On generalized-convex constrained multi-objective optimization*

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Abstract

In this paper, we consider multi-objective optimization problems involving not necessarily convex constraints and componentwise generalized-convex (e.g., semi-strictly quasi-convex, quasi-convex, or explicitly quasi-convex) vector-valued objective functions that are acting between a real linear topological pre-image space and a finite dimensional image space. For these multi-objective optimization problems, we show that the set of (strictly, weakly) efficient solutions can be computed completely by using at most two corresponding multi-objective optimization problems with a new feasible set that is a convex upper set of the original feasible set. Our approach relies on the fact that the original feasible set can be described using level sets of a certain real-valued function (a kind of penalization function). Finally, we apply our approach to problems where the constraints are given by a system of inequalities with a finite number of constraint functions.

Keywords: Multi-objective optimization, Pareto efficiency, Generalized convexity, Constrained optimization, Convex and nonconvex constraints, Slater constraint qualification

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1 Introduction

Convexity plays a crucial role in optimization theory (see, e.g., the books of convex analysis by Hiriart-Urruty and Lemaréchal [14], Rockafellar [28] and Zălinescu [31]). In the last decades several new classes of functions are obtained by preserving several

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fundamental properties of convex functions, for instance, the following classes of generalized-convex functions $f : \mathbb{R}^n \to \mathbb{R}$ (see, e.g., Cambini and Martein [5] and Popovici [25] for more details):

- *Quasi-convex functions*: The level sets of f are convex for each level (i.e., the set of minimal solutions of f on $\mathbb{R}^n$ is a convex set in $\mathbb{R}^n$).
- *Semi-strictly quasi-convex functions*: Each local minimum point of f on $\mathbb{R}^n$ is also a global minimum point.
- *Explicitly quasi-convex functions*: Each local maximum point of f on $\mathbb{R}^n$ is actually a global minimum point (see Bagdasar and Popovici [3]).
- *Pseudo-convex functions* (e.g., in the sense of Mäkelä *et al.* [19]): 0 belongs to the Clarke subdifferential of f at a point $x \in \mathbb{R}^n$ if and only if x is global minimum point of f on $\mathbb{R}^n$.

Of course, as generalization of convexity, every convex function is quasi-convex, semi-strictly quasi-convex, explicitly quasi-convex as well as pseudo-convex. Generalized convexity assumptions appear in several branches of applications, e.g., production theory, utility theory or location theory (see, e.g., Cambini and Martein [5, Sec. 2.4]).

Since the area of multi-objective optimization has gained more and more interest, many authors studied generalized-convexity in a multi-objective optimization setting (see, e.g., Günther and Tammer [12], Mäkelä *et al.* [19], Malivert and Boissard [20], Popovici [22–25], Puerto and Rodríguez-Chía [26]). In multi-objective optimization (see, e.g., the books by Ehrgott [7], Göpfert *et al.* [11], Jahn [15]), one considers optimization problems with several conflicting objective functions. Depending on the application in practice, these problems often involve certain constraints.

The aim of this paper is to study the relationships between multi-objective optimization problems involving not necessarily convex constraints and multi-objective optimization problems involving convex constraints. In the literature, there exist techniques for solving different classes of constrained multi-objective optimization problems using corresponding unconstrained problems with an objective function that involves certain penalization terms in the component functions (see, e.g., Apetrii *et al.* [2] and Ye [29]), and, respectively, additional penalization functions (see, e.g., Durea *et al.* [6], Günther and Tammer [12], Klamroth and Tind [17]).

In the paper by Günther and Tammer [12], multi-objective optimization problems with convex constraints in finite dimensional spaces are considered and a certain gauge distance function is used as an additional penalization function. Now, we will extend and generalize the results in [12] to problems with nonconvex constraints and a real topological linear pre-image space. In our approach, the vector-valued objective function of the considered multi-objective optimization problem is assumed to be componentwise generalized-convex (e.g., semi-strictly quasi-convex, quasi-convex, or explicitly quasi-convex). We show that the set of efficient solutions of a multi-objective optimization problem involving a nonempty closed (not necessarily convex) feasible set, can be computed completely by using at most two corresponding multi-objective optimization problems with a new feasible set that is a convex upper set of the original feasible set. This means, we can apply methods (see Robinson [27] for the scalar case) that use the special structure of convex feasible sets for solving at most two multi-objective problems with convex feasible sets in order to solve the original multi-objective optimization problem with a nonconvex feasible set. Our approach relies on the fact that the original feasible set can be described using level sets of a certain real-valued function (a kind of penalization function).

