$.  

\[ \square \]
The edge expansion $h(G)$ of a graph $G$ on $n$ vertices is defined as

$$h(G) = \min_{S \subseteq V(G): 0 < |S| \leq \frac{n}{2}} \frac{|\partial S|}{|S|},$$

where

$$\partial S := \{(u,v) \in E(G) : u \in S, v \in V(G) \setminus S\}.$$

**Theorem 3.2.** The edge expansion of the graph $G(\mathcal{P})$ is $h(G(\mathcal{P})) = 1$.

**Proof.** For the set $S = \{v^{j,d} : 1 \leq j \leq d - 1\} \cup \{0\}$,

$$\partial S = \{(v^{j,d}, v^{j,d-1}) : 1 \leq j \leq d - 2\} \cup \{(v^{d-1,d}, u^d), (0, w)\}$$

and hence

$$\frac{|\partial S|}{|S|} = \frac{d - 2 + 2}{d - 1 + 1} = 1.$$

Therefore,

$$h(G(\mathcal{P})) \leq 1.$$ 

Suppose now $|S| \leq n/2$. We will consider two cases.

**Case 1:** $w \in S$. The set $\partial S$ contains $|S^c \cap N_\mathcal{P}(w)|$ edges incident with $w$. Let $|S^c \cap N_\mathcal{P}(w)^c| = a$. Then $0 \leq a \leq d - 1$ and $\partial S$ contains $a(d - 1 - a)$ edges from the $(d - 1)$-clique $N_\mathcal{P}(w)^c$. If $0 \leq a \leq d - 2$, then $a(d - 1 - a) \geq a$ and

$$\frac{|\partial S|}{|S|} \geq \frac{|S^c \cap N_\mathcal{P}(w)| + a}{|S|} = \frac{|S^c \cap N_\mathcal{P}(w)| + |S^c \cap N_\mathcal{P}(w)^c|}{|S|} = \frac{|S^c|}{|S|} \geq 1. \quad (13)$$

If $a = d - 1$, then $\{v^{j,d} : 1 \leq j \leq d - 1\} \subseteq S^c$. Let $k \geq 2$ be the maximal so that $\{v^{j,k} : 1 \leq j \leq k - 1\} \cup \{u^{k+2}, \ldots, u^d\} \not\subseteq S^c$. If such a $k$ doesn’t exist then $S \subseteq \{w, \theta, u^3, 0\}$ and

$$\frac{|\partial S|}{|S|} \geq \frac{|S^c \cap N_\mathcal{P}(w)|}{|S|} = \frac{|S^c| - (d - 1)}{|S|} \geq \frac{n - 4 - (d - 1)}{4} \geq 1.$$

So, assume such a $k$ does exist and let

$$\left|\left\{(v^{j,k} : 1 \leq j \leq k - 1) \cup \{u^{k+2}, \ldots, u^d\}\right]\cap S\right| = b \geq 1.$$

Then $\partial S$ contains $b(d - 2 - b)$ edges from the $(d - 2)$-clique $\{v^{j,k} : 1 \leq j \leq k - 1\} \cup \{u^{k+2}, \ldots, u^d\}$ and, because of the maximality of $k$, $b$ edges of the type $(v^{j,k}, v^{j,k+1})$ and $(u^{k'}, v^{j, k'-1, k'})$ for some $j < k$ and $k' \geq k + 2$. For $1 < b < d - 2$,

$$b(d - 2 - b) + b \geq d - 1 = a$$

and (13) holds. If $b = d - 2$ then $v^{k-1,k} \in S$, $u^{k,k+1} \in S^c$ and $\partial S$ additionally contains one of the edges $(v^{k-1,k}, u^k)$, $(v^{k,k+1}, u^{k+1})$, $(u^k, u^{k+1})$. Then

$$b(d - 2 - b) + b + 1 = d - 1 = a$$

and (13) holds. Finally, let $b = 1$. Then either $\{u^2 := \theta, u^3, u^4, \ldots, u^d\} \subseteq S^c$ or $\partial S$ contains one of the edges $(v^{d-1,d}, u^d)$, $(u^k, u^{k_2})$ for some $k_1 \neq k_2$. So, suppose $\{u^2 := \theta, u^3, u^4, \ldots, u^d\} \subseteq S^c$.
and let $v^{i,k} \in S$. If $j = k-1$ then $(v^{i,k}, u^k) \in \partial S$. If $j < k-1$, then one of the edges $(v^{i,k}, v^{i,k-1})$, $(u^{k+1}, v^{i,k-1})$ is in $\partial S$. Either way, we have established that

$$|\partial S| \geq |S^c \cap \mathcal{N}_P(w)| + b(d - 2 - b) + b + 1 = |S^c \cap \mathcal{N}_P(w)| + d - 1 = |S^c \cap \mathcal{N}_P(w)| + a$$

and (13) holds.

