Risk-based Loan Pricing: Portfolio Optimization Approach With Marginal Risk Contribution

So Yeon Chun
McDonough School of Business, Georgetown University, Washington D.C. 20057, soyeon.chun@georgetown.edu

Miguel A. Lejeune
Department of Decision Sciences, George Washington University, Washington, DC 20052, mlejeune@gwu.edu

We consider a lender (bank) who determines the optimal loan price (interest rates) to offer to prospective borrowers under uncertain risk and borrower response. A borrower may or may not accept the loan at the price offered, and in the presence of default risk, both the principal loaned and the interest income become uncertain. We present a risk-based loan pricing optimization model, which explicitly takes into account marginal risk contribution, portfolio risk, and borrower's acceptance probability. Marginal risk assesses the amount a prospective loan would contribute to the bank's loan portfolio risk by capturing the interrelationship between a prospective loan and the existing loans in the portfolio and is evaluated with respect to the Value-at-Risk and Conditional-Value-at-Risk risk measures. We examine the properties and computational difficulties of the associated formulations. Then, we design a concavifiability reformulation method that transforms the nonlinear objective function of the loan pricing problems and permits to derive equivalent mixed-integer nonlinear reformulations with convex continuous relaxations. We discuss managerial implications of the proposed model and test the computational tractability of the proposed solution approach.

Key words: risk-based pricing, revenue management, loan portfolio, marginal risk contribution, Value-at-Risk, Conditional Value-at-Risk, willingness-to-pay, mixed-integer nonlinear stochastic programming

1. Introduction

Lending is the primary business activity for most commercial banks. The loan portfolio is usually the largest asset and the predominant source of revenue but also carries a significant exposure to credit risk (Mercer 1992, Stanhouse and Stock 2008). Although specific loan agreement terms and conditions vary, one of the most critical elements controlling the performance and risk of the loan portfolio is the interest rate, which can be referred to as the *price of a loan*. Importance and challenges of the structured loan pricing have been recognized by many practitioners (e.g., PwC 2012, BCG 2016, Sageworks 2016). For example, BCG points out that advanced pricing techniques embraced by an ever-expanding variety of businesses have not been yet adopted in commercial lending, causing forgone revenue of 7% to 10%. Sageworks also recognizes that banks could significantly increase the revenue if they had more structured pricing methodologies in place.

PWC mentions "As companies realize the inefficiency and unreliability of the traditional loan pricing strategies, we believe a trend toward a pricing optimization" in loan pricing.

Until the early 1990s, banks simply posted one price ("house rate") for each loan type and rejected most high-risk borrowers (Johnson 1992). Following the financial reforms[1], improvement of the underwriting technologies, and drop in data storage cost in recent years, however, banks started to manage their risks more effectively by adopting so called *risk-based pricing*: estimate the specific risk of each borrower and offer different prices (interest rates) to different borrowers and transactions (Bostic 2002, Thomas 2009).

The rationale behind risk-based pricing is rather straightforward. A lender should charge higher prices for borrowers with higher default risk and larger potential losses since they are more *costly*. Notably, the key element to such a pricing strategy is to identify the risks that are being priced. In a typical loan underwriting process, banks make approval and pricing decisions based on an estimate of the borrower's probability of default (PD), which is usually assessed by the credit rating/scores (e.g., Moody's, S&P and Fitch ratings) and loss given default (LGD), which is the amount of money a bank loses if a borrower defaults (see, e.g., Gupton et al. 2002, Phillips 2013 for more details). While recognizing the risk posed by each loan is essential for the optimal loan pricing, there are some limitations and challenges in the current application of this pricing strategy.

It should be first noted that the performance of the entire loan portfolio does not only depend on the risk of individual loans, but also on the interrelationships between the loans in the portfolio. In fact, the concept of portfolio management is not new in finance and optimization models have been widely used in the financial industry to build optimal portfolios of securities and to manage market risk (see, e.g., Cornuejóls and Tútúncú 2007, DeMiguel et al. 2009, Kawas and Thiele 2011, Dentcheva and Ruszczynski 2015, and references therein). However, the use of such optimization models remains limited in credit risk, in particular to determine the price at which to grant the loan (see Allen and Saunders 2002 and Kimber 2003 for general reviews on credit portfolio risk management). In terms of loan pricing, the portfolio optimization approach suggests that instead of relying on the "standalone risk", the individual risk of a borrower measured in isolation, the price of a loan should incorporate the change in the portfolio risk triggered by a new loan, or "risk contribution", the risk amount a prospective loan would contribute to the bank's current loan portfolio risk.

Evaluation of risk contributions requires the selection of risk measures in which the decomposition of the overall portfolio credit risk into individual loan risk contributions is attainable[2]. To this

[1] According to the database covering 91 countries over the 1973-2005 period, 74 countries in the sample had fully liberalized lending and deposit rates by 2005, compared to four countries in 1973. Most of the countries completed interest rate liberalization in the 1980s and 1990s (Abiad et al. 2008).

[2] This view is related to a modern asset allocation and investment style called "risk budgeting" that focuses on how risk is distributed throughout a portfolio.

end, we consider the "marginal risk" contribution, i.e., marginal impact of a particular loan on the overall portfolio risk, with the widely used Value-at-Risk (VaR) and Conditional-Value-at-Risk (CVaR, also known as expected shortfall) risk measures. VaR is the risk measure initially recommended by the internal rating-based approach preconized by the Basel Committee[3] and has been one of the most commonly used risk measures in the financial industry. On the other hand, CVaR, which accounts for losses exceeding VaR, is now becoming the risk measure of choice for the Basel Committee intending to "move from Value-at-Risk to Expected Shortfall" (Basel Committee on Banking Supervision 2013). These two risk measures are positive homogeneous (the risk of a portfolio scales proportionally to its size), and thus the marginal risk contributions with respect to VaR and CVaR can be represented as the conditional loss expectation of a loan provided that the losses of the entire portfolio reach a certain level (Glasserman 2006). Inclusion of such marginal risk contributions in the pricing optimization problem poses serious computational challenges (e.g., non-convexity); yet it would enable the lender to directly capture the interdependence between the prospective and existing loans in the portfolio and thus control the risk and profitability of the loan portfolio more effectively.

Another critical element to optimal loan pricing, which is often ignored in risk-based pricing, is the borrower's response, the propensity of a borrower to accept the offered loan price (acceptance probability). In many industries, such as retail and hospitality, the concepts of price response (demand function) and willingness to pay (WTP) have been well understood. Hence, the price optimization paradigm, which takes into account the impact of price changes on demand and profit, has been widely adopted (e.g., Phillips 2005, Cohen et al. 2016, and references therein). Under risk-based pricing, however, the price is still mainly based on the cost to provide a loan and to cover potential losses; banks usually set the loan price by adding a fixed margin to total cost to hit a target rate of return on capital (Caufield 2012, PwC 2012). While such strategy avoids the underpricing of high risk loans, the margin (and the final price) could be too high (too low) in the sense that lower (higher) prices maximize the total profit. This is because it overlooks variations in the value different borrowers place on a loan (different WTP and price sensitivity), and on bank's specific market position and relationships with different borrowers. As a result, banks lose opportunities to either make profitable loans (lose demand from potential borrowers with low WTP) or make even larger profits (by offering a higher price to a borrower who is willing to pay more). By incorporating borrowers' acceptance probability into risk-based pricing, banks will be

[3] The Basel Committee on Banking Supervision issues Basel Accords, recommendations on banking laws and regulations. Basel Accords are intended to amend international standards that control how much capital banks need to hold to guard against the financial and operational risks faced. Worldwide adoption of the Basel II Accords gave further impetus to the use of VaR. See e.g., Basel Committee on Banking Supervision 2006 for more details.

able to assess the expected profits more accurately as a function of borrowers' characteristics, price, as well as risks. Given that banks and financial services in general have far more information on their customers than most other industries (e.g., banks know whom they are doing business with and whether their offers were accepted or declined at a particular price and by which customer), they have a unique opportunity to take the price optimization practice even further.

In this paper, we consider a lender (bank) who determines the optimal loan price for prospective borrowers in order to maximize the net interest income of a loan portfolio under uncertain response and risk. A borrower may or may not accept the loan at the price offered, and in the presence of default risk, both the principal loaned and the interest payments become uncertain. Our risk-based loan price optimization model explicitly takes into account two critical elements discussed earlier: loan portfolio risk with marginal risk contribution and borrower's response (loan accept probability). The risk is evaluated with the VaR or CVaR measures, and we present specific optimization formulations employing commonly used forms of demand response functions (linear, exponential, and logit) with portfolio and marginal risk constraints. We first show that it is a non-convex stochastic programming optimization problem, which is very difficult to solve. We examine the properties and computational difficulties of the associated formulations. Then, we propose a reformulation approach based on the concavifiability concept, which allows for the derivation of equivalent mixed-integer nonlinear reformulations with continuous relaxation. We also extend the approach to the multi-loan pricing problem, which features explicit loan selection decisions in addition to the pricing ones. We discuss managerial implications of the proposed model and implementations.

1.1. Relevant Literature

Loan pricing problems have received significant attention in recent years from both industry and academia. Many practitioners recognize the importance of the structured loan pricing model in the financial industry (e.g., PwC 2012, BCG 2016, Sageworks 2016), and some academic researchers demonstrate that consumers' price elasticity/WTP (e.g., Gross and Souleles 2002) and a portfolio view (e.g., Musto and Souleles 2006) are pertinent to loan/credit pricing decision. However, most existing studies on the loan pricing have focused on the empirical evidence of risk-based pricing in various credit markets (e.g., Schuermann 2004, Edelberg 2006, Magri and Pico 2011). To our knowledge, there is no quantitative loan pricing model in the literature that explicitly takes into account both consumers' response and interdependence between the prospective loan and the loans in the existing portfolio. The paper attempts to fill this gap by proposing a risk-based loan pricing optimization models incorporating both aspects and further developing efficient reformulations of the proposed model.

There is a sparse yet growing body of literature that incorporates the pricing angle into the loan rate optimization problem. For example, Phillips (2013) establishes that optimizing the price for a consumer loan involves trade-offs: increasing the price for a prospective loan reduces the probability that the customer will accept the loan but increases profitability if the customer does accept. He then proposes a consumer credit pricing model determining the optimal rate, which maximizes the lender's expected net interest income. Similarly, Oliver and Oliver (2014) emphasize that the loan price controls not only the risk but also the profit by managing demand response, and describe the structural solution of the loan rate as a function of default and response risk (based on the acceptance probability of the loan given the price). Huang and Thomas (2015) use a linear response function to model the probability that a borrower will take the loan, and study how the Basel Accord impacts the optimal loan price, which maximizes the lender's profit. These papers, however, employ a standalone risk approach and do not incorporate interrelationships between loans.

As we consider the marginal risk contribution to explicitly capture interdependence between the prospective loan and the loans in the existing portfolio, our paper also contributes to the literature on the portfolio management with marginal risk contribution. The concept of marginal risk contribution has attracted increasing attention in recent years. Specifically, several researchers have investigated the properties of marginal VaR and CVaR marginal risk contributions (see Tasche 2000, Gourieroux et al. 2000, Kurth and Tasche 2003, Merino and Nyfeler 2004, Glasserman 2006, Liu 2015 and references therein). However, marginal risk contributions in the literature are often discussed only in the ex post analysis context rather than as an ex ante consideration. There are only a few very recent papers that attempt to incorporate the marginal risk concept into the portfolio selection problem (Zhu et al. 2010, Cui et al. 2016), but these are only specific to the mean-variance framework and overlook more computationally challenging issues associated with downside risk metrics, such as VaR, or CVaR. As we will show later, the inclusion of both price response function (acceptance probability) and the marginal VaR or CVaR constraint in the loan pricing optimization model considerably increases the complexity and tractability of the problem. Nevertheless, we propose a computationally tractable solution method that permits to concurrently handle the ex ante estimation (prior to granting the loan) of the marginal risk and determines the optimal interest rate. In particular, we use a concavifiability method that provides equivalent convex reformulations of the non-convex risk pricing problems.

2. Risk-based Loan Pricing Model

We consider a lender (bank) who needs to determine the optimal prices (interest rates) to offer to prospective borrowers to maximize the expected profit (net interest income) under uncertain borrowers' response and risk. We assume that there are $\bar{n}$ prospective loans under consideration and $\tilde{n} = n - \bar{n}$ granted (existing) loans in the current loan portfolio. For each loan $i = 1, 2, \ldots n$ (prospective and existing), let x_i be the price, which is written as the annual percentage rate (APR), a_i be the loan amount (principal), L_i be the Loss Given Default (LGD), q_i be the payment frequency (e.g., monthly, quarterly, etc.), and T_i be the term, which is in the unit of payment frequency[4].

When a certain price is offered, the prospective borrower may or may not accept the offer. The acceptance probability would be obviously dependent upon the price (e.g., as the price increases, ceteris paribus, the acceptance probability would decrease). It could also depend on other characteristics of the borrower (e.g., the borrower with higher risk/switching cost might be willing to pay more. Thus, we use a price response (acceptance probability) function $g(x_n, s_n)$, where x_n is the price offered and s_n describes the characteristics of the borrower (e.g., credit rating, age, income level, transaction history, etc.), to account for the likelihood of a borrower to accept the proposed interest rate[5].

Now, let us consider the case when the offer is accepted and the loan is granted. If the interest payments are made on the agreed upon dates and the principal on the loan is paid in full at maturity, the lender faces no default/credit risk and receives back the original principal amount lent plus an interest income. However, if the borrower defaults, both the principal loaned and the interest payments expected to be received are at risk, and the loss/risk magnitude depends on the time of default. We denote by $\chi_{i,t}, t = 1, \ldots, T_i, i = 1, \ldots, n$ a binary random variable taking value 0 if the borrower of loan i defaults at any time until t and taking value 1 otherwise, by $t_i^* = \min(\sum_{t=1}^{T_i} \chi_{i,t}, T_i), i = 1, \ldots, n$ the time at which the principal will be (possibly partially) repaid (i.e., t_i^* is the default time in case of default, and is otherwise the maturity of the loan i), and by δ the (one-period) discount factor.

Then we can write down the present value of a future uncertain stream of payments (discounted cash flow) for each loan, $\mathcal{G}(x_i), i = 1, \ldots, n$[6]:

$$\mathcal{G}_i(x_i) = \sum_{t=1}^{T_i} \chi_{i,t} \delta^t a_i (x_i/q_i) + a_i(1 - L_i - L_i \chi_{i,T_i}) \delta^{t_i^*}. \qquad (1)$$

The first term represents the discounted value of the interest payments and the second term is the discounted value of the repaid principal amount.

[4] For instance, for a 4-year loan with a quarterly payment schedule, $T = 48$.

[5] In Section 3.3, we discuss specific forms of the price response functions widely used in practice and provide reformulations of the corresponding optimization problems.

[6] Note that $\mathcal{G}(x_i)$ also depends on $\chi_{i,t},\ i = 1, 2, \ldots, n,\ t = 1, 2, \ldots, T_i$. To ease the notations, we omit denoting the full dependence.

We next formulate the risk constraint in terms of the stochastic loss of a portfolio with respect to the risk measure $\rho(\cdot)$[7]. In particular, we incorporate the constraint on each loan's marginal risk contribution to the pre-existing portfolio, which accounts for the correlation among loans in the portfolio. We denote the random loss of a portfolio $\zeta = \sum_{i=1}^{n} \zeta_i$ where $\zeta_i = a_i - \mathcal{G}_i$.

