REGULARIZED DECOMPOSITION METHODS FOR DETERMINISTIC AND STOCHASTIC CONVEX OPTIMIZATION AND APPLICATION TO PORTFOLIO SELECTION WITH DIRECT TRANSACTION AND MARKET IMPACT COSTS

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Abstract. We define a regularized variant of the Dual Dynamic Programming algorithm called REDDP (REgularized Dual Dynamic Programming) to solve nonlinear dynamic programming equations. We extend the algorithm to solve nonlinear stochastic dynamic programming equations. The corresponding algorithm, called SDDP-REG, can be seen as an extension of a regularization of the Stochastic Dual Dynamic Programming (SDDP) algorithm recently introduced which was studied for linear problems only and with less general prox-centers. We show the convergence of REDDP and SDDP-REG. We assess the performance of REDDP and SDDP-REG on portfolio models with direct transaction and market impact costs. In particular, we propose a risk-neutral portfolio selection model which can be cast as a multistage stochastic second-order cone program. The formulation is motivated by the impact of market impact costs on large portfolio rebalancing operations. Numerical simulations show that REDDP is much quicker than DDP on all problem instances considered (up to 184 times quicker than DDP) and that SDDP-REG is quicker on the instances of portfolio selection problems with market impact costs tested and much faster on the instance of risk-neutral multistage stochastic linear program implemented (8.2 times faster).

Key words. Stochastic Optimization, Stochastic Dual Dynamic Programming, Regularization, Portfolio Selection, Market Impact Costs

AMS subject classifications. 90C15, 90C90

1. Introduction. Multistage stochastic optimization problems are used to model many real-life applications where a sequence of decisions has to be made, subject to random costs and constraints arising from the observations of a stochastic process. Solving such problems is challenging and often requires some assumptions on the underlying stochastic process, on the problem structure, and some sort of decomposition. In this paper, we are interested in problems for which deterministic or stochastic dynamic programming equations can be written. In this latter case, we will focus on situations where the underlying stochastic process is discrete interstage independent, the number of stages is moderate to large, and the state vector is of small size.

Two popular solution methods to solve stochastic dynamic programming equations are Approximate Dynamic Programming (ADP) [37] and Stochastic Dual Dynamic Programming (SDDP) [32], which is a sampling-based variant of the Nested Decomposition (ND) algorithm [6, 7]. Several enhancements of SDDP have been proposed such as the extension to interstage dependent stochastic processes [24, 19], different sampling schemes [8], and recently the introduction and analysis of risk-averse variants [22, 23, 25, 34, 40, 41], cut selection strategies [33, 35], and convergence proofs of the algorithm for linear problems in [36], for nonlinear risk-neutral problems in [16], for nonlinear risk-averse problems in [20], and for linear problems without relatively complete recourse in [20]. However, a known drawback of the method is its convergence rate, making it difficult to apply to problems with moderate or large state vectors. To cope with this difficulty, a regularized variant of SDDP was recently proposed in [4] for Multistage Stochastic Linear Programs (MSLPs). This variant consists in computing in the forward pass of SDDP the trial points penalizing the objective with a quadratic term depending on a prox-center (called incumbent in [4]) shared between nodes of the same stage and updated at each iteration. On the tests reported in [4], the regularized method converges faster than the classical SDDP method on risk-neutral instances of MSLPs. On the basis of these encouraging numerical results, several natural extensions of this regularized variant can be considered:

a) When specialized to deterministic problems, how does the regularized method behave? For such problems, how to extend the method when nonlinear objective and constraints are present and under which assumptions? Can we show the convergence of the method applied to these problems under these assumptions?

b) How can the regularized algorithm be extended to solve Multistage Stochastic NonLinear Problems (MSNLPs) and under which assumptions? Can we show the convergence of this algorithm applied
to MSNLPs under these assumptions?
c) What other prox-centers and penalization schemes can be proposed? Find a MSLP for testing the new prox-centers and penalization schemes. Can we observe on this application a faster convergence of the regularized method, as for the application considered in [4]?
d) Find a relevant application, modeled by a MSNL, to test the regularized variant of SDDP.

The objective of this paper is to study items a)-d) above. Our findings on these topics are as follows:

a) **REDDP: REgularized Dual Dynamic Programming.** We propose a regularized variant of Dual Dynamic Programming (DDP, the deterministic counterpart of SDDP) called REDDP, for nonlinear optimization problems. For REDDP, in Theorem 2.4, we show the convergence of the sequence of approximate first step optimal values to the optimal value of the problem and that any accumulation point of the sequence of trial points is an optimal solution of the problem. The same proof, with weaker assumptions (see Remark 2.4) can be used to show the convergence of this regularized variant of DDP applied to linear problems. We then consider instances of a portfolio problem with direct transaction costs with a large number of stages and compare the computational time required to solve these instances with DDP and REDDP. In all experiments, the computational time was drastically reduced using REDDP. More precisely, we tested 6 different implementations of REDDP and for problems with $T = 10, 50, 100, 150, 200, 250, 300,$ and 350 time periods, the range (for these 6 implementations) of the reduction factor of the overall computational time with REDDP was respectively $[3.0, 3.0], [13.8, 17.3], [22.3, 33.5], [37.1, 65.0], [46.6, 76.7], [80.0, 114.3], [71.5, 171.6],$ and $[95.5, 184.4]$. Since DDP (eventually with cut selection as in [21]) can already outperform direct solution methods (such as interior point methods or simplex) on some instances of large scale linear problems (see the numerical experiments in [21]), REDDP could be a competitive solution method to solve some large-scale problems, in particular linear, for which dynamic programming equations with convex value functions and a large number of time periods, can be written.

b) **SREDA: A Stochastic REgularized Decomposition Algorithm to solve MSNLPS.** We define a Stochastic REgularized Decomposition Algorithm (SREDA) for MSNLPs which samples in the backward pass to compute cuts at trial points computed, as in [4], in a forward pass, penalizing the objective with a quadratic term depending on a prox-center shared between nodes of the same stage. In Theorem 4.2, we show the convergence of this algorithm and observe in Remark 4 that the proof allows us to obtain the convergence of a regularized variant of SDDP called SDDP-REG applied to the nonlinear problems we are interested in. More precisely, we show (i) the convergence of the sequence of the optimal values of the approximate first stage problems and that (ii) any accumulation point of the sequence of decisions can be used to define an optimal solution of the problem. It will turn out that (ii) improves already known results for SDDP.

c) On prox-centers, penalization parameters, and on the performance of the regularization for MSLPs. We propose new prox-centers and penalization schemes and test them on risk-neutral and risk-averse instances of portfolio selection problems.

d) **Portfolio Selection with Direct Transaction and Market Impact Costs.** The multistage optimization models studied in this paper are directly applicable in finance and in particular for the rebalancing of portfolios that incur transaction costs. Transaction costs can have a major impact on the performance of an investment strategy (see, e.g., the survey [11]). Two main types of transaction costs, implicit and explicit, can be distinguished. Explicit or direct transaction costs are directly observable (e.g., broker, custodial fees), are directly charged to the investor, and are generally modelled as linear [5, 12] or piecewise linear [9]. In reality, it is however not possible to trade arbitrary large quantities of securities at their current theoretical market price. Implicit or indirect costs, often called market impact costs, result from imperfect markets due for example to market or liquidity restrictions (e.g., bid-ask spreads), depend on the order-book situation when the order is executed, and are not itemized explicitly, thereby making it difficult for investors to
recognize them. Yet, for large orders, they are typically much larger than the direct transaction costs.

Market impact costs are equal to the difference between the transaction price and the (unperturbed) market price that would have prevailed if the trade had not occurred [42, 43, 45]. Market impact costs are typically nonlinear (see, e.g., [2, 3, 15, 17, 42]), and much more challenging to model than direct transaction costs. Market impact costs are particularly important for large institutional investors, for which they can represent a major proportion of the total transaction costs [28, 42]. They can be viewed as an additional price for the immediate execution of large trades.

There is a widespread interest in the modeling and analysis of market impact costs as they are (one of) the main reducible parts of the transaction costs [28]. In this study, we propose a series of dynamic - deterministic and stochastic (risk-neutral and risk-averse) - optimization models for portfolio optimization with direct transaction and market impact costs.

We compare the computational time required to solve with SDDP-REG and SDDP instances of risk-neutral and risk-averse portfolio problems with direct transaction costs.

We also compare the computational time required to solve with SDDP-REG and SDDP risk-neutral instances of portfolio problems with market impact costs using real data and $T = 48$ stages. To our knowledge, no dynamic optimization problem for portfolio optimization with conic market impact costs has been proposed so far. Also, we are not aware of other published numerical tests on the application of SDDP to a real-life application modelled by a multistage stochastic second-order cone program with a large (48 in our case, which already corresponds to a very challenging multistage stochastic nonlinear problem) number of stages.

The paper is organized as follows. In Section 2, we present a class of convex deterministic nonlinear optimization problems for which dynamic programming equations can be written. We propose the variant REDDP of DDP to solve these problems and show the convergence of this method in Theorem 2.4. Though this theorem is a special case of Theorem 4.2 given in Section 4, we thought it would be convenient for the reader to have this simpler proof in mind when considering the more complicated stochastic case since most arguments of the proof of Theorem 2.4 are re-used for the proof of Theorem 4.2. In Section 3, we introduce the type of stochastic nonlinear problems we are interested in and propose SREDA, a regularized decomposition algorithm to solve these problems. In Section 4, we show in Theorem 4.2 the convergence of SREDA. The portfolio selection models described in item d) above are discussed in Section 5. Finally, the last Section 6 presents the results of numerical simulations that illustrate our results. We show that REDDP is much quicker than DDP on all problem instances considered (up to 184 times quicker than DDP) and that SDDP-REG is quicker on the instances of nonlinear stochastic programs tested and much faster on the instance of risk-neutral multistage stochastic linear program implemented (8.2 times faster).

We use the following notation and terminology:
- The usual scalar product in $\mathbb{R}^n$ is denoted by $\langle x, y \rangle = x^T y$ for $x, y \in \mathbb{R}^n$. The corresponding norm is $\|x\| = \|x\|_2 = \sqrt{\langle x, x \rangle}$.
- $\text{ri}(A)$ is the relative interior of set $A$.
- $\mathbb{B}_n = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$.
- $\text{dom}(f)$ is the domain of function $f$.
- $N_A(x)$ is the normal cone to $A$ at $x$.
- $\text{AV}_\alpha$ is the Average Value-at-Risk with confidence level $\alpha$, [38].
- $D(\mathcal{X})$ is the diameter of set $\mathcal{X}$.
- The notation $[A; B]$ represents the matrix $\begin{pmatrix} A \\ B \end{pmatrix}$.

2. Regularized dual dynamic programming: Algorithm and convergence.

2.1. Problem formulation and assumptions. Consider the problem

$$
\begin{align*}
\min & \sum_{t=1}^T f_t(x_{t-1}, x_t) \\
x_t & \in X_t(x_{t-1}), \forall t = 1, \ldots, T,
\end{align*}
$$
where \( X_t(x_{t-1}) \subset \mathcal{X}_t \subset \mathbb{R}^n \) is given by
\[
X_t(x_{t-1}) = \{ x_t \in X_t : A_t x_t + B_t x_{t-1} = b_t, g_t(x_{t-1}, x_t) \leq 0 \},
\]
and \( f_t : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \cup \{ +\infty \} \) is a convex function, \( g_t : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^p \), and \( x_0 \) is given.

For this problem, we can write dynamic programming equations defining recursively the functions \( Q_t : \mathcal{X}_{t-1} \to \mathbb{R} \) as
\[
Q_t(x_{t-1}) := \min \{ f_t(x_{t-1}, x_t) + Q_{t+1}(x_t) : x_t \in X_t(x_{t-1}) \}, \quad t = T, T-1, \ldots, 1,
\]
with the convention that \( Q_{T+1} := 0 \). Clearly, \( Q_t(x_0) \) is the optimal value of (1). More generally, we have
\[
Q_t(x_{t-1}) = \min \left\{ \sum_{j=t}^{T} f_j(x_{j-1}, x_j) : x_j \in X_j(x_{j-1}), \; \forall j = t, \ldots, T \right\}.
\]

We make the following assumptions: setting
\[
\mathcal{X}_t^\varepsilon := \mathcal{X}_t + \varepsilon \mathbb{B}_n
\]
(H0) For \( t = 1, \ldots, T \),
(a) \( \mathcal{X}_t \subset \mathbb{R}^n \) is nonempty, convex, and compact.
(b) \( f_t \) is proper, convex, and lower semicontinuous.
(c) Setting \( g_t(x_{t-1}, x_t) = (g_{t,1}(x_{t-1}, x_t), \ldots, g_{t,p}(x_{t-1}, x_t)) \), for \( i = 1, \ldots, p \), the \( i \)-th component function \( g_{t,i} : (x_{t-1}, x_t) \) is a convex lower semicontinuous function.
(d) There exists \( \varepsilon > 0 \) such that \( \mathcal{X}_t^\varepsilon \times \mathcal{X}_t \subset \text{dom}(f_t) \) and for every \( x_{t-1} \in \mathcal{X}_t^\varepsilon \), there exists \( x_t \in \mathcal{X}_t \) such that \( g_t(x_{t-1}, x_t) \leq 0 \) and \( A_t x_t + B_t x_{t-1} = b_t \).
(e) If \( t \geq 2 \), there exists \( \bar{x}_{t,t} \in \mathcal{X}_t \) such that \( \bar{x}_{t,t} \in \mathcal{X}_t \), \( g_t(\bar{x}_{t,t}, \bar{x}_{t,t}) \leq 0 \) and \( A_t \bar{x}_{t,t} + B_t \bar{x}_{t,t} = b_t \).

