Lyapunov rank of polyhedral positive operators

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Abstract

If K is a closed convex cone and if L is a linear operator having $L(K) \subseteq K$, then L is a *positive operator* on K and L preserves inequality with respect to K. The set of all positive operators on K is denoted by $\pi(K)$. If K^* is the dual of K, then its *complementarity set* is

$$C(K) \coloneqq \{(x, s) \in K \times K^* \mid \langle x, s \rangle = 0\}.$$

Such a set arises as optimality conditions in convex optimization, and a linear operator L is *Lyapunov-like* on K if $\langle L(x), s \rangle = 0$ for all $(x, s) \in C(K)$. Lyapunov-like operators help us find elements of $C(K)$, and the more linearly-independent operators we can find, the better. The set of all Lyapunov-like operators on K forms a vector space and its dimension is denoted by $\beta(K)$.

The number $\beta(K)$ is the *Lyapunov rank* of K, and it has been studied for many important cones. The set $\pi(K)$ is itself a cone, and it is natural to ask if $\beta(\pi(K))$ can be computed, possibly in terms of $\beta(K)$ itself. The problem appears difficult in general. We address the case where K is both proper and polyhedral, and show that $\beta(\pi(K)) = \beta(K)^2$ in that case.

1 Introduction

Lyapunov rank was introduced by Rudolf, Noyan, Papp, and Alizadeh [12] under the name *bilinearity rank*. Their goal was to quantify the ease with which optimality conditions can be decomposed into a system of equations. One motivating example for this decomposition is the standard linear program in $\mathbb{R}^n$.

Example 1. A linear program consists of a linear objective function and a system of linear constraints. In the primal problem, we are asked to

$$\text{minimize } \langle b, x \rangle \text{ subject to } L(x) \geq c \text{ and } x \geq 0.$$

This problem has an associated dual, to

$$\text{maximize } \langle c, y \rangle \text{ subject to } L^*(y) \leq b \text{ and } y \geq 0.$$

The dual optimal value exists and equals that of the primal under certain conditions. If $(\bar{x}, \bar{y})$ is a primal-dual pair of solutions, then $\langle L(\bar{x}) - c, \bar{y} \rangle = 0$. This requirement is called *complementary slackness*. The slackness condition can be decomposed by noting that $\langle L(\bar{x}) - c, \bar{y}\rangle = 0$ if and only if $(L(\bar{x}) - c)_i \, \bar{y}_i = 0$ for $i = 1, 2, \ldots, n$. The resulting system of n equations is easier to solve than the single equation $\langle L(\bar{x}) - c, y\rangle = 0$.

In our linear program, the condition $x \geq 0$ says that x belongs to the proper cone $\mathbb{R}^n_+$. The ease with which $\langle L(\bar{x}) - c, \bar{y}\rangle = 0$ can be decomposed in that case turns out to be a property of the cone $\mathbb{R}^n_+$. Rudolf et al. consider whether or not there are other cones possessing the same property. If K is a proper cone with dual K^* in some finite-dimensional real Hilbert space, then the set of pairs satisfying complementary slackness in Example 1 has a generalization called the *complementarity set* of K, defined as

$$C(K) \coloneqq \{(x, s) \in K \times K^* \mid \langle x, s\rangle = 0\}.$$

Membership in $C(K)$ is a condition for optimality in some convex optimization and complementarity problems [7]. We say that a linear operator L is *Lyapunov-like* on K if $\langle L(x), s\rangle = 0$ for all $(x, s) \in C(K)$. The Lyapunov-like operators provide a general method for decomposing the condition $(x, s) \in C(K)$ into a system of equations, as we did with complementary slackness. The dimension of the space of all Lyapunov-like operators is called the *Lyapunov rank* of K. Lyapunov rank measures the number of independent equations that we can obtain from the condition $(x, s) \in C(K)$.

Example 2. In Example 1, the minimization or maximization takes place over the nonnegative orthant $\mathbb{R}^n_+$. If $\{E_{ij}\}_{i,j=1}^n$ is the standard basis in $\mathbb{R}^{n\times n}$, then E_{ij} is Lyapunov-like on $\mathbb{R}^n_+$ if and only if $i = j$. The span of said E_{ii} is the space of diagonal matrices. Write the identity matrix as $I = E_{11} + E_{22} + \cdots + E_{nn}$ and substitute; the complementary slackness condition $\langle I(L(\bar{x}) - c), \bar{y}\rangle = 0$ produces a system of equations $(L(\bar{x}) - c)_i \, \bar{y}_i = 0$ for $i = 1, 2, \ldots, n$.

