The $p$-cones in dimension $n \geq 3$ are not homogeneous when $p \neq 2$

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Abstract

Using the $T$-algebra machinery we show that the only strictly convex homogeneous cones in $\mathbb{R}^n$ with $n \geq 3$ are the 2-cones, also known as Lorentz cones or second order cones. In particular, this shows that the $p$-cones are not homogeneous when $p \neq 2$, $1 < p < \infty$ and $n \geq 3$, thus answering a problem proposed by Gowda and Trott.

Keywords: Homogeneous cone, $p$-cone, $T$-algebra.

1 Introduction

We prove that if $p \neq 2$ and $1 < p < \infty$, then the $p$-cones $\mathcal{L}_p^n$ in the $n$-dimensional space $\mathbb{R}^n$ are not homogeneous when $n \geq 3$. This solves a problem posed by Gowda and Trott in Section 6 of [5], where they proved the non-homogeneity of $\mathcal{L}_2^n$ and its dual $\mathcal{L}_\infty^n$. In fact, we will prove a more general statement and show that the only strictly convex homogeneous cones are the 2-cones, that is, the Lorentz cones.

Recently, in a published article, we saw an attempt to equip $\mathbb{R}^n$ with an inner product (depending on $p$) so that $\mathcal{L}_p^n$ becomes self-dual. In the same article, it was claimed that $\mathcal{L}_p^n$ is homogeneous, thus showing that it is a symmetric cone under an appropriate inner product. The result we prove here implies, in particular, that there is no inner product under which the $\mathcal{L}_p^n$ become symmetric cones, except for the well-known case of $p = 2$.

Our result also provides some intuition at why it is significantly harder to optimize over $\mathcal{L}_p^n$ than over $\mathcal{L}_2^n$, for $p \neq 2$. Consider, for instance, interior point methods. Their performance is highly dependent on the so-called complexity parameter of self-concordant barriers and for $\mathcal{L}_2^n$, the complexity of an optimal barrier is 2, regardless of $n$. In contrast, all the known barriers for $\mathcal{L}_p^n$ have complexity parameter proportional to $n$ and the best one so far has complexity $4n$, see Section 1 in the work by Nesterov [7]. If $\mathcal{L}_p^n$ were homogeneous of rank $r$, we would readily have access to a barrier with parameter $r$, due to a result by Güller and Tunçel, see Theorem 4.1 in [6] and the related paper by Truong and Tunçel [8]. As the rank of a homogeneous cone is no larger than its dimension, we would have been able to construct a barrier of parameter less or equal than $n$, which is something that, as far as we known, has never appeared in the literature. It seems that it is still an open problem to determine the optimal barrier parameter for $\mathcal{L}_p^n$.

The outline of the proof is simple. Using the theory of $T$-algebras, we first check that the closure of all homogeneous cones of rank $r \leq 2$ are isomorphic to either $\{0\}$, $\mathbb{R}_+$, $\mathbb{R}_+^2$ or $\mathcal{L}_2^n$. Then, we argue that if $\mathcal{L}_p^n$ were to be a homogeneous cone then its rank would be less or equal than 2, which is the main missing piece we supply in this article. Now, Gowda and Trott proved in [5] that $\mathcal{L}_2^n$ and $\mathcal{L}_p^n$ are not

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isomorphic for \( n \geq 3 \) and \( p \neq 2 \). So for \( n \geq 3 \) and \( 1 < p < \infty \), \( L^p_n \) cannot possibly be homogeneous, since it is not isomorphic to the lower dimensional cones \( \{0\}, \mathbb{R}_+, \mathbb{R}^2_+ \).

2 Preliminaries on convex cones

A convex cone is a set \( K \) contained in some real vector space \( A \), such that \( \alpha x + \beta y \in K \), for all \( x, y \in K \) and \( \alpha, \beta \in \mathbb{R}_+ \). If \( A \) is equipped with some inner product \( \langle \cdot, \cdot \rangle \) we can we define the dual cone of \( K \) as \( K^* = \{ x \in A \mid \langle x, y \rangle \geq 0, \forall y \in K \} \). We will write int \( K \), cl \( K \), dim \( K \) for the interior, closure and dimension of \( K \), respectively.