The DDP algorithm solves (1) exploiting the convexity of recourse functions \( Q_t \):

**Lemma 2.1.** Consider recourse functions \( Q_t, t = 1, \ldots, T+1 \), given by (2). Let Assumptions (H0)-(a), (H0)-(b), (H0)-(c), and (H0)-(d) hold. Then for \( t = 1, \ldots, T+1 \), \( Q_t \) is convex, finite on \( \mathcal{X}_t^\varepsilon \), and Lipschitz continuous on \( \mathcal{X}_{t-1} \).

**Proof:** We give the idea of the proof. For more details, we refer to the proof of Proposition 3.1 in [20] where similar value functions are considered. The proof is by backward induction on \( t \), starting with \( t = T + 1 \) where the statement holds by definition of \( Q_{T+1} \). Assuming for \( t \in \{ 1, \ldots, T \} \) that \( Q_{t+1} \) is convex, finite on \( \mathcal{X}_t^\varepsilon \), and Lipschitz continuous on \( \mathcal{X}_t \), then Assumptions (H0)-(a), (b), (c) imply the convexity of \( Q_t \), and assumptions (H0)(a), (b), (d) that \( Q_t \) is finite on \( \mathcal{X}_t^\varepsilon \) and therefore Lipschitz continuous on \( \mathcal{X}_t \).

The description of the sub-divisional of \( Q_t \) given in the following proposition will be useful for DDP, REDDP, and SREDA:

**Proposition 2.2.** Lemma 2.1 in [20]. Let Assumptions (H0) hold. Let \( x_t(x_{t-1}) \) be an optimal solution of (2). Then for every \( t = 2, \ldots, T \), for every \( x_{t-1} \in \mathcal{X}_t \), \( s \in \partial Q_t \), \( x_{t-1}, x_t(x_{t-1}) \) if and only if
\[
(s, 0) \in \partial f_t(x_{t-1}, x_t(x_{t-1}))) + \left\{ [A_t^T : B_t^T] \nu : \nu \in \mathbb{R}^q \right\}
+ \left\{ \sum_{i \in I(x_{t-1}, x_t(x_{t-1}))} \mu_i \partial g_{t,i}(x_{t-1}, x_t(x_{t-1})) : \mu_i \geq 0 \right\} + \{ 0 \} \times \mathcal{N}_{X_t}(x_t(x_{t-1}))
\]
where \( I(x_{t-1}, x_t(x_{t-1})) = \left\{ i \in \{ 1, \ldots, p \} : g_{t,i}(x_{t-1}, x_t(x_{t-1})) = 0 \right\} \).

**Proof:** See [20].

**2.2. Dual Dynamic Programming.** We first recall DDP method to solve (2). It uses relatively easy approximations \( Q_t^k \) of \( Q_t \). At iteration \( k \), let functions \( Q_t^k : \mathcal{X}_{t-1} \to \mathbb{R} \) such that
\[
Q_{T+1}^k = Q_{T+1}, \quad Q_t^k \leq Q_t \quad t = 2, 3, \ldots, T,
\]
be given and define for $t = 1, 2, \ldots, T$ the function $Q^k_t : \mathcal{X}_{t-1} \rightarrow \mathbb{R}$ as

$$Q^k_t(x_{t-1}) = \min \left\{ f_t(x_{t-1}, x_t) + Q^k_{t+1}(x_t) : x_t \in X_t(x_{t-1}) \right\} \quad \forall x_{t-1} \in \mathcal{X}_{t-1}.$$ 

Clearly, (4) implies that

$$Q^k_T = Q_T, \quad Q^k_t \leq Q_t \quad t = 1, 2, \ldots, T - 1.$$ 

It is assumed that the functions $Q^k_t$ can be evaluated at any point $x_{t-1} \in \mathcal{X}_{t-1}$. The DDP algorithm works as follows:

**DDP (Dual Dynamic Programming).**

Step 1) **Initialization.** Let $Q^0_t : \mathcal{X}_{t-1} \rightarrow \mathbb{R} \cup \{-\infty\}$, $t = 2, \ldots, T + 1$, satisfying (4) be given. Set $k = 1$.

Step 2) **Forward pass.** Setting $x^k_0 = x_0$, for $t = 1, 2, \ldots, T$, compute

$$x^k_t \in \arg \min \left\{ f_t(x^k_{t-1}, x_t) + Q^{k-1}_{t+1}(x_t) : x_t \in X_t(x^k_{t-1}) \right\}.$$ 

Step 3) **Backward pass.** Define $Q^k_{T+1} \equiv 0$. For $t = T, T - 1, \ldots, 2$, solve the problem

$$Q^k_t(x_{t-1}) = \min \left\{ f_t(x^k_{t-1}, x_t) + Q^k_{t+1}(x_t) : x_t \in X_t(x^k_{t-1}) \right\},$$

using Proposition 2.2 take a subgradient $\beta^k_t$ of $Q^k_t(\cdot)$ at $x^k_{t-1}$, and store the new cut

$$C^k_t(x_{t-1}) := Q^k_t(x^k_{t-1}) + \langle \beta^k_t, x^k_{t-1} - x^k_{t-1} \rangle$$

for $Q_t$, making up the new approximation $Q^k_t = \max\{Q^{k-1}_t, C^k_t\}$.

Step 4) Do $k \leftarrow k + 1$ and go to Step 2).

### 2.3. Regularized Dual Dynamic Programming.

For the regularized DDP to be presented in this section, we still define

$$Q^k_t(x_{t-1}) = \min \left\{ F^k_t(x_{t-1}, x_t) : x_t \in X_t(x_{t-1}) \right\} \quad \forall x_{t-1} \in \mathcal{X}_{t-1},$$

where

$$F^k_t(x_{t-1}, x_t) = f_t(x_{t-1}, x_t) + Q^k_{t+1}(x_t).$$

However, since the function $Q^k_{t+1}$ computed by regularized DDP is different from the function $Q^k_{t+1}$ computed by DDP, the functions $Q^k_t$ obtained with respectively regularized DDP and DDP are different. The regularized DDP algorithm is given below:

**RERegularized DDP (REDDDP).**

Step 1) **Initialization.** Let $Q^0_t : \mathcal{X}_{t-1} \rightarrow \mathbb{R} \cup \{-\infty\}$, $t = 2, \ldots, T + 1$, satisfying (4) be given. Set $k = 1$.

Step 2) **Forward pass.** Setting $x^k_0 = x_0$, for $t = 1, 2, \ldots, T$, compute

$$x^k_t \in \arg \min \left\{ \tilde{F}^{k-1}_t(x^k_{t-1}, x_t, x^P_t) : x_t \in X_t(x^k_{t-1}) \right\},$$

where the prox-center $x^P_t$ is any point in $\mathcal{X}_t$ and where $\tilde{F}^{k-1}_t : \mathcal{X}_{t-1} \times \mathcal{X}_t \times \mathcal{X}_t \rightarrow \mathbb{R}$ is given by

$$\tilde{F}^{k-1}_t(x_{t-1}, x_t, x^P_t) = f_t(x_{t-1}, x_t) + Q^{k-1}_{t+1}(x_t) + \lambda_{t,k} \| x_t - x^P_t \|^2$$

for some exogenous nonnegative penalization $\lambda_{t,k}$ with $\lambda_{t,k} = 0$ if $t = T$ or $k = 1$.

Step 3) **Backward pass.** Define $Q^k_{T+1} \equiv 0$. For $t = T, T - 1, \ldots, 2$, solve the problem

$$Q^k_t(x_{t-1}) = \min \left\{ f_t(x^k_{t-1}, x_t) + Q^k_{t+1}(x_t) : x_t \in X_t(x^k_{t-1}) \right\},$$

using Proposition 2.2 take a subgradient $\beta^k_t$ of $Q^k_t(\cdot)$ at $x^k_{t-1}$, and store the new cut

$$C^k_t(x_{t-1}) := Q^k_t(x^k_{t-1}) + \langle \beta^k_t, x^k_{t-1} - x^k_{t-1} \rangle$$

for $Q_t$, making up the new approximation $Q^k_t = \max\{Q^{k-1}_t, C^k_t\}$.

Step 4) Do $k \leftarrow k + 1$ and go to Step 2).
Observe that the backward passes of the regularized and non-regularized DDP are the same. The algorithms differ from the way the trial points are computed: for regularized DDP a proximal term is added to the objective function of each period to avoid moving too far from the prox-center.

2.4. Convergence analysis. The following lemma will be useful to analyze the convergence of regularized DDP:

**Lemma 2.3.** Let Assumptions (H0) hold. Then the functions $Q^k_t$, $t = 2, \ldots, T + 1, k \geq 1$, generated by REDDP are Lipschitz continuous on $X^*_t$, satisfy $Q^k_t \leq Q_t$, and $Q^k_t(x^*_{t-1})$ and $\beta^k_t$ are bounded for all $t \geq 2, k \geq 1$.

**Proof:** It suffices to follow the proof of Lemma 3.2 in [20].

Assuming for $t \in \{1, \ldots, T\}$ that $Q^k_{t+1}$ is Lipschitz continuous on $X^*_t$ with $Q^k_{t+1} \leq Q_{t+1}$, then $Q^k_t \leq Q_t$. Using Proposition 2.2, whose assumptions are satisfied because (H0)-(e) holds, we get $Q^k_t \geq Q_t$. Thus, $Q^k_t \leq Q_t$ and $Q^k_t(x^*_{t-1})$ and $\beta^k_t$ are finite and allow us to obtain a uniform upper bound on $\beta^k_t$, i.e., a Lipschitz constant valid for all functions $Q^k_t$, $t = 2, \ldots, T + 1, k \geq 1$.

**Theorem 2.4.** Consider the sequences of decisions $x^k_t$ and approximate recourse functions $Q^k_t$ generated by REDDP. Let Assumptions (H0) hold and assume that for $t = 1, \ldots, T - 1$, we have $\lim_{k \to +\infty} \lambda_{t,k} = 0$ and $\lambda_{T,k} = 0$ for every $k \geq 1$. Then we have

$$Q^k_T(x^k_{T-1}) = Q^k_T(x^k_{T-1}) = Q^k_T(x^k_{T-1}),$$

and for $t = 2, \ldots, T - 1$,

$$H(t) \equiv \lim_{k \to +\infty} Q_t(x^k_{t-1}) - Q^k_t(x^k_{t-1}) = \lim_{k \to +\infty} Q_t(x^k_{t-1}) - Q^k_t(x^k_{t-1}) = 0.$$  

Also, (i) $\lim_{k \to +\infty} Q^k_t(x_0) = \lim_{k \to +\infty} F^{k-1}_t(x_0, x^k_1, x^k_T) = Q_1(x_0)$, the optimal value of (1), and (ii) any accumulation point $(x^*_1, \ldots, x^*_T)$ of the sequence $(x^k_1, \ldots, x^k_T)_k$ is an optimal solution of (1).

**Proof:** Since $Q^k_{T+1} = Q_{T+1} = 0$, we have $Q^k_{T+1}(x^k_T) = Q^k_{T+1}(x^k_T)$. Next recall that $Q^k_T(x^k_{T-1}) \leq Q_T(x^k_{T-1})$ and

$$Q^k_T(x^k_{T-1}) \geq C^k_T(x^k_{T-1}) = Q^k_T(x^k_{T+1}) \neq Q_T(x^k_{T-1})$$

which shows (10). We prove $H(t), t = 2, \ldots, T - 1$, by backward induction on $t$. We have just shown that $H(T)$ holds. Assume that $H(t+1)$ holds for some $t \in \{2, \ldots, T - 1\}$. We want to show that $H(t)$ holds.