For the ease of exposition and demonstration, we first consider the case where $\bar{n} = 1$, i.e., there are $n-1$ granted loans and the lender now considers a prospective n^{th} loan and determines x_n[8]. Let ρ_n^M be the (marginal) risk contribution of a new n^{th} loan with risk metric ρ. For risk measures with positive homogeneity, it has been been demonstrated (see, e.g., Gourieroux et al. 2000, Kalkbrener et al. 2004, Tasche 2009) that Euler's theorem permits to define ρ_n^M as:

$$\rho_n^M = \rho(\zeta_i|\zeta) := \frac{\partial \rho(\zeta)}{\partial a_n} \ . \tag{2}$$

That is, the risk contribution can be calculated by obtaining the first order partial derivative of a risk measure with respect to the loan amount.[9]

Based on the discussion above, we can formulate the risk-based loan pricing optimization problem with a single prospective loan, denoted as **LPO** as follows:

$$\mathbf{LPO}: \quad \max_{x_n} \left\{ h(x_n) \ := g(x_n, s_n)\mathbb{E}\ \left[\left(\mathcal{G}_n(x_n) - a_n\right)\right] + \mathbb{E}\ \left[\sum_{i=1}^{n-1} \left(\mathcal{G}_i(x_i) - a_i\right)\right] \right\} \tag{3}$$

$$\text{s.t.} \ \ l \le x_n \le u, \tag{4}$$

$$\rho_n^M\left(\zeta(x_n)\right) \ \ \le \ \ \kappa_M a_n, \tag{5}$$

$$\rho\left(\zeta(x_n)\right) \ \ \le \ \ \kappa_P \sum_{i=1}^{n} a_i. \tag{6}$$

The lender's objective (3) is to maximize the expected profit taking into account uncertainties in both demand (response) and default risk[10]. The constraint (4) implements possible lower and upper bounds on the price possibly driven by the regulations (e.g., usury laws) and/or business practices with marketing or operational considerations (e.g., price stability is desirable). The equations (5)

[7] In Sections 3.1 and 3.2, we consider specific risk measures, namely VaR and CVaR and discuss reformulations in detail.

[8] This case may reflect the underwriting process of large-scale business loans (e.g., large corporate loans). In Section 4, we extend our model and provide general formulations for the case with $\bar{n} > 1$.

[9] Different methods of calculating risk contributions have been studied for different purposes. Standalone and incremental risk contribution (the change in total risk due to the inclusion of a component) are other alternatives popular in practice. However, those violate the desirable properties for credit risk management such as diversification and linear aggregation axioms and the sum of the incremental risk contributions across all components is generally not equal to the risk of the entire portfolio. See, Kalkbrener (2005) and Mausser and Rosen (2008) for general discussions.

[10] We could easily incorporate other costs such as funding costs (with risk-free rate), overhead/administrative expenses, which do not depend on the price of a loan. In our formulation, those other costs are fixed and normalized to zero.

and (6) represent the marginal and portfolio risk constraints with the thresholds κ_M and κ_P, which are defined as percentages value of the potential new loan and the overall portfolio. It is implicitly assumed that if the optimal objective value is negative, thereby indicating that it is not possible to find an admissible interest rate giving a positive expected profit, the optimal decision is to not extend any offer for the new loan and reject it. Later, this aspect is explicitly considered in the multiple prospective loan case discussed in Section 4.

In the next section, we discuss the challenges in solving this optimization problem and provide reformulations for both VaR and CVaR risk measures with several functional forms of the price-response function.

3. Reformulations and Properties of Risk-Based Loan Pricing Model

In this section, we derive the specific VaR and CVaR formulations for the risk-pricing model **LPO** presented in Section 2. First, we provide the formulation of the VaR and CVaR portfolio and marginal risk constraints, examine their complexity, and propose a linearization approach of the feasible set defined by the risk constraints. Next, we define several price-response functional forms, introduce them in the formulations of the **LPO** models with VaR and CVaR constraints, and study the complexity of the models. The resulting optimization problems take the form of mixed-integer nonlinear programming (MINLP) problems (see Burer and Letchford 2012, DAmbrosio and Lodi 2011, Krokhmal et al. 2011 for reviews of the MINLP field and risk measures). For some of the considered price-response functions, the corresponding MINLP problems are particularly complex, since their continuous relation is not convex. Therefore, we design a concavifiability approach that transforms the nonlinear objective function and permits to obtain convex MINLP reformulations with identical optimal solutions (all proofs are in the Appendix).

3.1. Portfolio and Marginal VaR Constraints: Properties and Reformulations

We first consider the portfolio risk constraint (6) for the VaR measure. The Value-at-Risk $\mathfrak{q}_\alpha$ of the random portfolio loss ζ at the level α is defined as:

$$\mathfrak{q}_\alpha = \inf\{z : \mathbb{P}(\zeta > z) \leq \alpha\} \ . \tag{7}$$

We decompose the portfolio loss ζ into the loss ζ_n due to the loan n under consideration and the loss $\sum_{i=1}^{n-1} \zeta_i$ due to the $(n-1)$ loans previously granted. While the interest rate for the $(n-1)$ granted loans is known, the losses that might be incurred with those loans is dependent on whether the borrowers will default or not. On the other hand, the loss ζ_n incurred with loan n depends on the annual percentage rate x_n that the institution has yet to determine in addition to whether the borrower will default.

The formulation of the portfolio risk constraint (6) for the VaR measure takes the form of a chance constraint with random technology matrix (Kataoka 1963):

$$\mathbb{P}\left(a_n \left(1 - \sum_{t=1}^{T_n} \chi_{n,t} \delta^t \frac{x_n}{q_n} - (1 - L_n + L_n \chi_{n,T_n}) \delta^{t_n^*} \right) + \sum_{i=1}^{n-1} \zeta_i \leq \kappa_P^{VaR} \sum_{i=1}^{n} a_i \right) \geq \alpha \ , \tag{8}$$

with $\mathbb{P}$ referring to a probability measure and the upper bound $\kappa_P^{VaR} \sum_{i=1}^{n} a_i$ on $\mathfrak{q}_\alpha$ limits the authorized risk. In the above formulation, the value of the threshold $\kappa_P^{VaR} \sum_{i=1}^{n} a_i$ is determined exogenously, prior to the optimization as it is customary, and is called target (Benati and Rizzi 2007, Yu et al. 2015) or "benchmark VaR" (Gourieroux et al. 2000)[11].

We now consider the marginal risk contribution constraint (5) for the VaR measure. Specifically, the equation (2) can be written as (Glasserman 2006):

$$\rho_n^M = \frac{\partial VaR_\alpha(\zeta)}{\partial a_n} = \mathbb{E}[\zeta_n | \zeta = VaR_\alpha(\zeta)] \ . \tag{9}$$

The marginal VaR contribution due to loan n is equal to the expectation of the loss associated to loan n conditional to the entire portfolio loss being equal to VaR. The magnitude of the marginal VaR contribution is limited with the following conditional expectation constraint (Prékopa 1995):

$$\mathbb{E}\left[a_n \left(1 - \sum_{t=1}^{T_n} \chi_{n,t} \delta^t \frac{x_n}{q_n} - (1 - L_n + L_n \chi_{n,T_n}) \delta^{t_n^*} \right) \middle| a_n \left(1 - \sum_{t=1}^{T_n} \chi_{n,t} \delta^t \frac{x_n}{q_n} - (1 - L_n + L_n \chi_{n,T_n}) \delta^{t_n^*} \right) + \sum_{i=1}^{n-1} \zeta_i = \kappa_P \sum_{i=1}^{n} a_i \right] \leq \kappa_M^{VaR} a_n \ . \tag{10}$$

If the conditional event has a positive probability to happen (Tasche 2009), the above conditional expectation constraints is given by:

$$\frac{\mathbb{E}\left[a_n \left(1 - \sum_{t=1}^{T_n} \chi_{n,t} \delta^t \frac{x_n}{q_n} - (1 - L_n + L_n \chi_{n,T_n}) \delta^{t_n^*} \right) \ , \ a_n \left(1 - \sum_{t=1}^{T_n} \chi_{n,t} \delta^t \frac{x_n}{q_n} - (1 - L_n + L_n \chi_{n,T_n}) \delta^{t_n^*} \right) + \sum_{i=1}^{n-1} \zeta_i = \kappa_P \sum_{i=1}^{n} a_i \right]}{\mathbb{P}\left(a_n \left(1 - \sum_{t=1}^{T_n} \chi_{n,t} \delta^t \frac{x_n}{q_n} - (1 - L_n + L_n \chi_{n,T_n}) \delta^{t_n^*} \right) + \sum_{i=1}^{n-1} \zeta_i = \kappa_P \sum_{i=1}^{n} a_i \right)} \leq \kappa_M^{VaR} a_n \ . \tag{11}$$

Optimization problems including constraints of form (8) and (11) cannot be solved analytically, nor by optimization solvers. We shall therefore derive equivalent reformulations of the constraints (8) and (11) that are amenable to a numerical solution.

We proceed in two steps to derive a mixed-integer linear programming (MILP) reformulation of the feasible set defined by the portfolio and marginal VaR constraints. The following parameter and set notations will be used. We denote by K the scenario index set. The vector $\omega^k = [\gamma_n^k, O^k]$ refers to the scenario with index $k, k \in K$. Each component $\gamma_{n,t}^k$ of $\gamma_n^k \in \{0,1\}^T$ is a Boolean indicator taking value 0 if the prospective loan n defaults at any time until t in scenario ω^k and taking value

[11] An alternative would be to let the model define endogenously the value of the threshold, which would then be a decision variable.

1 otherwise. Each component O_i^k of $O^k \in \mathbb{R}^{n-1}$ is the loss due to existing loan $i, i=1,\ldots,n-1$ in scenario ω^k over the entire planning horizon. The probability of scenario ω^k is p^k with $\sum_{k\in K} p^k = 1$, M^{k+} and M^{k-} are large positive scalars, and ϵ is an infinitesimal positive number.

The reformulation of the feasible set will require the lifting of the decisional space and the introduction of the following decision variables:

- z^{k+}: non-negative variable defining the portfolio loss in excess to $\kappa_P^{VaR} \sum_{i=1}^n a_i$ in scenario ω^k.
- z^{k-}: non-negative variable defining the portfolio loss below $\kappa_P^{VaR} \sum_{i=1}^n a_i$ in scenario ω^k.
- β^{k+}: binary decision variable indicating whether the loss is strictly larger than $\kappa_P^{VaR} \sum_{i=1}^n a_i$ in scenario ω^k ($\beta^{k+}=1$) or not ($\beta^{k+}=0$).
- β^{k-}: binary decision variable indicating whether the loss is strictly smaller than $\kappa_P^{VaR} \sum_{i=1}^n a_i$ in scenario ω^k ($\beta^{k-}=1$) or not ($\beta^{k-}=0$).

Theorem 1 demonstrates an intermediary step towards the linearization of the feasible set defined by the marginal and portfolio VaR constraints.

Theorem 1 *The feasible set defined by the stochastic risk constraints* (8) *and* (10) *can be equivalently represented by the following set of mixed-integer quadratic inequalities:*

$$a_n\left(1-\sum_{t=1}^{T_n}\gamma_{n,t}^k\delta^t\frac{x_n}{q_n}-(1-L_n+L_n\gamma_{n,T_n}^k)\delta^{t_n^*}\right)+\sum_{i=1}^{n-1}O_i^k-z^{k+}+z^{k-}=\kappa_P^{VaR}\sum_{i=1}^n a_i \qquad k\in K \tag{12}$$

$$\epsilon\ \beta^{k+}\le z^{k+}\le M^{k+}\beta^{k+} \qquad k\in K \tag{13}$$

$$\epsilon\ \beta^{k-}\le z^{k-}\le M^{k-}\beta^{k-} \qquad k\in K \tag{14}$$

$$\beta^{k+}+\beta^{k-}\le 1 \qquad k\in K \tag{15}$$

$$\sum_{k\in K}p^k\beta^{k+}\le 1-\alpha \tag{16}$$

$$\sum_{k\in K}\left(p^k(1-\beta^{k+}-\beta^{k-})a_n\left(1-\sum_{t=1}^{T_n}\gamma_{n,t}^k\delta^t\frac{x_n}{q_n}-(1-L_n+L_n\gamma_{n,T_n}^k)\delta^{t_n^*}\right)\right)$$
$$\le\kappa_M^{VaR}a_n\sum_{k\in K}p^k(1-\beta^{k+}-\beta^{k-}) \tag{17}$$

$$\beta^{k+},\beta^{k-}\in\{0,1\} \qquad k\in K \tag{18}$$

with

$$M^{k+}=a_n\left(1-\sum_{t=1}^{T_n}\gamma_{n,t}^k\delta^t(l/q_n)-(1-L_n+L_n\gamma_{n,T_n}^k)\delta^{t_n^*}\right)+\sum_{i=1}^{n-1}O_i^k-\kappa_P^{VaR}\sum_{i=1}^n a_i \qquad k\in K \tag{19}$$

$$M^{k-}=\kappa_P^{VaR}\sum_{i=1}^n a_i-a_n\left(1-\sum_{t=1}^{T_n}\gamma_{n,t}^k\delta^t(u/q_n)-(1-L_n+L_n\gamma_{n,T_n}^k)\delta^{t_n^*}\right)-\sum_{i=1}^{n-1}O_i^k \qquad k\in K. \tag{20}$$

Setting the constants $M^{k+}, M^{k-}, k\in K$ to the smallest possible value is important for enabling an efficient solution of the above problem. It is indeed well-known that assigning arbitrarily large values to these constants would result into a very loose continuous relaxation of the above MINLP problem and hinder its solution via a branch-and-bound algorithm (see, e.g., Feng et al. 2015).

Corollary 2 *The feasible area defined by the set of mixed-integer nonlinear inequalities* (12)-(18) *is nonconvex.*

The above result is due to the presence of the bilinear terms $\beta^{k+}x_n$ and $\beta^{k-}x_n, k \in K$ in (17). We shall now linearize the bilinear terms using the McCormick inequalities (McCormick 1976) and introducing the auxiliary variables $y^{k+}, y^{k-}, k \in K$. The following set of linear inequalities

$$y^{k+} \leq \beta^{k+}u \ , \ k \in K \tag{21}$$

$$y^{k+} \leq x_n \ , \ k \in K \tag{22}$$

$$y^{k+} \geq x_n + \beta^{k+}u - u \ , \ k \in K \tag{23}$$

$$y^{k+} \geq 0 \ , \ k \in K \tag{24}$$

$$y^{k-} \leq \beta^{k-}u \ , \ k \in K \tag{25}$$

$$y^{k-} \leq x_n \ , \ k \in K \tag{26}$$

$$y^{k-} \geq x_n + \beta^{k-}u - u \ , \ k \in K \tag{27}$$

$$y^{k-} \geq 0 \ , \ k \in K \tag{28}$$

ensure that $\beta^{k+}x_n = y^{k+}, k \in K$ and $\beta^{k-}x_n = y^{k-}, k \in K$. This allows for the substitution of y^{k+} for $\beta^{k+}x_n$ and of y^{k-} for $\beta^{k-}x_n$ in the nonlinear left-side of (17), and the modeling of the feasible set of the VaR portfolio and marginal risk constraints with a mixed-integer linear set of inequalities.

Lemma 3 *The feasible set defined by the portfolio and marginal VaR constraints* (8) *and* (10) *can be equivalently represented by the set of linear mixed-integer inequalities*

$$\begin{aligned}
&\sum_{k \in K} \left(p^k(1-\beta^{k+}-\beta^{k-}) \ (1-\delta^{t_n^*}(1-L_n+L_n\gamma^k_{n,T_n}))\right) - \frac{\delta^t}{q_n} \sum_{k \in K} \left(\sum_{t=1}^{T_n} \gamma^k_{n,t} \ (x_n - y^{k+} - y^{k-}) \ p^k \right) \\
&\leq \kappa_M^{VaR} \sum_{k \in K} p^k (1-\beta^{k+}-\beta^{k-}) \\
&(12)-(16) \ ; \ (18) \ ; \ (21)-(28) \ .
\end{aligned} \tag{29}$$

3.2. Portfolio and Marginal CVaR Constraints: Properties and Reformulations

Similarly, we first provide the compact formulation of the CVaR portfolio risk constraint (Glasserman 2006)

$$\mathbb{E}[\zeta \leq VaR_\alpha(\zeta)] = (1-\alpha)^{-1}\mathbb{E}[\zeta \mathbb{I}_{\{\zeta < \mathfrak{q}_\alpha\}}] \leq \kappa_P^{CVaR} \sum_{i=1}^{n} a_i \ , \tag{30}$$

and the CVaR marginal risk constraint for (2)

$$\rho_n^M = \frac{\partial CVaR_\alpha(\zeta)}{\partial a_n} = \mathbb{E}[\zeta^n | \zeta \leq \mathfrak{q}_\alpha] = (1-\alpha)^{-1}\mathbb{E}[\zeta^n \mathbb{I}_{\{\zeta < \mathfrak{q}_\alpha\}}] \leq \kappa_M^{CVaR} a_n \ . \tag{31}$$

Both equations above take the form of conditional expectation stochastic constraints. Recall that $\mathfrak{q}_\alpha$ is a decision variable denoting the VaR at the α level. We provide now the extended formulations of the portfolio (30) and marginal (31) CVaR constraints:

$$\mathbb{E}\left[a_n\left(1-\sum_{t=1}^{T_n}\chi_{n,t}\delta^t\frac{x_n}{q_n}-(1-L_n+L_n\chi_{n,T_n})\delta^{t_n^*}\right)+\sum_{i=1}^{n-1}\zeta_i\,\middle|\,a_n\left(1-\sum_{t=1}^{T_n}\chi_{n,t}\delta^t\frac{x_n}{q_n}-(1-L_n+L_n\chi_{n,T_n})\delta^{t_n^*}\right)+\sum_{i=1}^{n-1}\zeta_i\geq\mathfrak{q}_\alpha\right]$$
$$\leq\kappa_P^{CVaR}\sum_{i=1}^{n}a_i\ , \tag{32}$$

$$\mathbb{E}\left[a_n\left[(1-\sum_{t=1}^{T_n}\chi_{n,t}\delta^t\frac{x_n}{q_n}-(1-L_n+L_n\chi_{n,T_n})\delta^{t_n^*}\right)\,\middle|\,a_n\left(1-\sum_{t=1}^{T_n}\chi_{n,t}\delta^t\frac{x_n}{q_n}-(1-L_n+L_n\chi_{n,T_n})\delta^{t_n^*}\right)+\sum_{i=1}^{n-1}\zeta_i\geq\mathfrak{q}_\alpha\right]$$
$$\leq\kappa_M^{CVaR}a_n\ . \tag{33}$$

As for the portfolio and marginal VaR constraints, optimization problems with constraints of form (32) and (33) cannot be solved analytically, nor by a numerical optimization solvers. Next, we derive equivalent reformulations for (32) and (33). As shown below, the linearization process for the CVaR constraints differs from the one used for the VaR constraints and poses additional challenges, in particular for the linearization of the marginal CVaR constraint (33).