The paper is organized as follows. After some preliminaries in Section 2 and a short introduction of generalized-convexity and semi-continuity properties in Section 3, we recall solution concepts for the vector-valued minimization in our constrained multi-objective optimization problem in Section 4. Moreover, we present an extended multi-objective optimization problem where we add one additional objective function (a kind of penalization function) to the objective functions given in the original multi-objective optimization problem. We present important examples of such penalization functions.

In Section 5, we derive the main results of our paper, the relationships between the original multi-objective optimization problem with a generalized-convex objective function involving a not necessarily convex feasible set, and two corresponding multi-objective optimization problems with a new feasible set that is a convex upper set of the original feasible set.

In Section 6, after introducing local generalized-convexity concepts, we present sufficient conditions for the validity of certain assumptions that are important in our new penalization approach.

Furthermore, in Section 7 we apply our approach to problems where the constraints are given by a system of inequalities with a finite number of constraint functions.

Section 8 contains some concluding remarks.

2 Preliminaries

Throughout this article, let $\mathcal{V}$ be a real topological linear space. In certain results, we assume that $\mathcal{V}$ is a normed space equipped with the norm $||\cdot|| : \mathcal{V} \to \mathbb{R}$. In this case the topology of $\mathcal{V}$ should be generated by the metric induced by the norm $||\cdot||$. Moreover, let the q-dimensional normed Euclidean space denoted by $\mathbb{R}^q$, $q \in \mathbb{N}$. For a nonempty set $\Omega \subseteq \mathcal{V}$, the expressions $\operatorname{cl} \Omega$, $\operatorname{bd} \Omega$, $\operatorname{int} \Omega$ stand for the standard notions of closure, boundary, interior of Ω, respectively. The cardinality of the set Ω is denoted by $\operatorname{card}(\Omega)$. For two points $a, b \in \mathcal{V}$ we define $[a, b] := \{(1-\lambda)a + \lambda b \mid \lambda \in [0, 1]\}$, $(a, b) := [a, b] \setminus \{a, b\}$, $[a, b) := [a, b] \setminus \{b\}$ and $(a, b] := [a, b] \setminus \{a\}$.

Let $\Omega \subseteq \mathcal{V}$. Considering a metric $d : \Omega \times \Omega \to \mathbb{R}$, we define the open ball around $x^0 \in \mathcal{V}$ of radius $\varepsilon > 0$ by

$$B_d(x^0, \varepsilon) := \{x^1 \in \mathcal{V} \mid d(x^0, x^1) < \varepsilon\},$$

while the closed ball around $x^0 \in \mathcal{V}$ of radius $\varepsilon > 0$ is given by

$$\overline{B}_d(x^0, \varepsilon) := \{x^1 \in \mathcal{V} \mid d(x^0, x^1) \leq \varepsilon\}.$$

If d is induced by a norm $||\cdot||$, then we write $B_{||\cdot||}(x^0, \varepsilon)$ and $\overline{B}_{||\cdot||}(x^0, \varepsilon)$.

The core (algebraic interior) of a nonempty set Ω is given by

$$\operatorname{cor}\Omega := \{x^0 \in \Omega \,|\, \forall\, v \in \mathcal{V}\, \exists\, \delta > 0 :\ x^0 + [0,\delta] \cdot v \subseteq \Omega\}.$$

The statements given in the next lemma are well-known (for more details, see Jahn [15] or Popovici [25]).

Lemma 2.1 *Let Ω be a nonempty set in a real topological linear space $\mathcal{V}$.*

1°. *It holds that* $\operatorname{int}\Omega \subseteq \operatorname{cor}\Omega$.

2°. *If* $\operatorname{int}\Omega \neq \emptyset$ *and Ω is convex, then* $\operatorname{int}\Omega = \operatorname{cor}\Omega$.

In the proofs of Lemmata 6.4 and 6.10, we will use the following property for interior points of a nonempty set Ω in a real normed space $\mathcal{V}$.