To alleviate notation, we define the function $F_t : X^{*}_{t-1} \times X^*_t \to \mathbb{R}$ given by

$$F_t(x_{t-1}, x_t) = f_t(x_{t-1}, x_t) + \lambda_{t+1}(x_t).$$

We will denote by $\bar{x}^k_t$ an optimal solution of the problem defining $Q^k_{t}^{k-1}(x^k_{t-1})$, i.e.,

$$Q^k_{t}^{k-1}(x^k_{t-1}) = F^k_{t-1}(x^k_{t-1}, \bar{x}^k_t).$$

By definition of $Q^k_t$, we have that $Q^k_t(x^k_{t-1}) \geq C^k_t(x^k_{t-1})$ which implies $Q^k_t(x^k_{t-1}) \geq C^k_t(x^k_{t-1}) = Q^k_t(x^k_{t-1})$. We deduce that

$$0 \leq Q_t(x^k_{t-1}) - Q^k_t(x^k_{t-1}) \leq Q_t(x^k_{t-1}) - Q^k_t(x^k_{t-1}),$$

$$\leq Q_t(x^k_{t-1}) - Q^k_t(x^k_{t-1}) \text{ by monotonicity of } (Q^k_t)_k,$$

$$= Q_t(x^k_{t-1}) - F^k_{t-1}(x^k_{t-1}, \bar{x}^k_t) \text{ by definition of } \bar{x}^k_t,$$

$$= Q_t(x^k_{t-1}) - F^k_{t-1}(x^k_{t-1}, x^k_t) + F^k_{t-1}(x^k_{t-1}, x^k_t) - F^k_{t-1}(x^k_{t-1}, \bar{x}^k_t).$$

Now observe that

$$Q_t(x^k_{t-1}) - F^k_{t-1}(x^k_{t-1}, x^k_t) = Q_t(x^k_{t-1}) - f_t(x^k_{t-1}, x^k_t) - Q^{k-1}_{t+1}(x^k_{t-1}) \text{ by definition of } F^k_{t-1},$$

In [20], a forward, instead of a forward-backward algorithm, is considered. In this setting, finiteness of coefficients $Q^k_t(x^k_{t-1})$ and $\beta^k_t$ is not guaranteed for the first iterations (for instance $Q^k_t(x^k_{t-1})$ are $-\infty$ as long as the lower bounding functions $Q^k_t$, $t = 2, \ldots, T$, are set to $-\infty$) but the proof is similar.
where the last inequality comes from the fact that \( x^k_t \in \mathcal{X}_t(x^k_{t-1}) \), i.e., \( x^k_t \) is feasible for the optimization problem defining \( Q_t(x^k_{t-1}) \) with optimal value \( Q_t(x^k_{t-1}) \) and objective function \( F_t(x^k_{t-1}, \cdot) \). The induction hypothesis gives
\[
\lim_{k \to +\infty} Q_{t+1}(x^k_t) - Q_{t+1}^{k+1}(x^k_t) = 0.
\]
Since functions \((Q^k_{t+1}(\cdot))_k\) are Lipschitz-continuous on \( \mathcal{X}_t \), \( Q_{t+1} \geq Q_{t+1}^k \geq Q_{t+1}^{k+1} \), and \((x^k_t)_k\) is a sequence of the compact set \( \mathcal{X}_t \), using Lemma A.1 in \cite{16}, (15) implies that
\[
\lim_{k \to +\infty} Q_{t+1}(x^k_t) - Q_{t+1}^{k+1}(x^k_t) = 0.
\]
Next, we have
\[
0 \leq F_t^{k-1}(x^k_{t-1}, x^k_t) - F_t^{k-1}(x^k_{t-1}, \bar{x}^k_t) = F_t^{k-1}(x^k_{t-1}, x^k_t) - F_t^{k-1}(x^k_{t-1}, x_t^P, x_t^P) + F_t^{k-1}(x^k_{t-1}, x_t^P, x_t^P) - F_t^{k-1}(x^k_{t-1}, \bar{x}^k_t, x_t^P) + F_t^{k-1}(x^k_{t-1}, \bar{x}^k_t, x_t^P) - F_t^{k-1}(x^k_{t-1}, \bar{x}^k_t)
\]
where the above inequality comes from the fact \( \bar{x}^k_t \in \mathcal{X}_t(x^k_{t-1}) \), i.e., \( \bar{x}^k_t \) is feasible for the optimization problem \((8)\) with objective function \( F_t^{k-1}(x^k_{t-1}, x_t^P) \) and optimal solution \( x_t^\ast \). We obtain
\[
0 \leq F_t^{k-1}(x^k_{t-1}, x^k_t) - F_t^{k-1}(x^k_{t-1}, \bar{x}^k_t) \leq \lambda_t, k(\|x^k_t - x_t^P\| - \|\bar{x}^k_t - x_t^P\|)
\]
where \( D(\mathcal{X}_t) \) is the diameter of \( \mathcal{X}_t \) (since \( \mathcal{X}_t \) is compact \( D(\mathcal{X}_t) \) is finite), i.e.,
\[
\lim_{k \to +\infty} F_t^{k-1}(x^k_{t-1}, x^k_t) - F_t^{k-1}(x^k_{t-1}, \bar{x}^k_t) = 0.
\]
Combining (13), (14), (16), and (17), we obtain \( H(t) \).

(i) Proceeding as above for \( t = 1 \), we obtain for \( Q_1(x_0) - Q^k_1(x_0) \) the bounds
\[
0 \leq Q_1(x_0) - Q^k_1(x_0) \leq Q_1(x_0) - Q^{k-1}_1(x_0) \leq Q_2(x^k_1) - Q^{k-1}_2(x^k_1) + \lambda_1, kD(\mathcal{X}_1)^2.
\]
Since \( H(2) \) holds, since functions \((Q^k_2(\cdot))_k\) are Lipschitz-continuous on \( \mathcal{X}_1 \), \( Q_2 \geq Q_2^k \geq Q_2^{k-1} \), and \((x^k_t)_k\) is a sequence of the compact set \( \mathcal{X}_1 \), we obtain, using again Lemma A.1 in \cite{16}, that \( \lim_{k \to +\infty} Q_2(x^k_t) - Q_2^{k-1}(x^k_t) = 0 \) and passing to the limit in (18) when \( k \to +\infty \), we get \( \lim_{k \to +\infty} Q_1^k(x_0) = Q_1(x_0) \). The above computations also show that
\[
-\lambda_1, kD(\mathcal{X}_1)^2 \leq Q_1(x_0) - F_t^{k-1}(x_0, x^k_t) \leq Q_2(x^k_1) - Q_2^{k-1}(x^k_1)
\]
which implies that \( Q_1(x_0) = \lim_{k \to +\infty} F_t^{k-1}(x_0, x^k_t) = \lim_{k \to +\infty} \bar{F}_t^{k-1}(x_0, x^k_t, x_t^P) \).

(ii) Let \((x^k_t)_k\) be an accumulation point of \((x^k_T)_k\) and let \( K \) be an infinite set of integers such that \( \lim_{k \in K, k \to +\infty} (x^k_1, \ldots, x^k_T) = (x^\ast_1, \ldots, x^\ast_T) \). Take now \( t \in \{1, \ldots, T\} \). Setting \( x^0_0 = x_0 \), from (10), (13), (18), and using the continuity of \( Q_t \), we have
\[
Q_t(x^\ast_{t-1}) = \lim_{k \in K, k \to +\infty} Q_t^{k-1}(x^k_{t-1}) = \lim_{k \in K, k \to +\infty} F_t^{k-1}(x^k_{t-1}, x^k_t) = \lim_{k \in K, k \to +\infty} \bar{F}_t^{k-1}(x^k_{t-1}, x^k_t).
\]
We have shown that \( \lim_{k \in K, k \to +\infty} F_t^{k-1}(x^k_{t-1}, \bar{x}^k_t) - F_t^{k-1}(x^k_{t-1}, x^k_t) = 0 \) which implies
\[
Q_t(x^\ast_{t-1}) = \lim_{k \in K, k \to +\infty} F_t^{k-1}(x^k_{t-1}, x^k_t).
\]
Using the continuity of $Q_{t+1}$, the fact that $\lim_{k \to +\infty} Q_{t+1}(x_t^k) - Q_{t+1}(x_t^*) = 0$, and the lower semi-continuity of $f_t$, we obtain

\[
F_t(x_{t-1}^*, x_t^*) = f_t(x_{t-1}^*, x_t^*) + Q_{t+1}(x_t^*) \leq \lim_{k \to +\infty} F_t^{k-1}(x_{t-1}^*, x_t^k) = Q_t(x_t^*).
\]

Since $g_t$ is lower semicontinuous, its level sets are closed, which implies that $g_t(x_{t-1}^*, x_t^*) \leq 0$. Recalling that $x_t^k \in X_t$ with $X_t$ closed, we have that $x_t^*$ is feasible for the problem defining $Q_t(x_t^*)$. Combining this observation with (20), we have shown that $x_t^*$ is an optimal solution for the problem defining $Q_t(x_t^*)$, i.e., problem (2) written for $x_{t-1} = x_{t-1}^*$. This shows that $(x_1^*, \ldots, x_T^*)$ is an optimal solution to (1).

If convergence of REDDP holds for any sequence $(x_t^{P,k})_{k \geq 2}$ of prox-centers in $X_t$ and of penalty parameters $\lambda_{t,k}$ converging to zero for every $t$, the performance of the method depends on how these sequences are chosen. DDP is obtained taking $\lambda_{t,k} = 0$ for every $t, k$.

For all numerical experiments of Section 6.2, REDDP was much faster than DDP. Some natural candidates for $\lambda_{t,k}$ and $x_t^{P,k}$, used in our numerical tests, are the following:

- Weighted average of previous values: $x_t^{P,k} = \frac{1}{1 + \sum_{j=1}^{k-1} \gamma_{t,j}} \sum_{j=1}^{k-1} \gamma_{t,j} x_t^j$ with $\gamma_{t,j} \geq 0$ nonnegative weights and $\Gamma_{t,k} = \sum_{j=1}^{k-1} \gamma_{t,j}$.
- Note that $x_t^k \in X_t$ because all $x_t^j$ are in the convex set $X_t$. Special cases include $\lambda_{t,k} = 0$ and $\lambda_{t,k} = \frac{1}{k}$ for $t < T, k \geq 2$.

Remark: If for a given stage $t$, $X_t$ is a polytope and we do not have the nonlinear constraints given by constraint functions $g_t$ (i.e., the constraints for this stage are linear), then the conclusions of Lemmas 2.1, 2.3, and Theorem 2.4 hold under weaker assumptions. More precisely, for such stages $t$, we assume (H0)-(a), (H0)-(b), and instead of (H0)-(d), (H0)-(e), the weaker assumption (H0)-(c):

- (H0)-(c) There exists $\varepsilon > 0$ such that:
- (c')1. $X_{t-1} \times X_t \subset \text{dom } f_t$;
- (c')2. For every $x_{t-1} \in X_{t-1}$, the set $X_t(x_{t-1})$ is nonempty.


3.1. Problem formulation and assumptions. Consider a stochastic process $(\xi_t)$ where $\xi_t$ is a discrete random vector with finite support containing in particular as components the entries in $(b_t, A_t, B_t)$ in a given order where $b_t$ are random vectors and $A_t, B_t$ are random matrices.

We denote by $\mathcal{F}_t$ the sigma-algebra $\sigma(\xi_1, \ldots, \xi_t)$ and by $Z_t$ the set of $\mathcal{F}_t$-measurable functions, $E_{\mathcal{F}_{t-1}} : Z_{t-1} \to Z_t$ is the conditional expectation at $t$.

With this notation, we are interested in solving problems of form

\[
\inf_{x_t \in X_t(x_{t-1}, \xi_t)} f_t(x_0, x_1, \xi_1) + \mathbb{E}_{\mathcal{F}_1} \left( \inf_{x_2 \in X_2(x_1, \xi_2)} f_2(x_1, x_2, \xi_2) + \ldots \right.
\]

\[
\left. + \mathbb{E}_{\mathcal{F}_{t-2}} \left( \inf_{x_{T-1} \in X_{T-1}(x_{T-2}, \xi_{T-1})} f_{T-1}(x_{T-2}, x_{T-1}, \xi_{T-1}) \right) + \ldots \right)
\]

for some functions $f_t$ taking values in $\mathbb{R} \cup \{+\infty\}$, where $x_0$ is given and where

\[
X_t(x_{t-1}, \xi_t) = \left\{ x_t \in X_t : g_t(x_{t-1}, x_t, \xi_t) \leq 0, \ A_t x_t + B_t x_{t-1} = b_t \right\}
\]

for some vector-valued function $g_t$ and some nonempty compact convex set $X_t \subset \mathbb{R}^n$.

We make the following assumption on $(\xi_t)$:

- (H1) $\xi_t$ is interstage independent and for $t = 2, \ldots, T$, $\xi_t$ is a random vector taking values in $\mathbb{R}^K$ with discrete distribution and finite support $\Theta_t = \{\xi_{t,1}, \ldots, \xi_{t,M}\}$ while $\xi_1$ is deterministic.

To alleviate notation and without loss of generality, we have assumed that the number $M$ of possible realizations of $\xi_t$, the size $K$ of $\xi_t$, and $n$ of $x_t$ do not depend on $t$. 
Under Assumption (H1), \( \mathbb{E}_{x_{t-1}} \) coincides with its unconditional counterpart \( \mathbb{E}_t \) where \( \mathbb{E}_t \) is the expectation computed with respect to the distribution of \( \xi_t \). To ease notation, we will drop the index \( t \) in \( \mathbb{E}_t \). As a result, for problem (21), we can write the following dynamic programming equations: we set \( Q_{T+1} \equiv 0 \) and for \( t = 2, \ldots, T \), define

\[
Q_t(x_{t-1}) = \mathbb{E}\left( \Omega_t(x_{t-1}, \xi_t) \right)
\]

with

\[
\Omega_t(x_{t-1}, \xi_t) = \begin{cases} 
\inf_{x_t} F_t(x_{t-1}, x_t, \xi_t) := f_t(x_{t-1}, x_t, \xi_t) + Q_{t+1}(x_t) \\
\quad x_t \in X_t(x_{t-1}, \xi_t) = \{ x_t \in X_t : g_t(x_{t-1}, x_t, \xi_t) \leq 0, A_t x_t + B_t x_{t-1} = b_t \} \end{cases}
\]

Problem (21) can then be written

\[
\begin{cases} 
\inf_{x_0} F_0(x_0, x_1, \xi_1) := f_0(x_0, x_1, \xi_1) + Q_1(x_1) \\
\quad x_1 \in X_1(x_0, \xi_1) = \{ x_1 \in X_1 : g_1(x_0, x_1, \xi_1) \leq 0, A_1 x_1 + B_1 x_0 = b_1 \}
\end{cases}
\]

with optimal value denoted by \( Q_1(x_0) = \Omega_1(x_0, \xi_1) \).