The portfolio CVaR constraint (32) can be equivalently written as:

$$\mathfrak{q}_\alpha+\frac{1}{1-\alpha}\sum_{k\in K}p^k\left[a_n\left(1-\sum_{t=1}^{T_n}\gamma_{n,t}^k\delta^t\frac{x_n}{q_n}-(1-L_n+L_n\gamma_{n,T_n}^k)\delta^{t_n^*}\right)+\sum_{i=1}^{n-1}O_i^k-\mathfrak{q}_\alpha\right]^+\leq\kappa_P^{CVaR}\sum_{i=1}^{n}a_i\ . \tag{34}$$

The expression $[a]^+$ refers to the positive part of a. Let $s^{k-}, k\in K$ be the loss in excess of the VaR at the α level in scenario ω^k. The VaR at the α level is itself a decision variable $\mathfrak{q}_\alpha$. It was demonstrated (see, e.g., Andersson et al. 2001) that the feasible set defined by the portfolio CVaR constraint (32) can be modelled with the following set of continuous linear inequalities:

$$s^{k-}\geq a_n\left(1-\sum_{t=1}^{T_n}\gamma_{n,t}^k\delta^t\frac{x_n}{q_n}-(1-L_n+L_n\gamma_{n,T_n}^k)\delta^{t_n^*}\right)+\sum_{i=1}^{n-1}O_i^k-\mathfrak{q}_\alpha \qquad k\in K \tag{35}$$

$$\mathfrak{q}_\alpha+\frac{\sum_{k\in K}p^k s^{k-}}{1-\alpha}\leq\kappa_P^{CVaR}\sum_{i=1}^{n}a_i \tag{36}$$

$$s^{k-}\geq 0 \qquad k\in K\ . \tag{37}$$

The linearization of the marginal CVaR constraint (33) cannot be done the same way as for (32) and requires an additional step involving the introduction of McCormick inequalities. We denote by s^{k+} the portfolio loss under the VaR at the α level in scenario ω^k, and by $N^{k-}, N^{k+}, k\in K$ two vectors of sufficiently large positive numbers.

Theorem 4 *The feasible set defined by the CVaR marginal risk constraint (33) can be modelled with the following set of mixed-integer linear inequalities:*

$$s^{k-} - s^{k+} = a_n \left(1 - \sum_{t=1}^{T_n} \gamma_{n,t}^k \delta^t \frac{x_n}{q_n} - (1 - L_n + L_n \gamma_{n,T_n}^k) \delta^{t_n^*}\right) + \sum_{i=1}^{n-1} O_i^k - \mathfrak{q}_\alpha \qquad k \in K \qquad (38)$$

$$s^{k-} \leq N^{k-} \eta^{k-} \qquad k \in K \qquad (39)$$

$$s^{k+} \leq N^{k+} \eta^{k+} \qquad k \in K \qquad (40)$$

$$s^{k-}, s^{k+} \geq 0 \qquad k \in K \qquad (41)$$

$$\eta^{k-} \in \{0,1\} \qquad k \in K \qquad (42)$$

$$\eta^{k+} \in \{0,1\} \qquad k \in K \qquad (43)$$

$$\eta^{k-} + \eta^{k+} \leq 1 \qquad k \in K \qquad (44)$$

$$\sum_{k \in K} p^k \eta^{k-} (1 - \delta^{t_n^*} (1 - L_n + L_n \gamma_{n,T_n}^k)) - \frac{1}{q_n} \sum_{k \in K} p^k \sum_{t=1}^{T_n} \delta^t \gamma_{n,t}^k h^k \leq (1-\alpha) \kappa_M^{CVaR} \qquad (45)$$

$$h^k \leq \eta^{k-} u \qquad k \in K \qquad (46)$$

$$h^k \leq x_n \qquad k \in K \qquad (47)$$

$$h^k \geq x_n + \eta^{k-} u - u \qquad k \in K \qquad (48)$$

$$h^k \geq 0 \qquad k \in K \qquad (49)$$

It is important to note that using the set of constraints $\{(35)-(37);(39);(42)\}$ is not sufficient to properly reformulate the marginal CVaR constraint. We must use the extended system of inequalities $\{(38)-(49)\}$ to calculate exactly the portfolio loss in excess of the VaR $\mathfrak{q}_\alpha$ at the α level in each scenario k and to enforce the requirements of the marginal CVaR constraint. Assume that the portfolio loss in scenario ω^k is smaller than $\mathfrak{q}_\alpha$. If we use $\{(35)-(37);(39);(42)\}$ instead of $\{(38)-(49)\}$ to define s^{k-}, we could very well have that s^{k-} takes a positive value, forcing $\eta^{k-}=1$ due to (39). That is likely to happen if the new loan n generates a profit in scenario ω^k, which means that $(1-\sum_{t=1}^{T_n} \gamma_{n,t}^k \delta^t \frac{x_n}{q_n} - (1-L_n+L_n\gamma_{n,T_n}^k)\delta^{t_n^*}) \leq 0$. Therefore, with $\eta^{k-}=1$, the corresponding term $\eta^{k-}(1-\sum_{t=1}^{T_n} \gamma_{n,t}^k \delta^t \frac{x_n}{q_n} - (1-L_n+L_n\gamma_{n,T_n}^k)\delta^{t_n^*})$ in (114) is also negative (instead of being equal to 0), which makes the satisfaction of (114) and its equivalent reformulation (45) easier. In contrast, using $\{(38)-(49)\}$ ensures that $s^{k-} \geq 0$ and $\eta^{k-}=1$ if and only if the portfolio loss in scenario k exceeds $\mathfrak{q}_\alpha$.

3.3. Reformulations and Properties of Risk-Based Loan Pricing Problems

In this section, we evaluate the computational difficulties posed by the two types of risk-based loan pricing optimization problems studied in this paper. Problem $\mathbf{LPO_{VaR}}$ with portfolio and marginal

VaR constraints

$$\begin{aligned}\mathbf{LPO_{VaR}}: \quad & \max \ \ g(x_n,s_n)\mathbb{E}\left[\left(\mathcal{G}_n(x_n)-a_n\right)\right]+\mathbb{E}\left[\sum_{i=1}^{n-1}\left(\mathcal{G}_i(x_i)-a_i\right)\right] \\ & \text{s.t.} \qquad (x_n,z^+,z^-,y^+,y^-,\beta^+,\beta^-)\in\mathcal{X}_{VaR}\ ,\end{aligned} \tag{50}$$

where

$$\mathcal{X}_{VaR}=\{(x_n,z^+,z^-,y^+,y^-,\beta^+,\beta^-)\in\mathbb{R}^{1+4|K|}\times\{0,1\}^{2|K|}:(12)-(16)\ ;\ (18)\ ;\ (21)-(29)\}\ , \tag{51}$$

and problem $\mathbf{LPO_{CVaR}}$ with portfolio and marginal CVaR constraints

$$\begin{aligned}\mathbf{LPO_{CVaR}}: \quad & \max \ \ g(x_n,s_n)\mathbb{E}\left[\left(\mathcal{G}_n(x_n)-a_n\right)\right]+\mathbb{E}\left[\sum_{i=1}^{n-1}\left(\mathcal{G}_i(x_i)-a_i\right)\right] \\ & \text{s.t.} \qquad (x_n,\mathfrak{q}_\alpha,s^-,s^+,h,\eta^-,\eta^+)\in\mathcal{X}_{CVaR}\ ,\end{aligned} \tag{52}$$

where

$$\mathcal{X}_{CVaR}=\{(x_n,\mathfrak{q}_\alpha,s^-,s^+,h,\eta^-,\eta^+)\in\mathbb{R}^{2+3|K|}\times\{0,1\}^{2|K|}:(36)\ ;(38)-(49)\} \tag{53}$$

are both MINLP problems with mixed-integer linear feasible sets as shown by Lemma 3 and Theorem 4 .

The difficulty to solve problems $\mathbf{LPO_{VaR}}$ and $\mathbf{LPO_{CVaR}}$ is strongly impacted by the properties of their objective function, which takes the form of a product of the two functions $g(x_n,s_n)$ and $\mathbb{E}\left[\left(\mathcal{G}_n(x_n)-a_n\right)\right]$. The former denotes the price-response function or probability of the borrower to accept the loan at the conditions set by the lender, and the latter is the expected profit for the lender. We observe that each term

$$p^k\ a_n\left(\sum_{t=1}^{T_n}\gamma^k_{n,t}\delta^t\frac{x_n}{q_n}+(1-L_n+L_n\gamma^k_{n,T_n})\delta^{t^*_n}-1\right) \tag{54}$$

in the expectation $\mathbb{E}\left[\mathcal{G}_n(x_n)-a_n\right)]$ in (50) is linear in x_n. For the acceptance probability, we consider three different price-response functional forms, which are widely used in the literature and in practice (see, e.g., Emmons and Gilbert 1998, Petruzzi and Dada 1999, Lau and Lau 2003, Besbes et al. 2010, Besbes and Zeevi 2015).

1. Linear acceptance probability

$$g^1(x_n,s_n)=\frac{\nu-\tau(s_n)x_n}{\nu} \tag{55}$$

for $x_n\in[l,\min(u,\nu/\tau(s_n))]$ for $\tau(s_n)$ is non-increasing in s_n. A larger value $\tau(s_n)$ indicates a lower price elasticity. That is, when $\tau(s_n)$ is large, the borrowers are more sensitive to interest rates.

2. Exponential acceptance probability

$$g^2(x_n,s_n)=\frac{e^{\nu-\tau(s_n)x_n}}{e^\nu} \tag{56}$$

for $x\in[l,u]$.

3. Logit acceptance probability

$$g^3(x_n, s_n) = \frac{e^{\nu-\tau(s_n)x_n}}{1+e^{\nu-\tau(s_n)x_n}} \tag{57}$$

for $x_n \in [l, u]$.

We now study for each price-response function to see whether the continuous relaxation of $\mathbf{LPO_{VaR}}$ and $\mathbf{LPO_{CVaR}}$ are convex programming problems. This is the case if their objective functions are concave as the maximization of a concave (nonlinear) objective function over a convex feasible set is a convex programming problem. To ease the notation, we thereafter denote the expected profit due to the new loan (linear in x_n) by dx_n, where d is a positive constant, and also the notation τ instead of $\tau(s_n)$. As the expected profit $\mathbb{E}\left[\sum_{i=1}^{n-1}\left(\mathcal{G}_i(x_i) - a_i\right)\right]$ due to the existing portfolio is a constant, we omit it in Theorem 5.

Theorem 5 *i) The continuous relaxations of the mixed-integer nonlinear optimization problems*

$$\mathbf{LPO^1_{VaR}}: \quad \left\{\max \ dx_n \frac{\nu - \tau x_n}{\nu} : (x_n, z^+, z^-, y^+, y^-, \beta^+, \beta^-) \in \mathcal{X}_{VaR}\right\}, \tag{58}$$

$$\mathbf{LPO^1_{CVaR}}: \quad \left\{\max \ dx_n \frac{\nu - \tau x_n}{\nu} : (x_n, \mathfrak{q}_\alpha, s^-, s^+, h, \eta^-, \eta^+) \in \mathcal{X}_{CVaR}\right\} \tag{59}$$

are convex programming problems.

ii) The continuous relaxations of the mixed-integer nonlinear optimization problems

$$\mathbf{LPO^2_{VaR}}: \quad \left\{\max \ dx_n \frac{e^{\nu-\tau x_n}}{e^{\nu}} : (x_n, z^+, z^-, y^+, y^-, \beta^+, \beta^-) \in \mathcal{X}_{VaR}\right\},$$

$$\mathbf{LPO^2_{CVaR}}: \quad \left\{\max \ dx_n \frac{e^{\nu-\tau x_n}}{e^{\nu}} : (x_n, s^k, h, \eta^+, \eta^-) \in \mathcal{X}_{CVaR}\right\},$$

$$\mathbf{LPO^3_{VaR}}: \quad \left\{\max \ dx_n \frac{e^{\nu-\tau x_n}}{1+e^{\nu-\tau x_n}} : (x_n, z^+, z^-, y^+, y^-, \beta^+, \beta^-) \in \mathcal{X}_{VaR}\right\},$$

$$\mathbf{LPO^3_{CVaR}}: \quad \left\{\max \ dx_n \frac{e^{\nu-\tau x_n}}{1+e^{\nu-\tau x_n}} : (x_n, s^k, h, \eta^+, \eta^-) \in \mathcal{X}_{CVaR}\right\}$$

are not convex programming problems.

3.4. Concavifiability

In this section, we utilize the concavifiability concept to derive convex programming formulations whose optimal solution is identical to the optimal solution of the mixed-integer nonconvex programming problems $\mathbf{LPO^2_{VaR}}$, $\mathbf{LPO^2_{VVaR}}$, $\mathbf{LPO^3_{VaR}}$, and $\mathbf{LPO^3_{CVaR}}$. In particular, we examine whether the nonlinear objective functions of these problems are concavifiable, i.e., can be transformed into concave functions. First, we briefly review the main concavifiability techniques, comment under which conditions they have been used, and introduce the concept of G-concavity. Next, we develop a concavifiability method for the risk-based loan pricing problems with exponential and logit willingness-to-pay functions and derive the resulting convex programming formulations for the continuous relaxations of these problems.

Concavifiability can sometimes be achieved by means of an increasing functional form transformation, called range transformation (see, e.g., Avriel et al. 1988, Horst 1984), through variable substitution (such as in fractional linear programming with Charnes-Cooper's transformation or in geometric programming), called domain transformation, or by a combination of both range and domain transformations (see, e.g., Li et al. 2005).

Consider the following optimization problem

$$\begin{aligned} \mathbf{P}: \qquad & \min \ f_0(x) \\ & f_j(x) \leq 0, \ j = 1, \dots, m \end{aligned} \tag{60}$$

with the real-valued functions $f_j, j = 0, \dots, m$ defined on a convex subset C of $\mathbb{R}^n$ and feasible set $M \subseteq \mathbb{R}^n$. Let $F_j, j = 0, \dots, m$ be a series of continuous strictly monotone increasing real-valued functions defined on the range I_j on the function $f_j, j = 0, \dots, m$. It is evident that the problem

$$\begin{aligned} \mathbf{TP}: \qquad & \min \ F(f_0(x)) \\ & F(f_j(x)) \leq 0, \ j = 1, \dots, m \end{aligned} \tag{61}$$

is equivalent to $\mathbf{P}$. Equivalence between two optimization problems is here understood as the two problems admitting the same optimal solution(s).