**Case 2:** $v \notin S$. If $S \subseteq \mathcal{N}_P(w)$ then clearly $|\partial S| \geq |S|$. Suppose $|S \cap \mathcal{N}_P(w)^c| = a > 1$. The $a \leq d + 1$ and $\partial S$ contains $a(d - 1) - a)$ edges from the $(d - 1)$-clique $\mathcal{N}_P(w)^c$. If $1 \leq a \leq d - 2$, then $a(d - 1) - a) \geq a$ and

$$|\partial S| - |S^c \cap \mathcal{N}_P(w)| + a = |S \cap \mathcal{N}_P(w)|(|S \cap \mathcal{N}_P(w)^c) = |S| = 1. \quad (14)$$

So, let $a = d - 1$. Then $\{v^{j,d}: 1 \leq j \leq d - 1\} \subseteq S$. Let $k \geq 2$ be the maximal so that

$$\{v^{i,k}: 1 \leq j \leq k - 1\} \cup \{u^{k+2}, \ldots, u^d\} \not\subseteq S.$$

Note that such a $k$ exists because otherwise $|S| \geq n/2$. Let

$$\left|\left\{v^{i,k}: 1 \leq j \leq k - 1\right\} \cup \{u^{k+2}, \ldots, u^d\}\right| \cap S^c = b \geq 1.$$ 

Reasoning as in Case 1, where the role of $S$ and $\partial S$ are swapped, we conclude that $\partial S$ contains at least $a$ more edges and therefore (14) holds. \hfill $\square$

## 4 The grevlex polytope $\mathcal{Q}$

In this section we will describe the main properties of the polytope $\mathcal{Q}$. Throughout, $\preceq$ will denote the grevlex order.

### 4.1 \(\mathcal{Q}\)-polytope

Consider the following integral points:

$$\bar{u}^k := (\tilde{b}_{k-1} \mathbf{e}_{k-1}, \theta_k, \theta_{k+1}, \ldots, \theta_d) \quad 2 \leq k \leq d + 1 \quad (15a)$$

$$\bar{v}^{j,k} := ((\tilde{b}_{j-1} - 1) \mathbf{e}_j, 0, \theta_0 + 1, \theta_{k+1}, \ldots, \theta_d) \quad 1 \leq j < k - 1 \leq d. \quad (15b)$$

In particular, $\bar{u}^2 = \theta$, $\bar{u}^{d+1} = b \mathbf{e}_d$, and $\bar{v}^{d+1} = (b - 1) \mathbf{e}_j$. By construction, we have

**Observation 4.1.** $\bar{u}^k \in H_0$ for all $k$, $\bar{v}^{j,k} \in H_0$ for $k \leq d$, $\bar{v}^{j,d+1} \not\in H_0$ for $1 \leq j \leq d - 1$.

**Proposition 4.1.** The vertices of $\mathcal{Q}$ are

$$\text{vert}(\mathcal{Q}) = \{0\} \bigcup \{\bar{u}^k: 2 \leq k \leq d + 1\} \bigcup \{\bar{v}^{j,k}: 1 \leq j \leq k - 1 \leq d\}.$$

**Proof.** It is clear that 0 cannot be written as a nontrivial convex combination of integral points in $\mathcal{Q}$. Suppose $\bar{u}^k = \sum_{i=1}^s \lambda_i x^i$ is a nontrivial convex combination of some $x^i \in \mathcal{Q} \cap \mathbb{Z}^d$. Since $H_0$ is a facet of $\mathcal{Q}$, $x^i \in H_0$. Let

$$m = \max \{j: x^i_j \neq \theta_i \text{ for some } i\}.$$ 

Then $\theta \preceq x^i_i \implies x^i_m \geq \theta_m$, leading to $\sum_{i=1}^s \lambda_i x^i_m > \theta_m$. Therefore, $m = k - 1$. Also, $x^i_j = 0$ for all $i$ and $j < k - 2$. So, the only possibility is $x^i = \bar{u}^k$ for all $i$.

Now suppose $\bar{v}^{j,k} = \sum_{i=1}^s \lambda_i x^i$, $k \leq d$ is a nontrivial convex combination of some $x^i \in \mathcal{Q} \cap \mathbb{Z}^d$. As before, we conclude $x^i \in H_0$. Also, $x^i_r = 0$ for $r \in \{1, 2, \ldots, j - 1, j + 1, \ldots, k - 1\}$. Reasoning
the same way as in the case of $\bar{u}^k$, we conclude that $x^i_r = \theta_r$ for $r > k$ and, therefore, $x^i_k \geq \theta_k + 1$. This in turn implies that $x^i_k = \theta_k + 1$ and thus $x^i = \bar{v}^j k$ for all $i$.