Recalling definition (3) of the \( \varepsilon \)-fattening of a set, we make the following Assumption (H2) for \( t = 1, \ldots, T \):

1. \( X_t \subset \mathbb{R}^n \) is nonempty, convex, and compact.
2. For every \( x_{t-1}, x_t \in \mathbb{R}^n \) the function \( f_t(x_{t-1}, x_t, \cdot) \) is measurable and for every \( j = 1, \ldots, M \), the function \( f_t(\cdot, \cdot, \xi_{t,j}) \) is proper, convex, and lower semicontinuous.
3. For every \( j = 1, \ldots, M \), each component of the function \( g_t(\cdot, \cdot, \xi_{t,j}) \) is a convex lower semicontinuous function.
4. There exists \( \varepsilon > 0 \) such that:
   4.1) for every \( j = 1, \ldots, M \), \( X_{t-1}^\varepsilon \times X_t \subseteq \text{dom } f_t(\cdot, \cdot, \xi_{t,j}) \);
   4.2) for every \( j = 1, \ldots, M \), for every \( x_{t-1} \in X_{t-1}^\varepsilon \), the set \( X_t(x_{t-1}, \xi_{t,j}) \) is nonempty.
5. If \( t \geq 2 \), for every \( j = 1, \ldots, M \), there exists

\[
\tilde{x}_{t,j} = (\tilde{x}_{t,j,t-1}, \tilde{x}_{t,j,t}) \in X_t(\tilde{x}_{t,j,t-1}, \xi_{t,j}) 
\]

such that \( \tilde{x}_{t,j} \in X_t(\tilde{x}_{t,j,t-1}, \xi_{t,j}) \).

The following proposition, proved in [20], shows that Assumption (H2) guarantees that for \( t = 2, \ldots, T \), recourse function \( Q_t \) is convex and Lipschitz continuous on the set \( X_{t-1}^\varepsilon \) for every \( 0 < \varepsilon \) < \( \varepsilon \). SREDA and its convergence analysis are based on this proposition.

**Proposition 3.1.** Under Assumption (H2), for \( t = 2, \ldots, T + 1 \), for every \( 0 < \varepsilon < \varepsilon \), the recourse function \( Q_t \) is convex, finite and Lipschitz continuous on \( X_{t-1}^\varepsilon \).

**Proof:** We refer to the proof of Proposition 3.1 in [20] where similar value functions are considered.

Assumption (H2) will also be used to derive explicit formulas for the cuts to be built for recourse functions \( Q_t \) in SREDA applied to the nonlinear problems we are interested in.

**3.2. Algorithm.** Recalling Assumption (H1), the distribution of \( (\xi_2, \ldots, \xi_T) \) is discrete and the \( M^{T-1} \) possible realizations of \( \xi_2, \ldots, \xi_T \) can be organized in a finite scenario tree with the root node \( n_0 \) associated to a stage 0 (with decision \( x_0 \) taken at that node) having one child node \( n_1 \) associated to the first stage (with \( \xi_1 \) deterministic).

To describe SREDA, we need some notation: \( N \) is the set of nodes, \( \text{Nodes}(t) \) is the set of nodes of the scenario tree for stage \( t \) and for a node \( n \) of the tree, we denote by:

- \( C(n) \) the set of its children nodes (the empty set for the leaves);
- \( x_n \) a decision taken at that node;
- \( p_n \) the transition probability from the parent node of \( n \) to \( n \);
- \( \xi_n \) the realization of process \( (\xi_t) \) at node \( n \): for a node \( n \) of stage \( t \), this realization \( \xi_n \) contains in

(Note that to simplify notation, the same notation \( \xi_{\text{Index}} \) is used to denote the realization of the process at node \( \text{Index} \) of the scenario tree and the value of the process \( (\xi_t) \) for stage \( \text{Index} \). The context will allow us to know which concept is being referred to. In particular, letters \( n \) and \( m \) will only be used to refer to nodes while \( t \) will be used to refer to stages.)
The Stochastic REgularized Decomposition Algorithm (SREDA) is given below:

\[ F_t(x_{t-1}, \xi_{t}) = \min_{x_{t}} F_t(x_{t}, x_{m}, \xi_{m}) \]

where \( F_t(x_{t}, x_{m}, \xi_{m}) = f_t(x_{t}, x_{m}, \xi_{m}) + Q_{t+1}^{k}(x_{m}) \) is the function given by

\[ Q_{t}^{k}(x_{m}) = \min_{0 \leq k \leq k_{t}} C_{t}^{k}(x_{t-1}) \]

To describe and analyze the algorithm, it is convenient to introduce the function \( Q_{t}^{k} : \mathcal{X}_{t-1} \times \Theta_{t} \rightarrow \mathbb{R} \) given by

\[ Q_{t}^{k}(x_{n}, x_{m}, \xi_{m}) = \inf_{x_{m}} F_{t}^{k}(x_{n}, x_{m}, \xi_{m}) \]

where

\[ F_{t}^{k}(x_{n}, x_{m}, \xi_{m}) = f_{t}(x_{n}, x_{m}, \xi_{m}) + Q_{t+1}^{k}(x_{m}) \]

The Stochastic REgularized Decomposition Algorithm (SREDA) is given below:

**SREDA (Stochastic REgularized Decomposition Algorithm).**

Step 1) **Initialization.** Let \( Q_{t}^{0} : \mathcal{X}_{t-1} \rightarrow \mathbb{R} \cup \{-\infty\}, t = 2, \ldots, T, \) satisfying \( Q_{t}^{0} \leq Q_{t} \) be given and \( Q_{T+1}^{0} \equiv 0. \) Set \( C_{t}^{0} = Q_{t}^{0}, t = 2, \ldots, T+1, k = 1. \)

Step 2) **Forward pass.**

For \( t = 1, \ldots, T, \)

For every node \( n \) of stage \( t-1, \)

For every child node \( m \) of node \( n, \) compute

an optimal solution \( x_{m}^{k} \) of

\[ \inf_{x_{m}} F_{t}^{k-1}(x_{n}, x_{m}, x_{t}^{P,k}, \xi_{m}) \]

where \( x_{n}^{k} = x_{0}, x_{t}^{P,k} \) is any point in \( \mathcal{X}_{t} \) and where \( F_{t}^{k-1} \) is the function given by

\[ F_{t}^{k-1}(x_{n}, x_{m}, x_{t}^{P}, \xi_{m}) = f_{t}(x_{n}, x_{m}, \xi_{m}) + Q_{t+1}^{k-1}(x_{m}) + \lambda_{t,k} \| x_{m} - x_{t}^{P} \|^2, \]

with \( \lambda_{t,k} = 0 \) if \( t = T \) or \( k = 1. \)

End For

End For

Step 3) **Backward pass.**

Select a set of nodes \( (n_{1}^{k}, n_{2}^{k}, \ldots, n_{T}^{k}) \) with \( n_{1}^{k} \) a node of stage \( t (n_{1}^{k} = n_{1}) \) and for \( t \geq 2, n_{t}^{k} \) a child node of \( n_{t-1}^{k} \) corresponding to a sample \( (\xi_{1}^{k}, \xi_{2}^{k}, \ldots, \xi_{T}^{k}) \) of \( (\xi_{1}, \xi_{2}, \ldots, \xi_{T}) \).
Set $\theta_{T+1}^k = 0$ and $\beta_{T+1}^k = 0$.

For $t = T, \ldots, 2$,

For every child node $m$ of $n = n_{t-1}^k$, solve

$$\mathcal{Q}_t^k(x_n, \xi_m) = \left\{ \inf_{x_m} F_t^k(x_n, x_m, \xi_m) \right\} \quad x_m \in \mathcal{X}_t(x_n, \xi_m)$$

and compute, using Proposition 2.2, a subgradient $\pi_m^k \in \partial \mathcal{Q}_t^k(x_n, \xi_m)$ at $x_n$.

End For

The new cut $C_t^k$ is obtained computing

$$\theta_t^k = \sum_{m \in C(n)} p_m \mathcal{Q}_t^k(x_n, \xi_m), \quad \beta_t^k = \sum_{m \in C(n)} p_m \pi_m^k.$$

End For

Step 4) Do $k \leftarrow k + 1$ and go to Step 2).

### 3.3. On the prox-centers and penalizations.

Though $x_t^{P,k}$ are now random variables, the remarks of Section 2.4 on the choice of the prox-centers for REDDP still apply for SREDA. Indeed, convergence of SREDA holds for any sequence $(x_t^{P,k})_{k \geq 2}$ of prox-centers in $\mathcal{X}_t$ and of penalty parameters $\lambda_{t,k}$ converging to zero for every $t$, but the performance of the method depends on how these sequences are chosen. The following choices for $\lambda_{t,k}$ and $x_t^{P,k}$ will be used in our numerical tests of SREDA:

- Weighted average of previous values: $x_t^{P,k} = \sum_{j=1}^{k-1} \gamma_{t,k,j} x_t^j$ with $\gamma_{t,k,j}$ nonnegative weights and $\Gamma_{t,k} = \sum_{j=1}^{k-1} \gamma_{t,k,j}$.
- $\lambda_{t,k} = \rho_t^k$ where $0 < \rho_t < 1$ or $\lambda_{t,k} = \frac{1}{t}$ for $t < T$, $k \geq 2$.

### 4. Convergence analysis of SREDA.

We will assume that the sampling procedure in SREDA satisfies the following property:

(H3) for every $j = 1, \ldots, M$, for every $t = 2, \ldots, T$, and for every $k \in \mathbb{N}^*$, $\mathbb{P}(\xi_t^k = \xi_{t,j}) = \mathbb{P}(\xi_t = \xi_{t,j}) > 0$.

For every $t = 2, \ldots, T$, and $k \geq 1$,

$$\xi_t^k$$

is independent of $\sigma(\xi_1^k, \xi_T^k, \xi_2^k, \ldots, \xi_{T-1}^k, \xi_{T-2}^k, \ldots, \xi_{t-1}^k)$.

The following lemma will be useful in the sequel:

**Lemma 4.1.** Consider the sequences $Q_t^k, \theta_t^k,$ and $\beta_t^k$ generated by SREDA. Under Assumptions (H2), then almost surely, for $t = 2, \ldots, T+1$, the following holds:

(a) $Q_t^k$ is convex with $Q_t^k \leq Q_t$ on $\mathcal{X}_{t-1}^r$ for all $k \geq 1$;

(b) the sequences $(\theta_t^k)_{k \geq 1}$ and $(\beta_t^k)_{k \geq 1}$ are bounded;

(c) for $k \geq 1$, $Q_t^k$ is Lipschitz continuous on $\mathcal{X}_{t-1}^r$.

**Proof.** It suffices to follow the proof of Lemma 3.2 in [20].\(^5\) We give the main steps of the proof which is by backward induction on $t$ starting with $t = T + 1$ where the statement holds by definition of $Q_{T+1}$. Assuming for $t \in \{2, \ldots, T\}$ that $Q_t^k$ is Lipschitz continuous on $\mathcal{X}_{t-1}^r$ with $Q_{t+1}^k \leq Q_{t+1}$, then setting $n = n_t$, for every $m \in C(n)$ we have $Q_t(x_n, \xi_m) \geq Q_t^k(x_n, \xi_m)$ which gives

$$Q_t(x_{t-1}) = \sum_{m \in C(n)} p_m Q_t(x_{t-1}, \xi_m) \geq \sum_{m \in C(n)} p_m Q_t^k(x_{t-1}, \xi_m) \geq \sum_{m \in C(n)} p_m \left( \mathcal{Q}_t^k(x_n, \xi_m) + \langle \pi_m^k, x_{t-1} - x_n^k \rangle \right) = C_t^k(x_{t-1}).$$

\(^4\)Note that the proposition can be applied because Assumption (H2) holds and thus the assumptions of the proposition are satisfied for value function $Q_t^k(\cdot, \xi_m)$.

\(^5\)In [20] a forward algorithm is considered. In this setting, finiteness of coefficients $\theta_t^k$ and $\beta_t^k$ is not guaranteed for the first iterations (for instance $\theta_1^k$ are $-\infty$) but the proof is similar.
Observing that for every $F$ we have $\lambda$ on $X_C$, we have $\sum_{k=0}^{\infty} C^k$ defines a valid cut for $Q_t$ and $Q_t \geq C^k$. Assumptions (H2)-1)-4) and finiteness of $Q_t$ on $X_C$ imply that $Q^k(x_n^k, \xi_m)$ and $\pi^k_m$ are bounded for every $m \in C(n)$, and allow us to obtain a uniform upper bound on $\beta^k$, i.e., a Lipschitz constant valid for all functions $Q^k_t, t = 2, \ldots, T + 1, k \geq 1$.

