The $p$-cones in dimension $n \geq 3$ are not homogeneous when $p \neq 2$

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Abstract

Using the $T$-algebra machinery we show that, up to linear isomorphism, the only strictly convex homogeneous cones in $\mathbb{R}^n$ with $n \geq 3$ are the 2-cones, also known as Lorentz cones or second order cones. In particular, this shows that the $p$-cones are not homogeneous when $p \neq 2$, $1 < p < \infty$ and $n \geq 3$, thus answering a problem proposed by Gowda and Trott.

Keywords: Homogeneous cone, $p$-cone, $T$-algebra.

1 Introduction

We prove that if $p \neq 2$ and $1 < p < \infty$, then the $p$-cones $\mathcal{L}_p^n$ in the $n$-dimensional space $\mathbb{R}^n$ are not homogeneous when $n \geq 3$. This solves a problem posed by Gowda and Trott in Section 6 of [5], where they proved the non-homogeneity of $\mathcal{L}_1^n$ and its dual $\mathcal{L}_\infty^n$. In fact, we will prove a more general statement and show that, up to isomorphism, the only strictly convex homogeneous cones are the 2-cones, that is, the Lorentz cones.

Recently, in a published article, we saw an attempt to equip $\mathbb{R}^n$ with an inner product (depending on $p$) so that $\mathcal{L}_p^n$ becomes self-dual. In the same article, it was claimed that $\mathcal{L}_p^n$ is homogeneous, thus showing that it is a symmetric cone under an appropriate inner product. For a discussion of its flaws, we refer to the paper by Miao, Lin, Chen [8]. The result we prove here implies, in particular, that for $n \geq 3$ and $p \neq 2$, $\mathcal{L}_p^n$ is never a symmetric cone. This, however, does not rule out the possibility of $\mathcal{L}_p^n$ being self-dual under an appropriate inner product. We remark that it can be shown that $\mathcal{L}_p^n$ is not self-dual under “reasonable” inner products, see Section 3 in [8].

The discussion here also provides some intuition at why it seems to be significantly harder to optimize over $\mathcal{L}_p^n$ than over $\mathcal{L}_2^n$, for $p \neq 2$ and $1 < p < \infty$. The culprit is probably not the (apparent) absence of self-duality but, instead, the lack of homogeneity. Consider, for instance, interior point methods. Their performance is highly dependent on the so-called complexity parameter of self-concordant barriers and for $\mathcal{L}_2^n$, the complexity of an optimal barrier is 2, regardless of $n$. In contrast, all the barriers for $\mathcal{L}_p^n$ with known closed-form expressions have complexity parameter proportional to $n$ and the best one so far has complexity $4n$, see Section 1 in the work by Nesterov [9]. Recently, Hildebrand proved [7] that a regular $n$-dimensional convex cone admits a barrier with parameter not exceeding $n$, which implies that $\mathcal{L}_p^n$ also has a barrier with parameter $n$, although it could be hard to compute it or to obtain a closed form expression. If $\mathcal{L}_p^n$ were homogeneous of rank $r$, Proposition 5 would imply $r \leq 2$. Then, we would readily have access to a barrier with parameter not larger than 2, due to a result by Güller and Tunçel, see Theorem 4.1 in [6] and the related paper by Truong and Tunçel [10]. A barrier for $\mathcal{L}_p^n$ not depending on $n$ has never appeared previously in the literature and

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would be a truly remarkable object. It seems that it is still an open problem to determine the optimal barrier parameter for $L^n_p$.

The outline of the proof is simple. Using the theory of $T$-algebras, we first check that the closure of an homogeneous cone of rank $r \leq 2$ is isomorphic to either $\{0\}$, $\mathbb{R}^+$, $\mathbb{R}^2_+$ or $L^n_2$. Then, we argue that if $L^n_p$ were to be a homogeneous cone then its rank would be less or equal than 2, which is the main missing piece we supply in this article. Now, Gowda and Trott proved in [5] that $L^n_2$ and $L^n_p$ are not isomorphic for $n \geq 3$ and $p \neq 2$. So for $n \geq 3$ and $1 < p < \infty$, $L^n_2$ cannot possibly be homogeneous, since it is not isomorphic to the lower dimensional cones $\{0\}$, $\mathbb{R}^+$, $\mathbb{R}^2_+$.