It is well-known that if each function f_j is convex, then the functional transformations $F_j(f_j(x))$ preserve convexity if each $F_j : \mathbb{R} \to \mathbb{R}$ is convex and increasing. Another related but more challenging question is to investigate whether there exists a functional transformation $F_j : \mathbb{R} \to \mathbb{R}$ such that $F_j(f_j(x))$ is convex assuming that f_j is non-convex. This result was provided by Li et al. (2001) when f_j is nonconvex and monotone.

The above conditions under which concavifiability results are available are however not satisfied by problems $\mathbf{LPO^2_{VaR}}$, $\mathbf{LPO^2_{VVaR}}$, $\mathbf{LPO^3_{VaR}}$, and $\mathbf{LPO^3_{CVaR}}$, since these are maximization problems in which the objective function is not concave and not monotone in x. The next task is thus to find a monotone increasing function F_0 such that the functional transformation $F_0 : \mathbb{R} \to \mathbb{R}$ of the objective function f_0 (denoted by f^2 in $\mathbf{LPO^2_{VaR}}$ and $\mathbf{LPO^2_{VVaR}}$, and by f^3 in $\mathbf{LPO^3_{VaR}}$ and $\mathbf{LPO^3_{CVaR}}$) is concave. Stated differently, we must determine whether the function f_0 is *G-concave* (Avriel et al. 1988), a concept also known as concave transformable or transconcave first studied by De Finetti (1949) and Fenchel (1951).

Definition 6 *Let $f: C \to \mathbb{R}$ defined on $C \subseteq \mathbb{R}^n$ and with range $I_f(C)$. The function f is said to be G-concave if there is a continuous real-valued monotone increasing function $G: I_f(C) \to \mathbb{R}$ such that $G(f(x))$ is concave over C:*

$$G(f(\lambda x^1 + (1-\lambda)x^2)) \geq \lambda G(f(x^1)) + (1-\lambda)G(f(x^2)) \tag{62}$$

holds for any $x^1, x^2 \in C, 0 \leq \lambda \leq 1$.

Let G^{-1} be the inverse of G. The inequality (62) is equivalent to:

$$f(\lambda x^1 + (1-\lambda)x^2)) \geq G^{-1}(\lambda G(f(x^1)) + (1-\lambda)G(f(x^2))) \ . \tag{63}$$

As G is increasing, G^{-1} must be too. Therefore, the equivalence between (63) and the concavity of $G(f(x))$ is immediate.

The next question to settle concerns the minimal requirements or properties that a function f must own to qualify as G-concave. First, we recall the definitions of quasi-convavity and semistrict quasi-concavity.

Definition 7 *Let $0 < \lambda < 1$ and x^1, x^2 be two arbitrary points in C.*
A function $f: C \to \mathbb{R}$ is quasi-concave if and only if

$$f(\lambda x^1 + (1-\lambda)x^2)) \geq min(f(x^1), f(x^2)) \ . \tag{64}$$

A function $f: C \to \mathbb{R}$ is semistrictly quasi-concave if and only if

$$f(x^2) \geq f(x^1) \ \Rightarrow \ f(\lambda x^1 + (1-\lambda)x^2)) > f(x^1) \ . \tag{65}$$

Theorem 8 *Every G-concave function f on a convex set C is quasi-concave.*

Note that the converse does hot hold since a quasi-concave function f is not necessarily G-concave. If f is quasi-concave but not G-concave, it might be appropriate to combine range and domain transformations (see, e.g., Avriel et al. 1988, Li et al. 2005). This approach rests on two steps. First, it proceeds to a one-to-one transformation of the domain of a function so that its upper sets are transformed into convex ones, which implies that the "transformed" function is quasi-concave. Second, a monotone increasing range transformation of the quasi-concave function is carried out in order to obtain a concave function.

Theorem 9 *The functions*

$$g^2(x_n, s_n) = x\frac{e^{\nu - \tau x_n}}{e^{\nu}} \qquad \textit{and} \qquad g^3(x_n, s_n) = x\frac{e^{\nu - \tau x_n}}{1 + e^{\nu - \tau x_n}}$$

are quasi-concave.

We shall now demonstrate that the logarithmic function can be used to transform the continuous relaxations of problems $\mathbf{LPO^2_{VaR}}$, $\mathbf{LPO^2_{CVaR}}$, $\mathbf{LPO^3_{VaR}}$, and $\mathbf{LPO^3_{CVaR}}$ into convex programming problems.

Theorem 10 *The continuous relaxations of the MINLP problems*

$$\mathbf{RLPO^2_{VaR}} : \left\{\max \ d\ ln(x_n) - \tau x_n : (x_n, z^+, z^-, y^+, y^-, \beta^+, \beta^-) \in \mathcal{X}_{VaR}\right\}, \tag{66}$$

$$\mathbf{RLPO^2_{CVaR}} : \left\{\max \ d\ ln(x_n) - \tau x_n : (x_n, \mathfrak{q}_\alpha, s^-, s^+, h, \eta^-, \eta^+) \in \mathcal{X}_{CVaR}\right\}, \tag{67}$$

$$\mathbf{RLPO^3_{VaR}} : \left\{\max \ d\ ln(x_n) + \nu - \tau x_n - ln(1 + e^{\nu - \tau x_n}) : (x_n, z^+, z^-, y^+, y^-, \beta^+, \beta^-) \in \mathcal{X}_{VaR}\right\}, \tag{68}$$

$$\mathbf{RLPO^3_{CVaR}} : \left\{\max \ d\ ln(x_n) + \nu - \tau x_n - ln(1 + e^{\nu - \tau x_n}) : (x_n, \mathfrak{q}_\alpha, s^-, s^+, h, \eta^-, \eta^+) \in \mathcal{X}_{CVaR}\right\} \tag{69}$$

are convex programming problems and their optimal solutions are identical to those of $\mathbf{LPO^2_{VaR}}$*,* $\mathbf{LPO^2_{CVaR}}$*,* $\mathbf{LPO^3_{VaR}}$*, and* $\mathbf{LPO^3_{CVaR}}$*, respectively.*

Corollary 11 *The functions* $g^2(x_n, s_n) = \frac{e^{\nu-\tau x_n}}{e^{\nu}}$ *and* $g^3(x_n, s_n) = \frac{e^{\nu-\tau x_n}}{1+e^{\nu-\tau x_n}}$ *are G-concave.*

The above results follow immediately from Theorem 10 and Definition 6.

4. Extension: Multiple Prospective Loans

We now extend our study to the situation when the lender (bank) simultaneously considers multiple prospective loans, i.e., $\bar{n} > 1$ with $\tilde{n} = n - \bar{n}$ granted loans in the existing portfolio. In this case, the bank concurrently selects the prospective loans to which it will extend an offer and the price (interest rate) that it will charge while accounting for the probability that each borrower will accept the proposed interest rate. Thus, the interdependencies between the new loans and those in the existing portfolio as well as the interdependence across the new loans must be factored in the loan granting and pricing decisions.

4.1. Generic Formulation

In Section 3, when a single loan is under consideration, the decision to grant the loan was implicitly responded to by the sign of the optimal value of the loan pricing problem/feasibility of the model. In the multi-loan context, we proceed in a different fashion, since we must take a loan granting decision for each prospective loan as well. Consequently, for the loan selection process, we incorporate an additional set of binary decision variables $\theta_i, i = \tilde{n}+1, \ldots, n$ taking value 1 if the bank decides to grant new loan i and taking value 0 otherwise, and the price x_i of each new loan $i = \tilde{n}+1, \ldots, n$ are also selected. A prospective new loan i should not be granted if it is not possible to find an admissible interest rate (i.e., satisfying all the constraints of the multi-loan risk-pricing model introduced below) that leads to a positive profit expected value for loan i. Clearly, from equation (54), if it is not possible to find x_i such that

$$a_i \sum_{k\in K} p^k \left(\sum_{t=1}^{T_i} \gamma_{i,t}^k \delta^t \frac{x_i}{q_i} + (1 - L_i + L_i \gamma_{i,T_i}^k) \delta^{t_i^*} - 1 \right) \geq 0 \ , \tag{70}$$

then loan i should not be granted, since granting it would lead to a negative profit expected value and be detrimental to the profitability of the bank (and to the value of the objective function). If the bank does not extend an offer for a new loan, we have that $\theta_i = 0 = \zeta_i$. The random loss of a portfolio is $\zeta = \sum_{i=1}^{n} \zeta_i$, where $\zeta_i = a_i - \mathcal{G}_i, i = 1 \ldots, \tilde{n}$ is the random loss due to a loan in the existing portfolio and $\zeta_i = \theta_i(a_i - \mathcal{G}_i), i = \tilde{n}+1, \ldots, n$, is the random loss due to a prospective new loan. As explained below, the loan selection process introduces additional combinatorics and nonconvexity elements that further compound the complexity of the loan pricing problem.

The generic formulation of the risk-based loan pricing model **LPO-M** with simultaneous consideration of multiple loans can be written as:

LPO-M :

$$\max_{x_{\tilde{n}+1},\ldots,x_n,\theta} \quad \left\{ h(x_{\tilde{n}+1},\ldots,x_n,\theta) := \sum_{i=\tilde{n}+1}^{n} \theta_i g(x_i, s_i) \mathbb{E}\big[\big(\mathcal{G}_i(x_i) - a_i\big)\big] + \mathbb{E}\big[\sum_{i=1}^{\tilde{n}} \big(\mathcal{G}_i(x_i) - a_i\big)\big] \right\} \tag{71}$$

$$\text{s.t.} \quad l \leq x_i \leq u, \qquad i = \tilde{n}+1, \ldots, n \tag{72}$$

$$\rho_i^M \left(\zeta(x_{\tilde{n}+1},\ldots,x_n,\theta)\right) \ \leq \ \kappa_M a_i, \qquad i = \tilde{n}+1, \ldots, n \tag{73}$$

$$\rho\left(\zeta(x_{\tilde{n}+1},\ldots,x_n,\theta)\right) \ \leq \ \kappa_P \sum_{i=1}^{n} a_i \ . \tag{74}$$

The objective (71) is to maximize the expected profit taking into account uncertainties in both the borrowers' reaction to the proposed interest rate and the default risk. Constraints (72) underline that a price must be determined for each prospective loan. The inequalities (73) and (74) represent the $\bar{n}$ marginal and the portfolio risk constraints with respective risk thresholds κ_M and κ_P.

In the next section, we develop the explicit mathematical formulations of the multi-loan risk-pricing problems with VaR and CVaR measures, discuss their complexity, and develop reformulations amenable to a numerical solution.

4.2. Marginal and Portfolio VaR Constraints with Multiple New Loans

The portfolio VaR constraint (74) is modelled as a chance constraint with random technology matrix:

$$\mathbb{P}\left(\sum_{i=\tilde{n}+1}^{n} \left(\theta_i a_i \left(1 - \sum_{t=1}^{T_i} \chi_{i,t} \delta^t \frac{x_i}{q_i} - (1 - L_i + L_i \chi_{i,T_i}) \delta^{t_i^*} \right) \right) + \sum_{i=1}^{\tilde{n}} \zeta_i \leq \kappa_P^{VaR} \left(\sum_{i=1}^{\tilde{n}} a_i + \sum_{i=\tilde{n}+1}^{n} \theta_i a_i \right) \right) \geq \alpha. \tag{75}$$

However, while the stochastic inequality that must hold with probability α is linear in the single loan pricing problem (6), the stochastic inequality

$$\sum_{i=\tilde{n}+1}^{n}\left(\theta_i a_i\left(1-\sum_{t=1}^{T_i}\chi_{i,t}\delta^t\frac{x_i}{q_i}-(1-L_i+L_i\chi_{i,T_i})\delta^{t_i^*}\right)\right)+\sum_{i=1}^{\tilde{n}}\zeta_i\leq\kappa_P^{VaR}\left(\sum_{i=1}^{\tilde{n}}a_i+\sum_{i=\tilde{n}+1}^{n}\theta_i a_i\right)$$

is nonlinear and includes binary variables and bilinear nonconvex terms $\theta_i\ x_i, i=\tilde{n}+1,\ldots,n$ (see, e.g., Lejeune and Margot 2016) in the multi-loan context.

Two differences in the modeling of the marginal VaR constraint must be noted between the single loan and multi-loan contexts. First, we need to include one VaR marginal constraint for each new loan, thereby increasing the size of the constraint space. Second, each marginal VaR constraint (conditional expectation constraint) will involve nonlinear stochastic inequalities with binary decision variables and bilinear terms:

$$\mathbb{E}\left[\theta_j\ a_j\left(1-\sum_{t=1}^{T_j}\chi_{j,t}\delta^t\frac{x_j}{q_j}-(1-L_j+L_j\chi_{j,T_j})\delta^{t_j^*}\right)\middle|\sum_{i=\tilde{n}+1}^{n}\left(\theta_i\ a_i\left(1-\sum_{t=1}^{T_i}\chi_{i,t}\delta^t\frac{x_i}{q_i}-(1-L_i+L_i\chi_{i,T_i})\delta^{t_i^*}\right)\right)+\sum_{i=1}^{\tilde{n}}\zeta_i=\kappa_P^{VaR}\left(\sum_{i=1}^{\tilde{n}}a_i+\sum_{i=\tilde{n}+1}^{n}\theta_i a_i\right)\right]$$
$$\leq\kappa_M^{VaR}a_j\ ,\ j=\tilde{n}+1,\ldots,n\ . \tag{76}$$

If the conditional event has a positive probability to happen (Tasche 2009), (76) becomes:

$$\frac{\mathbb{E}\left[\theta_j\ a_j\left(1-\sum_{t=1}^{T_j}\chi_{j,t}\delta^t\frac{x_j}{q_j}-(1-L_j+L_j\chi_{j,T_j})\delta^{t_j^*}\right)\ ,\ \sum_{i=\tilde{n}+1}^{n}\left(\theta_i\ a_i\left(1-\sum_{t=1}^{T_i}\chi_{i,t}\delta^t\frac{x_i}{q_i}-(1-L_i+L_i\chi_{i,T_i})\delta^{t_i^*}\right)\right)+\sum_{i=1}^{\tilde{n}}\zeta_i=\kappa_P^{VaR}\left(\sum_{i=1}^{\tilde{n}}a_i+\sum_{i=\tilde{n}+1}^{n}\theta_i a_i\right)\right]}{\mathbb{P}\left(\sum_{i=\tilde{n}+1}^{n}\left(\theta_i\ a_i\left(1-\sum_{t=1}^{T_i}\chi_{i,t}\delta^t\frac{x_i}{q_i}-(1-L_i+L_i\chi_{i,T_i})\delta^{t_i^*}\right)\right)+\sum_{i=1}^{\tilde{n}}\zeta_i=\kappa_P^{VaR}\left(\sum_{i=1}^{\tilde{n}}a_i+\sum_{i=\tilde{n}+1}^{n}\theta_i a_i\right)\right)}$$
$$\leq\kappa_M^{VaR}a_j\ ,\ j=\tilde{n}+1,\ldots,n\ . \tag{77}$$

Lemma 12 follows immediately.

Lemma 12 *The feasible areas defined by the continuous relaxation of the portfolio and marginal VaR constraints* (75) *and* (77) *are nonconvex.*

Next, we derive equivalent reformulations for the constraints (75) and (77) that can be handled by optimization solvers. Theorem 13 presents the mixed-integer linear reformulation of the feasible set defined by the portfolio (75) and marginal (77) VaR constraints and uses the following argument. Let $\upsilon_i, i=\tilde{n}+1,\ldots,n$ be a semi-continuous variable which takes value 0 if new loan i is not granted and which is otherwise equal to the interest rate proposed for new loan i in scenario k. A prospective new loan i should not be granted if it is not possible to find an admissible interest rate (i.e., satisfying all the constraints of the model) that leads to a positive expected prodit for loan i. If there is no x_i such that (70) holds, loan i should not be granted, and the corresponding binary θ_i and semi-continuous υ_i variables must all take value 0. The following parameters M_i, M_m^{k+}, and M_m^{k-} are large positive numbers; M_i denotes the lower bound on the average profit generated by loan i and M_m^{k+} (resp., M_m^{k-}) is the lower (resp., upper) bound on the profit generated by the entire portfolio (existing and prospective loans together) in scenario k.