Theorem 4.2 shows the convergence of the sequence $Q^k_t(x_0, \xi_1)$ to the optimal value $Q_1(x_0)$ of (21) and that any accumulation point of the sequence $((x^k_n)_{n \in N})_{k \geq 1}$ can be used to define an optimal solution of (21).

**Theorem 4.2 (Convergence analysis of SREDA).** Consider the sequences of stochastic decisions $x^k_n$ and of recourse functions $Q^k_t$ generated by SREDA to solve dynamic programming equations (22)-(23)-(24). Let Assumptions (H1), (H2), and (H3) hold and assume that $\lambda_{T, k} = 0$ and that for every $t = 1, \ldots, T - 1$, we have $\lim_{k \to +\infty} \lambda_{t, k} = 0$. Then:

(i) almost surely, for $t = 2, \ldots, T + 1$, the following holds:

$$\mathcal{H}(t): \forall n \in \text{Nodes}(t - 1), \lim_{k \to +\infty} Q_t(x^k_n) - Q^k_t(x^k_n) = 0.$$ 

(ii) Almost surely, the limit of the sequence $(\bar{F}_{t-1}(x_0, x^k_n, x_{P,k}^k, \xi))_k$ of the approximate first stage optimal values and of the sequence $(Q^k_t(x_0, \xi_1))_k$ is the optimal value $Q_1(x_0)$ of (21). Also, let $(x^k_n)_{n \in N}$ be any accumulation point of the sequence $((x^k_n)_{n \in N})_{k \geq 1}$. Now define $x_1, \ldots, x_T$ with $x_t : \mathbb{Z} \to \mathbb{R}^n$ by $x_t(\xi_1, \ldots, \xi_t) = x^*_{m}$ where $m$ is given by $\xi_{[m]} = (\xi_1, \ldots, \xi_t)$. Then $(x_1, \ldots, x_T)$ is an optimal solution to (21).

**Proof:** In this proof, all equalities and inequalities hold almost surely. We show $\mathcal{H}(2), \ldots, \mathcal{H}(T + 1)$, by induction backwards in time. $\mathcal{H}(T + 1)$ follows from the fact that $Q_{T+1} = Q^k_{T+1} = 0$. Now assume that $\mathcal{H}(t + 1)$ holds for some $t \in \{2, \ldots, T\}$. We want to show that $\mathcal{H}(t)$ holds. Take a node $n \in \text{Nodes}(t - 1)$. Let $\mathcal{S}_n = \{k \geq 1 : n^k_{t-1} = n\}$ be the set of iterations such that the sampled scenario passes through node $n$. Due to Assumption (H3), the set $\mathcal{S}_n$ is infinite. We first show that

$$\lim_{k \to +\infty, k \in \mathcal{S}_n} Q_t(x^k_n) - Q^k_t(x^k_n) = 0.$$ 

Take $k \in \mathcal{S}_n$. We have $n^k_{t-1} = n, x^k_{n^k_{t-1}} = x^k_n$ and recalling (25), we have $Q^k_t(x^k_n) = \theta^k_t$. Using definition (30) of $\theta^k_t$, it follows that

$$Q^k_t(x^k_n) \geq Q^k_t(x^k_n) = \theta^k_t = \sum_{m \in C(n)} p_m(Q^k_t(x^k_m, \xi_m)).$$

Now let $\bar{x}^k_t$ such that $F_{t-1}^k(x^k_n, \bar{x}^k_m, \xi_m) = Q^k_{t-1}(x^k_n, \xi_m)$ where $Q^k_{t-1}$ is defined by (26) with $k$ replaced by $k - 1$. Using (32) and the definition of $Q_t$, we get

$$0 \leq Q_t(x^k_n) - Q^k_t(x^k_n) \leq \sum_{m \in C(n)} p_m \left[ Q_t(x^k_m, \xi_m) - Q^k_t(x^k_m, \xi_m) \right],$$

$$\leq \sum_{m \in C(n)} p_m \left[ Q_t(x^k_m, \xi_m) - Q^k_{t-1}(x^k_m, \xi_m) \right] \
= \sum_{m \in C(n)} p_m \left[ Q_t(x^k_m, \xi_m) - F_{t-1}^k(x^k_m, \bar{x}^k_m, \xi_m) \right],$$

$$= \sum_{m \in C(n)} p_m \left[ Q_t(x^k_m, \xi_m) - F_{t-1}^k(x^k_m, \xi_m) \right] + \sum_{m \in C(n)} p_m \left[ F_{t-1}^k(x^k_m, \xi_m) - F_{t-1}^k(x^k_m, \bar{x}^k_m, \xi_m) \right].$$

Now using the definitions of $F_{t-1}^k$ and $F_t$, we obtain

$$Q_t(x^k_n, \xi_m) - F_{t-1}^k(x^k_m, \xi_m) = Q_t(x^k_n, \xi_m) - f_t(x^k_n, x^k_m, \xi_m) - Q^k_{t-1}(x^k_m)$$

$$= Q_t(x^k_n, \xi_m) - f_t(x^k_n, x^k_m, \xi_m) + Q_{t+1}(x^k_m) - Q^k_{t+1}(x^k_m).$$

Observing that for every $m \in C(n)$ the decision $x^k_m \in X_t(x^k_n, \xi_m)$, we obtain, using definition (23) of $Q_t$,
that
\[ F_t(x^k, x^k, \xi_m) \geq \Omega_t(x^k, \xi_m). \]
Combining this relation with (34) gives for \( k \in S_n \)
\[ \Omega_t(x^k, \xi_m) - F_t^{k-1}(x^k, x^k, \xi_m) \leq \Omega_{t+1}(x^k) - \Omega_{t+1}^{k-1}(x^k). \]
Next,
\[
\begin{align*}
F_t^{k-1}(x^k, x^k, \xi_m) - F_t^{k-1}(x^k, x^k, \xi_m) \\
+ \bar{F}_t^{k-1}(x^k, x^k, x^k, \xi_m) - \bar{F}_t^{k-1}(x^k, x^k, x^k, \xi_m) \\
+ \overline{F}_t^{k-1}(x^k, x^k, x^k, \xi_m) &- \overline{F}_t^{k-1}(x^k, x^k, x^k, \xi_m) \\
\leq F_t^{k-1}(x^k, x^k, \xi_m) - F_t^{k-1}(x^k, x^k, x^k, \xi_m) \\
+ \bar{F}_t^{k-1}(x^k, x^k, x^k, \xi_m) &- \bar{F}_t^{k-1}(x^k, x^k, x^k, \xi_m),
\end{align*}
\]
where the above inequality comes from the fact \( \bar{x}^k \in X_t(x^k, \xi_m), \) i.e., \( \bar{x}^k \) is feasible for optimization problem (28) with objective function \( \bar{F}_t^{k-1}(x^k, x^k, \xi_m) \) and optimal solution \( \bar{x}^k. \) We get
\[ 0 \leq F_t^{k-1}(x^k, x^k, \xi_m) - F_t^{k-1}(x^k, x^k, \xi_m) \leq \lambda_{t,k}(\|x^k - x^k P_k\|^2 - \|x^k - x^k P_k\|^2) \leq \lambda_{t,k}\|x^k - x^k\|^2 - \lambda_{t,k}D(X_t)^2, \]
where \( D(X_t) \) is the diameter of \( X_t \) (finite, since \( X_t \) is compact). Plugging (37) and (35) into (33) yields for any \( k \in S_n \)
\[ 0 \leq \Omega_t(x^k) - Q_t^k(x^k) \leq \lambda_{t,k}D(X_t)^2 + \sum_{m \in C(n)} p_m \left( \Omega_{t+1}(x^k) - \Omega_{t+1}^{k-1}(x^k) \right). \]
Using the induction hypothesis \( \mathcal{H}(t+1), \) we have for every child node \( m \) of node \( n \) that
\[ \lim_{k \rightarrow +\infty} \Omega_{t+1}(x^k) - \Omega_{t+1}^{k-1}(x^k) = 0. \]
Now recall that \( \Omega_{t+1} \) is convex on the compact set \( X_t \) (Proposition 3.1), \( x^k \in X_t \) for every child node \( m \) of node \( n, \) and the functions \( \Omega_{t+1}^{k-1}, k \geq 1, \) are Lipschitz continuous with \( \Omega_{t+1}^{k-1} \geq \Omega_{t+1}^{k-1} \) on \( X_t \) (Lemma 4.1). It follows that we can use Lemma A.1 in [16] to deduce from (39) that for every \( m \in C(n) \)
\[ \lim_{k \rightarrow +\infty} \Omega_{t+1}(x^k) - \Omega_{t+1}^{k-1}(x^k) = 0. \]
Combining this relation with (38) and using the fact that \( \lim_{k \rightarrow +\infty} \lambda_{t,k} = 0, \) we obtain
\[ \lim_{k \rightarrow +\infty, k \in S_n} \Omega_t(x^k) - Q_t^k(x^k) = 0. \]
To show \( \mathcal{H}(t), \) it remains to show that
\[ \lim_{k \rightarrow +\infty, k \in S_n} \Omega_t(x^k) - Q_t^k(x^k) = 0. \]
To show (41), we proceed similarly to the end of the proof of Theorem 4.1 in [20], by contradiction and using the Strong Law of Large Numbers (the same arguments were first used in a similar context in Theorem 3.1 of [16] but for a different problem formulation and sampling scheme). However, aside the regularization aspect, SREDA builds the cuts in a backward pass, using, at iteration \( k, \) recourse functions \( Q_t^{k-1} \) instead of \( Q_t^{k-1} \) used in [20], [16]. For the sake of completeness, we check in Lemma 7.1 in the Appendix that relation (41) holds for SREDA, the key to the proof being the fact that the sampled nodes for iteration \( k \) are independent on the decisions computed at the nodes of the scenario tree for that iteration and on recourse functions \( Q_t^{k-1}. \) This achieves the proof of (i).
(ii) Recalling that the root node \( n_0 \) with decision \( x_0 \) taken at that node has a single child node \( n_1 \) with
corresponding decision $x_{n_1}^k$ at iteration $k$, the computations in (i) show that for every $k \geq 1$, we have

$$
0 \leq \Omega_1(x_0, \xi_1) - \Omega_k(x_0, \xi_1) \leq \Omega_1(x_0, \xi_1) - F_t^{k-1}(x_0, x_{n_1}^k, \xi_1) + \lambda_1 k D(\lambda_1)^2,
$$

$$
\leq \Omega_2(x_{n_1}^k) - \Omega_{k-1}^{k-1}(x_{n_1}^k) + \lambda_1 k D(\lambda_1)^2.
$$

We have shown in (i) that $\lim_{k \to \infty} \Omega_k(x_{n_1}^k) - \Omega_{k-1}^{k-1}(x_{n_1}^k) = 0$. Plugging this relation into (42) shows that

$$
\lim_{k \to \infty} \Omega_1(x_0, \xi_1) = \lim_{k \to \infty} F_t^{k-1}(x_0, x_{n_1}^k, \xi_1) = \lim_{k \to \infty} F_t^{k-1}(x_0, x_{n_1}^k, \xi_1) = \Omega_1(x_0, \xi_1).
$$

Now take an accumulation point $(x_{n_k}^\ast)_{n \in \mathcal{N}}$ of the sequence $(x_{n_k}^k)_{n \in \mathcal{N}}_{k \geq 1}$ and let $K$ be an infinite set of iterations such that for every $n \in \mathcal{N}$, $\lim_{k \to \infty, k \in K} x_{n_k}^k = x_{n}^\ast$. Using once again computations from (i), we get for any $k \geq 1$, $t = 1, \ldots, T$, $n \in \text{Nodes}(t-1)$, $m \in C(n),

$$
0 \leq \Omega_t(x_n^k, \xi_m) - \Omega_t^{k-1}(x_n^k, \xi_m) \leq \Omega_t(x_n^k, \xi_m) - F_t^{k-1}(x_n^k, x_m^k, \xi_m) + \lambda_t k D(\lambda_t)^2,
$$

$$
\leq \Omega_{t+1}(x_m^k) - \Omega_{t+1}^{k-1}(x_m^k) + \lambda_t k D(\lambda_t)^2,
$$

which can be written

$$
-\lambda_t k D(\lambda_t)^2 \leq \Omega_t(x_n^k, \xi_m) - F_t^{k-1}(x_n^k, x_m^k, \xi_m) \leq \Omega_{t+1}(x_m^k) - \Omega_{t+1}^{k-1}(x_m^k).
$$

Since $\lim_{k \to \infty} \Omega_{t+1}(x_m^k) - \Omega_{t+1}^{k-1}(x_m^k) = 0$ (due to (i)), the above relation shows that

$$
\lim_{k \to \infty} \Omega_t(x_n^k, \xi_m) - F_t^{k-1}(x_n^k, x_m^k, \xi_m) = 0.
$$

We will now use the continuity of $\Omega_t(\cdot, \xi_m)$ which follows from (H2) (see Lemma 3.2 in [20] for a proof). We have

$$
\Omega_t(x_n^\ast, \xi_m) = \lim_{k \to \infty, k \in K} \Omega_t(x_n^k, \xi_m) \text{ using the continuity of } \Omega_t(\cdot, \xi_m),
$$

$$
= \lim_{k \to \infty, k \in K} F_t^{k-1}(x_n^k, x_m^k, \xi_m) \text{ using (43),}
$$

$$
= \lim_{k \to \infty, k \in K} f_t(x_n^k, x_m^k, \xi_m) + \Omega_{t+1}^{k-1}(x_m^k),
$$

$$
\geq f_t(x_n^\ast, x_m^\ast, \xi_m) + \lim_{k \to \infty, k \in K} \Omega_{t+1}(x_m^k) \text{ using (i) and lsc of } f_t,
$$

$$
\geq f_t(x_n^\ast, x_m^\ast, \xi_m) + \Omega_{t+1}(x_m^\ast) = F_t(x_n^\ast, x_m^\ast, \xi_m)
$$

where for the last inequality we have used the continuity of $\Omega_{t+1}$. To achieve the proof of (ii) it suffices to observe that the sequence $(x_n^k, x_m^k)_{k \in K}$ belongs to the set

$$
\tilde{X}_{t,m} = \{ (x_{t-1}, x_t) \in \mathcal{X}_{t-1} \times \mathcal{X}_t : g_t(x_{t-1}, x_t, \xi_m) \leq 0, A_m x_t + B_m x_{t-1} = b_m \}
$$

and this set is closed since $g_t$ is lower semicontinuous and $\mathcal{X}_t$ is closed. Therefore $x_n^\ast \in X_t(x_n^\ast, \xi_m)$, which, together with (44), shows that $x_n^\ast$ is an optimal solution of $\Omega_t(x_n^\ast, \xi_m) = \inf \{ F_t(x_n^\ast, x_m^\ast, \xi_m) : x_m \in X_t(x_n^\ast, \xi_m) \}$ and achieves the proof of (ii).