2 Preliminaries on convex cones

A convex cone is a set $K$ contained in some real vector space $A$, such that $\alpha x + \beta y \in K$, for all $x, y \in K$ and $\alpha, \beta \in \mathbb{R}_+$. If $A$ is equipped with some inner product $\langle \cdot, \cdot \rangle$ we can define the dual cone of $K$ as $K^* = \{ x \in A \mid \langle x, y \rangle \geq 0, \forall y \in K \}$. We will write $\text{int } K, \text{cl } K, \text{dim } K$ for the interior, closure and dimension of $K$, respectively.

A convex cone $K$ is said to be pointed if $\text{cl } K \cap -\text{cl } K = \{0\}$ and it is said to be full-dimensional if $\text{int } K \neq \emptyset$. Note that all convex cones can be made to be full-dimensional if we substitute the underlying space by the span of $K$. An automorphism of $K$ is a linear bijection $Q$ such that $QK = K$. Then, $K$ is said to be homogeneous if it is a full-dimensional pointed convex cone such that its group of automorphisms acts transitively on $\text{int } K$. This means that for every $x, y \in \text{int } K$, there is an automorphism of $K$ for which $Q(x) = y$.

In some works on convex cones it is common to consider open convex cones, that is, convex cones satisfying $K = \text{int } K$. In fact, the definition of “convex cone” by Vinberg included the requirement that $K$ should be open. In optimization, however, it is common to consider closed convex cones. Suppose that $K$ is full-dimensional, then $\text{int } K = \text{int } (\text{int } K) = \text{int } (\text{cl } K)$. Therefore, for the study of homogeneity, it does not matter whether we study $\text{int } K, \text{cl } K$ or $\text{cl } K$.

Two convex cones $K_1, K_2$ are said to be isomorphic if there is a linear bijection $Q$ such that $QK_1 = K_2$. Note that if $K_1$ and $K_2$ are full-dimensional, then $\text{cl } K_1$ and $\text{cl } K_2$ are isomorphic if and only if $\text{int } K_1$ and $\text{int } K_2$ are isomorphic.

Let $C$ be a convex set. A convex set $F \subseteq C$ is said to be a face of $C$ if the following condition holds: if $x, y \in C$ and $\alpha x + (1 - \alpha)y \in F$ for some $0 < \alpha < 1$, then $x, y \in F$. A face $F$ is said to be proper if $F \neq C$. An extreme point is a face consisting of a single point.

2.1 Strictly convex cones

A compact convex set $C$ with nonempty interior is said to be strictly convex if every proper face of $C$ is an extreme point. Similarly, a pointed closed convex cone $K$ with nonempty interior is said to be a strictly convex cone if every proper face of $K$ has dimension 0 or 1.

A norm $\|\cdot\|$ on a real vector space is said to be a strictly convex norm if

$$\|x + y\| < 2 \text{ whenever } \|x\| = \|y\| = 1, \ x \neq y$$

or, equivalently,

$$\|\alpha x + (1 - \alpha)y\| < 1 \text{ whenever } \|x\| = \|y\| = 1, \ x \neq y, \ \alpha \in (0, 1).$$

The relations between these notions is as follows.

**Proposition 1.** Let $\|\cdot\|$ be a norm on a real vector space $A$. Then, the following are equivalent.

(i) $\|\cdot\|$ is a strictly convex norm.

(ii) $B = \{ x \in A \mid \|x\| \leq 1 \}$ is a strictly convex set.
(iii) \( \mathcal{K} = \{(t, x) \in \mathbb{R} \times \mathcal{A} \mid \|x\| \leq t\} \) is a strictly convex cone.

**Proof.** Note that \( B \) is strictly convex if and only if \( \alpha x + (1 - \alpha) \in \text{int} B \) whenever \( x, y \in B \) and \( \alpha \in (0, 1) \). This proves the equivalence (i) \( \iff \) (ii).

Now, it is straightforward to check that, for a compact convex set \( C \), the map

\[
F \mapsto \text{cone} \left( \{1\} \times F \right) = \{ (t, tx) \mid t \geq 0, \ x \in F \}
\]

is a bijection from the set of faces of \( C \) onto the set of nonzero faces of \( \text{cone} \left( \{1\} \times C \right) \). Hence, the equivalence (ii) \( \iff \) (iii) immediately follows because \( K = \text{cone} \left( \{1\} \times B \right) \) holds. \( \square \)

We are particularly interested in the case of \( p \)-norms that is, \( \|x\|_p = \left( \sum_{i=1}^n |x_i|^p \right)^{1/p} \) for \( p \in [1, \infty) \) and \( \|x\|_\infty = \max_{1 \leq i \leq n} |x_i| \). Then, the \( p \)-cone is defined as

\[
\mathcal{L}^n_p = \{(t, x) \in \mathbb{R} \times \mathbb{R}^{n-1} \mid t \geq \|x\|_p \}.
\]

It follows by Proposition 1 that only the \( p \)-cones \( \mathcal{L}^n_p \) for \( 1 < p < \infty \) are strictly convex, since those are the values that correspond to strictly convex norms.

