On Matroid Parity and Matching Polytopes*

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Abstract

The *matroid parity* (MP) problem is a natural extension of the matching problem to the matroid setting. It can be formulated as a $0-1$ linear program using the so-called *rank* and *line* constraints. We call the associated family of polytopes *MP polytopes*. We then prove the following: (i) when the matroid is a *gammoid*, each MP polytope is a projection of a perfect matching polytope into a suitable subspace; (ii) when the matroid is *laminar*, each MP polytope is affinely congruent to a perfect matching polytope; (iii) even if the matroid is laminar, MP polytopes can have facets that are defined by inequalities with non-ternary left-hand side coefficients; (iv) for any matroid, the elementary closure of the continuous relaxation of the rank-and-line formulation is equal to its $\{0, \frac{1}{2}\}$-closure.

1 Introduction

Matchings and matroids have played a key role in combinatorial optimisation right from the early days of the subject. In particular, Edmonds' seminal works on polyhedra associated with matchings and matroids [5, 6] essentially marked the birth of what is now known as polyhedral combinatorics (see, e.g., [3]). Moreover, the *matroid parity* problem, an elegant generalisation of the matching and matroid intersection problems, has been studied in depth (e.g., [10, 12, 13, 14, 15]). On the other hand, the polyhedra associated with the matroid parity problem have received far less attention. These polyhedra, and their connections with matching polyhedra, are the subject of this paper.

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Before explaining our contribution, we will need some definitions. A *c-capacitated b-matching* in an undirected graph $G(V, E)$ is a vector $x \in \mathbb{Z}_+^E$ such that

$$x(\delta(i)) \le b_i \qquad \forall\, i \in V, \tag{1}$$
$$0 \le x_e \le c_e \qquad \forall\, e \in E, \tag{2}$$

where, as usual, $\delta(i)$ denotes the set of edges incident on node i and, for any $E' \subset E$, we let $x(E')$ denote $\sum_{e \in E'} x_e$. The associated *c-capacitated b-matching polytope* is defined as

$$\mathscr{P}_{b,c} = \text{conv}\left\{x \in \mathbb{Z}_+^E : \ (1), (2) \text{ hold}\right\}.$$

One can optimise linear functions over $\mathscr{P}_{b,c}$ in polynomial time [4, 8]. Edmonds [5] and Pulleyblank [18] showed that a linear description of $\mathscr{P}_{b,c}$ is given by (1), (2) and the *blossom* inequalities

$$x(E(H)) + x(T) \le \left\lfloor \tfrac{b(H)+c(T)}{2} \right\rfloor, \forall H \subset V, T \subseteq \delta(H) : b(H) + c(T) \text{ odd}, \tag{3}$$

where $E(H)$ denotes the set of edges with both end-nodes in H, and $b(H)$ and $c(T)$ denote $\sum_{i \in H} b_i$ and $\sum_{e \in T} c_e$, respectively.

A c-capacitated b-matching is *perfect* if it satisfies the constraints (1) at equality. We will also call the associated polytope *perfect*.

A *matroid* $M(F, \mathcal{I})$ is a collection $\mathcal{I}$ of *independent* subsets of a ground set F that is downward closed and also satisfies the *exchange property*: for any two subsets in $\mathcal{I}$ of different size, there is an element in the larger subset that can be added to the smaller one while preserving independence. The convex hull of the incidence vectors $x \in \{0, 1\}^F$ of subsets in $\mathcal{I}$ is the *matroid polytope*; its linear description is given [6] by the non-negativity inequalities and the *rank inequalities*

$$x(S) \le r_M(S) \qquad \forall S \subseteq F, \tag{4}$$

where

$$r_M(S) = \max_{A \subseteq S} \{|A| : A \in \mathcal{I}\}.$$

is the rank function of $M(F, \mathcal{I})$. One can optimise linear functions over the matroid polytope in polynomial time [6].

The *matroid parity* (MP) problem [12] assumes a matroid M with a ground set F of even cardinality, along with a partition of F into two-element subsets called *lines* and indexed by $\mathcal{L}$; without loss of generality, each line forms an independent set. The problem calls for a set of lines of maximum cardinality (or weight) such that their union is independent in M. To define the associated polytope, we initially consider the incidence vectors of both independent sets $x \in \{0,1\}^F$ and lines $y \in \{0,1\}^{\mathcal{L}}$, where (x, y) must satisfy the rank inequalities (4) plus the *line constraints*

$$x_i = x_j = y_{ij} \qquad \forall \{i,j\} \in \mathcal{L}. \tag{5}$$

Using (5) to project out the x variables, we derive the *projected rank* inequalities, apparently first discovered by Vande Vate [21]:

$$\sum_{\ell \in \mathcal{L}} |S \cap \ell| \, y_\ell \leq r_M(S) \qquad \forall \, S \subseteq F. \tag{6}$$

The *matroid parity (MP) polytope* is then:

$$\mathscr{P}_{M,\mathcal{L}} \quad = \quad \text{conv}\left\{ y \in \{0,1\}^{\mathcal{L}} : (6) \text{ hold} \right\}.$$

Optimising linear functions over $\mathscr{P}_M$ is difficult in general. It is $\mathcal{NP}$-hard when the matroid M has a compact description [13] and, if M is given only implicitly via an independence oracle, it can take exponential time [11]. It can however be done efficiently under certain conditions, e.g., if M is given by a linear representation [10, 19], or if M is a *gammoid* [20]. Moreover, Lee *et al.* [15] have provided some nice approximation results.

In this paper, then, we investigate MP polytopes and their connections to matching polytopes. Specifically, we prove the following: (i) when M is a *gammoid*, the MP polytope is a projection of a perfect matching polytope into a suitable subspace; (ii) when M is *laminar*, the MP polytope is affinely congruent to a perfect matching polytope; (iii) even if M is laminar, the MP polytope can have facets that are defined by inequalities with non-ternary left-hand side coefficients; (iv) for any M, the *elementary closure* of the polytope defined by non-negativity and projected rank inequalities is equal to its $\{0, \frac{1}{2}\}$*-closure* (all terms in italics will be defined later on).

The rest of the paper has a simple structure. Section 2 presents some additional background information, notation and terminology. The remaining four sections present the four results mentioned above, occasionally presenting minor additional results along the way.

2 Background

2.1 Matroids

Most matroids of interest can be represented by a matrix having one column per element $f \in F$ and the property that $S \in \mathcal{I}$ if and only if the corresponding columns are linearly independent in some field. Such matroids are called *linear* and include among others the subclasses portrayed in Figure 1 (a quite similar figure appears in [9]).

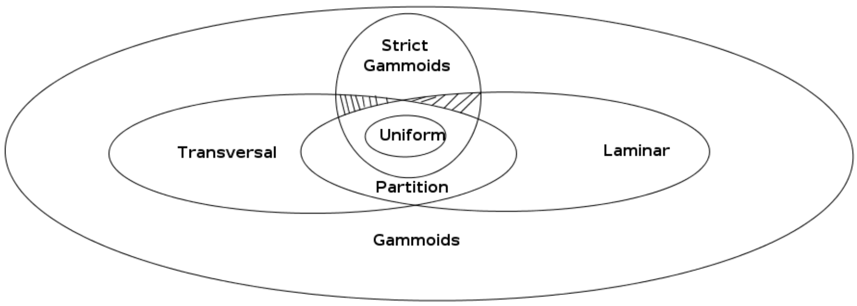


Figure 1: Representation of the inclusion relationships among some classes of linear matroids. Note that areas do not represent the relative size of the corresponding sets, while shaded parts represent empty sets.

Definition 1 *Given a set system* $(F, \mathcal{A})$ *where* $\mathcal{A} = \{A_j \subset F : j \in J\}$*, a 'transversal' of* $\mathcal{A}$ *is a set* $S \subset F$ *such that there exists a bijection* $\pi : J \to S$ *with* $\pi(j) \in A_j$ *for all* $j \in J$*.*

A transversal defined over a subset of $\mathcal{A}$ is called a *partial transversal.*

Definition 2 *The collection of partial transversals of* $\mathcal{A}$ *forms the independent set* $\mathcal{I}$ *of a matroid with ground set* F*, called 'transveral matroid'.*

In what follows we write $M[\mathcal{A}]$ to denote any such matroid.

It is known that transversals can be associated with matchings in a bipartite graph $G(V_1, V_2, E)$ such that V_1 and V_2 have one node per member of F and $\mathcal{A}$ respectively, while $E = \{e_{ij} : i \in A_j, i \in V_1, j \in V_2\}$. It becomes easy to see that $S \subset F$ is a partial transversal of $(F, \mathcal{I})$ if and only if it defines a matching in G.