A cone is said to be pointed if \( \text{cl} K \cap -\text{cl} K = \{0\} \) and it is said to be full-dimensional if \( \text{int} K \neq \emptyset \). Note that all cones can be made to be full-dimensional if we substitute the underlying space by the span of \( K \). An automorphism of \( K \) is a linear bijection \( Q \) such that \( QK = K \). Then, \( K \) is said to be homogeneous if it is a full-dimensional pointed convex cone such that its group of automorphisms acts transitively on int \( K \). This means that for every \( x, y \in \text{int} K \), there is an automorphism of \( K \) for which \( Q(x) = y \).

In some works on convex cones its common to consider open cones, that is, cones satisfying \( K = \text{int} K \). In fact, the definition of “convex cone” by Vinberg included the requirement that \( K \) should be open. In optimization, however, it is common to consider closed cones. Suppose that \( K \) is full-dimensional, then \( \text{int} K = (\text{int} K) = (\text{cl} K) \). Therefore, for the study of homogeneity, it does not matter whether we study int \( K \), \( K \) or cl \( K \).

Two cones \( K_1, K_2 \) are said to be isomorphic if there is a linear bijection \( Q \) such that \( QK_1 = K_2 \). Note that if \( K_1 \) and \( K_2 \) are full-dimensional, then \( K_1 \) and \( K_2 \) are isomorphic if and only if int \( K_1 \) and int \( K_2 \) are isomorphic.

Let \( C \) be a convex set. A convex set \( F \subseteq C \) is said to be a face of \( C \) if the following condition holds: if \( x, y \in C \) and \( ax + (1 - \alpha)y \in F \) for some \( 0 < \alpha < 1 \), then \( x, y \in F \). A face \( F \) is said to be proper if \( F \neq C \). An extreme point is a face consisting of a single point.

2.1 Strictly convex cones

A compact convex set \( C \) with nonempty interior is said to be strictly convex if every proper face of \( C \) is an extreme point. Similarly, a pointed closed convex cone \( K \) with nonempty interior is said to be a strictly convex cone if every proper face of \( K \) has dimension 0 or 1.

A norm \( \|\cdot\| \) on a real vector space is said to be a strictly convex norm if

\[
\|x + y\| < 2 \text{ whenever } \|x\| = \|y\| = 1, \ x \neq y
\]

or, equivalently,

\[
\|\alpha x + (1 - \alpha)y\| < 1 \text{ whenever } \|x\| = \|y\| = 1, \ x \neq y, \ \alpha \in (0, 1).
\]

The relations between these notions is as follows.

**Proposition 1.** Let \( \|\cdot\| \) be a norm on a real vector space \( A \). Then, the following are equivalent.

(i) \( \|\cdot\| \) is a strictly convex norm.

(ii) \( B = \{ x \in A \mid \| x \| \leq 1 \} \) is a strictly convex set.

(iii) \( K = \{ (t, x) \in \mathbb{R} \times A \mid \| x \| \leq t \} \) is a strictly convex cone.

**Proof.** Note that \( B \) is strictly convex if and only if \( \alpha x + (1 - \alpha) \in \text{int} B \) whenever \( x, y \in B \) and \( \alpha \in (0, 1) \). This proves the equivalence \( (i) \Leftrightarrow (ii) \).

Now, it is straightforward to check that, for a compact convex set \( C \), the map

\[
\mathcal{F} \mapsto \text{cone}(\{1\} \times \mathcal{F}) = \{ (t, tx) \mid t \geq 0, \ x \in \mathcal{F} \}
\]
is a bijection from the set of faces of $C$ onto the set of nonzero faces of cone $(\{1\} \times C)$. Hence, the equivalence $(ii) \Leftrightarrow (iii)$ immediately follows because $K = \text{cone} \{1\} \times B$ holds.

We are particularly interested in the case of $p$-norms that is,

$$\|x\|_p = \left(\sum_{i=1}^{n}|x_i|^p\right)^{1/p}$$

for $p \in [1, \infty)$ and $\|x\|_\infty = \max_{1 \leq i \leq n}|x_i|$. Then, the $p$-cone is defined as

$$\mathcal{L}_p^n = \{(t,x) \in \mathbb{R} \times \mathbb{R}^{n-1} \mid t \geq \|x\|_p\}.$$ 

It follows by Proposition 1 that only the $p$-cones $\mathcal{L}_p^n$ for $1 < p < \infty$ are strictly convex, since those are the values that correspond to strictly convex norms.