### 3 T-algebras

T-algebras were proposed by Vinberg [11] as a natural framework for the study of homogeneous convex cones. For a more recent treatment, including its connections to optimization, see the work of Chua [1, 2]. We recall that an algebra \( \mathcal{A} \) is a vector space \( \mathcal{A} \) over some field \( K \) such that \( \mathcal{A} \) is equipped with a product \( \times : \mathcal{A} \times \mathcal{A} \to \mathcal{A} \) that satisfies

\[
(a + b) \times c = a \times c + b \times c \quad \text{and} \quad c \times (a + b) = c \times a + c \times b
\]

\[
(\alpha a) \times (\beta b) = (\alpha \beta) a \times b,
\]

for all \( a, b, c \in \mathcal{A}, \alpha, \beta \in K \). In the cases we discuss here, we will always consider the real number field \( \mathbb{R} \) and write \( a \times b = ab \), for \( a, b \in \mathcal{A} \). Then, a matrix algebra of rank \( r \) is an algebra over \( \mathbb{R} \) that is equipped with a decomposition as a direct sum \( \mathcal{A} = \bigoplus_{i,j=1}^r \mathcal{A}_{ij} \) satisfying the following properties:

\[
\mathcal{A}_{ij} \mathcal{A}_{jk} \subseteq \mathcal{A}_{ik} \quad \text{and} \quad \mathcal{A}_{ij} \mathcal{A}_{ki} = 0 \quad \text{if} \ j \neq k.
\]

This decomposition is called a bigradation. Therefore, in a matrix algebra we can represent an element \( a \in \mathcal{A} \) as a generalized matrix \( a = (a_{ij})_{i,j=1}^r \), where \( a_{ij} \in \mathcal{A}_{ij} \), for all \( i, j \). With that, the multiplication in \( \mathcal{A} \) follows the usual matrix multiplication rules \( (ab)_{ij} = \sum_{k=1}^r a_{ik} b_{kj} \).

A matrix algebra with involution is a matrix algebra equipped with a linear bijection \( * : \mathcal{A} \to \mathcal{A} \) such that

\[
a** = a, \quad (ab)^* = b^*a^* \quad \text{and} \quad A_{ij}^* = A_{ji} \quad \text{for all} \ i, j.
\]

With that, we have \( (a^*)_ij = a_{ji}^* \).

Finally, a T-algebra is a matrix algebra with involution satisfying the following properties, see Definition 4 in [2].

(i) For each \( i \), \( \mathcal{A}_{ii} \) is a subalgebra isomorphic to \( \mathbb{R} \).
Let $\rho_i : A_{ii} \to \mathbb{R}$ denote the algebra isomorphism and let $e_i$ denote the unit element in $A_{ii}$, i.e., the element satisfying $\rho_i(e_i) = 1$. Furthermore, define the function $\text{tr} : A \to \mathbb{R}$ by $\text{tr}(a) := \sum_{i=1}^{r} \rho_i(a_{ii})$.

(iii) For all $a \in A$ and all $i, j \in \{1, \ldots, r\}$ we have $e_i a_{ij} = a_{ij}$ and $a_{ji} e_i = a_{ji}$.

Proposition 3.

Proof. (i) The restriction of the involution $* : A_{ii} \to A_{ii}$ becomes an automorphism of $A_{ii}$. Since $A_{ii}$ is isomorphic to $\mathbb{R}$ and the only automorphism of $\mathbb{R}$ is the identity map, we conclude that the restriction of $*$ to $A_{ii}$ must be the identity map as well.

(ii) The map $\rho_i$ is an algebra isomorphism, so it satisfies $\rho_i(\alpha x) = \alpha \rho_i(x)$ for every $\alpha \in \mathbb{R}$ and $x \in A_{ii}$. Since $\rho_i(\rho_i(a_{ii}) e_i) = \rho_i(a_{ii}) 1$ and $\rho_i$ is a bijection, we conclude that $a_{ii} = \rho_i(a_{ii}) e_i$.

(iii) Note that $(tt^*)_{ii} = \sum_{j=1}^{r} t_{ij} t_{ji}^*$. Therefore, $\rho_i((tt^*)_{ii}) = \sum_{j=1}^{r} \rho_i(t_{ij} t_{ji}^*)$. From Axiom (v), every term inside the summation is nonnegative, so that $\rho_i((tt^*)_{ii}) \geq 0$.