Example 1 (based on [22]). Let $F = \{1, \ldots, 6\}$ and $\mathcal{A} = \{A_1, A_2, A_3, A_4\}$ where $A_1 = \{1, 2\}$, $A_2 = \{1, 2, 3, 4\}$, $A_3 = \{3, 4, 5\}$ and $A_4 = \{3, 5, 6\}$. Then $M[\mathcal{A}]$ is a transversal matroid. The set $S = \{1, 3, 4, 6\}$ is a transversal of $\mathcal{A}$ while $S' = \{3, 4, 5\}$ is a partial transversal. Figure 2a shows the bipartite graph associated with $M[\mathcal{A}]$.

Let us extend the relation of transversals to matchings to include also *linkings*. For a digraph $G(V, A)$ and two subsets S, T of V (not necessarily distinct) with $|S| = |T|$, we say that S can be linked *onto* T if there are $|S|$ node-disjoint paths in G with sources in S and sinks in T; S is linked *into* T if it is linked onto some subset of T. Observe that such paths might consist of a single node hence allowing S to be linked onto itself.

Let us define the node sets $N_v = v \cup \{u \in V : (v, u) \in S\}$ for all $v \in V$ and the set family $\mathcal{N}_S = \{N_v : v \in S\}$ for any $S \subseteq V$. That is, N_v is the node-set of the star graph having $v \in V$ as internal node and $u \in V$ such that $(v, u) \in A$ as 'leaves'. The relationship between linkings and transversals arises from the *linkage lemma* of Ingleton and Piff [18]: for any two node-subsets S and T of a digraph $G(V, A)$, S is linked into T if and only if $V \setminus S$ is a transversal of $\mathcal{N}_{V \setminus T}$.

A set $S \subseteq F$ is a transversal of some set family over F if and only if S defines at least one matching in the associated bipartite graph. In general then many matchings might correspond to the same transversal and thus given some transversal $V \setminus S$ of $\mathcal{N}_{V \setminus T}$, there might be alternative ways to link $V \setminus S$ onto $\mathcal{N}_{V \setminus T}$. The implication concerning the digraph $G(V, A)$ is that there might be alternative sets of node-disjoint paths from S to T. Therefore, the mapping from linkings to matchings is undefined.

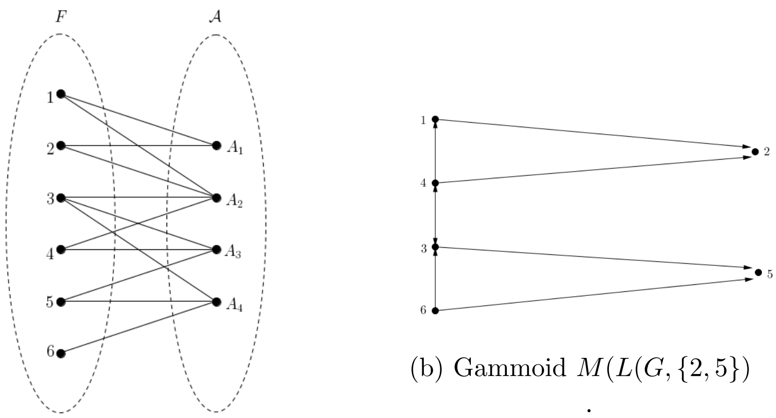
Example 2 Consider the digraph given in Figure 2b. The set $\{4, 6\}$ is linked *onto* $\{2, 5\}$ because of the node-disjoint paths $\{4, 1, 2\}$ and $\{6, 3, 5\}$; an alternative set of node-disjoint paths is $\{4, 2\}$ and $\{6, 5\}$. The set $\{4, 6\}$ is linked *into* $\{1, 2, 5\}$ because there exist node-disjoint paths linking $\{4, 6\}$ onto a subset of $\{1, 2, 5\}$. Based on the linkage lemma, the linking $\{4, 6\}$ onto $\{2, 5\}$ exists if and only if $\{1, 2, 3, 5\}$ is a transversal of the set family $\mathcal{N}_S$ for $S = \{1, 3, 4, 6\}$; this is indeed the case since $1 \in N_1$, $2 \in N_4$, $3 \in N_3$ and $5 \in N_6$. Intuitively, the linkage lemma suggests that reaching our destination nodes 2 and 5 through node-disjoint paths implies that such paths should contain exactly one edge with starting node in $1, 3, 4, 6$ and ending node in N_1, N_3, N_4, N_6 respectively. Observe the relationship between the graphs in Figures 2a and 2b. By setting $A_1 = N_1$, $A_2 = N_4$, $A_3 = N_3$ and $A_4 = N_6$, the transversal derived in Example 1 is exactly the transversal $\{1, 2, 3, 5\}$. This is a consequence of the linkage lemma that yields a mapping from transversals to linkings and thus from matchings in a bipartite graph to linkings on a directed graph.

2.2 Gammoids

The subject of this sub-section is a class of matroids traditionally defined over digraphs $G(V, A)$. We use A to denote the arc-set of a digraph, instead of E that denotes the edge-set of an undirected graph. When necessary, we will write V^G, (instead of V) to denote the node-set of a graph (or digraph) G. Analogously we will occasionally use E^G and A^G.

Perfect [17] introduced a broad class of matroids called *gammoids*, defined over linkings in a digraph $G(V, A)$. For a fixed node-subset T of $G(V, A)$, let $L(G, T)$ denote the set family of nodes that can be linked into T, i.e., $S \subseteq V$ is in $L(G, T)$ if and only if it is linked onto some $X \subseteq T$.

Definition 3 *A 'strict gammoid' is a matroid $M(F, \mathcal{I})$ such that $F = V$ and $\mathcal{I} = L(G, T)$.*



(b) Gammoid $M(L(G, \{2, 5\})$

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(a) Transversal matroid $M[\mathcal{A}]$.

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Figure 2: Graphical representation of the transversal matroid (figure a) and the gammoid (figure b) associated with example 1 and 2 respectively.

It is known that for any matroid $M(F, \mathcal{I})$, the complements of maximal subsets in $\mathcal{I}$ define the *dual* matroid of M denoted by M^*. Our discussion up to this point then implies that a matroid is a strict gammoid if and only if its dual matroid is transversal [16].

Definition 4 *A 'gammoid' is a matroid that is isomorphic to a 'restriction of a strict gammoid'.*

We denote as $M[L(G, T)]$ a *strict gammoid* and following standard notation we write $M[L(G, T)]|F$ to denote a *gammoid* (i.e. a *restriction* of the matroid $M[L(G, T)]$ by the set $V \setminus F$). We should note that transversal matroids are gammoids hence gammoids are closed under duality. In fact, gammoids are also closed under taking minors and coincide with transversal matroids plus their *contractions* (i.e. the dual operation of restriction).

Let us now relate gammoids to matchings using the linkage lemma. For a strict gammoid $M[L(G, T)]$, one can derive a representation of its dual that is transversal (and vice-versa). Moreover, we know that $(M[L(G, T)])^* = M[\mathcal{N}_{V \setminus T}]$. Take a transversal matroid $M[\mathcal{A}]$ on a ground set F plus some transversal $T = \{v_1, \ldots, v_q\}$ of $\mathcal{A}$. For each $i = \{1, \ldots, q\}$ draw a directed edge (v_i, j) for $j \in A_i \setminus v_i$. In this way we yield a digraph $G(V, A)$ such that $V^G = F$ and $(M[\mathcal{A}])^* = M[L(G, V \setminus T)]$.

Example 3 Define a transversal matroid $M[\mathcal{A}]$ based on the digraph $G(V, A)$ given in Figure 2b and $T = \{2, 5\}$, by setting $\mathcal{A} = \mathcal{N}_{V \setminus T}$, i.e., $\mathcal{A} = \{N_1, N_4, N_3, N_6\}$. This yields the transversal matroid represented by the bipartite graph of Figure 2a for $A_1 = N_1$, $A_2 = N_4$, $A_3 = N_3$ and $A_4 = N_6$ (for the reverse transformation see example 2).