Now we argue that if $v \in \text{vert}(Q) \setminus \{0\}$, then $v$ must be equal to some $\bar{u}^k$ or $\bar{v}^j k$. Since

$$Q = \text{conv} \left( \left\{ x \in \mathbb{R}^d : \sum_{i=1}^d x_i \leq b - 1 \right\} \cup \text{conv} \left( \left\{ x \in \mathbb{Z}^d : \sum_{i=1}^d x_i = b, \theta \leq \text{lex} x \right\} \right) \right),$$

we have

$$\text{vert}(Q) \subseteq \text{vert}\{ x \in \mathbb{R}^d : \sum_i x_i \leq b - 1 \} \cup \text{vert}(Q \cap H_0).$$

Note that $(b - 1)e_j$ is not a vertex of $Q$ because $0, be_d \in Q$ and we have already shown that all the other vertices of the simplex $\{ x \in \mathbb{R}^d : \sum_i x_i \leq b - 1 \}$ are also vertices of $Q$. Suppose now that $x \in Q \cap H_0$ is a vertex of $Q$. Let $k = \max\{ i : x_j \neq \theta_i \}$. Then $k \geq 2$ and $x_k \geq \theta_k + 1$.

- Suppose first $x_k \geq \theta_k + 2$ and there is $j < k$ such that $x_j > 0$. Let $x' = x + e_j - e_k$ and $x'' = x - e_j + e_k$. Then $x = (x' + x'')/2$ and since $x', x'' \in Q$, we conclude $x$ is not a vertex of $Q$. If there is no $j < k$ such that $x_j > 0$ then $x_k = \bar{b}_k$ and $x = \bar{u}^{k+1}$.

- Suppose now $x_k = \theta_k + 1$. If there exist $i < j < k$ such that $x_i, x_j > 0$ then $x = ((x + e_i - e_j) + (x - e_i + e_j))/2$ and, therefore, $x$ is not a vertex of $Q$. Otherwise either $x = \bar{v}^j k$ for some $j < k - 1$ or $x = ((\bar{b}_k - 1)e_{k-1}, \theta_k + 1, \theta_{k+1}, \ldots, \theta_d)$. But in the latter case

$$x = \bar{b}_{k-1} - 1 \bar{b}_{k-1} (\bar{b}_{k-1} e_{k-1}, \theta_k, \theta_{k+1}, \ldots, \theta_d) + \frac{1}{b_{k-1}} (\bar{b}_k e_k, \theta_{k+1}, \ldots, \theta_d)$$

and, therefore, $x$ is not a vertex of $Q$. \[\square\]

As in Observation 3.2, the coordinate planes define facets of $Q$. We refer to all other facets of $Q$ as nontrivial facets. The grading plane $H_0$ defines a nontrivial facet since the face $Q \cap H_0$ contains $d$ affinely independent vertices $\theta, \bar{u}^2, \ldots, \bar{u}^{d+1}$. We show that the coefficients of any nontrivial facet-defining inequality are nonnegative and nonincreasing.

Lemma 4.1. Suppose $Q \subseteq \{ x : cx \leq c_0 \}$ and let $F = Q \cap \{ x : cx = c_0 \}$ be a nontrivial face of $Q$. If $F$ is not contained in $x_i = 0$ for some $i \in \{ 1, \ldots, d - 1 \}$ then $0 \leq c_{i+1} \leq c_i$. Consequently, if $F$ is a nontrivial facet then $0 \leq c_{i+1} \leq c_i$ for all $1 \leq i \leq d - 1$.

Proof. Let $\bar{x}$ be a vertex on $F$ with $\bar{x}_i \geq 1$. Consider first $x' = \bar{x} - e_i$. Since $0 \leq x' \leq \bar{x}$, we have $c x' \leq c_0 = c \bar{x}$, which implies $c_{i+1} \geq 0$. The point $x'' = \bar{x} - e_i + e_{i+1}$ has the property $0 \leq x'' \leq \bar{x}$. Therefore, we have $c x'' \leq c_0 = c \bar{x}$, which yields $c_{i+1} \leq c_i$. \[\square\]

The above property is useful for characterizing the neighbors of $\theta$.

Proposition 4.2. The neighbors of $\theta$ and $0$ are

$$\mathcal{N}_Q(\theta) = \{ \bar{v}^3 \} \cup \{ \bar{v}^{1,k} : 3 \leq k \leq d + 1 \}, \quad \mathcal{N}_Q(0) = \{ \bar{u}^{d+1} \} \cup \{ \bar{v}^{j,d+1} : 1 \leq j \leq d - 1 \}.$$

Proof. Let $F = Q \cap \{ x : cx = c_0 \}$ be an edge of $Q$ defined by $cx \leq c_0$ such that $\theta \in F$. We argue by contradiction that this edge contains exactly one of the proposed neighboring vertices in $\mathcal{N}_P(Q)$, or equivalently that none of the other vertices belong to this edge. First suppose that $\bar{v}^{j,k} \in F$ for some $2 \leq j < k - 1 \leq d$. Now $c \bar{v}^{j,k} - c \bar{v}^{j,k} = (\bar{b}_{k-1} - 1)(c_{j-1} - c_j)$, which is nonnegative because of Lemma 4.1. Therefore $\bar{v}^{j-1,k} \in F$, a contradiction to the edge property of $F$. For $\bar{u}^k$ with $k \geq 4$, similar reasoning carries through by considering $\bar{u}^{k-1}$. For $\mathcal{N}_Q(0)$, noting that the proposed neighbors are coordinate vectors, the proof is exactly the same as that in Proposition 3.2. \[\square\]
Thus $\mathcal{Q}$ has at least $2d$ facets, $d$ of which are coordinate planes that contain $0$ and the other $d$ contain $\theta$. We show in Theorem 4.1 that there are no other facets. The following properties about nontrivial facets will be useful.