Gowda and Tao [7] showed that the space of all Lyapunov-like operators on K is the Lie algebra of the automorphism group of K. Another reason for studying Lyapunov-like operators is thus as a means to understanding the automorphism groups of cones. The Lyapunov rank has been computed for a growing number of cones: the moment cone [12], symmetric cones [7], completely-positive and copositive cones [6], special Bishop-Phelps cones [8], and extended second-order cones [15]. An upper bound is known for all proper cones [10].

Our focus will be on the cone of *positive operators*. If L is linear with $L(K) \subseteq K$, then L is a positive operator on K. Positive operators arose from the study of integral operators and matrices with nonnegative entries [1]; they preserve inequality with respect to a cone. The famous Krein-Rutman theorem extends Perron-Frobenius and connects positive operators to the theory of dynamical systems [13], to game theory [5], and more.

Example 3. If K is $\mathbb{R}^n_+$ and if $L \in \mathbb{R}^{n\times n}$ with $L(K) \subseteq K$, then one can consider the action of L on the standard basis to show that L has nonnegative entries. Such matrices are precisely the positive operators on $\mathbb{R}^n_+$.

The set of all positive operators on K is denoted by $\pi(K)$. If K is a closed convex cone, then $\pi(K)$ is itself a closed convex cone and one can consider the Lyapunov rank of $\pi(K)$. Positive operators are difficult to characterize in general. Computing the Lyapunov rank of $\pi(K)$ also appears to be problematic without additional assumptions, so we restrict our attention to proper polyhedral K. This represents a generalization of what is known for $\mathbb{R}^n_+$.

2 Preliminaries

In what follows, V and W will always be finite-dimensional real Hilbert spaces, and K and H will always be closed convex cones in V or W.

Definition 1. A nonempty subset K of V is a *cone* if $\lambda K = K$ for all $\lambda \geq 0$. A *closed convex cone* is a cone that is closed and convex as a subset of V. The *conic hull* of a nonempty subset X of V is

$$\operatorname{cone}(X) := \left\{ \sum_{i=1}^{m} \alpha_i x_i \;\middle|\; x_i \in X,\ \alpha_i \geq 0,\ m \in \mathbb{N} \right\}.$$

If $K = \operatorname{cone}(X)$ for some finite set X, then K is *polyhedral*.

Definition 2. The dimension of $K \subseteq V$ is $\dim(K) := \dim(\operatorname{span}(K))$. A convex cone K is *solid* if $\operatorname{span}(K) = V$, and *pointed* if $-K \cap K = \{0\}$. A pointed, solid, and closed convex cone is *proper*.

We prove our main result by decomposing a reducible cone into a direct sum of irreducible cones. Beware that the terms "decomposable" and "indecomposable" are used by various authors as synonyms for "reducible" and "irreducible."

Definition 3. A closed convex cone K is *reducible* if $K = K_1 + K_2$ where K_1 and K_2 are nonzero closed convex cones such that $\operatorname{span}(K_1) \cap \operatorname{span}(K_2) = \{0\}$. A cone is *irreducible* if it is not reducible. We will use the direct sum notation $K = K_1 \oplus K_2$ for reducible cones.

Our definition of reducibility is due to Gowda and Tao [7]. Barker and Loewy define decomposability slightly differently [2], not requiring K_1 and K_2 to be closed convex cones. However if $K = K_1 \oplus K_2$ for nonzero nonempty K_1 and K_2, then $K = \operatorname{cone}(K_1) \oplus \operatorname{cone}(K_2)$. Thus the definitions are equivalent.