Theorem 13 *The feasible set defined by the portfolio (75) and marginal (77) VaR constraints can be equivalently represented by the following set of mixed-integer linear inequalities:*

$$a_i \sum_{k\in K} p^k \left(\sum_{t=1}^{T_i} \gamma^k_{i,t} \delta^t \frac{x_i}{q_i} + (1 - L_i + L_i \gamma^k_{i,T_i}) \delta^{t^*_i} - 1 \right) + M_i(1-\theta_i) \geq 0 \qquad i = \tilde{n}+1,\dots,n \quad (78)$$

$$\upsilon_i \geq 0 \qquad i = \tilde{n}+1,\dots,n \quad (79)$$

$$\upsilon_i \leq \theta_i u \qquad i = \tilde{n}+1,\dots,n \quad (80)$$

$$\upsilon_i \leq x_i \qquad i = \tilde{n}+1,\dots,n \quad (81)$$

$$\upsilon_i \geq x_i + \theta_i u - u \qquad i = \tilde{n}+1,\dots,n \quad (82)$$

$$\sum_{i=\tilde{n}+1}^{n} \left(-\upsilon_i \frac{a_i}{q_i} \sum_{t=1}^{T_i} \gamma^k_{i,t} \delta^t - \theta_i \; a_i \left(\sum_{t=1}^{T_i} (1 - L_i + L_i \gamma^k_{i,T_i}) \delta^{t^*_i} - 1 \right) \right)$$
$$+ \sum_{i=1}^{\tilde{n}} O^k_i - z^{k+} + z^{k-} = \kappa^{VaR}_P \left(\sum_{i=1}^{\tilde{n}} a_i + \sum_{i=\tilde{n}+1}^{n} \theta_i a_i \right) \qquad k \in K \quad (83)$$

$$\epsilon \; \beta^{k+} \leq z^{k+} \leq M^{k+}_m \beta^{k+} \qquad k \in K \quad (84)$$

$$\epsilon \; \beta^{k-} \leq z^{k-} \leq M^{k-}_m \beta^{k-} \qquad k \in K \quad (85)$$

$$y^{k+}_i \leq \beta^{k+} u \qquad k \in K, i = \tilde{n}+1,\dots,n \quad (86)$$

$$y^{k+}_i \leq x_i \qquad k \in K, i = \tilde{n}+1,\dots,n \quad (87)$$

$$y^{k+}_i \geq x_i + \beta^{k+} u - u \qquad k \in K, i = \tilde{n}+1,\dots,n \quad (88)$$

$$y^{k+}_i \geq 0 \qquad k \in K, i = \tilde{n}+1,\dots,n \quad (89)$$

$$y^{k-}_i \leq \beta^{k-} u \qquad k \in K, i = \tilde{n}+1,\dots,n \quad (90)$$

$$y^{k-}_i \leq x_i \qquad k \in K, i = \tilde{n}+1,\dots,n \quad (91)$$

$$y^{k-}_i \geq x_i + \beta^{k-} u - u \qquad k \in K, i = \tilde{n}+1,\dots,n \quad (92)$$

$$y^{k-}_i \geq 0 \qquad k \in K, i = \tilde{n}+1,\dots,n \quad (93)$$

$$\sum_{k\in K} \left(p^k (1 - \beta^{k+} - \beta^{k-}) \; (1 - \delta^{t^*_i} (1 - L_i + L_i \gamma^k_{i,T_i})) \right) -$$
$$\frac{\delta^t}{q_i} \sum_{k\in K} \left(\sum_{t=1}^{T_i} \gamma^k_{i,t} (x_i - y^{k+}_i - y^{k-}_i) p^k \right) \leq \kappa^{VaR}_M \sum_{k\in K} p^k (1 - \beta^{k+} - \beta^{k-}) + (1-\theta_i) N \qquad i = \tilde{n}+1,\dots,n \quad (94)$$

$$\theta_i \in \{0,1\} \qquad i = \tilde{n}+1,\dots,n \quad (95)$$

(15) – (16); (18)

4.3. Marginal and Portfolio CVaR Constraints with Multiple New Loans

As for the VaR constraints in the multi-loan context, the marginal and portfolio CVaR constraints involve additional binary decision variables and bilinear terms. We must also include one marginal CVaR constraint for each prospective new loan. The portfolio (96) and marginal (97) CVaR constraints are conditional expectation stochastic constraints:

$$\mathbb{E}\left[\sum_{i=\tilde{n}+1}^{n} \left(\theta_i \; a_i \left(1 - \sum_{t=1}^{T_i} \chi_{i,t} \delta^t \frac{x_i}{q_i} - (1 - L_i + L_i \chi_{i,T_i}) \delta^{t^*_i} \right) \right) + \sum_{i=1}^{\tilde{n}} \zeta_i \middle| \sum_{i=\tilde{n}+1}^{n} \left(\theta_i \; a_i \left(1 - \sum_{t=1}^{T_i} \chi_{i,t} \delta^t \frac{x_i}{q_i} - (1 - L_i + L_i \chi_{i,T_i}) \delta^{t^*_i} \right) \right) + \sum_{i=1}^{\tilde{n}} \zeta_i \geq \mathfrak{q}_\alpha \right]$$

$$\leq \kappa_P^{CVaR} \left(\sum_{i=1}^{\tilde{n}} a_i + \sum_{i=\tilde{n}+1}^{n} \theta_i a_i \right) \ , \tag{96}$$

$$\mathbb{E}\left[\theta_j \ a_j \left(1 - \sum_{t=1}^{T_j} \chi_{j,t} \delta^t \frac{x_j}{q_j} - (1 - L_j + L_j \chi_{n,T_j}) \delta^{t_j^*} \right) \Bigg| \sum_{i=\tilde{n}+1}^{n} \left(\theta_i \ a_i \left(1 - \sum_{t=1}^{T_i} \chi_{i,t} \delta^t \frac{x_i}{q_i} - (1 - L_i + L_i \chi_{i,T_i}) \delta^{t_i^*} \right) \right) + \sum_{i=1}^{\tilde{n}} \zeta_i \geq \mathfrak{q}_\alpha \right]$$
$$\leq \kappa_M^{CVaR} a_j \ , \ j = \tilde{n}+1, \ldots, n \tag{97}$$

with $\mathfrak{q}_\alpha$ being a decision variable representing the VaR at the α level. Lemma 14 follows immediately.

Lemma 14 *The feasible area defined by the continuous relaxation of the portfolio (96) and marginal (97) CVaR constraints is nonconvex.*

Next, we derive equivalent reformulations for (96) and (97) that are amenable to a numerical solution.

Theorem 15 *The feasible set defined by the portfolio (96) and marginal (97) CVaR constraints can be modelled with the following set of mixed-integer linear inequalities:*

$$\mathfrak{q}_\alpha + \frac{\sum_{k \in K} p^k s^{k-}}{1-\alpha} \leq \kappa_P^{CVaR} \left(\sum_{i=1}^{\tilde{n}} a_i + \sum_{i=\tilde{n}+1}^{n} \theta_i a_i \right) \tag{98}$$

$$s^{k-} \geq \sum_{i=1}^{\tilde{n}} \left(-\upsilon_i \frac{a_i}{q_i} \sum_{t=1}^{T_i} \gamma_{i,t}^k \delta^t - \theta_i \ a_i \left(\sum_{t=1}^{T_i} (1 - L_i + L_i \gamma_{i,T_i}^k) \delta^{t_i^*} - 1 \right) \right)$$
$$+ \sum_{i=1}^{\tilde{n}} O_i^k - \mathfrak{q}_\alpha \qquad k \in K \tag{99}$$

$$s^{k-} - s^{k+} = \sum_{i=\tilde{n}+1}^{n} \left(-\upsilon_i \frac{a_i}{q_i} \sum_{t=1}^{T_i} \gamma_{i,t}^k \delta^t - \theta_i \ a_i \left(\sum_{t=1}^{T_i} (1 - L_i + L_i \gamma_{i,T_i}^k) \delta^{t_i^*} - 1 \right) \right)$$
$$+ \sum_{i=1}^{\tilde{n}} O_i^k - \mathfrak{q}_\alpha \qquad k \in K \tag{100}$$

$$\sum_{k \in K} p^k \eta^{k-} (1 - \delta^{t_i^*} (1 - L_i + L_i \gamma_{i,T_i}^k)) - \frac{1}{q_i} \sum_{k \in K} p^k \sum_{t=1}^{T_i} \gamma_{i,t}^k \delta^t h_i^k$$
$$\leq (1-\alpha) \kappa_M^{CVaR} + (1 - \theta_i) N \qquad i = \tilde{n}+1, \ldots, n \tag{101}$$

$$h_i^k \leq \eta^{k-} u \qquad k \in K, i = \tilde{n}+1, \ldots, n \tag{102}$$

$$h_i^k \leq x_i \qquad k \in K, i = \tilde{n}+1, \ldots, n \tag{103}$$

$$h_i^k \geq x_i + \eta^{k-} u - u \qquad k \in K, i = \tilde{n}+1, \ldots, n \tag{104}$$

$$h_i^k \geq 0 \qquad k \in K, i = \tilde{n}+1, \ldots, n \tag{105}$$

(39) − (44); (78) − (82); (95)

4.4. Properties of Risk-Based Multi-Loan Pricing Problems

We evaluate now the convexity of the risk-based multi-loan pricing optimization problems. Besides the combinatorial restrictions, the difficulty to solve these problems stem from their objective function, which is nonlinear, including bilinear and trilinear terms, and involves the sum (over each prospective loan) of the products of the acceptance probability function by the expected profit:

$$\sum_{i=\tilde{n}+1}^{n}\left(\theta_i a_i \frac{\nu-\tau x_i}{\nu}\sum_{k\in K}p^k\left(\sum_{t=1}^{T_i}\gamma_{i,t}^k\delta^t\frac{x_i}{q_i}+(1-L_i+L_i\gamma_{i,T_i}^k)\delta^{t_i^*}-1\right)\right)-\sum_{k\in K}\sum_{i=1}^{\tilde{n}}p^kO_i^k \tag{106}$$

$$\sum_{i=\tilde{n}+1}^{n}\left(\theta_i a_i \frac{e^{\nu-\tau x_i}}{e^{\nu}}\sum_{k\in K}p^k\left(\sum_{t=1}^{T_i}\gamma_{i,t}^k\delta^t\frac{x_i}{q_i}+(1-L_i+L_i\gamma_{i,T_i}^k)\delta^{t_i^*}-1\right)\right)-\sum_{k\in K}\sum_{i=1}^{\tilde{n}}p^kO_i^k \tag{107}$$

$$\sum_{i=\tilde{n}+1}^{n}\left(\theta_i a_i \frac{e^{\nu-\tau x_i}}{1+e^{\nu-\tau(s_i)x_i}}\sum_{k\in K}p^k\left(\sum_{t=1}^{T_i}\gamma_{i,t}^k\delta^t\frac{x_i}{q_i}+(1-L_i+L_i\gamma_{i,T_i}^k)\delta^{t_i^*}-1\right)\right)-\sum_{k\in K}\sum_{i=1}^{\tilde{n}}p^kO_i^k \tag{108}$$

Let $\mathcal{X}_{VaR-M}$ and $\mathcal{X}_{CVaR-M}$ be the mixed-linear feasible sets (see Theorems 13 and 15) of the multi-loan pricing problems with the VaR and CVaR metrics:

$$\mathcal{X}_{VaR-M}=\left\{(x,z^+,z^-,y^+,y^-,\upsilon,\beta^+,\beta^-,\theta)\in\mathbb{R}^{2\tilde{n}+2|K|(1+\tilde{n})}\times\{0,1\}^{\tilde{n}+2|K|}:(15)-(16);(18);(78)-(95)\right\}\ .$$

$$\mathcal{X}_{CVaR-M}=\left\{(x,\mathfrak{q}_\alpha,s^-,s^+,h,\upsilon,\eta^+,\eta^-,\theta)\in\mathbb{R}^{1+2\tilde{n}+|K|(2+\tilde{n})}\times\{0,1\}^{2|K|+\tilde{n}}:(39)-(44);(78)-(82);(98)-(105)\right\}\ .$$

Lemma 16 *The objective functions of the optimization problems*

$$\begin{aligned}
\mathbf{LPO^1_{VaR-M}}: &\quad \{\max\ (106):(x,z^+,z^-,y^+,y^-,\upsilon,\beta^+,\beta^-,\theta)\in\mathcal{X}_{VaR-M}\}\ ,\\
\mathbf{LPO^1_{CVaR-M}}: &\quad \{\max\ (106):(x,\mathfrak{q}_\alpha,s^-,s^+,h,\upsilon,\eta^+,\eta^-,\theta)\in\mathcal{X}_{CVaR-M}\}\ ,\\
\mathbf{LPO^2_{VaR-M}}: &\quad \{\max\ (107):(x,z^+,z^-,y^+,y^-,\upsilon,\beta^+,\beta^-,\theta)\in\mathcal{X}_{VaR-M}\}\ ,\\
\mathbf{LPO^2_{CVaR-M}}: &\quad \{\max\ (107):(x,\mathfrak{q}_\alpha,s^-,s^+,h,\upsilon,\eta^+,\eta^-,\theta)\in\mathcal{X}_{CVaR-M}\}\ ,\\
\mathbf{LPO^3_{VaR-M}}: &\quad \{\max\ (108):(x,z^+,z^-,y^+,y^-,\upsilon,\beta^+,\beta^-,\theta)\in\mathcal{X}_{VaR-M}\}\ ,\\
\mathbf{LPO^3_{CVaR-M}}: &\quad \{\max\ (108):(x,\mathfrak{q}_\alpha,s^-,s^+,h,\upsilon,\eta^+,\eta^-,\theta)\in\mathcal{X}_{CVaR-M}\}
\end{aligned}$$

are not concave. The continuous relaxations of the above mixed-integer nonlinear optimization problems are not convex.

The result follows directly from Theorem 5.

Theorem 17 *i) The quadratic mixed-integer nonlinear optimization problems* $\mathbf{LPO^1_{VaR-M2}}$

$$\max\sum_{i=\tilde{n}+1}^{n}\left(a_i\sum_{k\in K}p^k\left(\sum_{t=1}^{T_i}\gamma_{i,t}^k\delta^t\left(\frac{\upsilon_i}{q_i}-\frac{\tau(\upsilon_i)^2}{\nu q_i}\right)+\frac{\nu\theta_i-\tau\upsilon_i}{\nu}(1-L_i+L_i\gamma_{i,T_i}^k)\delta^{t_i^*}-\frac{\nu\theta_i-\tau\upsilon_i}{\nu}\right)\right)-\sum_{k\in K}\sum_{i=1}^{\tilde{n}}p^kO_i^k \tag{109}$$

$$\text{s.t.}\quad (x,z^+,z^-,y^+,y^-,\upsilon,\beta^+,\beta^-,\theta)\in\mathcal{X}_{VaR-M}$$

and $\mathbf{LPO^1_{CVaR-M2}}$

$$\begin{aligned}&\max\ (109)\\ &\quad\text{s.t.}\quad (x,\mathfrak{q}_\alpha,s^-,s^+,h,\upsilon,\eta^+,\eta^-,\theta)\in\mathcal{X}_{CVaR-M}\end{aligned}$$

are equivalent to $\mathbf{LPO^1_{VaR-M}}$ *and* $\mathbf{LPO^1_{CVaR-M}}$*, respectively.*
ii) The continuous relaxations of $\mathbf{LPO^1_{VaR-M2}}$ *and* $\mathbf{LPO^1_{CVaR-M2}}$ *are convex.*

Theorem 9 shows that the objective functions of the single loan risk-pricing problems $\mathbf{LPO^2_{VaR-M}}$, $\mathbf{LPO^2_{CVaR-M}}$, $\mathbf{LPO^3_{VaR-M}}$, and $\mathbf{LPO^3_{CVaR-M}}$ associated with the exponential and logit acceptance probability functions are quasi-concave, which subsequently allows for their concavifiability. However, in case of multiple loans, the objective functions involve the sum of quasi-concave functions and it is well-known that the quasi-concavity property does not carry over the summation operation, which impedes their concavifiability and the derivation of convex continuous relaxations.