**Lemma 4.2.** Let $F$ be a nontrivial facet of $\mathcal{Q}$ defined by $cx \leq c_0$.

1. $\tilde{v}^{i,k} \in F$ implies $\tilde{v}'^{j,k} \in F$ for all $1 \leq j' \leq j$ and $c_1 = c_2 = \cdots = c_j$.

2. If $F$ is not defined by $H_0$, then $F$ contains some coordinate vertex, i.e., there exists some $1 \leq j \leq d - 1$ such that $\tilde{v}^{j,d+1} \in F$.

3. $\bar{u}^k \notin F$ implies $\bar{u}'^k \notin F$ for $k + 1 \leq k' \leq d + 1$.

**Proof.** (1) This is because $c\bar{v}^{j,t} - c\bar{v}'^{j',t} = (c_j - c_{j'}) (\bar{b}_{t-1} - 1) \leq 0$ and so $c\bar{v}^{j,t} = c_0$ implies $c\bar{v}'^{j,t} = c_0$. Consequently, we also get $c_1 = c_2 = \cdots = c_j$.

(2) To argue this, recall from Observation 4.1 that $\text{vert}(\mathcal{Q}) \setminus \{0, \bar{v}^{1,d+1}, \ldots, \bar{v}^{d,d+1}\} \subseteq H_0$. So if $\tilde{v}^{j,d+1} \notin F$ for all $j$, the nontriviality of $F$ (i.e., $0 \notin F$) then implies that $F = \mathcal{Q} \cap H_0$.

(3) We know from Lemma 4.1 that $F = \{x \in \mathcal{Q}: \alpha x = \alpha_0\}$ with $0 \leq \alpha_n \leq \alpha_{n-1} \leq \cdots \leq \alpha_1$. Then for $2 \leq t \leq d$, we have

$$\alpha \bar{u}'^t - \alpha \bar{u}'^{t+1} = \alpha_{t-1} \bar{b}_{t-1} + \alpha_t (\theta_t - \bar{b}_t) = (\alpha_{t-1} - \alpha_t) \bar{b}_{t-1} \geq 0.$$  

Therefore, $\alpha \bar{u}'^{d+1} \leq \alpha \bar{u}'^d \leq \cdots \leq \alpha \bar{u}'^2 = \alpha \theta$. If $\bar{u}'^k \notin F$, then $\alpha \bar{u}'^k < \alpha_0$ and the claim follows.  

**Theorem 4.1.** $\mathcal{Q} = \mathcal{D}(\mathcal{Q}, 0, \theta)$.

**Proof.** We need to prove that every facet of $\mathcal{Q}$ contains either $0$ or $\theta$. Let $F$ be a facet of $\mathcal{Q}$ induced by the valid inequality $cx \leq c_0$. If $F$ doesn’t contain $\theta$ nor any of the vertices $\bar{v}^{1,k}$, then it is contained in the subspace $x_1 = 0$ and hence is equal to the trivial facet defined by $x_1 \geq 0$ and therefore contains $0$. For an arbitrary nontrivial facet $F$, Lemma 4.1 tells us

$$0 \leq c_n \leq c_{n-1} \leq \cdots \leq c_1,$$  

(16a)

Now assume $\bar{v}^{1,k} \in F$ for some $3 \leq k \leq d - 1$ and $0 \notin F$. Then it must be that $F$ is a nontrivial facet. Suppose for contradiction that $\theta \notin F$. This means $F \neq \mathcal{Q} \cap H_0$ and also has two other implications. First, we have $c_1 > c_{k-1}$. Suppose this is not true, which by (16a) means that $c_1 = c_2 = \cdots = c_{k-1}$. Then

$$0 < c_0 - c\theta = c\bar{v}^{1,k} - c\theta = c_k + c_1 (\bar{b}_{k-1} - 1) - \sum_{j=1}^{k-1} c_1 \theta_j = c_k - c_1,$$

which contradicts $c_1 \geq c_k$ from (16a). Second, we have $\bar{u}'^k \notin F$ for all $k$ due to $\theta = \bar{u}'^2$, the assumption $c\theta < c_0$ and the third item in Lemma 4.2.