The set of all linear operators from V to W is $\mathcal{B}(V, W)$, and we abbreviate $\mathcal{B}(V, V)$ by $\mathcal{B}(V)$. The identity operator on V is $\mathrm{id}_V \in \mathcal{B}(V)$. Given $x \in V$ and $s \in W$, we define $s \otimes x \in \mathcal{B}(V, W)$ as the map $y \mapsto \langle x, y \rangle s$. If $F \in \mathcal{B}(W)$ and $G \in \mathcal{B}(V)$, then we define $F \odot G \in \mathcal{B}(\mathcal{B}(V, W))$ to be the map $s \otimes x \mapsto F(s) \otimes G(x)$. We will use the shorthand notation $S \otimes X$ or $S \odot X$ on sets S and X to mean $\{s \otimes x \mid s \in S, x \in X\}$ or $\{s \odot x \mid s \in S, x \in X\}$. Any $L \in \mathcal{B}(V, W)$ has an adjoint $L^* \in \mathcal{B}(W, V)$ such that $\langle L(x), y \rangle = \langle x, L^*(y) \rangle$ for all $x \in V$ and $y \in W$. The adjoint of $s \otimes x$ is $x \otimes s$ for vectors $x \in V$ and $s \in W$. We adopt the trace inner-product $\langle L_1, L_2 \rangle := \operatorname{trace}(L_1 L_2^*)$ on $\mathcal{B}(V)$, and "trace" can be taken to mean "sum of eigenvalues." To simplify the notation, composition of linear operators is indicated by juxtaposition. An invertible linear operator that preserves inner-products is an *isometry*.

Definition 4. The operator $L \in \mathcal{B}(V)$ is a *positive operator* on K if $L(K) \subseteq K$. The set of all such operators is denoted by $\pi(K)$. To generalize, we allow that $L \in \mathcal{B}(V, W)$, and that $L(K) \subseteq H$ for subsets $K \subseteq V$ and $H \subseteq W$. The set of all such operators is $\pi(K, H)$, and $\pi(K)$ is the special case where $H = K$.

Definition 5. If K is a subset of V, then the *dual cone* K^* of K is

$$K^* \coloneqq \{y \in V \mid \langle x, y\rangle \geq 0 \text{ for all } x \in K\}.$$

The *complementarity set* of K is $C(K) \coloneqq \{(x, s) \in K \times K^* \mid \langle x, s\rangle = 0\}$ and $L \in \mathcal{B}(V)$ is *Lyapunov-like* on K if $\langle L(x), s\rangle = 0$ for all $(x, s) \in C(K)$. By $\mathbf{LL}(K)$ we denote the set of all Lyapunov-like operators on K. The *Lyapunov rank* of K is $\beta(K) \coloneqq \dim(\mathbf{LL}(K))$.

We will briefly discuss a third family, the completely-positive operators.

Definition 6. If K is a closed cone, then $\mathbf{CP}(K) \coloneqq \operatorname{cone}(\{x \otimes x \mid x \in K\})$ is the *completely-positive* cone of K.

The completely-positive cone of K is proper when K is solid [6]. Its dual $\mathbf{CP}(K)^*$ is the cone of *copositive* operators that appears in nonconvex-quadratic and combinatorial optimization. Both contain only self-adjoint operators.

Definition 7. A nonempty convex subset F of a convex cone K is a *face* of K if $x, y \in K$ and $\alpha x + (1 - \alpha) y \in F$ for $0 < \alpha < 1$ together imply that $x, y \in F$. If in addition $\dim(F) = 1$, then F is an *extreme ray* of K. The *extreme directions* of K are representatives of its extreme rays defined by,

$$\operatorname{Ext}(K) \coloneqq \{x \mid x \text{ belongs to an extreme ray of } K \text{ and } \|x\| = 1\}.$$

If K is a proper cone, then $K = \operatorname{cone}(\operatorname{Ext}(K))$ by a conic version of the Krein-Milman theorem—Fenchel's Theorem 14, for example [4]. It then follows that K is polyhedral if and only if $\operatorname{Ext}(K)$ is finite. Moreover, we need only consider the elements of $\operatorname{Ext}(K)$ and $\operatorname{Ext}(K^*)$ to show that $L \in \mathbf{LL}(K)$ [12].

3 Positive operators

The goal in this section is to compute the Lyapunov rank of $\pi(K)$ when K is a proper polyhedral cone. To motivate this, we will see what happens when K is the nonnegative orthant $\mathbb{R}^n_+$.

Example 4. We showed in Example 2 that $\mathbf{LL}\left(\mathbb{R}^n_+\right)$ is the space of all diagonal matrices in $\mathbb{R}^{n \times n}$. As a result, $\beta\left(\mathbb{R}^n_+\right) = n$. We saw in Example 3 that $\pi\left(\mathbb{R}^n_+\right)$ is the set of nonnegative matrices in $\mathbb{R}^{n \times n}$. There is an obvious isometry between $\pi\left(\mathbb{R}^n_+\right)$ and $\mathbb{R}^{n^2}_+$, so it follows that $\beta\left(\pi\left(\mathbb{R}^n_+\right)\right) = \beta\left(\mathbb{R}^{n^2}_+\right) = n^2 = \beta\left(\mathbb{R}^n_+\right)^2$.