5. Illustrative Numerical Examples

In this section, we perform Monte Carlo simulations and present numerical examples to illustrate our results and provide additional insight. To demonstrate how the marginal risk and price-response (acceptance probability) considerations could affect the optimal price and the profit (portfolio performance), we first consider a portfolio with one new loan $\bar{n} = 1$, and $n - \bar{n} = n - 1 = 49$ (later, we also consider $n = 500$). Wet $q_i = 4$, $T_i = 20$, $i = 1, 2, \ldots n$, and vary the loan characteristics by randomly generating x_i (for the existing loans $i = 1, \ldots, n-1$) and a_i from uniform distributions: $x_i \sim \mathcal{U}(0.0062, 0.0092)$[12], $a_i \sim \mathcal{U}(400, 1200)$, and L_i from a beta distribution: $L_i \sim Beta(2, 1)$[13] for $i = 1, 2, \ldots n$. To capture the (default probability) correlations between loans, we assume that each loan in the portfolio belongs to one of three groups $S_j, j = 1, 2, 3$, which represent different industry sectors or categories of borrowers with a specific correlation structure given in Table 1. Note that the correlations between loans in a given group are high, while correlations across groups are low. For example, the correlation between loans in S_3 is 0.55 but the correlation between loans in S_1 and S_3 is 0.10.

Table 1 Correlation structure

	S_1	S_2	S_3
S_1	0.45	0.15	0.10
S_2	0.15	0.35	0.20
S_3	0.10	0.20	0.55

For simulating default, we implement a widely used structural approach[14] based on the normal copula model explicitly capturing default correlations (e.g., see Li (1999), Cherubini et al. (2004), Glasserman and

[12] These values reflect existing loan rates from the "Survey of Terms of Business Lending" (`http://www.federalreserve.gov/releases/e2/Current/default.htm`).

[13] Beta distributions are widely used to model LGD since is has support [0, 1] and flexible to reflect highly asymmetric nature of the distribution (Gupton et al. 2002, Bruche and Gonzalez-Aguado 2010). The senior commercial loan portfolio manager of a major US financial institution confirmed in a private conversation the validity of our simulation settings. While the empirical distributions of the loan rates and amounts might change over time (i.e., may differ from the uniform distribution), the results and insights remain the same with different distribution settings.

[14] A well-known risk modeling approach dates back to Merton (1974) and is based on the principles of option pricing Black and Scholes (1973). In such a framework, default is driven by an underlying process describing the asset value of the firm and occurs when the process hits a certain threshold. See, e.g., Lipton and Rennie (2013) and Morgan (1997) for more details.

Li (2005), Glasserman et al. (2008)) as follows. Let $p_{i,t}$ be the marginal probability that the i^{th} loan defaults at time t and $D_{i,t}$ be the default indicator taking value 1 if i^{th} loan defaults at time t and value 0, otherwise. In the normal copula model, correlation between the default indicators is captured through a multivariate normal vector $\mathbf{Z} = (Z_1, Z_2, \ldots, Z_n)$ of latent variables (we use the correlation matrix based on the structure in Table 1) where $D_{i,t} = \mathbf{1}\{Z_{i,t} > z_{i,t}\}, i = 1, \ldots, n$, and the thresholds $z_{i,t}$ are chosen to match the marginal default probability $p_{i,t}$. The marginal default probabilities used in our simulation are based on credit ratings and credit quality migration (upgrades or downgrades), which is modeled with transition rates estimated from historical data of credit qualities. We consider eight credit rating categories (AAA, AA, A, BBB, BB, B, CCC, and D), where AAA corresponds to the highest credit quality and D represents default. We set the marginal default probabilities for each loan with a specific rating from the credit transition matrices estimated by Standard & Poor's annual rating migrations (Vazza et al. 2014). Table 2 presents an example of such a credit transition matrix, which exhibits higher default risk for lower quality ratings[15].

Table 2 Example of one-year credit migration transition matrix based on S&P rating histories, 1981-2013 (%).

From/To	AAA	AA	A	BBB	BB	B	CCC/C	D
AAA	87.1	8.88	0.53	0.05	0.08	0.03	0.05	0
AA	0.55	86.39	8.26	0.56	0.06	0.07	0.02	0.02
A	0.03	1.87	87.33	5.48	0.35	0.14	0.02	0.07
BBB	0.01	0.12	3.59	85.22	3.82	0.59	0.13	0.21
BB	0.02	0.04	0.15	5.2	76.28	7.09	0.69	0.8
B	0	0.03	0.11	0.22	5.48	73.89	4.46	4.11
CCC/C	0	0	0.15	0.23	0.69	13.49	43.81	26.87
D	0	0	0	0	0	0	0	0

We randomly generate credit ratings for the granted loans, but set the credit rating for the prospective loan as BBB and assume it belongs to S_3 without loss of generality. To investigate how the existing loans in the same group affect the price of the prospective loan, we consider two different cases: the solution $x^*_{n,G}$ from good existing loan rating (average rating A) and the solution $x^*_{n,B}$ from bad existing loan rating (average rating BB). For each case, we generate $m = 5000$ scenarios (i.e. $\mathbf{Z} \in \mathbb{R}^{m \times n} = \mathbb{R}^{5000 \times 50}$) and solve the risk-based loan pricing optimization problem **LPO** given in Section 2 with $\delta = 0.995, l = 0.006, u = 0.011$ with different acceptance probability functions (i.e., solve the three corresponding convex reformulations) discussed in Section 3.3 and confidence levels ($\alpha = 0.9, 0.95, 0.99$) for both VaR and CVaR risk measures.

The AMPL modeling language is used to formulate the mathematical programming problems and we use the Cplex 12.6.1 and Bonmin solvers to solve the instances. The machine used for the tests is a 64 bits PowerEdge R515 with twelve AMD Opteron 4176 2.4GHz processors, of which we used only one, with 64GB of memory.

[15] Recall that we set $q_i = 4$ and $T_i = 20$. Thus, we utilize multi-year transition matrices (one-year to five year) as sources of cumulative probabilities of default and interpolate marginal default probabilities for each time period.

Implication of Marginal and Portfolio Risk Consideration. To demonstrate the impact of the marginal and portfolio risk constraints, we consider the case with standalone VaR and CVaR constraints as benchmark:

- Standalone VaR constraint:

$$\mathbb{P}\left(a_n\left(1-\sum_{t=1}^{T_n}\chi_{n,t}\delta^t\frac{x_n}{q_n}-(1-L_n+L_n\chi_{n,T_n})\delta^{t_n^*}\right)+\sum_{i=1}^{n-1}\zeta_i\leq\kappa_S^{VaR}a_n\right)\geq\alpha, \tag{110}$$

- Standalone CVaR constraint:

$$\mathbb{E}\left[a_n\left(1-\sum_{t=1}^{T_n}\chi_{n,t}\delta^t\tfrac{x_n}{q_n}-(1-L_n+L_n\chi_{n,T_n})\delta^{t_n^*}\right)\middle|a_n\left(1-\sum_{t=1}^{T_n}\chi_{n,t}\delta^t\tfrac{x_n}{q_n}-(1-L_n+L_n\chi_{n,T_n})\delta^{t_n^*}\right)\geq\mathfrak{q}_{\alpha_n}\right]\leq\kappa_S^{CVaR}a_n, \tag{111}$$

where κ_S is the standalone risk threshold and $\mathfrak{q}_{\alpha_n}$ is the VaR of the new loan at the α confidence level.

We reformulate and solve the risk-pricing problems **M-LPO** and **S-LPO** that respectively include a marginal and a standalone risk constraint with different levels of κ_M and κ_S[16]:

$$\begin{aligned}\mathbf{M-LPO:}\quad & \max\ (3)\\ & \text{s.t.}\quad (4);(33)\ .\\ \mathbf{S-LPO:}\quad & \max\ (3)\\ & \text{s.t.}\quad (4);(111)\ .\end{aligned}$$

Comparing optimal solutions $x^*_{n,G}$ and $x^*_{n,B}$, we should expect $x^*_{n,G}<x^*_{n,B}$ under the problem with the marginal risk constraint (**M-LPO**), but $x^*_{n,G}=x^*_{n,B}$ under the problem with the standalone constraint (**S-LPO**).

Table 3 Marginal Risk Constraint versus Standalone Risk Constraints With CVaR

Marginal			Standalone		
κ_M	$x^*_{n,G}$	$x^*_{n,B}$	κ_S	$x^*_{n,G}$	$x^*_{n,B}$
0.065	0.00835	0.00858	0.275	0.00835	0.00835
0.060	0.00848	0.00894	0.270	0.00852	0.00852
0.055	0.00884	0.00931	0.265	0.00935	0.00868
0.050	0.00920	0.00967	0.260	0.01018	0.00885
0.045	0.00956	0.01004	0.255	0.01102	0.00902

Table 3 presents the results that we expected, which underscores the importance of considering the interdependence between the loans in the existing portfolio and the new loans when making the pricing decision[17].

[16] Thresholds κ_M and κ_S given in Table 3 are chosen as follows. We first identify the smallest value for each threshold at which the corresponding risk constraint is non-binding and the solutions of both problems (i.e., with marginal and standalone risk constraints) are identical. These values are respectively $\kappa_M=0.065$ and $\kappa_S=0.275$ (see first row of Table 3) and the optimal interest rate is 0.835%.

[17] The reported results are obtained with the linear acceptance probability function (55) with parameters $\nu=1$ and $\tau=90/4$ and a confidence level $\alpha=0.99$ for the CVaR risk measure. The same results and insights prevail under different parameters and settings. In a complementary manner, the results also indicate that the optimal solution of the problem with the standalone risk constraint is particularly sensitive to the value of the threshold parameter κ_S. A 1.8% decrease (from 0.275 to 0.270) in the value of κ_S leads to an 2% increase in the optimal interest rate, while a 7.7% decrease (from 0.065 to 0.060) in the value of κ_M results into 1.6% increase in the optimal interest rate of the problem with marginal risk constraint.

We observe the same result when we solve the following problem **P-LPO** that includes a portfolio risk constraint (30):

$$\mathbf{P-LPO}: \quad \max \ (3)$$
$$\text{s.t.} \quad (4); (32) \ .$$

That is, by considering the marginal or portfolio constraint, the lender can take into account the quality of the existing loans as well as the interdependence between existing loans and the prospective loan when making pricing decisions.

However, there is a subtle different between the marginal and portfolio risk constraints. To illustrate this, we solve problem **P-LPO**, in which we add a marginal risk constraint with fixed marginal risk threshold $\kappa_M = 0.055$, for various values of the portfolio risk threshold κ_P.

Table 4 **Marginal Risk Constraint versus Portfolio Risk Constraints With CVaR ($\alpha = 0.99$)**

κ_M	κ_P	$x^*_{n,G}$	$Slack_{M,G}$	$Slack_{P,G}$	$x^*_{n,B}$	$Slack_{M,B}$	$Slack_{P,B}$
0.055	0.2500	0.00884	0	2939.7	0.00931	0	6.1
0.055	0.2502	0.00884	0	2909.7	0.01105	23.9	0
0.055	0.1809	0.00886	0.3	0		Infeasible	
0.055	0.1800	0.01101	30.3	0		Infeasible	

Table 4 demonstrates that as the threshold κ_P decreases, the portfolio constraint becomes binding and hence the optimal interest rate increases. The notations $Slack_{M,G}$ and $Slack_{M,B}$ (resp., $Slack_{P,G}$ and $Slack_{P,B}$) denote the slack of the marginal (resp., portfolio) risk constraint in the optimal solution of the problem. A positive slack indicates that the constraint is not binding. By comparing cases under good and bad existing loan ratings, we can see that the portfolio constraint could be utilized to more effectively control the risk of existing loans. If the risk of existing loan is already large, it would be optimal to charge a higher price or not to accept the loan even when the risk contribution of the new loan is small. On the other hand, the marginal risk constraint could be employed to more effectively control the risk of the new loan. For example, the lender might want to set a tighter threshold for a specific prospective loan depending on its industry category, and in this case, the portfolio constraint would not be binding.

Implication of acceptance probability. We also investigate the gain from incorporating the price response of the borrower. We first solve the problem **M-LPO** and obtain the optimal solution $x^*_{n,G}$ and compute $y^* = \mathbb{E}\big[g(x^*_{n,G}, s_n)\big(\mathcal{G}_n(x^*_{n,G}) - a_n\big)\big]$, which is the value of the objective function associated with the new loan (we also do the same for $x^*_{n,G}$ with the bad existing loan ratings). Then, we solve $\max\limits_{x_n} \sum\limits_{i=1}^{n} \big(\mathcal{G}_i(x_i) - a_i\big)$ subject to the same constraints and compute $\bar{y} = \mathbb{E}\big[g(\tilde{x}, s_n)\big(\mathcal{G}_n(\tilde{x}) - a_n\big)\big]$, which is the value of the objective function associated with the new loan when the acceptance probability is ignored. Not surprisingly, the optimal solutions obtained without the acceptance probability is equal to 0.011, which is the upper bound on the interest rate. The relative gain is then calculated as $rel = (y^* - \bar{y})/a_n \times 100$. As shown in Table 5, given the parameters considered and threshold κ_M values, the relative gain is between 0.81% and 1.29% (exact values of relative gain would be dependent on the estimated parameters in practice). Without considering the acceptance probability, it is more likely to overprice when the existing loan ratings are good, and thus the lender benefits more with the good existing loan ratings. Also, we observe that relative gain increases as κ_M increases for the same reason. We obtain similar results under standalone and portfolio constraints as well.

Table 5 Relative Gain From Considering the Acceptance Probability (%)

κ_M	$rel(x^*_{n,G})$	$rel(x^*_{n,B})$
0.065	1.290	1.282
0.060	1.287	1.231
0.055	1.250	1.135
0.050	1.168	0.995
0.045	1.004	0.809

Efficiency and Scalability of the solution approach. Lastly, we report the average solution times for three different acceptance probability functions (linear, exponential, logit) and varying number of scenarios and confidence levels. More precisely, we consider instances that differ in terms of the value of the confidence level α (i.e. α=0.9, 0.95, 0.99) and the number of scenarios $|K|$ (i.e., $|K| = 500, 1000, 5000, 10000$). For each problem instance, we generate six data sets (with $n = 500$), thereby obtaining 72 problem instances. We consider the three acceptance probability functions (i.e., linear (55) , exponential (56) and logit (57)) and solve the three corresponding convex reformulations (59), (67), and (69) of the loan pricing problem with marginal and portfolio CVaR risk constraints for the 72 problem instances. The results displayed in the third column of Table 6 indicate the average solution across 18 problem instances (i.e., 6 data sets, 3 confidence levels) for each considered acceptance probability function and number of scenarios. Column 4 provides the average solution time across the 72 problem instances for each acceptance probability function. We use the Cplex 12.6.1 to solve the quadratic MINLP optimization models involving the linear acceptance probability function. For the other acceptance probability functions (exponential and logit), the nonlinearity of the objective function is not of a quadratic form and therefore, the Cplex 12.6.1 can not be used. For those instances, we employ the Bonmin solver specialized for MINLP problems with convex continuous relaxations.

As shown, formulations based on the linear acceptance probability functions are the most quickly solved (i.e., about 4 seconds on average). The average solution time for the logit acceptance probability function amounts to 150 seconds and is about 15% lower than the one for the exponential probability function. While the solution time is an increasing function of the number of scenarios, it still remains reasonable (does not exceed 360 seconds on average with up to 10000 scenarios). Therefore, these results attest the scalability and computational tractability of the proposed convex programming reformulations.

We report now the computational results for the multi-loan risk-pricing problem with marginal and portfolio CvaR constraints and with linear acceptance probability function. We econsider 24 problem instances with varying number of prospective loans (from 2% to 10% of the total number of loans). For each problem instance, we generate 12 data sets that differ in terms of the value of α (i.e., α=0.9, 0.95, 0.99) and $|K|$

Table 6 **Single Loan Risk-Pricing - Computation Times (in seconds)**

$g(x_n, s_n)$	$\lvert K \rvert$	Time	Average Time
Linear	500	0.24	3.91
	1000	0.26	
	5000	3.41	
	10000	11.74	
Exponential	500	3.23	171.43
	1000	10.04	
	5000	101.43	
	10000	571.03	
Logit	500	2.21	150.05
	1000	7.90	
	5000	92.77	
	10000	497.33	

(i.e., $|K| = 500, 1000, 5000, 10000$), thereby obtaining 288 problem instances. The results displayed in the third column of Table 7 indicate the average solution time across 72 problem instances (i.e., 24 data sets, 3 confidence levels) for different number of scenarios. We use the Cplex 12.6.1 solver.