Now let $j$ be the maximal index such that $\tilde{v}^{j,d+1} \in F$; we know such a $j$ exists because of $F \neq \mathcal{Q} \cap H_0$ and the second item in Lemma 4.2. If $j \geq k - 1$, then applying the first item in Lemma 4.2 to $\bar{v}^{j,d+1}$ would imply $c_1 = c_{k-1}$, a contradiction to $c_1 > c_{k-1}$. Hence $1 \leq j \leq k - 2$ and (16a) and maximality of $j$ lead us to

$$c_1 = \cdots = c_j, \quad c_1 > c_i, \quad i = j + 1, \ldots, d.$$  

(16b)

Using above and $c\theta < c\bar{v}^{1,k}$ by assumption, gives us

$$c_k - c_1 > \sum_{i=j+1}^{k-1} (c_i - c_1) \theta_i.$$  

(16c)
Since $F$ is not contained in any coordinate plane and $\bar{u}^k \notin F \forall k$, we know that for every $t = j + 1, \ldots, k - 1$, there exist some $(i_t, k_t)$ such that $\bar{v}^{i_t, k_t} \in F$ and either $k_t \leq t$ or $i_t = t$. The second possibility $i_t = t$ can be ruled out since applying the first item in Lemma 4.2 to $\bar{v}^{i_t, k_t}$ would imply $c_1 = c_t$, a contradiction to (16b) due to $t \geq j + 1$. Therefore $1 \leq i_t < k_t - 1 \leq t - 1$. Now since $\bar{v}^{i_t, k_t} \in F$, the first item in Lemma 4.2 implies $\bar{v}^{i_t, k_t}$, which upon simplification yields $c_k - c_{k_t} = \sum_{i=j+1}^{k-1} (c_i - c_1)\theta_i$ for $j + 1 \leq t \leq k - 1$. Choosing $t = j + 1$ gives us

$$c_k - c_{k+1} = \sum_{i=j+1}^{k-1} (c_i - c_1)\theta_i = \sum_{i=j+1}^{k-1} (c_i - c_1)\theta_i,$$

where the second equality is due to $k_{j+1} \leq j + 1$ by construction, and $c_1 = \cdots = c_j$ from (16b). Since $k_{j+1} \leq j + 1$, (16b) tells us $c_{k_{j+1}} \leq c_1$. Substituting this into (16d) leads to $c_k - c_1 \leq \sum_{i=j+1}^{k-1} (c_i - c_1)\theta_i$, but this is a contradiction to (16c).

Similar to $\mathcal{P}$, for $d \geq 4$ the only antipodal vertex pair of $Q$ is $(0, \theta)$. We will argue this separately in Corollary 4.4.

### 4.2 $H$-polytope

By Proposition 2.1, we need to invert

$$M = \begin{bmatrix} \bar{v}^{3} - \theta & \bar{v}^{1,3} - \theta & \bar{v}^{1,4} - \theta & \cdots & \bar{v}^{1,d} - \theta & \bar{v}^{1,d+1} - \theta \end{bmatrix}$$

$$= \begin{bmatrix} -\theta_1 & \theta_2 - 1 & \theta_2 + \theta_3 - 1 & \theta_2 + \theta_3 + \theta_4 - 1 & \cdots & \theta_2 + \cdots + \theta_{d-1} - 1 & \theta_2 + \cdots + \theta_d - 1 \\ \theta_1 & -\theta_2 & -\theta_2 & -\theta_2 & \cdots & -\theta_2 & -\theta_2 \\ 0 & 1 & -\theta_4 & \cdots & -\theta_4 & -\theta_4 \\ \vdots & 0 & 1 & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & \cdots & 1 & -\theta_d \end{bmatrix}$$

Let

$$q_i^j = \begin{cases} \theta_i \theta_j \prod_{k=i+1}^{j-1} (\theta_k + 1) & j > i \\ \theta_i & j = i \\ 1 & j < i. \end{cases}$$

Proposition 4.3. $Q = \{ x \geq 0 : N x \geq N \theta \}$ where $N = M^{-1}$ with

$$N_{i,d} = \begin{cases} -1 & i = d \\ -\theta_d + 1 & i = d - 1 \\ -q_i^d & 2 \leq i \leq d - 2 \\ -q_i^d \frac{\theta_i}{\theta_1} & i = 1 \\ \frac{1}{\theta_1} & j = 1 \\ \frac{\theta_{j-1}}{\theta_1} & j = 2 \\ \frac{q_j^j}{\theta_1} & 3 \leq j \leq d - 1 \end{cases}$$

and for $i \geq 2$

$$N_{i,j} = N_{i,j+1} + \begin{cases} 0 & j < i \\ -1 & j = i \\ -\theta_j + 1 & j = i + 1 \\ -q_i^{j+1} & i + 2 \leq j \leq d - 1 \end{cases}$$
Proof. Suppose \( i \neq 1, j \neq 1 \). Then
\[
N_i.M_j = \sum_{l=1}^{j+1} N_{i,l}M_{l,j+1}
\]
\[
= N_{i,1}(\theta_2 + \cdots + \theta_j - 1) - \sum_{l=2}^{j} N_{i,l}\theta_j + N_{i,j+1}
\]
\[
= \sum_{l=2}^{j} (N_{i,1} - N_{i,l})\theta_l + (N_{i,j+1} - N_{i,1})
\] (17)