We will relax two restrictions in the previous example. The cone $\mathbb{R}^n_+$ is self-dual and *simplicial*—it has exactly $\dim(\mathbb{R}^n)$ extreme directions. By extending the result to a proper polyhedral cone K in V, we allow for $K \neq K^*$, and for K to possess more than $\dim(V)$ extreme directions.

To compute $\beta(\pi(K))$, we will ultimately need to find the more-general quantity $\beta(\pi(K,H))$. Some features of $\pi(K,H)$ depend on those of K and H.

Proposition 1 (Schneider and Vidyasagar [14])**.** *If K and H are proper (polyhedral) cones in finite-dimensional real Hilbert spaces V and W respectively, then $\pi(K,H)$ is a proper (polyhedral) cone in $\mathcal{B}(V,W)$.*

It therefore makes sense to consider the Lyapunov rank of $\pi(K,H)$. If $\beta(\pi(K,H))$ can be expressed in terms of K and H, then $\beta(\pi(K))$ is obtained when $H = K$. However, the cone $\pi(K,H)$ is unwieldy; its dual $\pi(K,H)^*$ is more tractable and the extreme directions of that dual are known.

Proposition 2 (Berman and Gaiha [3])**.** *If K and H are proper polyhedral cones in finite-dimensional real Hilbert spaces, then*

$$\operatorname{Ext}\left(\pi(K,H)^*\right) = \operatorname{Ext}(H^*) \otimes \operatorname{Ext}(K).$$

When we compute $\pi(K,H)$, we would like to be able to assume that it is irreducible at first. In Theorem 1, we will prove that $\pi(K,H)$ is irreducible if both K and H are irreducible. The converse of that statement is known, and we are free to work with the dual of $\pi(K,H)$ instead.

Proposition 3 (Haynsworth, Fiedler, and Pták [9])**.** *If K and H are proper cones in finite-dimensional real Hilbert spaces and if either K or H is reducible, then $\pi(K,H)$ is reducible.*

Proposition 4 (Barker and Loewy [2])**.** *If K is a proper cone in some finite-dimensional real Hilbert space, then K is reducible if and only if K^* is reducible.*

The proof of our first theorem is a straightforward adaptation of Barker and Loewy's Lemma 2.2 to the case where $K \neq H$.

Theorem 1. *If K and H are proper cones in finite-dimensional real Hilbert spaces, then $\pi(K,H)$ is reducible if and only if either K or H is reducible.*

Proof. One implication is given by Proposition 3. For the other, it suffices by Proposition 4 to show that if $\pi(K,H)^*$ is reducible, then either K or H is reducible. So, suppose that

$$\pi(K,H)^* = \operatorname{cone}\left(\operatorname{Ext}\left(\pi(K,H)^*\right)\right) = \Delta_1 \oplus \Delta_2$$

where Δ_1 and Δ_2 satisfy the conditions in Definition 3. As a result,

$$x \in \operatorname{Ext}(K) \text{ and } s \in \operatorname{Ext}(H^*) \implies s \otimes x \in \Delta_i \text{ for a unique } i. \qquad (\star)$$

The implication ($\star$) follows from Proposition 2 which shows that for the given x and s we have $s \otimes x \in \operatorname{Ext}\left(\pi\left(K,H\right)^{*}\right)$. Definition 7 combined with the linear-independence of Δ_1 and Δ_2 shows that $s \otimes x$ cannot be a nontrivial sum. It follows that $s \otimes x \in \Delta_1 + \Delta_2$ belongs to exactly one of the Δ_i.

One consequence of ($\star$) is that both Δ_1 and Δ_2 must contain at least one element of $\operatorname{Ext}\left(\pi\left(K,H\right)^{*}\right)$. If not, then, for example, $\pi\left(K,H\right)^{*} \subseteq \Delta_1$ and $\Delta_2 = \{0\}$ contradicting Definition 3. Define functions

$$S_i\left(s\right) \coloneqq \{x \in \operatorname{Ext}\left(K\right) \mid s \otimes x \in \Delta_i\} \text{ for } i \in \{1,2\}$$

and consider the two possible cases.