Lemma 2.2 *Let Ω be a set in a real normed space $(\mathcal{V}, ||\cdot||)$ with* $\operatorname{int}\Omega \neq \emptyset$. *Consider $x^0 \in \operatorname{int}\Omega$, i.e., it exists $\varepsilon > 0$ such that $B_{||\cdot||}(x^0, \varepsilon) \subseteq \Omega$. Then, for all $v \in \mathcal{V}$ with $||v|| = 1$ and all $\delta \in (0, \varepsilon)$, we have*

$$[x^0 - \delta v, x^0 + \delta v] \subseteq B_{||\cdot||}(x^0, \varepsilon) \subseteq \Omega.$$

Remark 2.3 1°. *Notice that the statements given in Lemma 2.1 and Lemma 2.2 are not true in general metric spaces. Consider the metric space $(\mathbb{R}^2, d)$, where $d : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ represents the discrete metric on $\mathbb{R}^2$ that is defined by $d(x, y) = 1$ for all $x, y \in \mathbb{R}^2$ with $x \neq y$ and $d(x, y) = 0$ for $x = y$. The feasible set is given by $\Omega := [-1, 1] \times [-1, 1]$. Now, it can easily be seen that $x^0 := (1, 1) \in \operatorname{int}\Omega$, since we have $B_d(x^0, \varepsilon) = \{x^0\} \subseteq \Omega$ for $\varepsilon \in (0, 1)$. However, for $v := \frac{x^1 - x^0}{||x^1 - x^0||}$ with $x^1 := (2, 2) \neq x^0$, we have $x^0 + \delta v \in \mathbb{R}^2 \setminus \Omega$ for all $\delta > 0$.*

2°. *It is important to note that the metric space $(\mathbb{R}^2, d)$ with the discrete metric d is a topological space (considering the discrete metric topology associated with d) but not a real topological linear space as well as not a real normed space (d is not derived from a norm).*

3°. *If $\mathcal{V}$ is a real linear space and d is a metric on $\mathcal{V}$ that is invariant with respect to translation as well as homogeneous, then $d(\cdot, 0) =: ||\cdot||: \mathcal{V} \to \mathbb{R}$ defines a norm on $\mathcal{V}$.*

In what follows, we define further notions that will be used in the sequel.

Let $h : \mathcal{V} \to \mathbb{R}$ be a real-valued function, X be a nonempty set in $\mathcal{V}$ and $s \in \mathbb{R}$. Then, the (strict) lower-level set and the level line of h to the level s are defined in the usual way by

$$L_{\sim}(X, h, s) := \{x \in X \ \mid \ h(x) \sim s\} \quad \text{for all } \sim \in \{<, \leq, =\}.$$

Note, for any set Y with $X \subseteq Y \subseteq \mathcal{V}$, we have

$$L_{\sim}(X, h, s) = L_{\sim}(Y, h, s) \cap X \quad \text{for all } \sim \in \{<, \leq, =\}.$$

Moreover, the (strict) upper-level set of h to the level s are defined by

$$L_>(X, h, s) := L_<(X, -h, -s) \text{ and } L_\geq(X, h, s) := L_\leq(X, -h, -s).$$

For notational convenience, for any $m \in \mathbb{N}$, we introduce the index set

$$I_m := \{1, \ldots, m\}.$$

Consider a function $f = (f_1, \ldots, f_m)$ with $f_i : \mathcal{V} \to \mathbb{R}$ for all $i \in I_m$. For any $x^0 \in X$ we define the intersections of (strict) lower-level sets / level lines by

$$S_\sim(X, f, x^0) := \bigcap_{i \in I_m} L_\sim(X, f_i, f_i(x^0)) \quad \text{for all } \sim \in \{<, \leq, =\}.$$

3 Semi-continuity and generalized-convexity properties

In this section, we recall some definitions and facts about generalized-convex and semi-continuous functions (see, e.g., Cambini and Martein [5], Giorgi *et al.* [10], and Popovici [25]).

In order to operate with certain generalized-convexity and semi-continuity notions, we define, for any $(x^0, x^1) \in \mathcal{V} \times \mathcal{V}$, the function $l_{x^0,x^1} : [0, 1] \to \mathcal{V}$,

$$l_{x^0,x^1}(\lambda) := (1 - \lambda)x^0 + \lambda x^1 \quad \text{for all } \lambda \in [0, 1].$$

Consider a convex set $X \subseteq \mathcal{V}$. Recall that a function $h : \mathcal{V} \to \mathbb{R}$ is