So, if \( j < i \), then since \( N_{i,1} = N_{i,2} = \cdots = N_{i,i} \) we have \( N_i.M_j = 0 \). If \( j = i \), then \( N_i.M_j = N_{i,i+1} - N_{i,i} = 1 \). For \( j > i \),
\[
N_i.M_j = N_i.M_{j-1} + (N_{i,1} - N_{i,j})\theta_j + (N_{i,j+1} - N_{i,j}).
\]
Therefore, if \( j = 1 + 1 \), then
\[
N_i.M_{i+1} = 1 + (-1)\theta_{i+1} + (\theta_{i+1} - 1) = 0.
\]
For, \( j > i + 1 \), inductively we get
\[
N_i.M_j = 0 + (- \sum_{k=i+1}^{j-1} q_{i+1}^k)\theta_j + q_{i+1} = 0.
\]

The entries \( N_i.M_j \) and \( N_i.M_1 \) can be computed in a similar way. \( \square \)

4.3 Graph of the polytope

We derive some basic properties of the graph of \( Q \), denoted by \( G(Q) \). This graph has \( \frac{d^2 + d + 2}{2} \) vertices incidence enumerated in Proposition 4.1. To find all the edges of \( G(Q) \), we characterize the vertex-facet incidence for \( Q \) in Proposition 4.4. Adopting the same approach as in §3.3 to denote \( H_v \), for \( v \in N_Q(\theta) \), the only facet-defining hyperplane that contains \( \theta \) but not \( v \), we have
\[
\mathcal{H}_Q^\theta := \{ H_{\bar{u}^3}, H_{\bar{u}^i,\bar{u}}, \ldots, H_{\bar{u}^i,d+1} \}, \text{ with}
\]
\[
H_{\bar{u}^3} = \{ x : N_1.(x - \theta) = 0 \}, \quad H_{\bar{u}^i,k} = \{ x : N_{(k-1)}.(x - \theta) = 0 \} \quad 3 \leq k \leq d + 1,
\]
where \( N \) is the matrix inverse from Proposition 4.3. The hyperplanes incident to \( 0 \) are the coordinate planes denoted in (8b). As before, for any \( v \in \text{vert}(Q) \), let \( \psi_Q(v) \) denote the subset of facet-defining hyperplanes of \( Q \) that contain \( v \).

Proposition 4.4. We have
\[
\psi_Q(0) = \mathcal{H}_Q^0, \quad \psi_Q(\theta) = \mathcal{H}_Q^\theta,
\]
\[
\psi_Q(\bar{u}^k) = (\mathcal{H}_Q^0 \setminus \{ H_{\bar{u}^3}, H_{\bar{u}^i,\bar{u}}, \ldots, H_{\bar{u}^i,k-1} \}) \cup \mathcal{H}_Q^{0,k-2}, \quad 3 \leq k \leq d + 1
\]
\[
\psi_Q(\bar{v}^{j,k}) = (\mathcal{H}_Q^0 \setminus \{ H_{\bar{u}^3}, H_{\bar{u}^i,\bar{u}}, \ldots, H_{\bar{u}^j,\bar{u}}, H_{\bar{u}^j,k} \}) \cup (\mathcal{H}_Q^{0,k-1} \setminus \{ H_j \}), \quad 2 \leq j < k - 1 \leq d
\]
\[
\psi_Q(\bar{v}^{1,k}) = (\mathcal{H}_Q^0 \setminus \{ H_{\bar{u}^i,k} \}) \cup (\mathcal{H}_Q^{0,k-1} \setminus \{ H_1 \}), \quad 3 \leq k \leq d + 1.
\]

Proof. The coordinate planes are trivial to check due to \( \theta \geq 1 \), whereas \( \psi_Q(0) \) and \( \psi_Q(\theta) \) follow from the construction of \( \mathcal{H}_Q^\theta \) and \( \mathcal{H}_Q^0 \). It remains to argue the incidence of a hyperplane in \( \mathcal{H}_Q^\theta \) onto a \( \bar{u}^k \), for \( k \geq 3 \), or a \( \bar{v}^{j,k} \). The formula for \( N \) in Proposition 4.3 and the monotone property of facet coefficients in Lemma 4.1 tells us that
\[
-N_{t-1,1} = \cdots = -N_{t-1,t-1} > -N_{t-1,t} \geq -N_{t-1,t+1} \geq \cdots \geq -N_{t-1,d} \quad 2 \leq t \leq d.
\] (18)
Consider $\bar{u}^k$ for $k \geq 3$. Since $\bar{u}^3 \notin H_{\bar{q}^3}$ by construction, the third item in Lemma 4.2 implies that $\bar{u}^k \notin H_{\bar{q}^3}$. For any $3 \leq j \leq k - 1$, since $\theta \in H_{\bar{q}^1,j}$, we have $\bar{u}^k \notin H_{\bar{q}^1,j}$ if and only if $N_{(j-1),\bar{u}^k} < N_{(j-1),\theta}$. Now

$$N_{(j-1),\bar{u}^k} = N_{j-1,k-1}\bar{b}_{k-1} + \sum_{i=k}^{d} N_{j-1,i}\theta_i$$

$$= N_{j-1,k-1}\bar{b}_{j-1} + N_{j-1,k-1} \sum_{i=j}^{k-1} \theta_i + \sum_{i=k}^{d} N_{j-1,i}\theta_i$$