3 $T$-algebras

$T$-algebras were proposed by Vinberg [9] as a natural framework for the study of homogeneous cones. For a more recent treatment, including its connections to optimization, see the work of Chua [1, 2].

We recall that an algebra is a vector space $A$ over some field $\mathbb{K}$ such that $A$ is equipped with a product $\times : A \times A \to A$ that satisfies

$$(a + b) \times c = a \times c + b \times c$$

$$c \times (a + b) = c \times a + c \times b$$

$$(\alpha a) \times (\beta b) = (\alpha \beta) a \times b,$$

for all $a, b, c \in A, \alpha, \beta \in \mathbb{K}$. In the cases we discuss here, we will always consider the real number field $\mathbb{R}$ and write $a \times b = ab$, for $a, b \in A$. Then, a matrix algebra of rank $r$ is an algebra over $\mathbb{R}$ that is equipped with a decomposition as a direct sum $A = \bigoplus_{i,j=1}^{r} A_{ij}$ satisfying the following properties:

$$A_{ij}A_{jk} \subseteq A_{ik}$$

$$A_{ij}A_{kl} = 0 \quad \text{if } j \neq k.$$

This decomposition is called a bigradation. Therefore, in a matrix algebra we can represent an element $a \in A$ as a generalized matrix $a = (a_{ij})\rho_{i,j=1}^{r}$, where $a_{ij} \in A_{ij}$, for all $i, j$. With that, the multiplication in $A$ follows the usual matrix multiplication rules $(ab)_{ij} = \sum_{k=1}^{r} a_{ik}b_{kj}$.

A matrix algebra with involution is a matrix algebra equipped with a linear bijection $*: A \to A$ such that

$$a^{**} = a, \quad (ab)^* = b^*a^* \quad \text{and} \quad A_{ij}^* = A_{ji} \quad \text{for all } i, j.$$ 

With that, we have $(a^*)_{ij} = a_{ji}^*.$

Finally, a $T$-algebra is a matrix algebra with involution satisfying the following properties, see Definition 4 in [2].

(i) For each $i$, $A_{ii}$ is a subalgebra isomorphic to $\mathbb{R}$.

Let $\rho_i : A_{ii} \to \mathbb{R}$ denote the algebra isomorphism and let $e_i$ denote the unit element in $A_{ii}$, i.e., the element satisfying $\rho_i(e_i) = 1$. Furthermore, define the function $\text{tr} : A \to \mathbb{R}$ by $\text{tr}(a) := \sum_{i=1}^{r} \rho_i(a_{ii})$.

(ii) For all $a \in A$ and all $i, j \in \{1, \ldots, r\}$ we have $e_i a_{ij} = a_{ij}$ and $a_{ji} e_i = a_{ji}$.

(iii) For all $a \in A$ and all $i, j \in \{1, \ldots, r\}$, $\rho_i(a_{ij} b_{ji}) = \rho_j(b_{ji} a_{ij})$.

(iv) For all $a, b, c \in A$ and $i, j, k \in \{1, \ldots, r\}$, we have $a_{ij}(b_{jk} c_{ki}) = (a_{ij} b_{jk}) c_{ki}$.  

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(v) For all $a \in \mathcal{A}$ and $i, j \in \{1, \ldots, r\}$, we have $\rho_1(a_{ij}^*a_{ij}) \geq 0$, with equality if and only if $a_{ij} = 0$.

(vi) For all $a, b, c \in \mathcal{A}$ and $1 \leq i \leq j \leq k \leq l \leq r$, we have $a_{ij}(b_{jk}c_{kl}) = (a_{ij}b_{jk})c_{kl}$.

(vii) For all $a, b \in \mathcal{A}$ and $1 \leq i \leq j \leq k \leq r$ and $1 \leq l \leq k \leq r$, we have $a_{ij}(b_{jk}b_{lk}^*) = (a_{ij}b_{jk})b_{lk}^*$.