Tong *et al.* [20] represent a gammoid using a bipartite graph as follows. Assume a strict gammoid $M[L(G,T)]$ and consider its dual, that is the transversal matroid $M[\mathcal{N}_{V\setminus T}]$. Construct the bipartite graph $G'(V_1, V_2, E)$ that corresponds to $M[\mathcal{N}_{V\setminus T}]$ by setting $V_1^{G'} = V^G$, $V_2^{G'} = V^G \setminus T$ and $E^{G'} = \{(u,v) : u \in N_v, v \in V^G \setminus T\}$. Based on our discussion, $S \in L(G,T)$ if and only if there is some matching in G' that matches each node in $V_2^{G'}$ with exactly one node in $V_1^{G'} \setminus S$. In their construction, Tong *et al.* add another copy of V^G to represent the ground set of the strict gammoid. Specifically, the bipartite graph takes the form $\bar{G}(V_1, V_2 \cup V_3, E)$, where $V_1^{\bar{G}}$ and $\bar{V}_3^G$ are both copies of V^G, and $V_2^{\bar{G}} = V^G \setminus T$. Let us use v_i to denote a node $v \in V^G$ that is also a member of the node set $V_i^{\bar{G}}$, $i \in \{1,2,3\}$. The edge set of $\bar{G}$ can be written as $E^{\bar{G}} = E^{G'} \cup \{(v_1, v_3) : v_1 \in V_1^{\bar{G}}, v_3 \in V_3^{\bar{G}}, v \in F\}$. Then, $S \in L(G,T)$ if and only if there is some matching $\bar{G}$ that covers every node in $V_2^{\bar{G}}$, as well as every node in $S \cap V_3^{\bar{G}}$.

By Definition 4, a gammoid is a restriction of a strict gammoid. In the construction of Tong *et al.*, the 'restriction' is implemented by deleting nodes from the set $V_3^{\bar{G}}$. As a result, the nodes that we restricted our strict matroid upon can no longer act as source nodes (i.e. members of an $S \in L(G,T)$) but they should be intermediate nodes of the corresponding disjoint paths.

Example 4 Consider the digraph in Figure 2b and the corresponding strict gammoid $M[L(G,\{2,5\})]$. As we have already seen the bipartite graph in Figure 1 represents the dual of $M[L(G,\{2,5\})]$, which is the transversal matroid $M[\mathcal{N}_{V\setminus T}]$. The bipartite graph $\bar{G}(V_1, V_2 \cup V_3, E)$ of Tong *et al.* [20] introduces the nodes set $V_3^{\bar{G}} = \{1_3, 2_3 \ldots, 6_3\}$ which is simply a copy of V^G. Each node v_1 in $V_1^{\bar{G}}$ is connected with its copy in $V_3^{\bar{G}}$ (the construction is depicted in Figure 3a). We see that $\{4,6\}$ is in $M[L(G,\{2,5\})]$ since there exists a matching that covers all nodes in $V_2^{\bar{G}}$ and also covers the nodes $\{4_3, 6_3\}$ in $V_3^{\bar{G}}$. Moreover, in this particular case we observe that the matching in $\bar{G}$ is not unique. Two such matchings are:

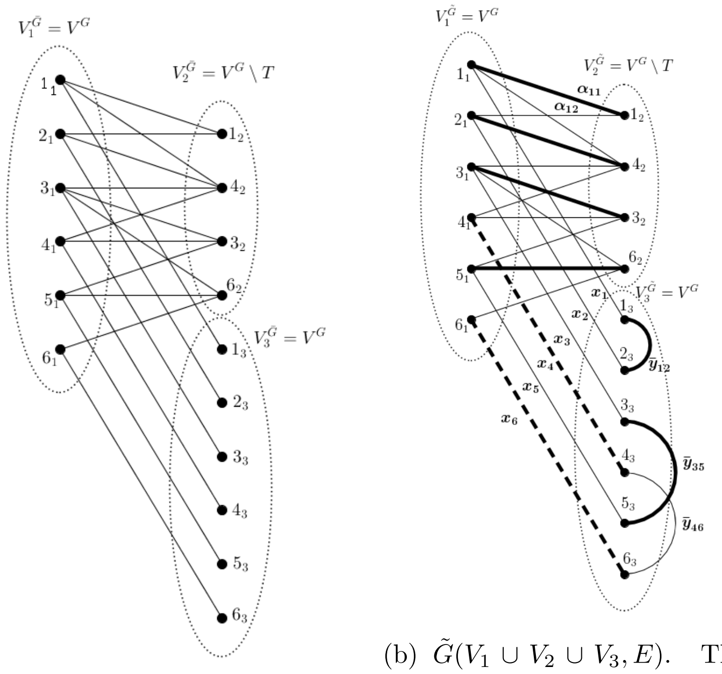
- $\{(1_2, 1_1), (4_2, 2_1), (3_2, 3_1), (6_2, 5_1), (4_3, 4_1), (6_3, 6_1)\}$
- $\{(4_2, 1_1), (1_2, 2_1), (6_2, 3_1), (3_2, 5_1), (4_3, 4_1), (6_3, 6_1)\}$

The first and second matching correspond respectively to the node disjoint paths $\{\{4,2\}, \{6,5\}\}$ and $\{\{4,1,2\}\ \{6,3,5\}\}$ in G.

2.3 Laminar matroids

Laminar families [7] give rise to another well-known class of matroids.

Definition 5 *Given a ground set F, a family $\mathcal{F} = \{F_i \subseteq F : i = 1, \ldots, k\}$ is called 'laminar' if, for all $F_i, F_j \in \mathcal{F}$ and $i \neq j$ exactly one of the following holds: $F_i \subset F_j$, $F_j \subset F_i$ or $F_i \cap F_j = \emptyset$.*



(a) $\bar{G}(V_1, V_2 \cup V_3, E)$.

(b) $\tilde{G}(V_1 \cup V_2 \cup V_3, E)$. The matching (thick edges) corresponds to the linking $\{4, 6\}$.

Figure 3: Figure (a) and (b) depict the graphs that correspond to the gammoid and gammoid parity polytope outlined in example 4 and 6 respectively.

We say that two sets $F_i, F_j \subset F$, $i \neq j$ *cross*, if and only if none of the three laminarity conditions hold. On the other hand, a *chain* is any laminar family $\mathcal{F}$ such that $F_i \cap F_j \neq \emptyset$ for all $F_i, F_j \in \mathcal{F}$, $i \neq j$. Also, a set family is laminar if and only if it can be represented graphically as an *arborescence (anti-arborescence)* [11]; i.e., a digraph where every node has at most one ingoing (outgoing) arc and the underlying undirected graph forms a tree (see also example 5).

Definition 6 *A matroid $M(F, \mathcal{I})$ is called 'laminar' if there is a laminar family $\mathcal{F}$ and positive integers $U = \{u_1, \ldots, u_k\}$, such that $S \in \mathcal{I}$ if and only if $|S \cap F_j| \leq u_k$ for $j = 1, \ldots, k$.*

We use the notation $M[F, \mathcal{F}, U]$ to denote a laminar matroid. Without loss of generality, we assume that $u_j \leq |F_j|$ for $j = 1, \ldots, k$, and that all sets F_j are non-redundant, in terms of sustaining the independent sets of $M[F, \mathcal{F}, U]$. It is not hard to see that the laminar matroids generalise partition matroids and cross with transversal matroids as shown in Figure 1 (Figure 2.4 in [9] shows a laminar matroid that is not transversal).

Moreover, laminar matroids are gammoids and to show this one can construct a suitable digraph, as in [9]. Let us first though introduce some notation to be also used in later sections. For a laminar matroid $M[F, \mathcal{F}, U]$ and $F_i, F_j \in \mathcal{F}$ such that $F_j \subset F_i$ and there is no $F_\ell \in \mathcal{F}$ satisfying $F_j \subset F_\ell \subset F_i$, we call j a 'child' of i. If $\chi(i)$ denotes the set of children of i, the set $F(i) = F_i \setminus \bigcup_{j \in \chi(i)} F_j$ contains any elements in F_i that are not elements of i's children.

We construct the digraph $G'(V, A)$ by introducing one node $V^{G'}(i)$ for each member i of the ground set F and one node $V^{G'}(F^i)$ for each member F_i of $\mathcal{F}$. The arc set is $A^{G'} = A^1 \cup A^2$, where

$$A^1 = \left\{ (V^{G'}(i), V^{G'}(F^j)) : i \in F, i \in F(j) \right\}$$

and

$$A^2 = \left\{ (V^{G'}(F^i), V^{G'}(F^j)) : F_i, F_j \in \mathcal{F}, i \in \chi(j) \right\}.$$

The capacity of each arc in A^1 is one, while for arcs $(V(F^i), V(F^j)) \in A^2$ the capacity is u_i. Note that the underlying undirected graph of G' is a forest with as many components as the maximal members of $\mathcal{F}$. We can easily convert this forest to a tree by introducing the pseudonode $V(F^{k+1})$ that represents the ground set F. If we take into account the direction of the edges, G' is an anti-arborescence and thus there is always a unique path linking every node in $V^{G'}(F)$ with the root node $V^{G'}(F^{k+1})$.