- *upper (lower) semi-continuous along line segments* on X if the composition $(h \circ l_{x^0,x^1}) : [0, 1] \to \mathbb{R}$ is upper (lower) semi-continuous on $[0, 1]$ for all $x^0, x^1 \in X$.
- *convex* on X if for all $x^0, x^1 \in X$ and for all $\lambda \in [0, 1]$ we have $h(l_{x^0,x^1}(\lambda)) \leq (1 - \lambda)h(x^0) + \lambda h(x^1)$.
- *quasi-convex* on X if for all $x^0, x^1 \in X$ and for all $\lambda \in [0, 1]$ we have $h(l_{x^0,x^1}(\lambda)) \leq \max\big(h(x^0), h(x^1)\big)$.
- *semi-strictly quasi-convex* on X if for all $x^0, x^1 \in X$, $h(x^0) \neq h(x^1)$, and for all $\lambda \in (0, 1)$ we have $h(l_{x^0,x^1}(\lambda)) < \max\big(h(x^0), h(x^1)\big)$.
- *explicitly quasi-convex* on X if h is both quasi-convex and semi-strictly quasi-convex on X.

A function $f : \mathcal{V} \to \mathbb{R}^m$ is called componentwise upper (lower) semi-continuous along line segments / convex / (semi-strictly, explicitly) quasi-convex / semi-strictly quasi-convex or quasi-convex on X if f_i is upper (lower) semi-continuous along line segments / convex / (semi-strictly, explicitly) quasi-convex / semi-strictly quasi-convex or quasi-convex on X for all $i \in I_m$.

Remark 3.1 *Notice that each convex function is explicitly quasi-convex and upper semi-continuous along line segments. Moreover, a semi-strictly quasi-convex function which is lower semi-continuous along line segments is explicitly quasi-convex. Counterexamples for the reverse implications are given in Example 3.2.*

Cambini and Martein [5] pointed out important applications of generalized-convexity. For instance, there are certain relationships between the field of generalized-convexity and fractional programming (see [5, Th. 2.3.8, Ch. 6, Ch. 7]). Moreover, in [5, Sec. 2.4] examples of quasi-concave classes of homogeneous functions that appear frequently in Economics (e.g., in utility and production theory) are provided. Since maximizing a generalized-concave function is equivalent to minimizing the negative of this function (a generalized-convex function), such functions from Economics (e.g., the Cobb-Douglas function) are important examples for our work.

Example 3.2 *Consider the set $X := \mathbb{R}$. The function $h : \mathbb{R} \to \mathbb{R}$ defined by $h(x) := x^3$ for all $x \in \mathbb{R}$ is explicitly quasi-convex and continuous but not convex on X. Furthermore, the function $h : \mathbb{R} \to \mathbb{R}$ given by*

$$h(x) := \begin{cases} (x-1)^3 & \text{for all } x > 1, \\ 0 & \text{for all } x \in [-1, 1], \\ (x+1)^3 & \text{for all } x < -1 \end{cases}$$

is quasi-convex and continuous but not semi-strictly quasi-convex on X. A semi-strictly quasi-convex functions which is upper semi-continuous along line segments must not be quasi-convex (e.g., consider the function $h_1 : \mathbb{R} \to \mathbb{R}$ given in Example 3.8).

It is well-known that quasi-convex functions are characterized by the convexity of its lower-level sets. Next, we present a useful equivalent characterization of semi-strictly quasi-convexity using level sets and level lines.

Lemma 3.3 *Let $h : \mathcal{V} \to \mathbb{R}$ be a function and X be a convex set in $\mathcal{V}$. Then, the following statements are equivalent:*

1°. *h is semi-strictly quasi-convex on X.*

2°. *For all $s \in \mathbb{R}$, $x^0 \in L_=(X, h, s)$, $x^1 \in L_<(X, h, s)$, we have $l_{x^0,x^1}(\lambda) \in L_<(X, h, s)$ for all $\lambda \in (0, 1]$.*

The next lemma (see Popovici [22, Prop. 2], [25, Prop. 2.1.2]) is important for the proofs of Lemmata 4.4 and 6.16.

Lemma 3.4 *Let $h : \mathcal{V} \to \mathbb{R}$ be a semi-strictly quasi-convex function on a nonempty convex set $X \subseteq \mathcal{V}$. Then, for every $(x^0, x^1) \in X \times X$, the set*

$$L_> \left((0, 1), (h \circ l_{x^0,x^1}), \max(h(x^0), h(x^1))\right)$$

is either a singleton set or the empty set.

In the following lemma, we recall useful equivalent characterizations of upper and lower semi-continuity.