For a $T$-algebra $\mathcal{A}$ of rank $r$, we write $\mathcal{T}$ for the set of “upper-triangular matrices” in $\mathcal{A}$, i.e., $\mathcal{T} = \{a \in \mathcal{A} \mid a_{ij} = 0 \text{ if } 1 \leq j < i \leq r\}$. We define $\mathcal{T}_+ = \{a \in \mathcal{T} \mid \rho_i(a_{ii}) \geq 0 \text{ if } 1 \leq i \leq r\}$ and $\mathcal{T}_{++} = \{a \in \mathcal{T} \mid \rho_i(a_{ii}) > 0 \text{ if } 1 \leq i \leq r\}$. With that, we define the cone associated to the $T$-algebra $\mathcal{A}$ as $\mathcal{K}(\mathcal{A}) = \{tt^* \mid t \in \mathcal{T}_{++}\}$.

We also have $\text{cl}\mathcal{K}(\mathcal{A}) = \{tt^* \mid t \in \mathcal{T}_+\}$, see the remarks before Proposition 1 in [2]. Vinberg proved in [9] the following landmark result.

**Theorem 2.** Let $\mathcal{K}$ be an open homogeneous convex cone. Then, there is a $T$-algebra $\mathcal{A}$ for which $\mathcal{K}(\mathcal{A}) = \mathcal{K}$. Conversely, if $\mathcal{K}(\mathcal{A}) = \mathcal{K}$ for some $T$-algebra $\mathcal{A}$, then $\mathcal{K}$ is an open homogeneous convex cone.

Following Theorem 2, we define the rank of a homogeneous cone as the rank of the underlying algebra. This is well-defined because if $\mathcal{A}$ and $\mathcal{A}'$ are two $T$-algebras such that $\text{int}\mathcal{K} = \mathcal{K}(\mathcal{A}) = \mathcal{K}(\mathcal{A}')$, then $\mathcal{A}$ and $\mathcal{A}'$ must be isomorphic due to Theorem 4 in Chapter 3 of [9]. We now state a few elementary observations about diagonal elements.

**Proposition 3.** Let $\mathcal{A}$ be a $T$-algebra and $a \in \mathcal{A}, t \in \mathcal{T}_+$, then for every $i$

(i) $a_{ii}^* = a_{ii}$,

(ii) $a_{ii} = \rho_i(a_{ii})e_i$,

(iii) $\rho_i((tt^*)_ii) \geq 0$.

**Proof.** (i) The restriction of the involution $^*$ to $\mathcal{A}_{ii}$ becomes an automorphism of $\mathcal{A}_{ii}$. Since $\mathcal{A}_{ii}$ is isomorphic to $\mathbb{R}$ and the only automorphism of $\mathbb{R}$ is the identity map, we conclude that the restriction of $^*$ to $\mathcal{A}_{ii}$ must be the identity map as well.

(ii) The map $\rho_i$ is an algebra isomorphism, so it satisfies $\rho_i(\alpha x) = \alpha \rho_i(x)$ for every $\alpha \in \mathbb{R}$ and $x \in \mathcal{A}_{ii}$. Since $\rho_i(\rho_i(a_{ii})e_i) = \rho_i(a_{ii})1$ and $\rho_i$ is a bijection, we conclude that $a_{ii} = \rho_i(a_{ii})e_i$.

(iii) Note that $(tt^*)_ii = \sum_{j=i}^{r} t_{ij}t_{ij}^*$. Therefore, $\rho_i((tt^*)_ii) = \sum_{j=i}^{r} \rho_i(t_{ij}t_{ij}^*)$. From Axiom (v), every term inside the summation is nonnegative, so that $\rho_i((tt^*)_ii) \geq 0$.

Finally, we define an inner product over $\mathcal{A}$ by taking $\langle x, y \rangle = \text{tr}(x^*y)$.

## 4 Main result

We will now gather a few results that will allow us to prove our main result. For what follows, recall that if $\mathcal{K}$ is a closed convex cone and $y \in \mathcal{K}^*$ then $\mathcal{K} \cap \{y\}^\perp = \{x \in \mathcal{K} \mid \langle x, y \rangle = 0\}$ is always a nonempty face of $\mathcal{K}$.

**Proposition 4.** The closure of a homogeneous cone $\mathcal{K}$ of rank $r \geq 1$ contains a proper face of dimension at least $r - 1$. 