Now, every collection of paths in G' that respects the capacities $u_1, \ldots, u_k$ and has source nodes in $V^{G'}(S) \subseteq V^{G'}(F)$ and destination the node $V^{G'}(F^{k+1})$ can be mapped into a collection of node-disjoint paths in a digraph $G(V, A)$ with source nodes $V^G(S)$ and sink nodes in $V^G(T)$. Hence, we set $V(G) = V(F) \cup V(f)$, where

$$f = \left\{ f_i^j : \text{ for } i \text{ such that } F_i \in \mathcal{F}, j \in \{1, \ldots, u_i\} \right\},$$

and draw, for each $F_i \in \mathcal{F}$ and $j \in F$ such that $j \in F(i)$, arcs from $V(j)$ to each $V(f_i^p)$, $p \in \{1, \ldots, u_i\}$. It is straightforward to check that $S \subseteq F$ is independent in M if and only if there are $|S|$ node-disjoint paths with source nodes in $V^G(F)$, and sink nodes in $V^G(T)$, $T = \bigcup_{j=1}^{j=u_k} f_k^j$. It follows that any laminar matroid $M[F, \mathcal{F}, U]$ has such a representation and thus is a gammoid $M[L(G, T)]|F$.

Example 5 Let $F_1 = \{1, 2\}$, $F_2 = \{1, \ldots, 4\}$, $F_3 = \{5, 6\}$, $F_4 = \{1, \ldots, 6\}$ and $u_1 = u_3 = 1, u_2 = 3, u_4 = 4$. The laminar matroid in this case can be represented via $G(V, A)$ or $G'(V, A)$ given in Figure 4. Note that thick arcs correspond to the linking $\{1, 3, 4, 5\}$.

3 The MP polytope for gammoids

Recall the representation of a gammoid $M[(L(G, T)]|F$ as the bipartite graph $\bar{G}(V_1, V_2 \cup V_3, E)$. A set $S \subseteq F$ is independent in M if and only if there is

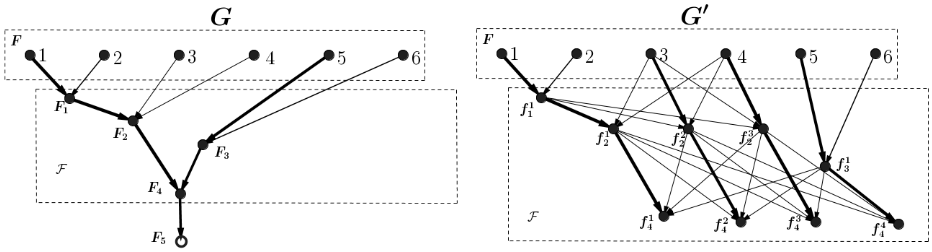


Figure 4: Digraphs G and G' representing the laminar matroid of Example 5.

a matching in $\bar{G}$ that covers all the nodes in $V_2^{\bar{G}}$ and nodes in $V_3^{\bar{G}}(S)$. To obtain a representation of the parity problem over a gammoid construct the graph $\tilde{G}$ such that $V^{\tilde{G}} = V^{\bar{G}}$ and $E^{\tilde{G}} = E^{\bar{G}} \cup \{(v,u) : \{v,u\} \in \ell, \ell \in \mathcal{L}\}$. That is, $\tilde{G}$ is a copy of $\bar{G}$ plus the edges corresponding to lines $\ell \in \mathcal{L}$. We observe that $\tilde{G}$ is not bipartite and that any matching $\tilde{G}$ defines an independent set of $M[(L(G,T)]|F$. Moreover, an edge $(v,u) \in E^{\tilde{G}}$ that corresponds to the line $\{v,u\} \in \mathcal{L}$ is in a matching of $\tilde{G}$ if and only if that line is *not* selected. The authors in [20] impose large weights to all edges adjacent to nodes in $V_2^{\tilde{G}}$ to satisfy the condition of covering that node set. Instead, one can drop these weights and formulate the problem as a *perfect matching* problem.

Proposition 1 *The parity problem over the gammoid $M[L(G,T)]|F$ and the set of lines $\mathcal{L}$ is reducible to a perfect matching problem on an undirected graph with $|F| + |V \setminus T| + |A| + |\mathcal{L}|$ edges and $|V| + |F| + |V \setminus T|$ nodes.*

Proof. We provide an integer programming (IP) formulation of the parity problem. For each $v \in F$, define the binary variable x_v that takes the value 1 if and only if v is to be included in the linking S. Define also the binary variable $\bar{y}_\ell$ taking the value 1 if and only if ℓ is *not* selected. For each $v = 1, \ldots, |V \setminus T|$ and $u \in F_v$, let the binary variable a_{vu} be 1 if and only if the arc $(v,u) \in A^G$ is included in the node-disjoint paths that link S and T. Finally, for each $v \in V^G$, define the node-set $C_v = \{u \in V \setminus T : v \in N_u\}$ representing the subset of nodes in V^G that are adjacent at the tails of arcs in A^G that have their head adjacent to node v.

Assuming S is independent in $M[L(G,T)]|F$ and by the definition of vectors x, $\bar{y}$ and a, it is easy to check that the incidence vector of S corresponds to a feasible solution to the following IP.

$$
\begin{array}{llll}
\max & \frac{1}{2}x(S) & & \\
\text{s.t.} & \sum_{u\in N_v} a_{vu} = 1 & v \in V\setminus T & (7)\\
& \sum_{u\in C_v} a_{vu} + x_v = 1 & v \in F & (8)\\
& \sum_{u\in C_v} a_{vu} = 1 & v \in V\setminus F & (9)\\
& x_v + \bar{y}_\ell = 1 & \forall \ell \in \mathcal{L},\ v\in \ell & (10)\\
& x_v,\ \bar{y}_\ell\ \in \mathbb{Z}_+ & \forall \ell \in \mathcal{L},\ v\in \ell & (11)\\
& a_{vu}\ \in \mathbb{Z}_+ & \forall v \in V\setminus T, u\in V. & (12)
\end{array}
$$

In particular, the constraints (7) are the degree equations for the nodes in $V_2^{\tilde{G}}$, the constraints (8) and (9) are the degree equations for the nodes in $V_1^{\tilde{G}}$, and the constraints (10) are the degree equations for the nodes in $V_3^{\tilde{G}}$. □

Proposition 1 has the following interesting corollary:

Corollary 1 *Every MP polytope for a gammoid is a projection of a perfect matching polytope.*

Proof. The convex hull of solutions to (7)–(12) is a perfect matching polytope. To obtain the corresponding MP polytope, it suffices to project the matching polytope into the space of the $\bar{y}$ variables, and then complement each $\bar{y}$ variable. □

Example 6 Consider the strict gammoid $M[L(G,\{2,5\})]$ defined over the digraph given in Figure 2b, along with the line set $\mathcal{L} = \{\{1,2\},\{3,5\},\{4,6\}\}$. The corresponding matroid parity polytope is a projection of the perfect 1-matching polytope defined over the graph $\tilde{G}(V,E)$ depicted in figure 3b.

4 The MP polytope for laminar matroids

Since laminar matroids are gammoids, Proposition 1 implies that the laminar matroid parity problem can be formulated as a perfect 1-matching problem. Laminarity allows for stronger results, i.e., the derivation of a *b-matching* formulation in a lower dimension and the proof that the MP polytope (i.e., the polytope defined only on the y-space) is affinely congruent to a perfect matching polytope.

Assume a laminar matroid $M[F,\mathcal{F},U]$ and the anti-arborescence representation $G'(V,A)$ described in Section 2.3. By definition, the laminar matroid $M[F,\mathcal{F},U]$ is equivalent to a gammoid represented by G' and thus it has a bipartite representation $\bar{G}(V_1,V_2\cup V_3,E)$. We now exploit the connection between graphs G' and $\bar{G}$. Members of $V_1^{\bar{G}}$ and $V_2^{\bar{G}}$, represent the sink and source nodes respectively of each arc in the anti-arborescence G'. Members of $V_3^{\bar{G}}$ on the other hand represent the potential source nodes of each collection of paths from a subset of $V^{G'}(F)$ to the root node $V^{G'}(F^{k+1})$. For any $i \in F$ then, nodes in $V_p^{G'}$, $p = \{1, 2, 3\}$, concern the leaves of G'. Moreover, the special structure of G' implies that for any $i \in F$:

- $V_1^{\bar{G}}(i)$ is adjacent to exactly two nodes, namely $V_3^{\bar{G}}(i)$ and $V_2^{\bar{G}}(i)$,
- $V_2^{\bar{G}}(i)$ is also adjacent to exactly two nodes; that is $V_1^{\bar{G}}(i)$ and $V_1^{\bar{G}}(F^j)$ for some $F_j \in \mathcal{F} \cup F_{k+1}$ such that $i \in F(j)$, and finally
- $V_3^{\bar{G}}(i)$ is adjacent to $V_1^{\bar{G}}(i)$ only.