$$\leq \sum_{i=j}^{k-1} N_{j-1,i}\theta_i$$

$$< N_{(j-1),\theta},$$

where the inequalities are due to equation (18). Therefore $\bar{u}^k \notin H_{\bar{q}^1,j}$ for $3 \leq j \leq k - 1$, and so $\bar{u}^k \in H$ for some $H \in H^\theta_\bar{q}$ only if $H$ is one of the $d - k + 2$ hyperplanes $H_{\bar{q}^1,k}, H_{\bar{q}^1,k+1}, \ldots, H_{\bar{q}^1,d+1}$. Since exactly $k - 2$ coordinate planes contain $\bar{u}^k$ and we know that $|\psi_Q(\bar{u}^k)| \geq d$ due to $\bar{u}^k$ being a vertex of the $d$-polytope $Q$, it follows that $\bar{u}^k \in H_{\bar{q}^1,t}$ for $k \leq t \leq d + 1$.

Now we derive $\psi_Q(\bar{v}^{j,k})$. Note that $k \geq 3$. The construction of $H^\theta_{\bar{q}}$ and $\bar{v}^{1:k} \in \mathcal{N}_P(\theta)$ implies that $\psi_Q(\bar{v}^{1:k}) \cap H^\theta_{\bar{q}} = \emptyset \setminus \{H_{\bar{q}^1,k}\}$. This leads to $\bar{v}^{j,k} \notin H_{\bar{q}^1,k}$ because otherwise the first item in Lemma 4.2 gives the contradiction $\bar{v}^{1,k} \in H_{\bar{q}^1,k}$. Consider the hyperplane $H_{\bar{q}^1,t} := \{x : N_{(t-1),x} = c_0\}$ for $3 \leq t \leq d + 1, t \neq k$, which contains $\bar{v}^{1,k}$. Then $\bar{v}^{j,k} \in H_{\bar{q}^1,t}$ if and only if $N_{(t-1),\bar{v}^{j,k}} = N_{(t-1),\bar{v}^{1,k}}$. Now, $N_{(t-1),\bar{v}^{j,k}} - N_{(t-1),\bar{v}^{1,k}} = (N_{(t-1,j} - N_{(t-1,1)})(\bar{b}_{t-1} - 1)$ and since $\bar{b}_{t-1} > 1$ for $k \geq 3$ due to $\theta \geq 1$, we have $\bar{v}^{j,k} \in H_{\bar{q}^1,t}$ if and only if $N_{t-1,j} - N_{t-1,1}$. Equation (18) tells us that $N_{t-1,1} = N_{t-1,j}$ if and only if $j < t - 1$, which, along with $t \neq k$, is equivalent to $t \in \{j + 1, \ldots, k - 1, k + 1, \ldots, d + 1\}$. The claim for $\psi_Q(\bar{v}^{j,k})$ follows. The arguments for $\bar{v}^{j,k} \notin H_{\bar{q}^3}$, for $j \geq 2$, are similar.

Since $(v, v')$ is an edge in $G(Q)$ if and only if $|\psi_Q(v) \cap \psi_Q(v')| \geq d - 1$, Proposition 4.4 implies a complete list of edges and thereby the degree of each vertex.

**Corollary 4.1.** $G(Q)$ has $\frac{1}{3}(d^2 + d + 2)$ vertices, the edges between which are as follows:

1. $(0, \bar{v}^d)$ and $(0, \bar{v}^{d+1})$ for $1 \leq j \leq d - 1$,
2. $(\bar{u}^k, \bar{u}^{k+1})$ for $2 \leq k \leq d$,
3. $(\bar{u}^k, \bar{v}^{j,k-1})$ for $3 \leq k \leq d + 1$,
4. $(\bar{u}^j, \bar{v}^{j,k})$ for $2 \leq j \leq d$,
5. $(\bar{v}^{j,k}, \bar{v}^{j,k+1})$ for $k_1 \neq k_2$,
6. $(\bar{v}^{j,k}, \bar{v}^{j,k})$ for $j_1 \neq j_2$,
7. $(\bar{v}^{j,k}, \bar{v}^{j,k})$ if $(j_1 < j_2 < k_1 < k_2$ and $k_1 \geq j_2 + 3)$ or $(j_2 < j_1 < k_1 < k_2$ and $k_1 \geq j_1 + 3)$.

**Corollary 4.2.** The degrees of the vertices of $G(Q)$ are

$$\deg(0) = \deg(\bar{u}^k) = d, \quad \deg(\bar{v}^{j,k}) = \begin{cases} \frac{d}{2(j-2)(k-2)(j-3)} & k = j + 2 \\ \frac{d}{2} & k \geq j + 3. \end{cases}$$

The total number of edges is $\frac{d^4}{12} \frac{3d^3}{4} + \frac{7d^2}{24} - \frac{2d}{4} + \frac{7}{2}$ and the average degree is $\frac{2d^3 - 9d^2 + 73d^2 - 126d + 84}{6(d^2 + d + 2)}$.

As a consequence we see that $P$ and $Q$ define two different families of Dantzig figures.

23
Corollary 4.3. For $\theta > 1$, $\mathcal{P}$ and $\mathcal{Q}$ have the same number of vertices but they are not combinatorially equivalent.