Proof. If $r = 1$, we take $\{0\}$ as the desired face. So suppose that $r \geq 2$. Consider a $T$-algebra $A$ of rank $r$ such that $\text{int} K = K(A)$. We will prove that $(cl K(A)) \cap \{e_1\}^\perp$ has dimension greater or equal than $r - 1$. We first argue that $e_1 \in K(A)^*$. Let $x \in K(A)$, then $(e_1 x)_{11} = x_{11}$. By Proposition 3, we have $\rho_1(x_{11}) \geq 0$, therefore $(e_1 x) = tr(e_1 x) = \rho_1(x_{11}) \geq 0$.

We have $cl K = cl K(A) = \{tt^* \mid t \in T_+\}$. For all $i$, we have $e_i \in T_+$, so that $e_i e_i^* = e_i e_i = e_i \in cl K$. Since $A$ is a matrix algebra, $e_i e_j = 0$ if $i \neq j$. Therefore, $(e_i, e_j) = 0$ for $i \neq j$. This shows that $\{e_2, \ldots, e_r\} \subseteq (cl K) \cap \{e_1\}^\perp$, so that the dimension of $(cl K) \cap \{e_1\}^\perp$ is at least $r - 1$.

\[\square\]

**Proposition 5.** Let $K$ be a convex cone such that the proper faces of $cl K$ have dimension 0 or 1. If $K$ is homogeneous and nonzero, then its rank is less or equal than 2.

**Proof.** It is an immediate consequence of Proposition 4. If $K$ were homogeneous of rank $r \geq 3$, then it would have a proper face of dimension at least 2. Therefore, $r \leq 2$.

It turns out that the closure of a homogeneous cone of rank $r \leq 2$ is self-dual under an appropriate inner product, which is something mentioned by Vinberg in [9], see Section 8 of Chapter 3. Using that, the next result follows from the classification of symmetric cones, see Chapter 5 in [3]. Nevertheless, we will give a constructive proof that explicitly exhibits the isomorphism.

**Proposition 6.** Let $K$ be a nonzero homogeneous cone of rank $r \leq 2$. Then $cl K$ is isomorphic to either $\mathbb{R}_+, \mathbb{R}_+^2$ or $\mathcal{L}_2^n$ for some $n$.

**Proof.** Let $A$ be a $T$-algebra of rank $r$ such that $K(A) = \text{int} K$. If $r = 1$, it is clear that $cl K$ must be isomorphic to $\mathbb{R}_+$. If $r = 2$, we consider two cases. If $A_{12} = A_{21} = \{0\}$ it is clear that $cl K$ must be isomorphic to $\mathbb{R}_+^2$. So now we consider the case where the dimension of $A_{12}$ is greater than zero and we identify $A_{12}$ with some $\mathbb{R}^m$ with $m > 0$.

We first establish necessary and sufficient conditions for $a \in A$ to belong in $K(A)$. Due to Proposition 3, we can write $a_{11} = \alpha e_1, a_{22} = \beta e_2$, where $\alpha = \rho_1(a_{11})$ and $\beta = \rho_2(a_{22})$.

Now, $a \in K(A)$ if and only if there is $t \in T_+$ such that $a = tt^*$. Similarly, we can write $t_{11} = \gamma_1 e_1, t_{22} = \gamma_2 e_2$, where $\gamma_1 = \rho_1(t_{11})$ and $\gamma_2 = \rho_2(t_{22})$. We have

\[
tt^* = t_{11} t_{11} + t_{12} t_{12}^* + t_{12} t_{22}^* + t_{22} t_{22}
= (\gamma_2^2 + \rho_1(t_{12} t_{12}^*)) e_1 + \gamma_2 \gamma_2 e_1 + \gamma_2 t_{12}^* + \gamma_2^2 e_2
= (\gamma_2^2 + \rho_1(t_{12} t_{12}^*)) e_1 + \gamma_2 t_{12} + \gamma_2^2 e_2 + \gamma_2 e_2,
\]

where the last equality follows from Axiom (ii). Then, comparing the expressions for $tt^*$ and $a$, we conclude that in order to have $tt^* = a$, we must have $a = a^*$ and

\[
\gamma_2 = \sqrt{\beta},
\]

\[
t_{12} = \frac{a_{12}}{\sqrt{\beta}},
\]

\[
\gamma_1^2 = \alpha - \frac{\rho_1(a_{12} a_{12}^*)}{\beta}.
\]