Lemma 3.5 *Let $h : \mathcal{V} \to \mathbb{R}$ be a function and X be a nonempty closed set in $\mathcal{V}$. Then, the following statements are equivalent:*

1°. *h is upper (lower) semi-continuous on X.*

2°. *$L_\geq(X, h, s)$ ($L_\leq(X, h, s)$) is closed for all $s \in \mathbb{R}$.*

Proof: A proof for the case $X = \mathcal{V}$ can be found in Barbu and Precupanu [4, Prop. 2.5].

Let I_X be the indicator function concerning the set X, i.e., $I_X(x)$ is 0 for x in X and $+\infty$ otherwise. Since X is closed, we know that I_X is lower semi-continuous on $\mathcal{V}$ (see Barbu and Precupanu [4, Cor. 2.7]). Then, the following statements are equivalent (compare Zeidler [32]):

- h is lower semi-continuous on X.
- $\tilde{h} := h + I_X$ is lower semi-continuous on $\mathcal{V}$.
- $L_{\leq}(\mathcal{V}, \tilde{h}, s)$ is closed for all $s \in \mathbb{R}$.
- $L_{\leq}(X, h, s)$ is closed for all $s \in \mathbb{R}$.

Notice that we have $L_{\leq}(\mathcal{V}, \tilde{h}, s) = L_{\leq}(X, h, s)$ for every $s \in \mathbb{R}$ because of $\tilde{h}(x) = +\infty$ for all $x \notin X$.

Moreover, we know that h is upper semi-continuous on X if and only if $-h$ is lower semi-continuous on X. Since $L_{\leq}(X, -h, s) = L_{\geq}(X, h, -s)$ for all $s \in \mathbb{R}$, the result for upper semi-continuity follows immediately. $\square$

Remark 3.6 *It is important to note that the assertion of Lemma 3.5 does not hold if X is not supposed to be closed, e.g., consider the continuous function $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) := 1$ for every $x \in \mathbb{R}$, and $X := (0, 1)$, then the set $L_{\leq}(X, f, 1) = X$ is not closed. The closedness assumption of X is missing in Günther and Tammer [12, Lem. 5], and hence must be added. Only one result, namely [12, Th. 4], uses the characterization given in [12, Lem. 5]. In this result a closed set of the space $\mathbb{R}$ is considered, consequently the result given in [12, Th. 4] still holds.*

In Section 7 we are interested in considering the function defined by the maximum of a finite number of scalar functions $h_i : \mathcal{V} \to \mathbb{R}$, $i \in I_l$, $l \in \mathbb{N}$. In the next lemma, we recall some important properties of this function.

Lemma 3.7 *Let a family of functions $h_i : \mathcal{V} \to \mathbb{R}$, $i \in I_l$, be given. Define the maximum of h_i, $i \in I_l$, by $h_{\max}(x) := \max_{i \in I_l} h_i(x)$ for all $x \in \mathcal{V}$. Suppose that X is a nonempty set in $\mathcal{V}$. Then, we have*

1°. *Assume that X is closed. If h_i, $i \in I_l$, are lower semi-continuous on X, then $h_{\max}$ is lower semi-continuous on X.*

2°. *Assume that X is convex. If h_i, $i \in I_l$, are convex on X, then $h_{\max}$ is convex on X.*

3°. *Assume that X is convex. If h_i, $i \in I_l$, are quasi-convex on X, then $h_{\max}$ is quasi-convex on X.*

In the next example, we show that an analogous statement to 2° and 3° of Lemma 3.7 does not hold for the concept of semi-strict quasi-convexity.

Example 3.8 *Consider the set $X := \mathbb{R}$ and two functions $h_i : \mathbb{R} \to \mathbb{R}$, $i \in I_2$, defined by $h_i(x) := 0$ for all $x \in X$, $x \neq i$, and $h_i(i) := 1$. Notice that h_1 and h_2 are semi-strictly quasi-convex on X. Let $h_{\max}$ be given by*

$$h_{\max}(x) := \begin{cases} 0 & \text{for all } x \in \mathbb{R} \setminus \{1, 2\}, \\ 1 & \text{for all } x \in \{1, 2\}. \end{cases}$$

Since $h_{\max}(0) = 0 < 1 = h_{\max}(1) = h_{\max}(2)$, we get that $h_{\max}$ is not semi-strictly quasi-convex on X.

4 A penalization approach in constrained multi-objective optimization

In this section, we present a new penalization approach for multi-objective optimization problems involving a not necessarily convex feasible set.