Case 1: there exists an $\bar{s} \in \operatorname{Ext}\left(H^{*}\right)$ with both $S_1\left(\bar{s}\right)$ and $S_2\left(\bar{s}\right)$ nonempty**.**

Apply ($\star$) to any $\bar{s} \otimes x$ with $x \in \operatorname{Ext}\left(K\right)$ to show that $x \in S_1\left(\bar{s}\right) \cup S_2\left(\bar{s}\right)$. It follows that $S_1\left(\bar{s}\right) \cup S_2\left(\bar{s}\right) = \operatorname{Ext}\left(K\right)$. Define $F_i \coloneqq \operatorname{cone}\left(S_i\left(\bar{s}\right)\right)$. Then,

$$\operatorname{cone}\left(S_1\left(\bar{s}\right) \cup S_2\left(\bar{s}\right)\right) \subseteq \operatorname{cone}\left(F_1 + F_2\right) = F_1 + F_2 \subseteq K.$$

We have $\operatorname{Ext}\left(K\right) = S_1\left(\bar{s}\right) \cup S_2\left(\bar{s}\right)$, so it follows that $\operatorname{cone}\left(S_1\left(\bar{s}\right) \cup S_2\left(\bar{s}\right)\right) = K$ and thus that $F_1 + F_2 = K$.

Take any $z \in \operatorname{span}\left(F_1\right) \cap \operatorname{span}\left(F_2\right)$. Each F_i is a convex cone, so $\operatorname{span}\left(F_i\right) = F_i - F_i$, and thus $z = z_1 - z_2 = w_1 - w_2$ for some $z_1, z_2 \in F_1$ and $w_1, w_2 \in F_2$. Expand $\bar{s} \otimes z$ to $\bar{s} \otimes z_1 - \bar{s} \otimes z_2$ and write $z_1 \in F_1 \coloneqq \operatorname{cone}\left(S_1\left(\bar{s}\right)\right)$ as $z_1 = \sum \alpha_j \sigma_j$ where $\alpha_j \geq 0$ and $\sigma_j \in S_1\left(\bar{s}\right)$. Expand $\bar{s} \otimes z_1$ to $\sum \alpha_j \left(\bar{s} \otimes \sigma_j\right)$. Each $\bar{s} \otimes \sigma_j$ belongs to Δ_1 by the definition of $S_1\left(\bar{s}\right)$, and since Δ_1 is a convex cone, the sum $\bar{s} \otimes z_1$ is also in Δ_1. A similar procedure shows that $\bar{s} \otimes z_2 \in \Delta_1$. Now,

$$\bar{s} \otimes z = \bar{s} \otimes z_1 - \bar{s} \otimes z_2 \in \Delta_1 - \Delta_1 = \operatorname{span}\left(\Delta_1\right).$$

Repeat the procedure with $z = w_1 - w_2$ to show that $\bar{s} \otimes z \in \operatorname{span}\left(\Delta_2\right)$.

The spans of Δ_1 and Δ_2 intersect trivially, so $\bar{s} \otimes z = 0$. But $\bar{s} \in \operatorname{Ext}\left(H^{*}\right)$ is nonzero (it has unit norm), so we must have $z = 0$. Since $z \in \operatorname{span}\left(F_1\right) \cap \operatorname{span}\left(F_2\right)$ was arbitrary, those two spaces have trivial intersection, and the sum $K = F_1 \oplus F_2$ is in fact a direct sum showing that K is reducible.

Case 2: either $S_1\left(s\right)$ or $S_2\left(s\right)$ is empty for all $s \in \operatorname{Ext}\left(H^{*}\right)$**.**

In this case, we will show that H^{*} is reducible. Define two new sets,

$$T_i \coloneqq \{s \in \operatorname{Ext}\left(H^{*}\right) \mid S_i\left(s\right) = \emptyset\} \text{ for } i \in \{1,2\}.$$

If T_1 is empty, then $S_1\left(s\right) \neq \emptyset$ for all $s \in \operatorname{Ext}\left(H^{*}\right)$. But then by assumption we have $S_2\left(s\right) = \emptyset$ for all $s \in \operatorname{Ext}\left(H^{*}\right)$, and thus $\Delta_2 = \{0\}$ which is not possible. It must therefore be the case that T_1 and (by the same reasoning) T_2 are nonempty.

Define $G_i \coloneqq \operatorname{cone}\left(T_i\right)$. Every $y \in \operatorname{Ext}\left(H^{*}\right)$ belongs to at least one of the T_i by construction; thus $\operatorname{Ext}\left(H^{*}\right) = T_1 \cup T_2$. As in the first case,

$$\operatorname{cone}\left(T_1 \cup T_2\right) \subseteq \operatorname{cone}\left(G_1 + G_2\right) = G_1 + G_2 \subseteq H^{*}.$$

Along with the fact that $\operatorname{Ext}(H^*) = T_1 \cup T_2$, this shows that $H^* = G_1 + G_2$.