For a given linking $S \in L(G', F^{k+1})|F$ and the corresponding matching in $\bar{G}$, we see that edge $(V_2^{\bar{G}}(i), V_1^{\bar{G}}(F^j))$ belongs to that matching for some $F_j \in \mathcal{F} \cup F_{k+1}$ such that $i \in F(j)$ if and only if $V^{G'}(i) \in S$. Then, because of laminarity and for any $i \in F$, we can identify the pairs of nodes ($V_2^{\bar{G}}(i)$, $V_3^{\bar{G}}(i)$) and ($V_1^{\bar{G}}(i)$, $V_1^{\bar{G}}(F^j)$) for $i \in F(j)$. We implement this transformation by simply deleting node sets $V_1^{\bar{G}}(F)$ and $V_2^{\bar{G}}(F)$.

The node $V^{G'}(F^{k+1})$ is a pseudo-node that corresponds to set F, thus the degree equation that corresponds to $V_1^{\bar{G}}(F^{k+1})$ is implied by the degree equations that correspond to the maximal members of $\mathcal{F}$ (i.e. the nodes $V^{\bar{G}}(\bar{\mathcal{F}})$). Without loss of generality then, we do not represent $V^{G'}(F^{k+1})$ explicitly in $\bar{G}$, and the multi-edges $(V_2^{\bar{G}}(F^i), \cdot)$ for all $F_i \in \bar{\mathcal{F}}$ have only one end point.

As in Section 3, we define the laminar matroid parity problem by introducing a line set $\mathcal{L}$ and construct the graph $\tilde{G}(V, E)$ such that $V^{\tilde{G}} = V^{\bar{G}}$ and $E^{\tilde{G}} = E^{\bar{G}} \cup \{(i, j) : \{i, j\} \in \ell, \ell \in \mathcal{L}\}$.

Lemma 1 *The parity problem on a laminar matroid $M = [F, \mathcal{F}, U]$ and a line set $\mathcal{L}$ is reducible to a perfect b-matching problem on a graph with $|F| + |\mathcal{L}|$ edges, $2|\mathcal{F}|$ multi-edges and $|F| + 2|\mathcal{F}|$ nodes.*

Proof. As in the proof of Proposition 1, for each $i \in F$, define the binary variable x_i, taking the value 1 if and only if i is to be included in the set $S \in \mathcal{I}$, and the binary variable $\bar{y}_\ell$, being 1 if and line ℓ is *not* selected. For $i = 1, \ldots, k$, define the variable $z_i \in \{0, \ldots, u_i\}$, representing the quantities $|S \cap F_i|$. Finally, for $i = 1, \ldots, k$, define the variable $\bar{z}_j \in \{0, \ldots, u_j\}$, representing the quantities $u_j - |S \cap F_j|$. Then, the incidence vector $(x, y, z, \bar{z})$ of an $S \in \mathcal{I}$ is a feasible solution to the following IP.

$$
\begin{array}{lll}
\max & \frac{1}{2}x(F) & \\
\text{s.t.} & x(F(i)) + \sum_{j\in\chi(i)} z_j + \bar{z}_i = u_i & i = 1,\ldots,k \quad (13)\\
& z_i + \bar{z}_i = u_i & i = 1,\ldots,k \quad (14)\\
& x_e + \bar{y}_\ell = 1 & \forall \ell \in \mathcal{L},\ e \in \ell \quad (15)\\
& x_e,\ \bar{y}_\ell,\ \in \mathbb{Z}_+ & \forall \ell \in \mathcal{L},\ e \in \ell\\
& z_i,\ \bar{z}_i \in \mathbb{Z}_+ & i = 1,\ldots,k.
\end{array}
$$

Constraints (13) and (14) are the degree equations of nodes in $V_1^{\bar{G}}$ and $V_2^{\bar{G}}$ respectively, while (15) are the degree equations of nodes in $V_3^{\bar{G}}$. □

Lemma 2 *The laminar MP polytope is affinely congruent to a perfect matching polytope.*

Proof. Let a laminar matroid parity instance be given by a matroid $M[F, \mathcal{F}, U]$ and a line set $\mathcal{L}$ and let $\mathscr{P}_M \subset \mathbb{R}^{|\mathcal{L}|}$ be the associated polytope. We prove that there is a perfect b-matching polytope in $\mathbb{R}^{|F|+|\mathcal{L}|+2|\mathcal{F}|}$ that is affinely congruent to $\mathscr{P}_M$.

Define the polytope

$$
\mathscr{P}_M^+ = \text{conv}\left\{(x, \bar{y}, z, \bar{z}) \in \mathbb{Z}_+^{|F|+|\mathcal{L}|+2|\mathcal{F}|} : (13) - (15) \text{ hold}\right\}.
$$

Lemma 1 implies that a vector y^* lies in $\mathscr{P}_M$ if and only if the corresponding vector $(x^*, \bar{y}^*, z^*, \bar{z}^*)$ lies in $\mathscr{P}_M^+$, where:

$$
\begin{aligned}
x_e^* = x_f^* &= y_{ef}^* && (\{e,f\} \in \mathcal{L}) && (16)\\
\bar{y}_\ell^* &= 1 - y_\ell^* && (\ell \in \mathcal{L}) && (17)\\
z_i^* &= \sum_{\ell\in\mathcal{L}} |\ell \cap F_i| y_\ell^* && (i = 1,\ldots,k) && (18)\\
\bar{z}_i^* &= u_i - \sum_{\ell\in\mathcal{L}} |\ell \cap F_i| y_\ell^* && (i = 1,\ldots,k). && (19)
\end{aligned}
$$

Since this mapping is affine and invertible, $\mathscr{P}_M$ is affinely congruent to $\mathscr{P}_M^+$. □

Example 7 Assume a laminar matroid defined over the ground set $F = \{1,\ldots,20\}$, the set family $\mathcal{F} = \{F_1,\ldots,F_5\}$ and the line set

$$
\begin{aligned}
\mathcal{L} = \quad & \{\{1,11\},\{2,12\},\{3,13\},\{4,14\},\{5,15\},\{6,16\},\\
& \{7,17\},\{8,18\},\{9,19\},\{10,20\}\}.
\end{aligned}
$$

The members of $\mathcal{F}$ along with their upper bounds are given in Table 1. Observe that each of the three columns of this table corresponds to a chain in

Chain		
1	2	3
$F_1 = \{1, \ldots, 9, 19\}$, $u_1 = 5$	$F_4 = \{11, 13, 17, 10\}$, $u_4 = 2$	$F_5 = \{12, 14, 18, 20\}$, $u_5 = 1$
$F_2 = \{5, \ldots, 9, 19\}$, $u_2 = 4$		
$F_3 = \{7, \ldots, 9, 19\}$, $u_3 = 3$		

Table 1: Members of the laminar family and bounds of Example

$\mathcal{F}$. The matroid parity polytope is the convex hull of the vectors $y_\ell \in \{0,1\}^{10}$ that satisfy the following five projected rank inequalities.

$$\sum_{i=1,\ldots,9} y_{i,10+i} + 2y_{9,19} \leq 5 \tag{20}$$

$$\sum_{i=5,\ldots,9} y_{i,10+i} + 2y_{9,19} \leq 4 \tag{21}$$

$$\sum_{i=7,8,9} y_{i,10+i} + 2y_{9,19} \leq 3 \tag{22}$$

$$y_{1,11} + y_{3,13} + y_{7,17} + y_{10,20} \leq 2 \tag{23}$$

$$y_{2,12} + y_{4,14} + y_{8,18} + y_{10,20} \leq 1. \tag{24}$$

By applying Lemma 1 we transform this laminar matroid parity polytope to a perfect b-matching polytope. The representation of the latter has 40 variables and 30 constraints and is given in the Appendix (equations (40) - (69)).

5 Facet-defining inequalities

The perfect matching polytope for a graph $G = (V, E)$ and vector $b \in \mathbb{Z}_+^V$ is completely described by the *degree equations* $x(\delta(i)) = b_i$ for all $i \in V$, the *non-negativity inequalities* $x_e \geq 0$ for all $e \in E$, and the *simple blossom inequalities*

$$x(E(H)) \leq \left\lfloor \frac{b(H)}{2} \right\rfloor, \tag{25}$$

where $H \subset V$ is such that $b(H)$ is odd. From this it follows that the only inequalities that can define facets of $\mathscr{P}_M^+$ are the non-negativity inequalities for the x, $\bar{y}$, z and $\bar{z}$ variables, together with the simple blossom inequalities, which can now involve combinations of those variables. Using this fact together with Lemma 2, one can derive a complete linear description of the laminar matroid parity polytope $\mathscr{P}_M$.