Throughout this article, we suppose that the following standard assumption is fulfilled:

$$\begin{cases} \text{Let } \mathcal{V} \text{ be a real topological linear space;} \\ \text{let } X \subseteq \mathcal{V} \text{ be a nonempty closed set with } X \neq \mathcal{V}; \\ \text{let } Y \subseteq \mathcal{V} \text{ be a convex set with } X \subseteq Y. \end{cases} \tag{4.1}$$

Remark 4.1 *Notice, under the assumptions given in* (4.1)*, we have that* $\operatorname{bd} X \neq \emptyset$*. This can be seen by the following observations:*

- *A topological space $\mathcal{V}$ is connected if and only if the only closed and open sets are the empty set and $\mathcal{V}$. Hence, the closedness of X and the assumption $\emptyset \neq X \neq \mathcal{V}$ imply that X is not open.*
- *Due to*

  $$\operatorname{int} X \subseteq X \subseteq \operatorname{cl} X = \operatorname{int} X \cup \operatorname{bd} X$$

  and the closedness of X (i.e., $X = \operatorname{cl} X$), it follows that X is not open if and only if $\operatorname{bd} X \neq \emptyset$.
- *Each real topological linear space $\mathcal{V}$ is connected.*

For the case $\operatorname{bd} X = \emptyset$ *(hence X is open), we refer to Corollary 4.6.*

4.1 The original multi-objective optimization problem $(\mathcal{P}_X)$

In this paper, we consider a multi-objective optimization problem involving m objective functions $f_1, \ldots, f_m : \mathcal{V} \to \mathbb{R}$ and a nonempty (not necessarily convex) feasible set $X \subseteq \mathcal{V}$ in the topological linear space $\mathcal{V}$:

$$f(x) := \begin{pmatrix} f_1(x) \\ \vdots \\ f_m(x) \end{pmatrix} \to \operatorname*{v-min}_{x \in X}. \tag{$\mathcal{P}_X$}$$

In the next definition, we recall solution concepts for the vector-valued minimization considered in problem $(\mathcal{P}_X)$ (see, e.g., Ehrgott [7], Jahn [15] and Khan *et al.*

[16] for more details). Notice that $f[X] := \{f(x) \in \mathbb{R}^m \mid x \in X\}$ denotes the image set of f over X, while $\mathbb{R}^m_+$ stands for the natural ordering cone in $\mathbb{R}^m$.

Definition 4.2 *Let $X \subseteq \mathcal{V}$ be a nonempty set. The set of Pareto efficient solutions of problem $(\mathcal{P}_X)$ with respect to $\mathbb{R}^m_+$ is defined by*

$$\mathrm{Eff}(X \mid f) := \{x^0 \in X \mid f[X] \cap (f(x^0) - (\mathbb{R}^m_+ \setminus \{0\})) = \emptyset\},$$

while that of weakly Pareto efficient solutions is given by

$$\mathrm{WEff}(X \mid f) := \{x^0 \in X \mid f[X] \cap (f(x^0) - (\operatorname{int} \mathbb{R}^m_+)) = \emptyset\}.$$

The set of strictly Pareto efficient solutions is defined by

$$\mathrm{SEff}(X \mid f) := \{x^0 \in \mathrm{Eff}(X \mid f) \mid \mathrm{card}(\{x \in X \mid f(x) = f(x^0)\}) = 1\}.$$

It can easily be seen that we have

$$\mathrm{SEff}(X \mid f) \subseteq \mathrm{Eff}(X \mid f) \subseteq \mathrm{WEff}(X \mid f).$$

In the next lemma, we recall useful characterizations of (strictly, weakly) efficient solutions using certain level sets and level lines of the component functions of f (see, e.g., Ehrgott [7, Th. 2.30]).

Lemma 4.3 *Let $X \subseteq \mathcal{V}$ be a nonempty set. For any $x^0 \in X$, we have*

$$\begin{aligned} x^0 \in \mathrm{Eff}(X \mid f) &\iff S_{\leq}(X, f, x^0) \subseteq S_{=}(X, f, x^0); \\ x^0 \in \mathrm{WEff}(X \mid f) &\iff S_{<}(X, f, x^0) = \emptyset; \\ x^0 \in \mathrm{SEff}(X \mid f) &\iff S_{\leq}(X, f, x^0) = \{x^0\}. \end{aligned}$$

In addition to the original problem $(\mathcal{P}_X)$, for a convex set Y with $X \subseteq Y \subseteq \mathcal{V}$, we consider a new multi-objective optimization problem $(\mathcal{P}_Y)$ that consists of minimizing the original objective function f of the problem $(\mathcal{P}_X)$ over the convex feasible set Y.