Table 7 **Multi-Loan Risk-Pricing - Computation Times (in seconds)**

$g(x, s)$	$\lvert K \rvert$	Time	Average Time
Linear	500	0.88	63.69
	1000	1.64	
	5000	64.89	
	10000	187.34	

As shown, the optimal solution can be obtained quickly. The average solution time increases with the number of scenarios considered and is about 64 seconds across the 288 instances. While the solution time is larger than for the single loan risk-pricing problem with linear probability function, it still remains reasonable, which highlights the applicability and scalability of the proposed approach.

6. Conclusion

Risk-based pricing enables lenders to charge different interest rates to different borrowers based on their risk. We present a genetic loan pricing optimization framework, which explicitly incorporates the interrelationship between prospective loans and the existing loans in the loan portfolio and borrowers' response to the prices offered. Interrelationship is captured by marginal risk contribution and portfolio risk with respect to the widely used Value-at-Risk and Conditional-Value-at-Risk downside risk measures. These risk measures are positive homogeneous and thus the marginal risk contributions can be represented as the conditional loss expectation of a loan provided that the losses of the entire portfolio reach a certain level. The acceptance probability incorporated into the optimization problem enables the lender to reflect heterogeneity in borrowers' willingness-to-pay in the loan price, which has been a well understood concept in the pricing literature.

The inclusion of marginal/portfolio risk constraints and price response functions leads to non-convex stochastic programming optimization problems, which pose serious computational challenges. To this end, we first consider a case where a single prospective loan is evaluated, and we develop a convex reformulation approach that allows us to derive equivalent deterministic reformulations taking the form of mixed-integer nonlinear programming problems by using a concavifiability method for different types of acceptance probability functions (linear, logit, and exponential). We then extend our model to the case where multiple prospective loans are evaluated simultaneously, and explicitly discuss the loan selection process along with pricing decisions. By analyzing the impact of the acceptance probability function on the tractability of the loan pricing problem, we demonstrate a potential benefit of a linear acceptance probability function, for which a convex reformulation can be obtained.

Future research can expand upon our study in several directions. For example, while we capture the borrowers' willingness-to-pay through the price-response function incorporating borrowers' risk and other characteristics, the price of a loan can also affect the riskiness (payment risk) of a loan in turn: charging a higher price for a particular type of loan might result in a higher rate of default. One can extend our mode to explicitly take into account such interplay of price and risk. In this study, we also assume the the estimated default probability, LGD, and price response function (acceptance probability) as inputs. Investigating how the calibration and estimation process of such inputs can be incorporated into the loan optimization modeling framework could be another interesting topic for the future study.

References

Abiad, A., E. Detragiache, T. Tressel. 2008. *A new database of financial reforms*. 2008-2266, International Monetary Fund.

Allen, L., A. Saunders. 2002. Credit risk measurement. new approaches to value at risk and other paradigms. *VJohn Wiley & Sons, .*

Andersson, F., H. Mausser, D. Rosen, S. Uryasev. 2001. Credit risk optimization with conditional value-at-risk criterion. *Mathematical Programming* **89** 273–291.

Avriel, M., W. E. Diewert, S. Schaible, T. Ziang. 1988. *Generalized Concavity*. SIAM.

Basel Committee on Banking Supervision. 2006. International convergence of capital measurement and capital standards: A revised framework. *Bank for International Settlements*.

Basel Committee on Banking Supervision. 2013. Fundamental review of the trading book: A revised market risk framework. *Bank for International Settlements*.

BCG. 2016. "moneyball" in commercial lending: From art to science in pricing. Boston Consulting Group Perspective Website. Goyal, D. and Karthikeyan, S. and Kulkarni, V. and Noguera, V. and Wachters, I. Retrieved on June 19 from https://www.bcgperspectives.com/content/articles/financial-institutions-moneyball-pricing-commercial-loans/.

Benati, S., R. Rizzi. 2007. A mixed integer linear programming formulations of the optimal mean/value-at-risk portfolio problem. *European Journal of Operational Research* **176** 423–434.

Besbes, O., R. Phillips, A. Zeevi. 2010. Testing the validity of a demand model: An operations perspective. *Manufacturing & Service Operations Management* **12**(1) 162–183.

Besbes, O., A. Zeevi. 2015. On the (surprising) sufficiency of linear models for dynamic pricing with demand learning. *Management Science* **61**(4) 723–739.

Black, F., M. Scholes. 1973. The pricing of options and corporate liabilities. *The journal of political economy* 637–654.

Bostic, R.W. 2002. Trends in equal access to credit products. *The Impact of Public Policy on Consumer Credit*. Springer, 171–208.

Bruche, M., C. Gonzalez-Aguado. 2010. Recovery rates, default probabilities, and the credit cycle. *Journal of Banking & Finance* **34**(4) 754–764.

Burer, S.A., A. Letchford. 2012. Non-convex mixed-integer nonlinear programming: A survey. *Surveys in Operations Research and Management Science* **17** 97–106.

Caufield, S. 2012. Consumer credit pricing. *The Oxford Handbook of Pricing Management, Oxford University Press, Oxford* 138–152.

Cherubini, U., E. Luciano, W. Vecchiato. 2004. *Copula Methods in Finance*. John Wiley & Sons.

Cohen, M. C., N. H. Z. Leung, K. Panchamgam, G. Perakis, A. Smith. 2016. The impact of linear optimization on promotion planning. *Forthcoming in Operations Research* .

Cornuejóls, G., R. Tútúncú. 2007. *Optimization Methods in Finance*. Cambridge, UK: Cambridge University Press.

Cui, X., S. Zhu, D. Li, X. Sun. 2016. Mean-variance portfolio optimization with parameter sensitivity control. *Optimization Methods and Software* **31**(4) 755–774.

DAmbrosio, C., A. Lodi. 2011. Mixed integer nonlinear programming tools: A practical overview. *4OR* **9**(4) 329–349.

De Finetti, B. 1949. Sulle stratificazioni convesse. *Annali di Matematica Pura ed Applicata* **30**(1) 123–183.

DeMiguel, V., L. Garlappi, F. J. Nogales, R. Uppal. 2009. A generalized approach to portfolio optimization: Improving performance by constraining portfolio norms. *Management Science* **55**(5) 798–812.

Dentcheva, D., A. Ruszczynski. 2015. Risk-averse portfolio optimization via stochastic dominance constraints. *Handbook of Financial Econometrics and Statistics*. Springer, 2281–2302.

Edelberg, W. 2006. Risk-based pricing of interest rates for consumer loans. *Journal of Monetary Economics* **53**(8) 2283–2298.

Emmons, H., S. M Gilbert. 1998. Note. the role of returns policies in pricing and inventory decisions for catalogue goods. *Management Science* **44**(2) 276–283.

Fenchel, W. 1951. *Convex Cones, Sets and Functions. Mimeographed lecture Notes*. Princeton University, Princeton, NJ, USA.

Feng, M., A. Wachter, J. Staum. 2015. Practice algorithms for value-at-risk portfolio optimization problems. *Quantitative Finance Letters* **3** 1–9.

Glasserman, P. 2006. Measuring marginal risk contributions in credit portfolios. *Journal of Computational Finance* **9**(2) 1–41.

Glasserman, P., W. Kang, P. Shahabuddin. 2008. Fast simulation of multifactor portfolio credit risk. *Operations Research* **56**(5) 1200–1217.

Glasserman, P., J. Li. 2005. Importance sampling for portfolio credit risk. *Management Science* **51**(11) 1643–1656.

Gourieroux, C., J. P. Laurent, O. Scaillet. 2000. Sensitivity analysis of values at risk. *Journal of Empirical Finance* **7** 225–245.

Gross, D., N. Souleles. 2002. Do liquidity constraints and interest rates matter for consumer behavior? evidence from credit card data. *The Quarterly Journal of Economics* **117**(1) 149–185.

Gupton, G.M., R.M. Stein, A. Salaam, D. Bren. 2002. LOSSCALCTM: Model for predicting loss given default (LGD). *Moodys KMV, New York* .

Horst, R. 1984. On the convexification of nonlinear programming problems: An applications-oriented survey. *European Journal of Operations Research* **15** 382–392.

Huang, B., L. C. Thomas. 2015. The impact of basel accords on the lender's profitability under different pricing decisions. *Journal of the Operational Research Society* **66** 1826–1839.

Johnson, R. W. 1992. Legal, social and economic issues in implementing scoring in the united states. J. N. Crook, D.B. Edelman, L.C. Thomas, eds., *Credit Scoring and Credit Control*. Oxford University Press, 5–15.

Kalkbrener, M. 2005. An axiomatic approach to capital allocation. *Mathematical Finance* **15** 425–437.

Kalkbrener, M., H. Lotter, L. Overbeck. 2004. Sensible and efficient asset allocation for credit portfolios. *Risk* **17** 19–24.

Kataoka, S. 1963. A stochastic programming model. *Econometrica* **31**(1-2) 181–196.

Kawas, B., A. Thiele. 2011. Short sales in log-robust portfolio management. *European Journal of Operational Research* **215**(3) 651–661.

Kimber, Andrew. 2003. *Credit Risk: From Transaction to Portfolio Management*. Butterworth-Heinemann.

Krokhmal, P., M. Zabarankin, S. Uryasev. 2011. Modeling and optimization of risk. *Surveys in Operations Research and Management Science* **16**(2) 49–66.

Kurth, A., D. Tasche. 2003. Contributions to credit risk. *Risk* **16** 84–88.

Lau, A. H. L., H.S. Lau. 2003. Effects of a demand-curves shape on the optimal solutions of a multi-echelon inventory/pricing model. *European Journal of Operational Research* **147**(3) 530–548.

Lejeune, M.A., F. Margot. 2016. Solving chance constrained problems with random technology matrix and stochastic quadratic inequalities. *Operations Research* **64**(4) 939–957.

Li, D., X. L. Sun, M. P. Biswal, F. Gao. 2001. Convexification, concavification and monotonization in global optimization. *Annals of Operations Research* **105** 213–226.

Li, D., Wu Z.-y., J. Lee H.-w., Yang X.-m., Zhang L.-s. 2005. Hidden convex minimization. *Journal og Global Optimization* **31** 211–233.

Li, D.X. 1999. On default correlation: A copula function approach. *Available at SSRN 187289* .

Lipton, A., A. Rennie. 2013. *The Oxford Handbook of Credit Derivatives*. Oxford University Press, Oxford.

Liu, G. 2015. Simulating risk contributions of credit portfolios. *Operations Research* **63**(1) 104–121.

Magri, S., R. Pico. 2011. The rise of risk-based pricing of mortgage interest rates in italy. *Journal of Banking & Finance* **35**(5) 1277–1290.

Mausser, H., D. Rosen. 2008. Economic credit capital allocation and risk contributions. *Handbooks in Operations Research & Management Science - Financial Engineering*. Eds: Birge J.R., Linetsky V. Volume 15. Elsevier, 681–726.

McCormick, G. P. 1976. Computability of global solutions to factorable nonconvex programs: Part i convex underestimating problems. *Mathematical Programming* **10**(1) 147–175.

Mercer, Z. C. 1992. *Valuing Financial Institutions*. Irwin Professional Pub.

Merino, S., M. A. Nyfeler. 2004. Applying importance sampling for estimating coherent credit risk contributions. *Quantitative Finance* **4**(2) 199–207.

Merton, R. C. 1974. On the pricing of corporate debt: The risk structure of interest rates. *The Journal of finance* **29**(2) 449–470.

Morgan, J. P. 1997. Creditmetrics^{TM}. *Technical Report.* New York, J.P. Morgan.

Musto, David K, Nicholas S Souleles. 2006. A portfolio view of consumer credit. *Journal of Monetary Economics* **53**(1) 59–84.

Oliver, B. V., R. M. Oliver. 2014. Optimal ROE loan pricing with or without adverse selection. *Journal of the Operational Research Society* **65**(3) 435–442.

Petruzzi, N. C., M. Dada. 1999. Pricing and the newsvendor problem: A review with extensions. *Operations Research* **47**(2) 183–194.

Phillips, R. 2005. *Pricing and Revenue Optimization*. Stanford University Press.

Phillips, R. 2013. Optimizing prices for consumer credit. *Journal of Revenue and Pricing Management* **12** 360–377.

Prékopa, A. 1995. *Stochastic Programming*. Kluwer. Boston, MA.

PwC. 2012. Enhancing origination revenue through price optimization modeling. PricewaterhouseCoopers Website. Retrieved on June 19 from http://www.pwc.com/us/en/consumer-finance/publications/loan-origination-price-optimization-modeling.html.

Sageworks. 2016. Loan-pricing models: What to consider for loan origination. Sageworks Blog. posted by Biery, M. E. Retrieved on May 20 from https://www.sageworks.com/blog/post/2016/01/29/loan-pricing-models-what-banks-need-to-consider.aspx.

Schuermann, Til. 2004. Why were banks better off in the 2001 recession? *Current Issues in Economics and Finance* **10**(1).

Stanhouse, B., D. Stock. 2008. Managing the risk of loan prepayments and the optimal structure of short term lending rates. *Annals of Finance* **4**(2) 197–215.

Tasche, D. 2000. Conditional expectation as quantile derivative. Tech. rep.

Tasche, D. 2009. Capital allocation for credit portfolios with kernel estimators. *Quantitative Finance* **9**(5) 581–595.

Thomas, L. C. 2009. *Consumer Credit Models*. Oxford University.

Vazza, D., N. Kraemer, N.M. Richhariya, A. Debnath, M. Sangoi. 2014. 2013 annual global corporate default study and rating transitions. *Standard and Poors, March* **1**.

Yu, J.-R., W.-J. Chiou, D.-R. Mu. 2015. A linearized value-at-risk model with transaction costs and short selling. *European Journal of Operational Research* **247** 872–878.

Zhu, S., D. Li, X. Sun. 2010. Portfolio selection with marginal risk control. *The Journal of Computational Finance* **14**(1) 3–28.

Appendix. Proofs

Proof of Theorem 1.

i) Portfolio VaR constraint: Each constraint (12) defines the amount $z^{k-} - z^{k+}$ by which the loss in scenario ω^k differs from the threshold loss level $\kappa_P^{VaR} \sum_{i=1}^{n} a_i$. Each constraint in (13) (resp., (14)) forces the binary variable β^{k+} (resp., β^{k-}) to take value 1 if the corresponding variable z^{k+} (resp., z^{k-}) is strictly positive. Since (15) allows at most one variable in each pair (β^{k+}, β^{k-}) to take value 1, it ensues that only one of z^{k+} and $z^{k-}, k \in K$ can be strictly positive and that z^{k-} (resp., z^{k+}) denotes the loss amount below (resp., above) the threshold $\kappa_P^{VaR} \sum_{i=1}^{n} a_i$. The expression $\sum_{k \in K} p^k \beta^{k+}$ is the probability of the set of scenarios with total losses exceeding $\kappa_P^{VaR} \sum_{i=1}^{n} a_i$. The knapsack constraint (16) stipulates that the sum of the probabilities of the violated scenarios must not exceed $1-\alpha$. As a result, the feasible set of (8) can be equivalently represented by the mixed-integer linear set defined by $\{(12)-(16); (18)\}$.

ii) Marginal VaR constraint: If the loss in scenario ω^k is equal to $\kappa_P^{VaR} \sum_{i=1}^{n} a_i$, (12)-(14) ensure: 1) $z^{k+} = z^{k-} = \beta^{k+} = \beta^{k-} = 0$ and 2) $(1 - \beta^{k+} - \beta^{k-}) = 1$. Therefore, $\sum_{k \in K} p^k (1 - \beta^{k+} - \beta^{k-})$ in (17) is the overall probability that the portfolio loss is equal to $\kappa_P^{VaR} \sum_{i=1}^{n} a_i$, and $p^k(1 - \beta^{k+} - \beta^{k-})$ is the joint probability that the portfolio loss is equal to $\kappa_P^{VaR} \sum_{i=1}^{n} a_i$ and that the marginal loss is equal to $a_n(1 - \sum_{t=1}^{T_n} \gamma_{n,t}^k \delta^t \frac{x_n}{q_n} - (1 - L_n + L_n \gamma_{n,T_n}^k)\delta^{t_n^*})$ in scenario ω^k. As a result, the conditional expectation constraint (10) on the marginal VaR exposure of loan n is rewritten as (17).

iii) Admissible value for M^{k+} and M^{k-}: In order to obtain the tightest continuous relaxation for the Big-M constraints (13)-(14), we derive the smallest values for each M^{k+} and $M^{k-}, k \in K$ that do not invalidate any of the constraints (13)-(14). They are obtained by computing the largest value that z^{k+} and z^{k-} can take. Since z^{k+} and z^{k-} only appear in (12), is is straightforward to see that z^{k+} (resp., z^{k-}) is upper-bounded by (19) (resp., (20)) in which x_n was replaced by its upper bound u (resp., lower bound l). ■

Proof of Lemma 3.