Since $\gamma_1$ and $\gamma_2$ must be positive, we conclude that $a \in K(A)$ if and only if $\alpha = a^*, \alpha > 0, \beta > 0$ and $\alpha \beta - \rho_1(a_{12} a_{12}^*) > 0$. The last inequality can be expressed equivalently as

\[
\left(\frac{\alpha + \beta}{2}\right)^2 > \left(\frac{\alpha - \beta}{2}\right)^2 + \rho_1(a_{12} a_{12}^*).
\]

Now, notice that we have $\rho_1(a_{12} a_{12}^*) = (a_{12}^*)^T (Q a_{12}) = (a_{12}, a_{12})$ when $a = a^*$. Restricting $\langle \cdot, \cdot \rangle$ as an inner product on $A_{12}$, we can write $(u, v) = u^T Q^T Q v$ for each $u, v \in A_{12}$ with some $m \times m$ nonsingular matrix $Q$. Then, we conclude that $a \in K(A)$ if and only if $a = a^*, \alpha + \beta > 0$ and

\[
\left(\frac{\alpha + \beta}{2}\right)^2 > \left(\frac{\alpha - \beta}{2}\right)^2 + (Q a_{12})^T (Q a_{12}),
\]

\[\square\]
which is equivalent to the statement that \( (\alpha + \frac{\beta}{2}, \alpha - \frac{\beta}{2}, Qa_{12}) \in \text{int } L^m_2 \).

Let \( \mathcal{H}(A) = \{ a \in A \mid a = a^* \} \) and consider the linear map \( S : \mathcal{H}(A) \to \mathbb{R}^{m+2} \) defined by
\[
S(a) := \left( \frac{\rho_1(a_{11}) + \rho_2(a_{22})}{2}, \frac{\rho_1(a_{11}) - \rho_2(a_{22})}{2}, Qa_{12} \right) = \left( \frac{\alpha + \beta}{2}, \frac{\alpha - \beta}{2}, Qa_{12} \right).
\]
Since \( Q \) is nonsingular, \( S \) is a bijection. Then, the discussion so far implies that \( S(K(A)) = \text{int } L^m_2 \).

Therefore, \( S \) is the desired isomorphism.

We thus arrive at the main result.

**Theorem 7.** Suppose that \( K \) is a cone such that \( \text{cl } K \) is strictly convex. Then, it is homogeneous if and only if \( \text{cl } K \) is isomorphic to \( \{0\} \), \( \mathbb{R}_+ \), \( \mathbb{R}_+^2 \) or \( L^n_2 \).

In particular, \( L^p_\alpha \) is not homogeneous if \( p \neq 2 \) and \( n \geq 3 \).

**Proof.** It is clear that \( \{0\} \), \( \mathbb{R}_+ \), \( \mathbb{R}_+^2 \) and \( L^n_2 \) are homogeneous cone. For the converse, if \( K = \{0\} \) we are done. Otherwise, by Proposition 6, \( K \) must be isomorphic to \( \mathbb{R}_+ \), \( \mathbb{R}_+^2 \) or \( L^n_2 \). This concludes the first half.

Now, suppose that \( K \) is homogeneous and \( K = L^p_n \) for \( p \neq 2 \), \( 1 < p < \infty \) and \( n \geq 3 \). Then \( K \) is strictly convex and, therefore, must be isomorphic to one of the four cones listed above. The only possible candidate is \( L^n_2 \), since all the others have dimension less or equal than 2. However, the results by Gowda and Trott in [5] imply that \( L^n_2 \) and \( L^p_n \) are not isomorphic if \( p \neq 2 \) and \( n \geq 3 \), since an invariant known as “Lyapunov rank” is \( \frac{n^2 - p + 2}{2} \) for the former and 1 for the latter. Furthermore, isomorphic cones have the same Lyapunov rank. See Section 1 and Theorem 5 in [5] and the related paper [4], for more details. This gives a contradiction, so \( K \) is not homogeneous.

We remark that the Gowda and Trott already showed that \( L^n_1 \) and \( L^n_\infty \) are not homogeneous for \( n \geq 3 \), see Theorem 7 and Section 6 of [5].

**References**