Finally, we define an inner product over $A$ by taking $\langle x, y \rangle = \text{tr}(x^* y)$. 

4 Main result

We will now gather a few results that will allow us to prove our main result. For what follows, recall that if $K$ is a closed convex cone and $y \in K^*$ then $K \cap \{y\}^\perp = \{x \in K \mid \langle x, y \rangle = 0 \}$ is always a nonempty face of $K$.

Proposition 4. The closure of a homogeneous convex cone $K$ of rank $r \geq 1$ contains a proper face of dimension at least $r - 1$.

Proof. If $r = 1$, we take $\{0\}$ as the desired face. So suppose that $r \geq 2$. Consider a $T$-algebra $A$ of rank $r$ such that $\text{int} \ K = K(A)$. We will prove that $(\text{cl} K(A)) \cap \{e_1\}^\perp$ has dimension greater or equal than $r - 1$. We first argue that $e_1 \in K(A)^*$. Let $x \in K(A)$, then $(e_1 x)_{11} = x_{11}$. By Proposition 3, we have $\rho_1(x_{11}) \geq 0$, therefore $\langle e_1, x \rangle = \text{tr}(e_1 x) = \rho_1(x_{11}) \geq 0$.

We have $\text{cl} K = \text{cl} K(A) = \{tt^* \mid t \in T_+\}$. For all $i$, we have $e_i \in T_+$, so that $e_i e_i^* = e_i e_i^* = e_i \in \text{cl} K$. Since $A$ is a matrix algebra, $e_i e_j = 0$ if $i \neq j$. Therefore, $\langle e_i, e_j \rangle = 0$ for $i \neq j$. This shows that $\{e_2, \ldots, e_r\} \subseteq (\text{cl} K) \cap \{e_1\}^\perp$, so that the dimension of $(\text{cl} K) \cap \{e_1\}^\perp$ is at least $r - 1$.

Proposition 5. Let $K$ be a convex cone such that $\text{cl} K$ is strictly convex. If $K$ is homogeneous and nonzero, then its rank is less or equal than 2.

Proof. It is an immediate consequence of Proposition 4. If $K$ were homogeneous of rank $r \geq 3$, then its closure would have a proper face of dimension at least 2 contradicting the strict convexity. Therefore, $r \leq 2$.

It turns out that the closure of a homogeneous convex cone of rank $r \leq 2$ is self-dual under an appropriate inner product, which is mentioned by Vinberg in [11], see Section 8 of Chapter 3. Using that, the next result follows from the classification of symmetric cones, see Chapter 5 in [3]. Nevertheless, we will give a constructive proof that explicitly exhibits the isomorphism.

Proposition 6. Let $K$ be a nonzero homogeneous convex cone of rank $r \leq 2$. Then $\text{cl} K$ is isomorphic to either $\mathbb{R}_+^2$ or $\mathbb{L}_2^n$ for some $n$.

Proof. Let $A$ be a $T$-algebra of rank $r$ such that $K(A) = \text{int} K$. If $r = 1$, it is clear that $\text{cl} K$ must be isomorphic to $\mathbb{R}_+$. If $r = 2$, we consider two cases. If $A_{12} = A_{21} = \{0\}$ it is clear that $\text{cl} K$ must be isomorphic to $\mathbb{R}_+^2$. So now we consider the case where the dimension of $A_{12}$ is greater than zero and we identify $A_{12}$ with some $\mathbb{R}^m$ with $m > 0$.

We first establish necessary and sufficient conditions for $a \in A$ to belong in $K(A)$. Due to Proposition 3, we can write $a_{11} = \alpha e_1, a_{22} = \beta e_2$, where $\alpha = \rho_1(a_{11})$ and $\beta = \rho_2(a_{22})$.

Now, $a \in K(A)$ if and only if there is $t \in T_+$ such that $a = tt^*$. Similarly, we can write $t_{11} = \gamma_1 e_1, t_{22} = \gamma_2 e_2$, where $\gamma_1 = \rho_1(t_{11})$ and $\gamma_2 = \rho_2(t_{22})$. We have

$$tt^* = t_{11} t_{11}^* + t_{12} t_{12}^* + t_{22} t_{22}^* + t_{22} t_{12}^* + t_{22} t_{12}$$

$$= (\gamma_1^2 + \rho_1(t_{12} t_{12}^*)) e_1 + \gamma_2 t_{12} e_2 + \gamma_2 t_{12} e_2 + \gamma_2^2 e_2$$