Fix an $\bar{x} \in \operatorname{Ext}(K)$ and let $w \in \operatorname{span}(G_1) \cap \operatorname{span}(G_2)$ be arbitrary. Write $w = w_1 - w_2 = z_1 - z_2$ for $w_1, w_2 \in G_1$ and $z_1, z_2 \in G_2$. Expand $w \otimes \bar{x}$ to $w_1 \otimes \bar{x} - w_2 \otimes \bar{x}$, and write $w_1 \in G_1 \coloneqq \operatorname{cone}(T_1)$ as $w_1 = \sum \alpha_j \tau_j$ for $\alpha_j \geq 0$ and $\tau_j \in T_1$. Expand $w_1 \otimes \bar{x}$ to $\sum \alpha_j (\tau_j \otimes \bar{x})$, and notice that no $\tau_j \otimes \bar{x}$ can belong to Δ_1 by definition of T_1 and $S_1(\tau_j)$. Consequently each $\tau_j \otimes \bar{x}$ belongs to the convex cone Δ_2 by ($\star$), and the sum $w_1 \otimes \bar{x}$ does too. The same reasoning shows that $w_2 \otimes \bar{x} \in \Delta_2$, and thus that $w = w_1 \otimes \bar{x} - w_2 \otimes \bar{x} \in \operatorname{span}(\Delta_2)$.

Repeat the argument with $w = z_1 - z_2$ to find that $w \otimes \bar{x} \in \operatorname{span}(\Delta_1)$ as well. Deduce that $w = 0$, that $\operatorname{span}(G_1) \cap \operatorname{span}(G_2) = \{0\}$, and finally that $H^* = G_1 \oplus G_2$ is reducible. Proposition 4 shows that H is reducible. □

The Lyapunov rank of an irreducible proper polyhedral cone is known, and every proper cone is (in an obvious way) a direct sum of irreducible closed convex cones. Combined with Theorem 1, these two observations form the base case to which we will reduce a general polyhedral proper cone.

Proposition 5 (Gowda and Tao [7])**.** *If K is a proper polyhedral cone in a finite-dimensional real Hilbert space, then $\beta(K) = 1$ if and only if K is irreducible.*

Lemma 1. *If K and H are two irreducible proper polyhedral cones in finite-dimensional real Hilbert spaces, then $\beta(\pi(K, H)) = \beta(K)\beta(H)$.*

Proof. Both K and H are irreducible, so Proposition 5 shows $\beta(K)\beta(H) = 1$. But Proposition 1 and Theorem 1 imply that $\pi(K, H)$ is also an irreducible proper polyhedral cone, and thus $\beta(\pi(K, H)) = 1$ by the same proposition. □

It remains to prove the full result for reducible cones. We will suppose that K and H are reducible, respectively, into m and n components.

Theorem 2. *If K and H are proper polyhedral cones in finite-dimensional real Hilbert spaces, then $\beta(\pi(K, H)) = \beta(K)\beta(H)$.*

Proof. If $K = \oplus_{i=1}^{m} K_i$ and $H = \oplus_{j=1}^{n} H_j$ satisfy Definition 3 with K_i and H_j irreducible, then there exist invertible linear operators A and B such that $A(K) = K_1 \times K_2 \cdots \times K_m$ and $B(H) = H_1 \times H_2 \times \cdots \times H_n$. It is easy to check that $\pi(A(K), B(H)) = B\pi(K, H)A^{-1}$. The Lyapunov rank is invariant under invertible linear operators [12], so for our purposes, we can disregard A and B everywhere and pretend that $K = K_1 \times K_2 \cdots \times K_m$ and $H = H_1 \times H_2 \times \cdots \times H_n$. This will be beneficial, because Lyapunov rank is additive on a cartesian product of proper cones [12]. By expanding, we find that

$$\beta(K)\beta(H) = \sum_{i=1}^{m} \sum_{j=1}^{n} \beta(K_i)\beta(H_j) = mn. \tag{†}$$

The last equality follows from Lemma 1 and the fact that each K_i and H_j is an irreducible proper polyhedral cone.