Remark: [Application to the convergence proof of Regularized SDDP.] In SREDA, decisions are computed at every iteration for all the nodes of the scenario tree. However, in practice, decisions will only be computed for the nodes of the sampled scenarios $(\xi_1^k, \ldots, \xi_T^k)$ (to compute the trial points) and their children nodes, to compute the cuts (such is the case of SDDP). This variant of SREDA, referred to as SDDP-REG in what follows, will build the same cuts and compute the same decisions for the nodes of the sampled scenarios as SREDA. For SDDP-REG, for a node $n$, the decision variables $(x_n^k)_{k}$ are defined for an infinite subset $\mathcal{S}_n$ of iterations where the sampled scenario passes through the parent node of node $n$, i.e., $\mathcal{S}_n = S_{P(n)}$. With this notation, for SDDP-REG, applying Theorem 4.2-(i), we get for $t = 2, \ldots, T + 1$,

$$
\lim_{k \to \infty, k \in \mathcal{S}_{P(n)}} \Omega_k(x_n^k) - \Omega_{k-1}^{k-1}(x_n^k) = 0
$$

almost surely. Also almost surely, the limit of the sequence $(F_t^{k-1}(x_0, x_{n_1}, x_1^{P,k}, \xi_1))_k$ of the approximate first stage optimal values is the optimal value $\Omega_1(x_0)$ of (21).

---

6Though when deriving these relations in (i) we had fixed $k \in \mathcal{S}_n$, the inequalities we now re-use for (ii) are valid for any $k \geq 1$.

7The existence of an accumulation point comes from the fact that the decisions belong almost surely to a compact set.
Remark: [Extension of SREDA to risk-averse nonlinear problems.] Using [20], SREDA algorithm can be extended to nested risk-averse formulations of risk-averse multistage stochastic nonlinear programs of form

\[
\inf_{x_1 \in X_1(x_0, \xi_1)} f_1(x_1, \xi_1) + \rho_1 f_2(x_1, \xi_2) + \ldots + \rho_{T-1} f_{T-1}(x_{T-1}, \xi_{T-1}) + \rho_T f_T(x_T, \xi_T),
\]

where \( \rho_t : Z_{t+1} \rightarrow Z_t \) is a coherent and law invariant conditional risk measure. The convergence proof of SREDA can be easily obtained combining the convergence proof of risk-averse decomposition methods from [20] with the convergence proof of Theorem 4.2.

For SREDA, we have an analogue of Remark 2.4 for REDDP:

Remark: If for a given stage \( t \), \( X_t \) is a polytope and we do not have the nonlinear constraints given by constraint functions \( g_t \) (i.e., the constraints for this stage are linear), then the conclusions of Proposition 3.1, Lemma 4.1, and Theorem 4.2 hold under weaker assumptions. More precisely, for such stages \( t \), we assume (H2)-1), (H2)-2), and instead of (H2)-4), (H2)-5), the weaker assumption (H2)-3):

(H2)-3' There exists \( \varepsilon > 0 \) such that:

3.1' for every \( j = 1, \ldots, M, X_{t-1}^c \times X_t \subset \text{dom} f_t (\cdot, \xi_{t,j}) ; \\
3.2' for every \( j = 1, \ldots, M, \) for every \( x_{t-1} \in X_{t-1} \), the set \( X_t(x_{t-1}, \xi_{t,j}) \) is nonempty.

5. Multistage portfolio optimization models with direct transaction and market impact costs.

5.1. Multistage portfolio selection models with direct transaction costs. This section presents risk-neutral and risk-averse multistage portfolio optimization models with direct transaction costs over a discretized horizon of \( T \) stages. We model the direct transaction costs incurred by selling and purchasing securities as being proportional to the amount of the transaction [5, 12].

Let \( n \) be the number of risky assets and asset \( n + 1 \) be cash. Next \( x^i_t \) is the dollar value of asset \( i = 1, \ldots, n+1 \) at the end of stage \( t = 1, \ldots, T \), \( \xi_t \) is the return of asset \( i \) at \( t \), \( y^i_t \) is the amount of asset \( i \) sold at the end of \( t \), \( z^i_t \) is the amount of asset \( i \) bought at the end of \( t \), \( \eta_t > 0 \) and \( \nu_t > 0 \) are respectively the proportional selling and purchasing transaction costs. Each component \( x^i_0, i = 1, \ldots, n+1, \) of \( x_0 \) is a known parameter. The expression \( \sum_{i=1}^{n+1} \xi^i_{t-1} x^i_t \) is the budget available at the start of the investment planning horizon. The notation \( u_i \) is a parameter defining the maximal amount that can be invested in each financial asset \( i \).

For \( t = 1, \ldots, T \), given a portfolio \( x_{t-1} = (x^1_{t-1}, \ldots, x^{n+1}_{t-1}) \) and \( \xi_t \), we define the set \( X_t(x_{t-1}, \xi_t) \) as the set of \( (x_t, y_t, z_t) \in \mathbb{R}^{n+1} \times \mathbb{R}^{n} \times \mathbb{R}^n \) satisfying

\[
x^{n+1}_t = \xi^{n+1}_{t-1} x^{n+1}_{t-1} + \sum_{i=1}^{n} ((1 - \eta_i) y^i_t - (1 + \nu_i) z^i_t),
\]

and for \( i = 1, \ldots, n, \)

\[
\begin{align*}
x^i_t &= \xi^i_t x^i_{t-1} - y^i_t + z^i_t, \\
y^i_t &\leq \xi^i_t x^i_{t-1}, \\
x^i_t &\leq u_i \sum_{i=1}^{n+1} \xi^i_{t-1}, \\
x^i_t &\geq 0, \\
y^i_t &\geq 0, \\
z^i_t &\geq 0.
\end{align*}
\]
Constraints (48a) define the amount of security \( i \) hold at each stage \( t \) and take into account the proportional transaction costs. Constraints (47) are the cash flow balance constraints and define how much cash is available at each stage. Constraints (48b) preclude selling an amount larger than the amount hold. Constraints (48c) do not allow the position in security \( i \) at time \( t \) to exceed a specified limit \( u_i \). Constraints (48d) prevent short-selling. Constraints (48e) and (48f) enforce the non-negativity of the amounts purchased and sold.

**Risk-neutral model.** With this notation, the dynamic programming equations of a risk-neutral portfolio model of form (22), (23), (24) can be written\(^8\): for \( t = T \), setting \( Q_{T+1}(x_T) = E[\sum_{i=1}^{n+1} \xi_{i+1}^T x_i^T] \) we solve the problem
\begin{equation}
Q_T(x_{T-1}, \xi_T) = \begin{cases} 
\text{Max } Q_{T+1}(x_T) \\
(x_T, y_T, z_T) \in X_T(x_{T-1}, \xi_T),
\end{cases}
\end{equation}
while at stage \( t = T - 1, \ldots, 1 \), we solve
\begin{equation}
Q_t(x_{t-1}, \xi_t) = \begin{cases} 
\text{Max } Q_{t+1}(x_t) \\
(x_t, y_t, z_t) \in X_t(x_{t-1}, \xi_t),
\end{cases}
\end{equation}
where for \( t = 2, \ldots, T \), \( Q_t(x_{t-1}) = E[Q_t(x_{t-1}, \xi_t)] \). With this model, we maximize the expected return of the portfolio taking into account the transaction costs, non-negativity constraints, and bounds imposed on the different securities.

**Risk-averse model.** As we recall from Remark 4, SREDA can be easily extended to solve risk-averse problems of form (46). We can therefore define a nested risk-averse counterpart of the risk-neutral portfolio problem we have just introduced and solve it with SREDA. This model is obtained replacing the expectation in the risk-neutral portfolio problem above by the (unconditional, due to Assumption (H1)) risk measure \( \rho_t : Z_t \rightarrow \mathbb{R} \) given by
\begin{equation}
\rho_t[Z] = (1 - \lambda_t)E[Z] + \lambda_t AV@R_{\alpha_t}[Z],
\end{equation}
where \( \lambda_t \in (0, 1) \), \( \alpha_t \in (0, 1) \) is the confidence level of the Average Value-at-Risk, and \( \rho_t \) is computed with respect to the distribution of \( \xi_t \). Therefore, a risk-averse portfolio problem with direct transaction costs can be written as follows: at stage \( T \), setting \( Q_{T+1}(x_T) = \rho_{T+1}[\sum_{i=1}^{n+1} \xi_{i+1}^T x_i^T] \), we solve
\begin{equation}
Q_T(x_{T-1}, \xi_T) = \begin{cases} 
\text{Max } Q_{T+1}(x_T) \\
(x_T, y_T, z_T) \in X_T(x_{T-1}, \xi_T),
\end{cases}
\end{equation}
while at stage \( t = T - 1, \ldots, 1 \), we solve
\begin{equation}
Q_t(x_{t-1}, \xi_t) = \begin{cases} 
\text{Max } Q_{t+1}(x_t) \\
(x_t, y_t, z_t) \in X_t(x_{t-1}, \xi_t),
\end{cases}
\end{equation}
where for \( t = 2, \ldots, T \), \( Q_t(x_{t-1}) = \rho_t[Q_t(x_{t-1}, \xi_t)] \).

### 5.2. Conic quadratic models for multistage portfolio selection with market impact costs.

Due to market imperfections, securities can seldom be traded at their current theoretical market price, which leads to additional costs, called market impact costs. If the trade is very large and involves the purchase (resp., selling) of a security, the price of the share may rise (resp., drop) between the placement of the trade and the completion of its execution [30]. As more of a security is bought or sold, the proportional cost increases due to the scarcity effect. Market impact costs are therefore particularly important for large institutional investors, for which they can represent a major proportion of the total transaction costs [28, 42]. Often, large trading orders are not executed at once, but are instead split into a sequence of smaller orders executed within a given time window. Taken individually, these small orders will exert little or no pressure on the market [45], which can curb market impact costs. The downside of such a delayed approach is that the execution of the entire trade order is postponed, which may lead to a loss in opportunities caused by

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\(^8\)It is indeed immediately seen that (49)-(50) is of form (22), (23), (24), writing the maximization problems as minimization problems and introducing the extended state \( s_1 = (x_t, y_t, z_t) \).
The change in the price of a security is impacted by the size of the transaction and is often modelled as a concave monotonically increasing function of the trade size [1, 44]. In that vein, Lillo et al. [26] and Gabaix et al. [14] model market impact costs as a concave power law function of the transaction size. Bouchaud et al. [10] use a logarithmic function of the transaction size and assert that the market impact is temporary and decays as a power law. Moazeni et al. [29] propose linear market impact costs and evaluate the sensitivity of optimal execution strategies with respect to errors in the estimation of the parameters. Mitchell and Braun [28] study the standard portfolio selection problem in which they incorporate convex transaction costs, including market impact costs, incurred when rebalancing the portfolio. They rescale the budget available after paying transaction costs, which results into a fractional programming problem that show that market impact costs are a function of the square root of the amount traded. Similarly, Torre [42] models the price change as proportional to a 3/2 power law of the transacted amount (see (54)).

The risk-neutral multistage portfolio optimization problem with market impact costs writes as follows:

For \( t = 1, \ldots, T \), given a portfolio \( x_{t-1} = (x_{t-1}^1, \ldots, x_{t-1}^n, x_{t-1}^n) \) and \( \xi_t \), we now define the set

\[
X^{MT}_t(x_{t-1}, \xi_t) = \left\{ (x_t, y_t, z_t, q_t, g_t) \in \mathbb{R}^{n+1}_+ \times \mathbb{R}^n_+ \times \mathbb{R}^n_+ : (48a) - (48c), i = 1, \ldots, n, \right. \\
x_t^{n+1} = \xi_t^{n+1} x_{t-1}^{n+1} + \sum_{i=1}^n (y_t^i - z_t^i - q_t^i), \quad \text{for } a, \\
g_t^i = y_t^i + z_t^i, \quad i = 1, \ldots, n, \quad \text{for } b, \\
m_t^i g_t^i \sqrt{g_t^i} \leq q_t^i, \quad i = 1, \ldots, n \quad \text{for } c
\]

Constraints (55)-(a) define how much cash is held at each period and take into account the market impact costs. Constraints (55)-(b) define the total amount \( g_t^i \) of security \( i \) traded at time \( t \). The nonlinear constraints (55)-(c) follow from (54) and permit to define the total market impact costs \( q_t^i \) incurred for security \( i \) at time \( t \).