The non-negativity inequalities are the easiest to handle:

- For each $\{e, f\} \in \mathcal{L}$, both inequalities $x_e \geq 0$ and $x_f \geq 0$ for $\mathscr{P}_M^+$ map to the inequality $y_{ef} \geq 0$ for $\mathscr{P}_M$.

- For each $\ell \in \mathcal{L}$, the inequality $\bar{y}_\ell \geq 0$ for $\mathscr{P}_M^+$ maps to the upper bound inequality $y_\ell \leq 1$ for $\mathscr{P}_M$.
- For each $i = 1, \ldots, k$, the inequality $z_i \geq 0$ is redundant, in light of equations (14), which imply $z_i = x(F_i)$ for all $i \in 1, \ldots, k$.
- For each $i = 1, \ldots, k$, the inequality $\bar{z}_i \geq 0$ for $\mathscr{P}_M^+$ is equivalent to $x(F_i) \leq u_i$, due to equations (14). This latter inequality in turn maps to the projected rank inequality

$$\sum_{\ell \in \mathcal{L}} |F_i \cap \ell| y_\ell \leq u_i.$$

The simple blossom inequalities for $\mathscr{P}_M^+$ map to a new and non-trivial family of valid inequalities for $\mathscr{P}_M$, which we call *projected* blossom inequalities. It is known that blossom inequalities (25) can be derived by summing together the degree inequalities for all nodes in H and the upper bounds for all edges in T, dividing the resulting inequality by two, and rounding down; i.e., they are "$\{0, \frac{1}{2}\}$-cuts" in the sense of Caprara & Fischetti [1]. Now, recall the perfect b-matching polytope $\mathscr{P}_M^+$, defined in Section 4 and let us introduce the corresponding undirected graph $G^+ = (V^+, E^+)$ that has one edge for each variable $x, \bar{y}, z, \bar{z}$ and one node for each degree equation (13)-(15). We also define the sets $T = \{1, \ldots, k\}$, $U = \{k+1, \ldots, 2k\}$ and $S = \{2k+1, \ldots, 2k+|F|\}$, which index the equations (13), (14) and (15), respectively. (By construction, T, U and S form a partition of V^+.)

Note that any set $F_i \in \mathcal{F}$ is associated with two equations in our IP formulation: one of the form (13), indexed by $i \in T$, and the other of the form (14), indexed by $(i+k) \in U$. Furthermore, any element $f \in F$ is associated with one degree equation of the form (15), while any line $\ell \in \mathcal{L}$ is associated with two of them.

Now, for a given simple blossom inequality, let $\bar{T} \subseteq T$, $\bar{U} \subseteq U$ and $\bar{S} \subseteq S$ denote the index sets of the equations that are used in their derivation as a $\{0, \frac{1}{2}\}$-cut. (By construction, $\bar{T}$, $\bar{U}$ and $\bar{S}$ form a partition of H.) We can now state the following lemma.

Lemma 3 *If a simple blossom inequality defines a facet of $\mathscr{P}_M^+$, then the corresponding sets $\bar{T}$, $\bar{U}$ and $\bar{S}$ satisfy the following conditions:*

1. $\sum_{i \in \bar{T}} u_i + \sum_{i \in \bar{U}} u_{i-k} + |\bar{S}|$ *is odd.*
2. $S = \{i + 2k : \exists \{i, j\} \in \mathcal{L}$ *such that* $i, j \in \bigcup_{n \in \bar{T}} F(n)\}$
3. *If* $(i+k) \in \bar{U}$, *then* $j \in \bar{T}$ *where* $i \in \chi(j)$.

Proof.

1. If condition 1 does not hold, no rounding down occurs on the right-hand side.

2. Suppose condition 2 does not hold. Then there is some element $i \in F$ and some line $\{i, j\} \in \mathcal{L}$ for which we are using the equation $x_i + \bar{y}_{ij} = 1$ in the derivation of the blossom inequality, yet for which the variable x_i does not appear in any other equation that we are using. Now consider two cases:

   (i) $j+2k$ does not lie in $\bar{S}$. Then both x_i and $\bar{y}_{ij}$ will receive a coefficient of zero in the blossom inequality. Then, the blossom inequality will be either unchanged or strengthened if we remove $i + 2k$ from $\bar{S}$.

   (ii) $j+2k$ does lie in $\bar{S}$. Then the net contribution of the two equations, before dividing by two and rounding down, is $x_i + x_j + 2\bar{y}_{ij} \leq 2$. After dividing by two and rounding down, the left-hand side coefficient of x_i will be zero. So the best possible scenario is that we have added $x_j + \bar{y}_{ij} \leq 1$ to the blossom inequality. There is no point doing this, since $x_j + \bar{y}_{ij} = 1$.

3. Suppose condition 3 does not hold. Then there is some degree equation $i + k$ in $\bar{U}$ for which the degree equation $j \in T$, corresponding to the unique parent of i, is not included in the derivation of the blossom inequality. We observe that the variable $\bar{z}_i$ appears in the degree equations $i + k$ and $i \in T$, while the variable z_i appears in $i + k$ and its parent $j \in T$. Again, we consider two cases:

   (i) the degree equation i is not in $\bar{T}$. Then, we are using the equation $z_i + \bar{z}_i = u_i$ in the derivation of the blossom inequality, even though neither z_i nor $\bar{z}_i$ appear in any other equation that is used. After dividing by two and rounding down, the left-hand side coefficients of both z_i and $\bar{z}_i$ will be zero. Then, we could get a stronger inequality by removing i from $\bar{T}$.

   (ii) i does lie in $\bar{T}$. Then, the net contribution of the equations $i + k$ and i, before dividing by two and rounding down, is

$$x(F(i)) + \sum_{j \in \chi(i)} z_j + 2\bar{z}_i + z_i \leq 2u_i.$$

   Thus, after dividing by two and rounding down, the contribution to the left-hand side of the blossom inequality is at most $x(F(i)) + \sum_{j \in \chi(i)} z_j + \bar{z}_i$, while the contribution to the right-hand side is exactly u_i. Since $x(F(i)) + \sum_{j \in \chi(i)} z_j + \bar{z}_i = u_i$, we could get a stronger blossom inequality by removing $i + k$ from $\bar{U}$ and i from $\bar{T}$. □

For given sets $\bar{S}$, $\bar{T}$ and $\bar{U}$ that respect the conditions of Lemma 3, we can derive a simple blossom inequality for $\mathscr{P}^+_M$. Before we present the general form of such an inequality, it is helpful to introduce some further index sets. We let $\bar{S}'$ denote a subset of $\bar{S}$ such that for any $\{i, j\} \in \mathcal{L}$ for which $(i+k) \in \bar{S}$ and $(j+k) \in \bar{S}$ we have either $(i+k) \in \bar{S}'$ or $(j+k) \in \bar{S}'$, but not both. In addition, we define the following two index sets associated with condition 3 of Lemma 3:

$$\begin{aligned} Z &= \left\{ i \in \{1,\ldots,k\} : i \notin \bar{T}, (i+k) \in \bar{U}, i \in \chi(j) \text{ for some } j \in \bar{T} \right\} \\ \tilde{Z} &= \left\{ i \in \{1,\ldots,k\} : i \in \bar{T}, (i+k) \in \bar{U}, i \in \chi(j) \text{ for some } j \in \bar{T} \right\} \end{aligned}$$

Note that the sets Z and $\tilde{Z}$ correspond to the possible scenarios for membership in $\bar{T}$ and $\bar{U}$. Using this notation we obtain the following general form of the simple blossom inequalities for $\mathscr{P}_M^+$:

$$x(\delta^+(\bar{T})) + \bar{y}(\delta^+(\bar{S}')) + z(\delta^+(Z \cup \tilde{Z})) + \bar{z}(\delta^+(\tilde{Z})) \leq \left\lfloor \frac{\sum_{i\in\bar{T}} u_i + \sum_{i\in\bar{U}} u_{i-k} + |\bar{S}|}{2} \right\rfloor . \qquad (26)$$

We can now state the main result of this section.