It is straightforward to show that any $L \in \pi(K, H)$ has the block form

$$\begin{aligned} &L = [L_{ji}], \text{ where} \\ &L_{ji} : \operatorname{span}(K_i) \to \operatorname{span}(H_j) \\ &L_{ji} \in \pi(K_i, H_j). \end{aligned} \tag{‡}$$

Moreover, any such L clearly satisfies $L \in \pi(K, H)$, so the two conditions are equivalent. Yet every block form operator is itself isometric to a cartesian product; if $L = [L_{ji}]$, then $L \cong L_{11} \times L_{12} \times \cdots \times L_{21} \times L_{22} \times \cdots \times L_{nm}$. Thus the set of all operators having the block form (‡), namely $\pi(K, H)$, is isometric to the cartesian product,

$$\bigtimes_{i=1}^{m} \bigtimes_{j=1}^{n} \pi(K_i, H_j) \cong \pi(K, H).$$

Apply Proposition 1, Theorem 1, and Proposition 5 to conclude in agreement with (†) that

$$\beta(\pi(K, H)) = \beta\left(\bigtimes_{i=1}^{m} \bigtimes_{j=1}^{n} \pi(K_i, H_j)\right) = \sum_{i=1}^{m} \sum_{j=1}^{n} 1 = mn. \qquad \square$$

Corollary 1. *If K is a proper polyhedral cone in a finite-dimensional real Hilbert space, then $\beta(\pi(K)) = \beta(K)^2$.*

Now that we know the dimension of $\mathbf{LL}(\pi(K, H))$, we would like to find a basis for it. Knowing its dimension, it suffices to find $\beta(\pi(K, H))$ linearly-independent Lyapunov-like operators on $\pi(K, H)$.

Theorem 3 (Gowda and Tao [7])**.** *If K is a proper polyhedral cone in a finite-dimensional real Hilbert space, then $L \in \mathbf{LL}(K)$ if and only if every $x \in \operatorname{Ext}(K)$ is an eigenvector of L.*

The extreme directions of $\pi(K, H)$ are not generally known. The next proposition relates the Lyapunov-like operators on a cone to those on its dual, and shows that we can work with whichever one is easier.

Proposition 6 (Rudolf et al. [12])**.** *If K is a closed convex cone, then L is Lyapunov-like on K if and only if L^* is Lyapunov-like on K^*.*

We aim to show that the Lyapunov-like operators on $\pi(K, H)$ are linear combinations of terms like $M \odot L$ where L and M are Lyapunov-like on K^* and H respectively. The next result is well-known [11].

Proposition 7. *If V and W are finite-dimensional real Hilbert spaces and if L and M are subsets of $\mathcal{B}(V)$ and $\mathcal{B}(W)$, then $\dim(L \odot M) = \dim(L)\dim(M)$.*

Lemma 2. *If K and H are proper polyhedral cones in finite-dimensional real Hilbert spaces V and W, then $\operatorname{span}(\mathbf{LL}(H^*) \odot \mathbf{LL}(K)) = \mathbf{LL}\left(\pi(K, H)^*\right)$.*

Proof. Take any $L \in \mathbf{LL}(K)$, $M \in \mathbf{LL}(H^*)$, and $s \otimes x \in \operatorname{Ext}\left(\pi(K,H)^*\right)$. Use Proposition 2 and Theorem 3 to see that $L(x) = \lambda x$ and $M(s) = \mu s$. Thus,

$$(M \odot L)(s \otimes x) = (M(s)) \otimes (L(x)) = \mu\lambda(s \otimes x).$$

Another application of Theorem 3 shows that $M \odot L \in \mathbf{LL}\left(\pi(K,H)^*\right)$. Compare the dimensions of $\mathbf{LL}(H^*) \odot \mathbf{LL}(K)$ and $\mathbf{LL}\left(\pi(K,H)^*\right)$ using Theorem 2 and Proposition 7. Conclude that the two spaces are equal. ☐

Theorem 4. *If K and H are proper polyhedral cones in finite-dimensional real Hilbert spaces, then $\mathbf{LL}(\pi(K,H)) = \operatorname{span}(\mathbf{LL}(H) \odot \mathbf{LL}(K^*))$.*

Proof. The adjoint of $M \odot L$ is $M^* \odot L^*$. That fact, along with Proposition 6 and Lemma 2, shows that $\mathbf{LL}(\pi(K,H)) = \operatorname{span}(\mathbf{LL}(H) \odot \mathbf{LL}(K^*))$. ☐

Corollary 2. *If K is a proper polyhedral cone in a finite-dimensional real Hilbert space, then $\mathbf{LL}(\pi(K)) = \operatorname{span}(\mathbf{LL}(K) \odot \mathbf{LL}(K^*))$.*

Theorem 2 clearly follows from Theorem 4, but the difficulty in determining $\mathbf{LL}(\pi(K,H))$ is to know when you are done—when all Lyapunov-like operators have been found. For that it was convenient to use the dimension of the space.