4.2 Relationships between the problems $(\mathcal{P}_X)$ and $(\mathcal{P}_Y)$

In this section, we present some useful relationships between the problems $(\mathcal{P}_X)$ and $(\mathcal{P}_Y)$. These relationships generalize the corresponding results derived by Günther and Tammer [12], where $\mathcal{V} = Y = \mathbb{R}^n$ and the topological interior instead of the algebraic interior of X is considered.

Lemma 4.4 *Let X be a nonempty set and Y be a convex set such that $X \subseteq Y \subseteq \mathcal{V}$. Then, we have*

1°. *It holds that*

$$\begin{aligned} X \cap \mathrm{Eff}(Y \mid f) &\subseteq \mathrm{Eff}(X \mid f); \\ X \cap \mathrm{WEff}(Y \mid f) &\subseteq \mathrm{WEff}(X \mid f); \\ X \cap \mathrm{SEff}(Y \mid f) &\subseteq \mathrm{SEff}(X \mid f). \end{aligned}$$

2°. *Assume that $f : \mathcal{V} \to \mathbb{R}^m$ is componentwise semi-strictly quasi-convex on Y. Then, it holds that*

$$(\operatorname{cor} X) \setminus \operatorname{Eff}(Y \mid f) \subseteq (\operatorname{cor} X) \setminus \operatorname{Eff}(X \mid f);$$
$$(\operatorname{cor} X) \setminus \operatorname{WEff}(Y \mid f) \subseteq (\operatorname{cor} X) \setminus \operatorname{WEff}(X \mid f).$$

3°. *If $f : \mathcal{V} \to \mathbb{R}^m$ is componentwise semi-strictly quasi-convex or quasi-convex on Y, then*

$$(\operatorname{cor} X) \setminus \operatorname{SEff}(Y \mid f) \subseteq (\operatorname{cor} X) \setminus \operatorname{SEff}(X \mid f).$$

Proof:

1°. Follows easily by Lemma 4.3.

2°. We are going to show the first inclusion. Consider $x^0 \in (\operatorname{cor} X) \setminus \operatorname{Eff}(Y \mid f)$. Since $x^0 \notin \operatorname{Eff}(Y \mid f)$, there exists $x^1 \in L_<(Y, f_j, f_j(x^0)) \cap S_\leq(Y, f, x^0)$ for some $j \in I_m$. We define the following two index sets

$$I^< := \{i \in I_m \mid x^1 \in L_<(Y, f_i, f_i(x^0))\},$$
$$I^= := \{i \in I_m \mid x^1 \in L_=(Y, f_i, f_i(x^0))\}.$$

Of course, we know that $I^< \neq \emptyset$ and $I^= \cup I^< = I_m$.

Clearly, for $x^1 \in X$, we get immediately $x^0 \in (\operatorname{cor} X) \setminus \operatorname{Eff}(X \mid f)$. Now, assume $x^1 \in Y \setminus X$. Since $x^0 \in \operatorname{cor} X$, by Lemma 2.1, we get $x^0 + [0, \delta] \cdot v \subseteq X$ for $v := x^1 - x^0 \neq 0$ and some $\delta > 0$. Obviously, since $x^1 \notin X$, it follows $\delta \in (0, 1)$. Hence, for $\lambda^* := \delta \in (0, 1)$, we have $x^\lambda := l_{x^0,x^1}(\lambda) \in X \cap (x^0, x^1)$ for all $\lambda \in (0, \lambda^*]$.

Now, consider two cases:

Case 1: Consider $i \in I^<$. The semi-strict quasi-convexity of f_i on Y implies $x^\lambda \in L_<(Y, f_i, f_i(x^0))$ for all $\lambda \in (0, 1]$ by Lemma 3.3. Because of $x^\lambda \in X$ for all $\lambda \in (0, \lambda^*]$, we get $x^\lambda \in L_<(X, f_i, f_i(x^0))$ for all $\lambda \in (0, \lambda^*]$.