Theorem 1 *The laminar matroid parity polytope $\mathscr{P}_M$ is completely described by the bound constraints $0 \leq y_\ell \leq 1$ for all $\ell \in \mathcal{L}$, the projected rank inequalities (6), and the set of $\left\{0, \frac{1}{2}\right\}$-cuts obtained from them.*

Proof. It suffices to show that every non-dominated simple blossom inequality for $\mathscr{P}_M^+$ becomes a $\left\{0, \frac{1}{2}\right\}$-cut for $\mathscr{P}_M$ when projected into $\mathbb{R}^{|\mathcal{L}|}$. So, consider a simple blossom inequality of the form (26). First, we use (16) and (17) to project out the $\bar{y}$ variables. Note that $2(\delta^+(\bar{S}')) = 2|\bar{S}'| = |\bar{S}|$, and therefore subtracting $|\bar{S}|$ from the right hand side of (26) does not change its parity. Thus, the inequality (26) is equivalent to:

$$x(\delta^+(\bar{T})) - x(\delta^+(\bar{S}')) + z(\delta^+(Z \cup \tilde{Z})) + \bar{z}(\delta^+(\tilde{Z})) \leq \left\lfloor \frac{\sum_{i\in\bar{T}} u_i + \sum_{i\in\bar{U}} u_{i-k}}{2} \right\rfloor .$$

Now, condition 2 of Lemma 3 implies that $x(\delta^+(\bar{T})) = x(\delta^+(\bar{S}')$, and therefore the inequality reduces to:

$$z(\delta^+(Z \cup \tilde{Z})) + \bar{z}(\delta^+(\tilde{Z}) \leq \left\lfloor \frac{\sum_{i\in\bar{T}} u_i + \sum_{i\in\bar{U}} u_{i-k}}{2} \right\rfloor .$$

Next, we eliminate the $\bar{z}$ variables, using (14), to obtain:

$$z(\delta^+(Z)) \leq \left\lfloor \sum_{i\in\bar{T}} u_i + \sum_{i\in\bar{U}} u_{i-k} \right\rfloor - \sum_{i\in\tilde{Z}} u_i. \qquad (27)$$

Now we simplify the right-hand side. Note that if $i \in \tilde{Z}$, then $i \in \bar{T}$ and $i + k \in \bar{U}$. Thus

$$\sum_{i\in\bar{T}} u_i + \sum_{i\in\bar{U}} u_{i-k} - 2\sum_{i\in\tilde{Z}} u_i = \sum_{i\in\bar{T}\setminus\tilde{Z}} u_i + \sum_{i\in\bar{U}\setminus\tilde{Z}} u_{i-k} = \sum_{i\in Z} u_i.$$

We can therefore re-write the inequality (27) in the following simplified form:

$$z(\delta^+(Z)) \leq \left\lfloor \sum_{i \in Z} u_i \right\rfloor. \qquad (28)$$

Finally, we will project out the z variables. To this end, we define the set family $\mathcal{Q} = \{F_i \in \mathcal{F} : i \in Z\}$ and let

$$\alpha_\ell = \frac{1}{2}\left(\sum_{\{i: F_i \in \mathcal{Q}\}} |\ell \cap F_i| \right), \quad \ell \in \mathcal{L},$$
$$\beta = \left\lfloor \frac{\sum_{\{i: F_i \in \mathcal{Q}\}} r_M(F_i)}{2} \right\rfloor.$$

Using equation (18), we project the inequality (28) into $\mathbb{R}^{\mathcal{L}}$ to yield:

$$\sum_{\ell \in \mathcal{L}} \alpha_\ell y_\ell \leq \beta. \qquad (29)$$

Inequality (29) is a $\left\{0, \frac{1}{2}\right\}$-cut for $\mathscr{P}_M$, derived from the projected rank inequalities (6) for the members of Q. □

This immediately yields the following corollary (see [2] for a definition of Chvátal rank):

Corollary 2 *The laminar matroid parity polytope has Chvátal rank 1.*

Recall once more that the c-capacitated b-matching polytope is completely described by (1)–(3). Thus, all its facet-defining inequalities have binary left-hand-side coefficients. On the other hand, the following statements imply a more elaborate structure for the laminar matroid parity polytope. The first two are shown via Example 8.

Proposition 2 *A projected rank inequality that defines a facet of the laminar matroid parity polytope may have ternary coefficients.*

Proposition 3 *A projected blossom inequality that defines a facet of the laminar matroid parity polytope may have non-ternary coefficients.*

Proposition 4 *The coefficient of a variable y_ℓ, $\ell \in L$, in a non-dominated projected blossom inequality for the laminar matroid parity polytope is at most* $\left\lfloor \frac{\sum_{i=1,\ldots,k} |\ell \cap F_i|}{2} \right\rfloor$.

Proof. Any non-dominated blossom inequality takes the form (29), i.e., a $\left\{0, \frac{1}{2}\right\}$-cut, hence the result. □

An $O(|F|)$ bound on the coefficients of a projected blossom inequality follows easily, while Example 8 shows that the bound of Proposition 4 is attainable.

Example 8 Consider the laminar matroid parity polytope described at Example 7 and its b-matching formulation given in the Appendix (equations (40)-(69)). We first note that the projected rank inequality (20) defines a facet of this polytope, since the ten extreme points presented in Table 2 (Appendix) satisfy it at equality. Yet, the left-hand side coefficients of this inequality are not all binary. Summing the five projected rank inequalities (20)-(24), dividing by two, and rounding down, we obtain the following valid inequality for this polytope:

$$\begin{aligned} y_{1,11} + y_{2,12} + y_{3,13} + y_{4,14} + y_{5,15} + \\ y_{6,16} + 2y_{7,17} + 2y_{8,18} + 3y_{9,19} + y_{10,20} \quad &\leq 7. \end{aligned} \tag{30}$$

The ten points presented in Table 3 (Appendix) are affinely independent and lie in the face defined by (30). Thus the latter inequality defines a facet of this laminar matroid polytope. We observe that the left-hand side coefficients of this inequality are not all ternary. Consider the blossom inequality defined by a set $H = \bar{S} \cup \bar{T} \cup \bar{U}$, such that $\bar{T} = \{(40), (42), (43), (44)\}$, $\bar{U} = \{(46)\}$ and $\bar{S} = \{\{(50), \ldots, (57)\} \cup \{(62), \ldots, (65)\} \cup \{(68), (69)\}\}$. Note that the selection of $\bar{S}$, $\bar{T}$ and $\bar{U}$ respects Lemma 3. The resulting blossom inequality takes the form (31). One can easily check that projecting this inequality into the y-space yields the $\left\{0, \frac{1}{2}\right\}$-cut (30).

$$\sum_{i \in I_1} (x_i + \bar{y}_{i,10+i}) + \sum_{i \in I_2} (x_i + \bar{y}_{i-10,i}) + \sum_{i=5}^{9} x_i + x_{19} + z_2 \leq 13, \tag{31}$$

where

$$\begin{aligned} I_1 &= \{1, \ldots, 4, 10\}, \\ I_2 &= \{\{11, \ldots, 14\} \cup \{17, 18, 20\}\}. \end{aligned}$$

6 Elementary closure

Now we return to the matroid parity problem for general (i.e., not necessarily laminar) matroids. Let $\mathscr{P}_1$ be the *elementary closure* of the feasible region of the LP relaxation of the matroid parity problem. That is, let $\mathscr{P}_1$ be the polytope defined by the intersection of bound and projected rank inequalities, together with the set of Chvátal-Gomory (C-G) cuts derived from them (see [2]). $\mathscr{P}_1$ is a natural polyhedral outer-approximation of $\mathscr{P}_M$. An apparently weaker such approximation is derived when the multipliers applied for the calculation of the C-G cut can only take the value 0 or 1/2.

We follow [1] in calling the corresponding polytope $\mathscr{P}_{1/2}$. We establish that $\mathscr{P}_1 = \mathscr{P}_{1/2}$ using the fact that the laminar matroid parity polytope is fully described by its $\{0, \frac{1}{2}\}$-cuts. For conciseness, let us call a set of inequalities (6) *laminar* if their supports form a laminar set.