4 Completely and copositive operators

The positive operators on K are the second family whose Lyapunov rank is known to be expressable in terms of the Lyapunov rank of K itself. Recall from Definition 6 that a linear operator L is completely-positive on the closed cone K if and only if $L = \sum_{i=1}^{m} x_i \otimes x_i$ for some $m \in \mathbb{N}$ and $\{x_i\}_{i=1}^{m} \subseteq K$.

Gowda, Sznajder, and Tao [6] showed that the Lyapunov rank of $\mathbf{CP}(K)$ is the same as that of K itself. The argument makes use of the transformation L_A defined on $\mathcal{B}(V)$ by $L_A(X) \coloneqq AX + XA^*$, and the authors show that the map $A \mapsto L_A$ is a Lie algebra isomorphism from $\mathbf{LL}(K)$ to $\mathbf{LL}(\mathbf{CP}(K))$. Thus the two spaces share the same dimension. We draw a comparison between their results and our Corollary 2. We will first show that there exists a simpler isomorphism between the aforementioned Lie algebras.

Lemma 3. *If K is a proper cone in a finite-dimensional real Hilbert space V and if ℓ_A is the "left-multiplication" operator defined on $\mathcal{B}(V)$ by $\ell_A(X) = AX$, then $A \mapsto \ell_A$ is a Lie algebra isomorphism between $\mathbf{LL}(K)$ and $\mathbf{LL}(\mathbf{CP}(K))$.*

Proof. If $A \in \mathbf{LL}(K)$, then the main result of Gowda et al. shows that $L_A \in \mathbf{LL}(\mathbf{CP}(K))$, where L_A is the transformation $L_A(X) \coloneqq AX + XA^*$. Suppose that $X \in \mathbf{CP}(K)$ and that $S \in \mathbf{CP}(K)^*$ with $\langle X, S\rangle = 0$. Then $L_A \in \mathbf{LL}(\mathbf{CP}(K))$ implies that $\langle L_A(X), S\rangle = \langle AX, S\rangle + \langle XA^*, S\rangle = 0$. By definition we have $\operatorname{trace}(AXS^*) + \operatorname{trace}(XA^*S^*) = 0$. Using basic properties of the trace and the fact that X and S are self-adjoint, deduce that $\operatorname{trace}(AXS^*) = 0$. It follows that $\langle \ell_A(X), S\rangle = 0$, or that ℓ_A is Lyapunov-like on $\mathbf{CP}(K)$.

We have shown that $A \mapsto \ell_A$ takes $\mathbf{LL}(K)$ to $\mathbf{LL}(\mathbf{CP}(K))$. It is clearly linear and injective, and surjectivity follows from the fact that $\dim(\mathbf{LL}(K)) = \dim(\mathbf{LL}(\mathbf{CP}(K)))$. The Lie bracket is easily seen to be preserved. $\square$

Theorem 5. *If K is a proper cone in a finite-dimensional real Hilbert space V, then $\mathbf{LL}(\mathbf{CP}(K)) = \mathbf{LL}(K) \odot \{\mathrm{id}_V\}$.*

Proof. If $\{v_i\}$ is a basis for V, then $\{v_j \otimes v_i\}$ forms a basis for $\mathcal{B}(V)$, and $\mathbf{LL}(K) \odot \{\mathrm{id}_V\} = \ell_{\mathbf{LL}(K)} = \mathbf{LL}(\mathbf{CP}(K))$ by Lemma 3. $\square$

From Theorem 5, one can also easily deduce $\beta(\mathbf{CP}(K)) = \beta(K)$ without reference to an isomorphism. Compare this result with Corollary 2 where the restriction to polyhedral cones is a serious one. If we try that corollary with nonpolyhedral K, then Theorem 3 fails and it is not clear why $\mathbf{LL}(K) \odot \mathbf{LL}(K^*)$ should be Lyapunov-like on $\pi(K)$. However, it is just as unclear from Definition 5 why $\mathbf{LL}(K) \odot \{\mathrm{id}_V\}$ should be Lyapunov-like on $\mathbf{CP}(K)$.

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