The risk-neutral multistage portfolio optimization problem with market impact costs writes as follows:
for \( t = T \), setting \( Q_{T+1}(x_T) = \mathbb{E}[\sum_{i=1}^{n+1} \xi_{i+1} x_i^T] \) we solve the problem

\[
(56) \quad \Omega_T (x_{T-1}, \xi_T) = \left\{ \begin{array}{l}
\max Q_{T+1}(x_T) \\
(x_T, y_T, z_T, q_T, g_T) \in \mathcal{X}_T^M(x_{T-1}, \xi_T),
\end{array} \right.
\]

while at stage \( t = T - 1, \ldots, 1 \), we solve

\[
(57) \quad \Omega_t (x_{t-1}, \xi_t) = \left\{ \begin{array}{l}
\max \mathbb{E}[Q_{t+1}(x_t)] \\
(x_t, y_t, z_t, q_t, g_t) \in \mathcal{X}_T^M(x_{t-1}, \xi_t),
\end{array} \right.
\]

where for \( t = 2, \ldots, T \), \( Q_t(x_{t-1}) = \mathbb{E}[\Omega_t (x_{t-1}, \xi_t)] \).

It is easy to see that the left-hand side of the constraints (55)-(c) are convex functions, which implies that Assumption (H2)-3) is satisfied. More generally, we check that Assumption (H2) holds for the above problem and SREDA and SDDP-REG can be applied to solve it. For implementation purposes, it is convenient to rewrite constraints (55)-(c) as a conic quadratic constraint:

**Theorem 5.1.** For \( t = 1, \ldots, T \), the convex feasible sets

\[
\mathcal{S}_t = \left\{ (g_t, q_t) = (g^1_t, \ldots, g^n_t, q^1_t, \ldots, q^n_t) \in \mathbb{R}^{n+1} \times \mathbb{R}^n : \right\}
\]

\[
m^i_t q^i_t \sqrt{q^i_t} \leq q^i_t, \quad i = 1, \ldots, n,
\]

\[
\text{can be equivalently represented with the rotated second-order constraints } (58a) \text{ and } (58b) \text{ and the linear constraints } (58c)-(58h):
\]

\[
(58a) \quad (\ell^i_t)^2 \leq 2s^i_t q^i_t/m^i_t \quad i = 1, \ldots, n,
\]

\[
(58b) \quad (w^i_t)^2 \leq 2v^i_t r^i_t \quad i = 1, \ldots, n,
\]

\[
(58c) \quad \ell^i_t = v^i_t \quad i = 1, \ldots, n,
\]

\[
(58d) \quad s^i_t = w^i_t \quad i = 1, \ldots, n,
\]

\[
(58e) \quad r^i_t = 0.125 \quad i = 1, \ldots, n,
\]

\[
(58f) \quad s^i_t, v^i_t \geq 0 \quad i = 1, \ldots, n,
\]

\[
(58g) \quad -g^i_t \leq \ell^i_t \quad i = 1, \ldots, n,
\]

\[
(58h) \quad g^i_t \leq \ell^i_t \quad i = 1, \ldots, n.
\]

**Proof:** This representation is proved in [3]. The proof is given in the appendix to make the presentation self-contained.

For \( t = 1, \ldots, T \), given \( x_{t-1} \in \mathbb{R}^{n+1} \) and \( \xi_t \), denoting

\[
(59) \quad \mathcal{X}_T^M(x_{t-1}, \xi_t) = \left\{ (x_t, y_t, z_t, q_t, g_t, \xi_t, s_t, v_t, w_t) \in \mathbb{R}^{n+1} \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n : \right\}
\]

\[
\begin{array}{l}
(x_{t+1} = \xi_t + x_{t-1} + \sum_{i=1}^{n+1} (y^i_t - z^i_t - q^i_t),
\end{array}
\]

and using Theorem 5.1, our portfolio optimization problem with market impact costs (56)-(57) can be rewritten substituting in (56)-(57) for \( t = 1, \ldots, T \), the constraints \((x_t, y_t, z_t, q_t, g_t) \in \mathcal{X}_T^M(x_{t-1}, \xi_t)\) by

\[
(x_t, y_t, z_t, q_t, g_t, \xi_t, s_t, v_t, w_t) \in \mathcal{X}_T^M(x_{t-1}, \xi_t).
\]

This formulation of the portfolio problem satisfies Assumption (H2) and can be solved using SDDP-REG with all subproblems of the forward and backward passes being conic quadratic optimization problems.

**6. Numerical experiments.** In this section, we evaluate the computational efficiency of the REDDP and SDDP-REG algorithms presented in Sections 2 and 3, and benchmark them with standard, non regularized versions of the deterministic and stochastic DDP algorithms. The analysis starts (Section 6.2) with the deterministic setting and the REDDP algorithm tested on a portfolio optimization problem with direct
Variant name | Prox-center $x^{P,k}_{t}$ for $t < T, k > 1$ | Penalization $\lambda_{t,k}$ for $t < T, k > 1$
--- | --- | ---
REDDP-PREV-REG1-\(\rho\) or SDDP-REG-PREV-REG1-\(\rho\) | $x_{t-1}^k$ | $\rho^k$ with $0 < \rho < 1$
REDDP-PREV-REG2 or SDDP-REG-PREV-REG2 | $x_{t-1}^k$ | $\frac{1}{k^2}$
REDDP-AVG-REG1-\(\rho\) or SDDP-REG-AVG-REG1-\(\rho\) | $\frac{1}{k} \sum_{j=1}^{k-1} x^j_t$ | $\rho^k$ with $0 < \rho < 1$
REDDP-AVG-REG2 or SDDP-REG-AVG-REG2 | $\frac{1}{k} \sum_{j=1}^{k} x^j_t$ | $\frac{1}{k^2}$

Table 1: Some variants of REDDP and SDDP-REG.

transaction costs, and continues in Section 6.3 with the stochastic case and the SDDP-REG algorithm tested on risk-neutral and risk-averse formulations involving either direct transaction or market impact costs. In practice, portfolio selection problem parameters (the returns) are not known in advance and stochastic optimization models are used for these applications. We use such models in Section 6.3. However, to compare DDP and REDDP, we assume that the parameters of the portfolio problems, namely the returns, are known over the optimization period. This allows us to easily generate feasible problem instances that can be solved with DDP and REDDP and to know what would have been the best return for these instances.

6.1. Data and parameter settings. The problem instances and the algorithms are modelled in Python and the problems are solved with MOSEK 8.0.0.50 solver [31]. The experiments are carried out using a single thread of an Intel(R) Core(TM) i5-4200M CPU @ 2.50GHz machine.

The following settings are used for the parameters of the portfolio optimization problems described in Section 5. The budget available is $1 billion and can be used to invest in $n = 6$ risky securities in addition to cash. The proportional direct transaction costs $\eta = \nu$ are set to 1%. The return data of six securities were collected from WRDS [39] for the period ranging from July 2005 to May 2016. The monthly fixed cash return is equal to 0.2%. The largest position in any security is set to $u_i = 20\%$. The parameters of the REDDP and SDDP-REG algorithms follow. We consider a number $T$ of stages ranging from 10 to 350. The sample size per stage, i.e., the cardinality of $\Theta_t$ (using the notation of Section 3), is set to $M = 60$.

As we recall from Section 2.4 for REDDP and from Subsection 3.3 for SDDP-REG, we need to define sequences $x^{P,k}_t$ of prox-centers and $\lambda_{t,k}$ of penalization parameters to define instances of REDDP and SDDP-REG. In our study, we will use the prox-centers and penalization parameters given in Table 1 (we recall that no penalization is used for $t = T$ and for $k = 1$, i.e., $\lambda_{T,k} = \lambda_{t,1} = 0$ for all $t, k$). This table also contains the names used for the corresponding REDDP and SDDP-REG variants. We recall that in [4], only the variant SDDP-REG-PREV-REG1-\(\rho\) was tested for linear programs. In this section, we test all deterministic variants from Table 1 for linear programs and variant SDDP-REG-PREV-REG2 for multistage stochastic linear and nonlinear programs, using the extension SDDP-REG of the SDDP algorithm used in [4] for linear programs.

6.2. Deterministic instances. In this section, we consider the deterministic counterpart of the portfolio optimization problem with direct transaction costs presented in Section 5.1 using the parameters given in the previous section and 8 different values for the number $T$ of time periods: $T = 10, 50, 100, 150, 200, 250, 300,$ and $350$. We solve these problems using DDP and the following 6 variants of REDDP (using the notation of Table 1): REDDP-PREV-REG1-0.2 (REDDP-PREV-REG1-\(\rho\) with $\rho = 0.2$), REDDP-PREV-REG1-0.9 (REDDP-PREV-REG1-\(\rho\) with $\rho = 0.9$), REDDP-PREV-REG2, REDDP-AVG-REG1-0.2 (REDDP-AVG-REG1-\(\rho\) with $\rho = 0.2$), REDDP-AVG-REG1-0.9 (REDDP-AVG-REG1-\(\rho\) with $\rho = 0.9$), and REDDP-AVG-REG2.

Stopping criterion. When studying the convergence of REDDP in Section 2, we have not discussed the stopping criterion. At each iteration, this algorithm (same as DDP) can compute a lower bound on
the optimal value of the problem which is given at iteration \( k \) by \( Q_k(x_0) \) (using the notation of Section 2), the optimal value of the approximate problem for the first time period. It can also compute at iteration \( k \) the upper bound \( \sum_{t=1}^{T} f_t(x_{k-1}^t, x_k^t) \) on the optimal value. Given a tolerance \( \varepsilon \) (taken equal to \( 10^{-6} \) in our experiments), the algorithm stops when the difference between the upper and lower bound is less than \( \varepsilon \) (in this case, we have computed an \( \varepsilon \)-solution to the problem). Note, however, that since our portfolio problems are maximization problems, the approximate first stage problem provides an upper bound on the optimal value and \( \sum_{t=1}^{T} f_t(x_{k-1}^t, x_k^t) \) provides a lower bound.

We have checked that on all instances, all algorithms correctly compute the same optimal value and that the upper and lower bounds were converging to this optimal value. For illustration, Figure 1 displays the evolution of the upper and lower bounds and of the optimality gap across the iterative process with DDP for the instance with \( T = 300 \).

![Fig. 1. DDP method: DDP lower and upper bounds (left plot) and gap (right plot) in % of the upper bound for \( T = 300 \).](image)

The CPU time needed to solve the different instances with DDP and our 6 variants of REDDP is given in Table 2 and the corresponding reduction factor in CPU time for these REDDP variants is given in Table 3. The number of iterations of the algorithms is given in Table 4. We observe that on all instances REDDP variants are much faster and need much less iterations than DDP. Most importantly, the benefits of regularization increase as the problem gets larger and the number of stages raises. When \( T \) is large there is a drastic improvement in CPU time with REDDP variants. For instance, for \( T = 250, 300, \) and \( 350 \), the reduction factor in CPU time varies (among the 6 REDDP variants) respectively in the interval \([80.0, 114.3]\), \([71.5, 171.6]\), and \([95.5, 184.4]\). Remarkably, the solution time with the regularized algorithm REDDP is not monotonically increasing with the number of stages, which points out to the scalability of the algorithm and the possibility to use it for even larger problems. As an illustration, the difference in time and number of iterations between DDP and REDDP-PREV-REG2 is shown in Figure 2, which highlights that the time and iteration differential increases with the number of stages.

### 6.3. Stochastic instances

In this section, we evaluate the computational efficiency of the SDDP-REG algorithm presented in Section 3, and benchmark it with the standard, non regularized version of the SDDP algorithm. We have implemented the regularization scheme SDDP-REG-PREV-REG2 given that penalization scheme REG2 performed best for the deterministic instances (see Section 6.2). The algorithms are tested on three types of problem instances with \( T = 12, 20, 24 \) periods: risk-neutral portfolio models of Subsection 5.1, risk-averse portfolio models of Subsection 5.1, and risk-neutral portfolio model with market impact costs from Subsection 5.2.