Lemma 4 *A C-G cut is a facet of $\mathscr{P}_1$ only if it is obtainable by a laminar set of rank inequalities.*

Proof. Let R be the set of inequalities defining $\mathscr{P}_M$. A facet of $\mathscr{P}_1$ is defined by a C-G cut, hence let $\lambda \in [0,1)^{|R|}$ be the corresponding multipliers and $R' = \{i \in R : \lambda_i > 0\}$. Assuming to the contrary that the set of rank inequalities indexed by R' is not laminar, there is a pair of projected rank inequalities $i, j \in R'$ whose supports S_i, S_j cross, i.e., $S_i \setminus S_j \neq \emptyset \neq S_j \setminus S_i$. The contribution of these two inequalities in the C-G cut, before rounding down, is the sum of

$$\lambda_i(\textstyle\sum_{\ell\in\mathcal{L}} |S_i \cap \ell|\, y_\ell) \leq \lambda_i r_M(S_i) \quad \text{and} \tag{32}$$

$$\lambda_j(\textstyle\sum_{\ell\in\mathcal{L}} |S_j \cap \ell|\, y_\ell) \leq \lambda_j r_M(S_j). \tag{33}$$

Assume without loss of generality that $\lambda_i \geq \lambda_j > 0$ and observe that (32) can alternatively be written as the sum of

$$(\lambda_i - \lambda_j)(\textstyle\sum_{\ell\in\mathcal{L}} |S_i \cap \ell|\, y_\ell) \leq (\lambda_i - \lambda_j) r_M(S_i) \quad \text{and} \tag{34}$$

$$\lambda_j(\textstyle\sum_{\ell\in\mathcal{L}} |S_i \cap \ell|\, y_\ell) \leq \lambda_j r_M(S_i). \tag{35}$$

Consider now the two rank inequalities derived by 'uncrossing' the sets S_i and S_j, i.e., the inequalities

$$\textstyle\sum_{\ell\in\mathcal{L}} |(S_i \cup S_j) \cap \ell|\, y_\ell \leq r_M(S_i \cup S_j) \quad \text{and} \tag{36}$$

$$\textstyle\sum_{\ell\in\mathcal{L}} |(S_i \cap S_j) \cap \ell|\, y_\ell \leq r_M(S_i \cap S_j). \tag{37}$$

It becomes easy to show that, for each $\ell \in \mathcal{L}$,

$$|(S_i \cup S_j) \cap \ell| + |(S_i \cap S_j) \cap \ell| = |S_i \cap \ell| + |S_j \cap \ell|, \tag{38}$$

by noticing the following partitions of $S_i \cap \ell$ and $(S_i \cup S_j) \cap \ell$:

$$S_i \cap \ell = ((S_i \setminus S_j) \cap \ell) \cup ((S_i \cap S_j) \cap \ell),$$
$$(S_i \cup S_j) \cap \ell = ((S_i \setminus S_j) \cap \ell) \cup ((S_i \cap S_j) \cap \ell) \cup ((S_j \setminus S_i) \cap \ell).$$

In more detail,

$$\begin{aligned} |S_i \cap \ell| + |S_j \cap \ell| = \;& |(S_i \setminus S_j) \cap \ell| + |(S_i \cap S_j) \cap \ell| + \\ & |(S_j \setminus S_i) \cap \ell| + |(S_i \cap S_j) \cap \ell| = \\ & (|(S_i \setminus S_j) \cap \ell| + |(S_i \cap S_j) \cap \ell| + \\ & |(S_j \setminus S_i) \cap \ell|) + |(S_i \cap S_j) \cap \ell| = \\ & |(S_i \cup S_j) \cap \ell| + |(S_i \cap S_j) \cap \ell|. \end{aligned}$$

Notice also that the submodularity of the rank function r_M implies

$$r_M(S_i \cup S_j) + r_M(S_i \cap S_j) \leq r_M(S_i) + r_M(S_j). \tag{39}$$

But then, the sum (36)-(37), each multiplied by λ_j, plus (34) provides an inequality with the same left-hand side with the sum of (32)-(33) (because of (38)) and a no-larger right-hand side (because of (39)). This suggests a substitution strategy for strengthening the facet-defining C-G cut, i.e., the substitution in R' of (32)-(33) with (36)-(37), each multiplied by λ_j, plus (34). By repeating this uncrossing argument for any pair of crossing inequalities in R', one can substitute every non-laminar subset of rank inequalities in R with a laminar one and derive a C-G cut that is at least as strong. □

Corollary 3 *For any matroid M and any set of lines $\mathcal{L}$, $\mathscr{P}_{1/2} = \mathscr{P}_1$.*

Proof. Lemma 4 implies that an inequality $\alpha y \leq \beta$ that is facet-defining for $\mathscr{P}_1$ can be derived as a C-G cut from a laminar set of projected rank inequalities. Let $R' \subset R$ be this laminar set and $A'y \leq b'$ be the system of linear inequalities that it defines. Then, $\{y \in \{0,1\}^{|E|} : A'y \leq b'\}$ is a laminar matroid parity polytope. By Theorem 1, this polytope is described by the bounds, projected rank inequalities and $\left\{0, \frac{1}{2}\right\}$-cuts. Hence, $\alpha y \leq \beta$ is also a $\left\{0, \frac{1}{2}\right\}$-cut. □

Since matroid parity is equivalent to matroid matching, it is interesting to recall that the matroid matching polytope is not equal to the elementary closure of the linear relaxation examined in [21], despite the fact that this relaxation admits some structural properties of the matching polytope like half-integral vertices.

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Appendix

$$\sum_{i=1,\ldots,9} x_i + x_{19} + \bar{z}_1 + z_2 = 5 \tag{40}$$

$$\sum_{i=5,\ldots,9} x_i + x_{19} + \bar{z}_2 + z_3 = 4 \tag{41}$$

$$\sum_{i=7,8,9} x_i + x_{19} + \bar{z}_3 = 3 \tag{42}$$

$$x_{11} + x_{13} + x_{17} + x_{10} + \bar{z}_4 = 2 \tag{43}$$

$$x_{12} + x_{14} + x_{18} + x_{20} + \bar{z}_5 = 1 \tag{44}$$

$$z_1 + \bar{z}_1 = 5 \tag{45}$$

$$z_2 + \bar{z}_2 = 4 \tag{46}$$

$$z_3 + \bar{z}_3 = 3 \tag{47}$$

$$z_4 + \bar{z}_4 = 2 \tag{48}$$

$$z_5 + \bar{z}_5 = 1 \tag{49}$$

$$x_1 + \bar{y}_{1,11} = 1 \tag{50}$$

$$x_{11} + \bar{y}_{1,11} = 1 \tag{51}$$

$$x_2 + \bar{y}_{2,12} = 1 \tag{52}$$

$$x_{12} + \bar{y}_{2,12} = 1 \tag{53}$$

$$x_3 + \bar{y}_{3,13} = 1 \tag{54}$$

$$x_{13} + \bar{y}_{3,13} = 1 \tag{55}$$

$$x_4 + \bar{y}_{4,14} = 1 \tag{56}$$

$$x_{14} + \bar{y}_{4,14} = 1 \tag{57}$$

$$x_5 + \bar{y}_{5,15} = 1 \tag{58}$$

$$x_{15} + \bar{y}_{5,15} = 1 \tag{59}$$

$$x_6 + \bar{y}_{6,16} = 1 \tag{60}$$

$$x_{16} + \bar{y}_{6,16} = 1 \tag{61}$$

$$x_7 + \bar{y}_{7,17} = 1 \tag{62}$$

$$x_{17} + \bar{y}_{7,17} = 1 \tag{63}$$

$$x_8 + \bar{y}_{8,18} = 1 \tag{64}$$

$$x_{18} + \bar{y}_{8,18} = 1 \tag{65}$$

$$x_9 + \bar{y}_{9,19} = 1 \tag{66}$$

$$x_{19} + \bar{y}_{9,19} = 1 \tag{67}$$

$$x_{10} + \bar{y}_{10,20} = 1 \tag{68}$$

$$x_{20} + \bar{y}_{10,20} = 1 \tag{69}$$

	$y_{1,11}$	$y_{2,12}$	$y_{3,13}$	$y_{4,14}$	$y_{5,15}$	$y_{6,16}$	$y_{7,17}$	$y_{8,18}$	$y_{9,19}$	$y_{10,20}$
1)	0	0	1	0	1	1	0	0	1	0
2)	0	0	1	0	1	1	0	0	1	1
3)	0	0	1	0	1	1	1	1	0	0
4)	0	0	1	1	1	0	0	0	1	0
5)	0	0	1	1	1	1	1	0	0	0
6)	0	1	0	0	1	1	0	0	1	0
7)	0	1	1	0	0	0	1	0	1	0
8)	0	1	1	0	0	1	0	0	1	0
9)	0	1	1	0	1	1	1	0	0	0
10)	1	0	0	0	0	1	0	1	1	0

Table 2: Affinely independent points that lie in the face defined by (20)

	$y_{1,11}$	$y_{2,12}$	$y_{3,13}$	$y_{4,14}$	$y_{5,15}$	$y_{6,16}$	$y_{7,17}$	$y_{8,18}$	$y_{9,19}$	$y_{10,20}$
1)	0	0	0	0	1	0	1	0	1	1
2)	0	0	0	1	0	1	1	0	1	0
3)	0	0	0	1	1	0	1	0	1	0
4)	0	0	1	0	1	0	1	0	1	0
5)	0	0	1	0	1	1	0	0	1	1
6)	0	0	1	0	1	1	1	1	0	0
7)	0	1	0	0	1	0	1	0	1	0
8)	0	1	1	0	0	0	1	0	1	0
9)	1	0	0	0	0	1	0	1	1	0
10)	1	0	0	0	0	1	1	0	1	0

Table 3: Affinely independent points that lie in the face defined by (30)