First, we divide both sides of (17) by a_n, which gives:

$$\sum_{k \in K} \left(p^k (1 - \beta^{k+} - \beta^{k-}) \; (1 - \delta^{t_n^*} (1 - L_n + L_n \gamma_{n,T_n}^k)) \right) - \frac{1}{q_n} \sum_{k \in K} \left(\sum_{t=1}^{T_n} \delta^t \gamma_{n,t}^k \; x_n \; (1 - \beta^{k+} - \beta^{k-}) \; p^k \right) \leq \kappa_M^{VaR} \sum_{k \in K} p^k (1 - \beta^{k+} - \beta^{k-}) \, . \tag{112}$$

Since $\beta^{k+} x_n = y^{k+}, k \in K$ and $\beta^{k-} x_n = y^{k-}, k \in K$ due to (21)-(28), the left-hand side of (112) can be rewritten as:

$$\sum_{k \in K} \left(p^k (1 - \beta^{k+} - \beta^{k-}) \; (1 - \delta^{t_n^*} (1 - L_n + L_n \gamma_{n,T_n}^k)) \right) - \frac{1}{q_n} \sum_{k \in K} \left(\sum_{t=1}^{T_n} \delta^t \gamma_{n,t}^k \; (x_n - y^{k+} - y^{k-}) \; p^k \right) \tag{113}$$

which provides the result that we set out to prove. ■

Proof of Theorem 4.

The constraints (38)-(44) define s^{k-} as the portfolio loss in excess of $\mathfrak{q}_\alpha$, and (39) forces $\eta^{k-} = 1$ if $s^{k-} > 0$. Therefore, $\frac{p^k \eta^{k-}}{1-\alpha}$ is the conditional probability of the loss in scenario ω^k given that that the portfolio loss exceeds $\mathfrak{q}_\alpha$, and the conditional expected loss with ω^k is $\frac{p^k\eta^{k-}}{1-\alpha}\left(1-\sum_{t=1}^{T_n}\gamma_{n,t}^k\delta^t\frac{x_n}{q_n}-(1-L_n+L_n\gamma_{n,T_n}^k)\delta^{t_n^*}\right)$. Summing the conditional expected losses over each k, we obtain

$$\sum_{k\in K}p^k\eta^{k-}\left(1-\sum_{t=1}^{T_n}\gamma_{n,t}^k\delta^t\frac{x_n}{q_n}-(1-L_n+L_n\gamma_{n,T_n}^k)\delta^{t_n^*}\right)\le(1-\alpha)\ \kappa_M^{CVaR}\ . \tag{114}$$

The inequality (114) is quadratic and define a nonconvex feasible area due to the bilinear terms $\eta^{k-}x_n, k\in K$. Since $\eta^{k-}x_n=h_k, k\in K$ due to (46)-(49), We can replace the bilinear terms $\eta^{k-}x_n$ by h_k in (114), thereby obtaining the linear constraint (45), which is the result we set out to prove. ■

Proof of Theorem 5.

The continuous relaxations of $\mathbf{LPO^1_{VaR}}$ and $\mathbf{LPO^1_{CVaR}}$ have a linear feasible region and are convex programming problems if their objective function is concave. The second derivative of their objective function is $\frac{-2db}{\nu}$. Since $x_n\in[0,\nu/\tau]$ due to (55), ν and τ must have the same sign. The loan is only granted if the resulting expected profit is non-negative, which implies that $d\ge 0$. Hence, the second derivative $\frac{-2d\tau}{\nu}$ is negative, and $dx_n\frac{\nu-\tau x_n}{\nu}$ is concave.

It is easy to see that the second derivative of the objective functions in $\mathbf{LPO^2_{VaR}}$, $\mathbf{LPO^2_{CVaR}}$, $\mathbf{LPO^3_{VaR}}$, and $\mathbf{LPO^3_{CVaR}}$ can be positive. Therefore, these problems are not convex in general. ■

Proof of Theorem 8.

Assume that f is not semistrictly quasi-concave on C. Then, (65) implies that one can find $0<\beta<1, x^1, x^2\in C$ with $f(x^1)\ne f(x^2)$ such that:

$$f(\beta x^1+(1-\beta)x^2))\le\min(f(x^1),f(x^2))\ . \tag{115}$$

Additionally, for any increasing G, Definition 6 implies that G^{-1} is also increasing and we have:

$$G^{-1}(min(G(f(x^1)),G(f(x^2)))<G^{-1}(\beta G(f(x^1))+(1-\beta)G(f(x^2)))\ . \tag{116}$$

Clearly, (115) and (116) contradict (63), which implies that f must be semistrictly quasi-concave and also quasi-concave due to the continuity of f. ■

Proof of Theorem 9.

The domain of definition of g^2 and g^3 is $C=[u,l]$ with $l>u>0$. Let $x^1,x^2\in C$, $0\le\lambda\le 1$ and assume $g^i(x^1)\le g^i(x^2), i=2,3$.

Part *i)*: Consider $\log\ g^2(x_n)$:

$$\log\ g^2(x_n)=\log(x)+(-\tau x_n)\ .$$

It is clear that $\log\ g^2(x)$ is concave as the logarithmic function is concave, $-\tau x_n$ is linear and thus concave, and concavity carries over the summation operation. This in turn implies that $g^2(x)$ is log-concave. Since $g^2(x)$ is both log-concave and positive, $g^2(x_n)$ is also quasi-concave.

Part *ii)*: The quasi-concavity of g^3 will be established by showing that:

$$g^3(\lambda x^1+(1-\lambda)x^2)=\frac{(\lambda x^1+(1-\lambda)x^2)e^{\nu-\tau(\lambda x^1+(1-\lambda)x^2))}}{1+(\lambda x^1+(1-\lambda)x^2)e^{\nu-\tau(\lambda x^1+(1-\lambda)x^2))}}\geq g^3(x^1)=\frac{x^1e^{\nu-\tau x^1}}{1+x^1e^{\nu-\tau x^1}}\ .$$

This is the case if:

$$(\lambda x^1+(1-\lambda)x^2)e^{\nu-\tau(\lambda x^1+(1-\lambda)x^2)})\ (1+x^1e^{\nu-\tau x^1})\ \geq \tag{117}$$

$$(x^1e^{\nu-\tau x^1})\ (1+(\lambda x^1+(1-\lambda)x^2)e^{\nu-\tau(\lambda x^1+(1-\lambda)x^2))} \tag{118}$$

$$\Leftrightarrow \quad (\lambda x^1+(1-\lambda)x^2)e^{\nu-\tau(\lambda x^1+(1-\lambda)x^2)}\ \geq\ x^1e^{\nu-\tau x^1}\ , \tag{119}$$

which is always true, since we have shown in Part *i)* that g^2 is quasi-concave, and thus that:

$$(\lambda x^1+(1-\lambda)x^2)e^{-\tau(\lambda x^1+(1-\lambda)x^2))}\geq x^1e^{-\tau x^1}\ , \tag{120}$$

as we assume that $g^2(g^1)=\min(g^2(x^1),g^2(x^2))$. ■

Proof of Theorem 10.

To obtain $\mathbf{RLPO^2_{VaR}}$, $\mathbf{RLPO^2_{CVaR}}$, $\mathbf{RLPO^3_{VaR}}$, and $\mathbf{RLPO^3_{CVaR}}$, we take the logarithm of the objective functions in $\mathbf{LPO^2_{VaR}}$, $\mathbf{LPO^2_{CVaR}}$, $\mathbf{LPO^3_{VaR}}$, and $\mathbf{LPO^3_{CVaR}}$. The logarithmic function is monotone and strictly increasing. Therefore, maximizing $ln\left(dx_n\frac{e^{\nu-\tau x_n}}{e^{\nu}}\right)$ and $ln\left(dx_n\frac{e^{\nu-\tau x_n}}{1+e^{\nu-\tau x_n}}\right)$ is the same as maximizing $dx_n\frac{e^{\nu-\tau x_n}}{e^{\nu}}$ and $dx_n\frac{e^{\nu-\tau x_n}}{1+e^{\nu-\tau x_n}}$, respectively.

The natural logarithm is a concave function, and d is non-negative. Therefore, $d\ ln(x_n)-\tau x_n$ is concave and the continuous relaxations of $\mathbf{PO^2_{VaR}}$ and $\mathbf{PO^2_{CVaR}}$ are convex programming problems.

The second derivative of $ln(1+e^{\nu-\tau x_n})$ is $\frac{\tau^2\ e^{\nu-\tau x_n}}{(1+e^{\nu-\tau x_n})^2}$ and is non-negative. Therefore, $-ln(1+e^{\nu-\tau x_n})$ is concave. Since the concavity property carries over the summation operation, $d\ ln(x_n)+\nu-\tau x_n-ln(1+e^{\nu-\tau x_n})$ is concave and $\mathbf{RLPO^2_{VaR}}$ and $\mathbf{RLPO^2_{CVaR}}$ are convex programming problems. ■

Proof of Theorem 13.

i) Portfolio VaR constraint: Each constraint (78) forces the corresponding binary variable θ_i (95) to take value 0 if the expected profit generated by loan i is not positive. The profit generated by a prospective loan in a certain scenario k depends on the interest rate x_i and the decision θ_i to grant the loan i and is given by the nonlinear expression

$$\theta_i a_i\left(\sum_{t=1}^{T_i}\gamma^k_{i,t}\delta^t\frac{x_i}{q_i}+(1-L_i+L_i\gamma^k_{i,T_i})\delta^{t_i^*}-1\right)\ \Leftrightarrow\ \frac{\theta_i x_i a_i}{q_i}\sum_{t=1}^{T_i}\gamma^k_{i,t}\delta^t+\theta_i a_i\left(\sum_{t=1}^{T_i}(1-L_i+L_i\gamma^k_{i,T_i})\delta^{t_i^*}-1\right)$$

with the nonconvex bilinear term $\theta_i x_i$. In order to linearize them, we lift the decisional and constraint spaces by introducing the decision variables υ_i and the constraints (79)-(82) that guarantee that $\upsilon_i=\theta_i x_i, i=\tilde{n}+1,\dots,n$. The profit stemming from loan i in scenario k can thus be written as

$$\upsilon_i\frac{a_i}{q_i}\ \sum_{t=1}^{T_i}\gamma^k_{i,t}\delta^t\ +\ \theta_i a_i\left(\sum_{t=1}^{T_i}(1-L_i+L_i\gamma^k_{i,T_i})\delta^{t_i^*}-1\right)\ , \tag{121}$$

and is equal to 0 if the loan i is rejected (i.e., $\theta_i=\upsilon_i=0$). We use (121) in (83) to express the loss due to i in scenario k. Constraints (83)-(85) have the same exact meaning as (12)-(14) in Theorem 1, and enforce, along with (15)-(16), that the portfolio VaR constraint (75) holds.

ii) Marginal VaR constraint: Constraints (86)-(93) are defined for each loan $i = \tilde{n}+1,\ldots,n$ and are similar to (21)-(28) in Theorem 1. They linearize the bilinear terms $\beta^{k+}x_i$ and $\beta^{k-}x_i$ in the marginal VaR constraint, and ensure that $y_i^{k+} = \beta^{k+}x_i$ and $y_i^{k-} = \beta^{k-}x_i$ for each $i = \tilde{n}+1,\ldots,n$. Note that we should have a (reformulated) marginal constraint *only* for the new loans that meet the positive expected profit condition (78). This is why we have the term $(1-\theta_i)N$ in (94). If the condition defined by (78) is not met, $\theta_i = 0$ and an arbitrarily large positive quantity N is added to the right-hand side of (94), which means that it is non-binding and has no impact on the feasible set and optimal solution. ■

Proof of Theorem 15.

i) Portfolio CVaR constraint: Constraints (79)-(82) ensure that υ_i represents the interest rate proposed for loan $i, i = \tilde{n}+1,\ldots,n$ (Theorem 13). Therefore, the loss due to i in scenario k is $-\upsilon_i \frac{a_i}{q_i}\sum_{t=1}^{T_i}\gamma_{i,t}^k\delta^t - \theta_i\ a_i\left(\sum_{t=1}^{T_i}(1-L_i+L_i\gamma_{i,T_i}^k)\delta^{t_i^*}-1\right)$. Constraints (100) is similar to (38) and, with (98) and (41), enforces the portfolio CVaR constraint.

i) Marginal CVaR constraint: Constraints (102)-(105) are defined for each loan $i = \tilde{n}+1,\ldots,n$ and linearize the bilinear terms $\eta^{k-}x_i$ in the marginal CVaR constraint, ensuring that $h_i^k = \eta^{k-+}x_i, i = \tilde{n}+1,\ldots,n$. Only new loans that satisfy the positive expected profit condition defined by (78) should be subjected to a marginal risk constraint. We proceed as in Theorem 13 and add the term $(1-\theta_i)N$ in each constraint (101). If the condition defined by (78) is not met, $\theta_i = 0$ and the corresponding constraint (101) is non-binding. ■

Proof of Theorem 17.

i) Since $\upsilon_i = \theta_i x_i, i = \tilde{n}+1,\ldots,n$ from (79)-(82), (106) becomes

$$\sum_{i=\tilde{n}+1}^{n}\left(a_i\sum_{k\in K}p^k\left(\sum_{t=1}^{T_i}\gamma_{i,t}^k\delta^t\left(\frac{\upsilon_i}{q_i}-\frac{\tau\theta_i(x_i)^2}{\nu q_i}\right)+\frac{\nu\theta_i-\tau\upsilon_i}{\nu}(1-L_i+L_i\gamma_{i,T_i}^k)\delta^{t_i^*}-\frac{\nu\theta_i-\tau\upsilon_i}{\nu}\right)\right)-\sum_{k\in K}\sum_{i=1}^{\tilde{n}}p^kO_i^k \tag{122}$$

Since each variable θ_i is binary, we have $\theta_i = (\theta_i)^2, i = \tilde{n}+1,\ldots,n$. Therefore, we can replace each term $\theta_i(x_i)^2$ in (122) by $(\theta_i)^2(x_i)^2 = (\upsilon_i)^2$ and (122) can be equivalently rewritten as (109), which gives result i) that we set out to prove.

To prove that the continuous relaxations of $\mathbf{LPO^1_{VaR-M2}}$ and $\mathbf{LPO^1_{CVaR-M2}}$ are convex, it is enough to show that each function

$$\sum_{t=1}^{T_i}\left(\gamma_{i,t}^k\delta^t\left(\frac{\upsilon_i}{q_i}-\frac{\tau(\upsilon_i)^2}{\nu q_i}\right)+\frac{\nu\theta_i-\tau\upsilon_i}{\nu}(1-L_i+L_i\gamma_{i,T_i}^k)\delta^{t_i^*}\right)-\theta_i+\frac{\tau\upsilon_i}{\nu}\ ,\ i=\tilde{n}+1,\ldots,n \tag{123}$$

is concave, since the sum of concave functions is a concave function itself and the feasible sets of $\mathbf{LPO^1_{VaR-M2}}$ and $\mathbf{LPO^1_{CVaR-M2}}$ are convex.

The Hessian matrix of (123) is

$$\mathbb{H} = \begin{Bmatrix} 0 & 0 \\ 0 & -2\sum_{t=1}^{T_i}\frac{\gamma_{i,t}^k\delta^t\tau}{\nu q_i} \end{Bmatrix}.$$

Since all the parameters in $\sum_{t=1}^{T_i} \frac{\gamma_{i,t}^k \delta^t \tau}{\nu q_i}$ are positive, the non-zero principal minor determinants of $\mathbb{H}$ have all the sign of $(-1)^k$, where k is the number of rows (and columns). Hence, the Hessian matrix $\mathbb{H}$ is negative semi-definite, thereby indicating that each function (123) and the objective function are concave, which proves ii). $\blacksquare$