$$= (\gamma_1^2 + \rho_1(t_{12} t_{12}^*)) e_1 + \gamma_2 t_{12} + \gamma_2 t_{12} + \gamma_2^2 e_2,$$

where the last equality follows from Axiom $(ii)$. Then, comparing the expressions for $tt^*$ and $a$, we conclude that in order to have $tt^* = a$, we must have $a = a^*$ and

$$\gamma_2 = \sqrt{\beta}$$

$$t_{12} = \frac{a_{12}}{\sqrt{\beta}}$$

$$\gamma_1^2 = \alpha - \frac{\rho_1(a_{12} a_{12}^*)}{\beta}.$$
Since $\gamma_1$ and $\gamma_2$ must be positive, we conclude that $a \in K(A)$ if and only if $a = a^*$, $\alpha > 0$, $\beta > 0$ and $\alpha \beta - \rho_1(a_{12}a_{12}^*) > 0$. The last inequality can be expressed equivalently as
\[
\left(\frac{\alpha + \beta}{2}\right)^2 > \left(\frac{\alpha - \beta}{2}\right)^2 + \rho_1(a_{12}a_{12}^*).
\]
(1)

Now, notice that we have $\rho_1(a_{12}a_{12}^*) = \langle a_{12}^*, a_{12}\rangle = \langle a_{12}, a_{12}\rangle$ when $a = a^*$. Restricting $\langle \cdot, \cdot \rangle$ as an inner product on $A_{12}$, we can write $(u, v) = u^T Q^T Q v$ for each $u, v \in A_{12}$ with some $m \times m$ nonsingular matrix $Q$. Then, we conclude that $a \in K(A)$ if and only if $a = a^*$, $\alpha + \beta > 0$ and
\[
\left(\frac{\alpha + \beta}{2}\right)^2 > \left(\frac{\alpha - \beta}{2}\right)^2 + (Qa_{12})^T (Qa_{12}),
\]
(2)
which is equivalent to the statement that $(\alpha + \beta, \frac{\alpha - \beta}{2}, Qa_{12}) \in \text{int} \ L_2^{m+2}$.

Let $H(A) = \{a \in A \mid a = a^*\}$ and consider the linear map $S : H(A) \to \mathbb{R}^{m+2}$ defined by
\[
S(a) := \left(\rho_1(a_{11}) + \rho_2(a_{22}), \frac{\rho_1(a_{11}) - \rho_2(a_{22})}{2}, Qa_{12}\right) = \left(\frac{\alpha + \beta}{2}, \frac{\alpha - \beta}{2}, Qa_{12}\right).
\]
Since $Q$ is nonsingular, $S$ is a bijection. Then, the discussion so far implies that $S(K(A)) = \text{int} \ L_2^{m+2}$. Therefore, $S$ is the desired isomorphism.

We thus arrive at the main result.

**Theorem 7.** Suppose that $K$ is a convex cone such that $\text{cl} \ K$ is strictly convex. Then, it is homogeneous if and only if $\text{cl} \ K$ is isomorphic to $\{0\}$, $\mathbb{R}_+$, $\mathbb{R}_+^2$ or $L_2^n$.

In particular, $L_p^n$ is not homogeneous if $p \neq 2$ and $n \geq 3$.

**Proof.** It is clear that $\{0\}$, $\mathbb{R}_+$, $\mathbb{R}_+^2$ and $L_2^n$ are homogeneous convex cone. For the converse, if $K = \{0\}$ we are done. Otherwise, by Propositions 5 and 6, $K$ must be isomorphic to $\mathbb{R}_+$, $\mathbb{R}_+^2$ or $L_2^n$. This concludes the first half.

Now, suppose that $K$ is homogeneous and $K = L_p^n$ for $p \neq 2$, $1 < p < \infty$ and $n \geq 3$. Then $K$ is strictly convex and, therefore, must be isomorphic to one of the four cones listed above. The only possible candidate is $L_2^2$, since all the others have dimension less or equal than 2. However, the results by Gowda and Trott in [5] imply that $L_2^2$ and $L_2^n$ are not isomorphic if $p \neq 2$ and $n \geq 3$, since an invariant known as “Lyapunov rank” is $\frac{n^2 - n + 2}{2}$ for the former and 1 for the latter. Furthermore, isomorphic cones have the same Lyapunov rank. See Section 1 and Theorem 5 in [5] and the related paper [4], for more details. This gives a contradiction, so $K$ is not homogeneous.

We remark that the Gowda and Trott already showed that $L_1^n$ and $L_\infty^n$ are not homogeneous for $n \geq 3$, see Theorem 7 and Section 6 of [5].

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**References**