**Stopping criterion.** For risk-neutral SDDP, we used the following stopping criterion. The algorithm stops if the gap is \(< 3\%\). The gap is defined as \( \frac{Ub-Lb}{Ub} \) where \( Ub \) and \( Lb \) correspond to upper and lower bounds, respectively. The upper bound \( Ub \) corresponds to the optimal value of the first stage problem (recall that we have a maximization problem). The lower bound \( Lb \) corresponds to the lower end of a 95% one-sided confidence interval on the optimal value for \( N = 500 \) policy realizations, see [40] for a detailed discussion on
Table 2

<table>
<thead>
<tr>
<th>$T$</th>
<th>10</th>
<th>50</th>
<th>100</th>
<th>150</th>
<th>200</th>
<th>250</th>
<th>300</th>
<th>350</th>
</tr>
</thead>
<tbody>
<tr>
<td>DDP</td>
<td>3</td>
<td>69</td>
<td>268</td>
<td>780</td>
<td>1304</td>
<td>2400</td>
<td>4289</td>
<td>5348</td>
</tr>
<tr>
<td>REDDP-PREV-REG2</td>
<td>1</td>
<td>4</td>
<td>8</td>
<td>13</td>
<td>17</td>
<td>30</td>
<td>25</td>
<td>29</td>
</tr>
<tr>
<td>REDDP-PREV-REG1-0.2</td>
<td>1</td>
<td>4</td>
<td>12</td>
<td>21</td>
<td>28</td>
<td>23</td>
<td>60</td>
<td>56</td>
</tr>
<tr>
<td>REDDP-PREV-REG1-0.9</td>
<td>1</td>
<td>4</td>
<td>8</td>
<td>12</td>
<td>17</td>
<td>21</td>
<td>25</td>
<td>29</td>
</tr>
<tr>
<td>REDDP-AVG-REG2</td>
<td>1</td>
<td>4</td>
<td>8</td>
<td>13</td>
<td>17</td>
<td>30</td>
<td>25</td>
<td>29</td>
</tr>
<tr>
<td>REDDP-AVG-REG1-0.2</td>
<td>1</td>
<td>5</td>
<td>8</td>
<td>12</td>
<td>17</td>
<td>21</td>
<td>47</td>
<td>55</td>
</tr>
<tr>
<td>REDDP-AVG-REG1-0.9</td>
<td>1</td>
<td>4</td>
<td>9</td>
<td>13</td>
<td>17</td>
<td>22</td>
<td>26</td>
<td>30</td>
</tr>
</tbody>
</table>

CPU time (in seconds) to solve instances of a portfolio problem of form (1), namely the deterministic counterpart of the portfolio models from Section 5.1, using DDP and various variants of REDDP.

Table 3

<table>
<thead>
<tr>
<th>$T$</th>
<th>10</th>
<th>50</th>
<th>100</th>
<th>150</th>
<th>200</th>
<th>250</th>
<th>300</th>
<th>350</th>
</tr>
</thead>
<tbody>
<tr>
<td>REDDP-PREV-REG2</td>
<td>3.0</td>
<td>17.3</td>
<td>33.5</td>
<td>60.0</td>
<td>76.7</td>
<td>80.0</td>
<td>171.6</td>
<td>184.4</td>
</tr>
<tr>
<td>REDDP-PREV-REG1-0.2</td>
<td>3.0</td>
<td>17.3</td>
<td>22.3</td>
<td>37.1</td>
<td>46.6</td>
<td>104.4</td>
<td>71.5</td>
<td>95.5</td>
</tr>
<tr>
<td>REDDP-PREV-REG1-0.9</td>
<td>3.0</td>
<td>17.3</td>
<td>33.5</td>
<td>65.0</td>
<td>76.7</td>
<td>114.3</td>
<td>171.6</td>
<td>184.4</td>
</tr>
<tr>
<td>REDDP-AVG-REG2</td>
<td>3.0</td>
<td>17.3</td>
<td>33.5</td>
<td>60.0</td>
<td>76.7</td>
<td>80.0</td>
<td>171.6</td>
<td>184.4</td>
</tr>
<tr>
<td>REDDP-AVG-REG1-0.2</td>
<td>3.0</td>
<td>13.8</td>
<td>33.5</td>
<td>65.0</td>
<td>76.7</td>
<td>114.3</td>
<td>91.3</td>
<td>97.2</td>
</tr>
<tr>
<td>REDDP-AVG-REG1-0.9</td>
<td>3.0</td>
<td>17.3</td>
<td>29.8</td>
<td>60.0</td>
<td>76.7</td>
<td>109.1</td>
<td>165.0</td>
<td>178.3</td>
</tr>
</tbody>
</table>

CPU time reduction factor for different REDDP variants.

Fig. 2. Difference in solution time and iteration number between REDDP-PREV-REG2 and DDP algorithms.

6.3.1. Risk-neutral multistage linear problem with direct transaction costs (49)-(50). We report in Table 5 the computational time and number of iterations required for SDDP and SDDP-REG-PREV-REG2 to solve the instance of portfolio problem (51)-(52) obtained taking $T = 12, 20, 24$ and the problem parameters given in Subsection 6.1. We observe that as in the deterministic case, the regularized decomposition method converges much faster (it is about twice as faster for $T = 24$) and requires much less iterations. We also refer to Figure 3 where the evolution of the upper and lower bounds and the gap (in % of the upper bound) are represented for SDDP and SDDP-REG-PREV-REG2 for $T = 24$. We see that the gap decreases much faster with SDDP-REG-PREV-REG2.

6.3.2. Risk-averse multistage linear problem with direct transaction costs (51)-(52). We implemented risk-averse models (51)-(52) taking $\lambda_t = 0.1$ and $\alpha_t = 0.1$, running the algorithms for 50 iterations. The CPU time is reported in Table 6. Since both problems are run for the same number
Table 4

<table>
<thead>
<tr>
<th>Variant</th>
<th>CPU time (s)</th>
<th>Number of iterations</th>
</tr>
</thead>
<tbody>
<tr>
<td>SDDP</td>
<td>14</td>
<td>4</td>
</tr>
<tr>
<td>SDDP-REG-PREV-REG2</td>
<td>6</td>
<td>2</td>
</tr>
<tr>
<td>SDDP</td>
<td>29</td>
<td>5</td>
</tr>
<tr>
<td>SDDP-REG-PREV-REG2</td>
<td>21</td>
<td>4</td>
</tr>
<tr>
<td>SDDP</td>
<td>40</td>
<td>6</td>
</tr>
<tr>
<td>SDDP-REG-PREV-REG2</td>
<td>18</td>
<td>3</td>
</tr>
</tbody>
</table>

Table 5

<table>
<thead>
<tr>
<th>T</th>
<th>Variant</th>
<th>CPU time (s)</th>
<th>Number of iterations</th>
</tr>
</thead>
<tbody>
<tr>
<td>12</td>
<td>SDDP</td>
<td>14</td>
<td>4</td>
</tr>
<tr>
<td></td>
<td>SDDP-REG-PREV-REG2</td>
<td>6</td>
<td>2</td>
</tr>
<tr>
<td>20</td>
<td>SDDP</td>
<td>29</td>
<td>5</td>
</tr>
<tr>
<td></td>
<td>SDDP-REG-PREV-REG2</td>
<td>21</td>
<td>4</td>
</tr>
<tr>
<td>24</td>
<td>SDDP</td>
<td>40</td>
<td>6</td>
</tr>
<tr>
<td></td>
<td>SDDP-REG-PREV-REG2</td>
<td>18</td>
<td>3</td>
</tr>
</tbody>
</table>

Fig. 3. Risk-neutral upper and lower bounds (left plot) and gap (right plot) in % of the upper bound for \( T = 24 \)

6.3.3. Conic risk-neutral multistage stochastic problem with market impact costs from Section 5.2. We consider two variants of the portfolio problem with market impact costs given in Section 5.2 in which we set the market impact unit cost \( m_i = 1, \ldots, n \), to respectively 3 basis points (we recall that a basis point is 0.01% = 10^{-4}) for the first model and 3% = 0.03 for the second. The CPU time and number of iterations to solve these problems with SDDP and SDDP-REG-PREV-REG2 are given in Table 7. The evolution of the upper and lower bounds and of the gap along the iterations of the algorithms are reported in Figures 5 and 6 for \( T = 24 \). We observe that when \( m_i \) are small the regularized variant is much quicker and the gap decreases much faster. When \( m_i \) increases, in particular for the value 3%, more money is invested in cash and the computational time and gap evolution with the non-regularized and regularized variants of SDDP are similar.
<table>
<thead>
<tr>
<th>Variant</th>
<th>CPU time (s)</th>
<th>Number of iterations</th>
</tr>
</thead>
<tbody>
<tr>
<td>SDDP</td>
<td>3895</td>
<td>50</td>
</tr>
<tr>
<td>SDDP-REG-PREV-REG2</td>
<td>3921</td>
<td>50</td>
</tr>
</tbody>
</table>

Table 6

CPU time and number of iterations to solve an instance of a portfolio problem of form (51)-(52) with $T = 48$ using SDDP and SDDP-REG-PREV-REG2.

Fig. 4. Risk-averse upper bounds, $\lambda_t = 0.1$, $\alpha_t = 0.1$.

Fig. 5. Upper and lower bounds (left plot) and gap (right plot) in % of the upper bound for the risk-neutral model with market costs, $m_i = 3bp$ for $T = 24$.

Fig. 6. Upper and lower bounds (left plot) and gap (right plot) in % of the upper bound for the risk-neutral model with market costs, $m_i = 3\%$ for $T = 24$.

7. Appendix.

7.1. Lemma used in the proof of Theorem 4.2. This lemma is essentially proved in the end of the proof of Theorem 4.1 in [20]. We provide the proof to make the presentation self-contained and check that the arguments apply to SREDA.


**Lemma 7.1.** Using the notation of Theorem 4.2, we have

\[ Q_t(x^k_n) - Q^k_t(x_n^k) = 0. \]

**Proof:** If (60) does not hold, there exists \( \varepsilon > 0 \) such that there is an infinite number of iterations \( k \in \mathbb{N} \) satisfying \( Q_t(x^k_n) - Q^k_t(x_n^k) \geq \varepsilon \). Since \( Q^k_t \geq Q^{-1}_t \), there is also an infinite number of iterations belonging to the set

\[ K_{n,\varepsilon} = \{ k \in \mathbb{N} : Q_t(x^k_n) - Q^k_t(x_n^k) \geq \varepsilon \}. \]

Consider the stochastic processes \((w^k_n)_{k \in \mathbb{N}}\) and \((y^k_n)_{k \in \mathbb{N}}\), where \( w^k_n = 1_{k \in K_{n,\varepsilon}} \) and \( y^k_n = 1_{k \in S_n} \), i.e., \( y^k_n \) takes the value 1 if node \( n \) belongs to the sampled scenario for iteration \( k \) (when \( n_{k-1} = n \)) and 0 otherwise. Assumption (H3) implies that random variables \((y^k_n)_{k \in \mathbb{N}}\) are independent and setting \( \tilde{F}_k = \sigma(w^1_n, \ldots, w^k_n, y^1_n, \ldots, y^n_{k-1}) \), by definition of \( x^k_n \) and \( Q^k_t \) that \( y^k_n \) is independent on \( (x^1_n, \ldots, x^k_n) \), using Lemma A.3 in [16], we obtain that random variables \( z^j \) are i.i.d. and have the distribution of \( y^1_n \). Using the Strong Law of Large Numbers, we get

\[ \lim_{N \to +\infty} \frac{1}{N} \sum_{j=1}^{N} z^j \xrightarrow{N \to +\infty} \mathbb{E}[z^1] = \mathbb{E}[y^1_n] = \mathbb{P}(y^1_n > 0) > 0. \tag{H3} \]

Relation (40) and Lemma A.1 in [16] imply that \( \lim_{k \to +\infty, k \in S_n} Q_t(x^k_n) - Q^k_t(x_n^k) = 0 \). It follows that the set \( K_{n,\varepsilon} \cap S_n = K_{n,\varepsilon} \cap \{ k \in \mathbb{N}^n : y^k_n = 1 \} \) is finite. This implies

\[ \lim_{N \to +\infty} \frac{1}{N} \sum_{j=1}^{N} z^j \xrightarrow{N \to +\infty} 0, \]

which yields the desired contradiction.

### 7.2. Proof of Theorem 5.1

The set

\[ S_t = \{ (g_i, q_i) = (g^1_i, \ldots, g^n_i, q^1_i, \ldots, q^n_i) \in \mathbb{R}_+^n \times \mathbb{R}_+^n : \}
\nt_i \leq q^i_i, i = 1, \ldots, n \}

can be equivalently written as

\[ |g^i| \leq \frac{q^i}{m^i}, i = 1, \ldots, n, \]
as well as with the following system of inequalities:

\[-\ell^i_t \leq g^i_t \leq \ell^i_t \quad i = 1, \ldots, n,\]

\[(\ell^i_t)^2 \leq \frac{q^i_t}{m^i_t} \sqrt{t^i} \quad i = 1, \ldots, n,\]

where \(\ell^i_t\) are auxiliary decision variables.

The nonlinear constraints (62) can in turn be equivalently represented with (58a)-(58f). Indeed, the inequalities (58b), (58d), and (58e) imply that:

\[(s^i_t)^2 \leq 2w^i_t v^i_t = 0.25\ell^i_t, \quad i = 1, \ldots, n.\]

Combining (58a) and (63), we have

\[(\ell^i_t)^2 \leq 2s^i_t \frac{q^i_t}{m^i_t} \leq \sqrt{t^i} \frac{q^i_t}{m^i_t}, \quad i = 1, \ldots, n.\]

Reciprocally, if \((g^i_t, q^i_t) \in S_i\), then \(\ell^i_t = v^i_t = |g^i_t|, s^i_t = w^i_t = \frac{1}{2}|q^i_t|\) defines a point satisfying (58a)-(58h), which achieves the proof of Theorem 5.1.

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