Proof. For $d = 3$, as can be seen from Figure 1, $G(\mathcal{P})$ has a pentagonal facet, while $G(\mathcal{Q})$ doesn’t. For $d \geq 4$, the highest degree vertex in $\mathcal{P}$ is $w$ with $\deg(w) = \frac{d^2-d+2}{2}$, while in $\mathcal{Q}$, the highest degree vertex is $\bar{v}^{1,d+1}$ with $\deg(\bar{v}^{1,d+1}) = \frac{d^2-3d+8}{2} < \frac{d^2-d+2}{2}$.

In $d = 3$, $\mathcal{Q} = \mathcal{D}(\mathcal{Q}, \bar{v}^{1,3}, \bar{v}^{2,4})$. But, for $d \geq 4$ $(0, \theta)$ are the only antipodal vertices of $\mathcal{Q}$.

Corollary 4.4. For $d \geq 4$, $(0, \theta)$ is the only antipodal vertex pair that generates the Dantzig figure $\mathcal{Q}$.

Proof. Since any antipodal vertex pair $(x, y)$ of a $d$-dimensional Dantzig figure must have $\deg(x) = \deg(y) = d$, Corollary 4.2 tells us that the only candidate vertices for forming an antipodal pair of $\mathcal{Q}$ are $0, \theta, \{\bar{u}^k: 3 \leq k \leq d+1\}, \{\bar{v}^{j,j+2}: 1 \leq j \leq d-1\}$. We also need $\psi_{\mathcal{Q}}(x) \cap \psi_{\mathcal{Q}}(y) = \emptyset$ for any antipodal pair. Proposition 4.4 gives us $H_1 \in \psi_{\mathcal{Q}}(\bar{u}^k) \cap \psi_{\mathcal{Q}}(\bar{v}^{j,j+2})$ for $k \geq 3, j \geq 2$, $\psi_{\mathcal{Q}}(\bar{u}^k) \cap H^j_\theta \neq \emptyset$ for $k \geq 3$, and $\psi_{\mathcal{Q}}(\bar{v}^{j,j+2}) \cap H^j_\theta \neq \emptyset$ for $j \geq 1$. The only remaining possibility is $\bar{v}^{1,3}$ but this is also easy to discard with similar arguments. □

Corollary 4.5. The graph of $\mathcal{Q}$ has the following properties.

(a) The radius of $G(\mathcal{Q})$ is $r(G(\mathcal{Q})) = 2$.
(b) The diameter of $G(\mathcal{Q})$ is $d(G(\mathcal{Q})) = 2$.
(c) $G(\mathcal{Q})$ is Hamiltonian.
(d) The chromatic number of $G(\mathcal{Q})$ is $\chi(G(\mathcal{Q})) \geq d$.

Proof. (a) Since no vertex is connected to everything $r(G(\mathcal{Q})) \geq 2$. The equality follows from the fact that $\bar{v}^{1,d+1}$ can be chosen as a center. Namely the vertices that are not neighbors of $\bar{v}^{1,d+1}$ are $\bar{u}^k$, $3 \leq k \leq d+1$ and $\bar{v}^{j,j+2}$, $2 \leq j \leq d-2$. But for these there are paths $\bar{v}^{1,d+1} - \bar{v}^{1,k-1} - \bar{u}^k$ and $\bar{v}^{1,d+1} - \bar{v}^{1,j+2} - \bar{v}^{j,j+2}$.

(b) It suffices to show any two non-neighbors of $\bar{v}^{1,d+1}$ have a common neighbor. For $j = k-1$, $(\bar{u}^j, \bar{u}^k)$ is an edge. For $j = k$, we have $\bar{u}^j - \bar{v}^{j-1,k-1} - \bar{u}^k$. Also, for $j < k$, we have $\bar{v}^{j,j+2} - \bar{v}^{j,k+2} - \bar{v}^{k,k+2}$. Finally, we have the paths $\bar{v}^{j,j+2} - \bar{v}^{j-1,k-1} - \bar{u}^k$ for $k \geq j + 3$, $\bar{v}^{j,j+2} - \bar{v}^{j-1,k-2} - \bar{u}^k$ for $k \geq j + 1$, and $\bar{v}^{j,j+2} - \bar{v}^{j-1,j+2} - \bar{u}^k$ for $k = j + 2$.

(c) For each $k$, $3 \leq k \leq d+1$, let $p_k$ be a Hamiltonian path in the clique $\{\bar{v}^{j,k}: 1 \leq j < k-1\}$ between $\bar{v}^{k-3,k}$ and $\bar{v}^{k-2,k}$. Then

$$0 - \bar{u}^{d+1} - \bar{u}^d - \cdots - \bar{u}^2 - p_3 - p_4 - \cdots - p_{d+1} - 0$$

is a Hamiltonian cycle in $G(\mathcal{Q})$.

(d) This is clear since $\{0\} \cup \{\bar{v}^{j,d+1}: 1 \leq j \leq d-1\}$ is a $d$-clique. □

References