Case 2: Consider $i \in I^=$. This means that $f_i(x^1) = f_i(x^0)$. By Lemma 3.4 (with Y in the role of X), we infer that

$$\operatorname{card}\big(L_>((0, 1), (f_i \circ l_{x^0,x^1}), f_i(x^0))\big) \leq 1.$$

In the case that $\operatorname{card}\big(L_>((0, 1), (f_i \circ l_{x^0,x^1}), f_i(x^0))\big) = 1$, we get that there exists $\lambda_i \in (0, 1)$ such that $f_i(l_{x^0,x^1}(\lambda_i)) > f_i(x^0)$. Otherwise we define $\lambda_i := 2\lambda^*$.

For $\overline{\lambda} := \min(\lambda^*, 0.5 \cdot \min\{\lambda_i \,|\, i \in I^=\})$, it follows that $x^{\overline{\lambda}} \in L_\leq(X, f_i, f_i(x^0))$ for all $i \in I^=$ as well as $x^{\overline{\lambda}} \in L_<(X, f_i, f_i(x^0))$ for all $i \in I^<$. So, we get $x^0 \in (\operatorname{cor} X) \setminus \operatorname{Eff}(X \mid f)$ by Lemma 4.3.

The proof of the second inclusion is analogous to the proof of the first inclusion in 2°. Notice that $I^< = I_m$ and $I^= = \emptyset$.

3°. Consider $x^0 \in (\operatorname{cor} X) \setminus \mathrm{SEff}(Y \mid f)$. Since $x^0 \notin \mathrm{SEff}(Y \mid f)$, it exists $x^1 \in Y \setminus \{x^0\}$ such that $x^1 \in S_{\leq}(Y, f, x^0)$. Of course, since $x^1 \in X$, we get $x^0 \in (\operatorname{cor} X) \setminus \mathrm{SEff}(X \mid f)$. Now, assume $x^1 \in Y \setminus X$. Analogously to the proof of statement 2° in this lemma, there exists $\lambda^* \in (0,1)$ such that $x^\lambda := l_{x^0,x^1}(\lambda) \in X \cap (x^0, x^1)$ for all $\lambda \in (0, \lambda^*]$.

Let $i \in I_m$ and consider two cases:

Case 1: Let f_i be semi-strictly quasi-convex on Y. Analogously to the proof of statement 2° of this lemma, we get that there exists $\lambda_i \in (0, \lambda^*]$ with $x^\lambda \in L_{\leq}(X, f_i, f_i(x^0))$ for all $\lambda \in (0, \lambda_i]$.

Case 2: Let f_i be quasi-convex on Y. By the convexity of the level sets of f_i, we conclude $[x^0, x^1] \subseteq L_{\leq}(Y, f_i, f_i(x^0))$ for $x^0, x^1 \in L_{\leq}(Y, f_i, f_i(x^0))$. We put $\lambda_i := \lambda^*$.

Hence, for $\overline{\lambda} := \min\{\lambda_i \mid i \in I_m\}$, it follows that $x^{\overline{\lambda}} \in S_{\leq}(X, f, x^0) \setminus \{x^0\}$. By Lemma 4.3, we get $x^0 \in (\operatorname{cor} X) \setminus \mathrm{SEff}(X \mid f)$.

□

Notice that the proof of Lemma 4.4 uses ideas given in Günther and Tammer [12]. The semi-strict quasi-convexity assumption with respect to f can not be replaced by a quasi-convexity assumption in 2° of Lemma 4.4 (see [12, Ex. 2]).

The following corollary gives useful bounds for the sets of (strictly, weakly) efficient solutions of the problem $(\mathcal{P}_X)$ under generalized-convexity assumption on f but without convexity assumption on the feasible set X.

Corollary 4.5 *Let X be a nonempty set and Y be a convex set such that $X \subseteq Y \subseteq \mathcal{V}$. Then, we have*

1°. *If $f : \mathcal{V} \to \mathbb{R}^m$ is componentwise semi-strictly quasi-convex on Y, then*

$$\begin{aligned} X \cap \mathrm{Eff}(Y \mid f) &\subseteq \mathrm{Eff}(X \mid f) \subseteq [X \cap \mathrm{Eff}(Y \mid f)] \cup \operatorname{bd} X; \\ X \cap \mathrm{WEff}(Y \mid f) &\subseteq \mathrm{WEff}(X \mid f) \subseteq [X \cap \mathrm{WEff}(Y \mid f)] \cup \operatorname{bd} X. \end{aligned}$$

2°. *If $f : \mathcal{V} \to \mathbb{R}^m$ is componentwise semi-strictly quasi-convex or quasi-convex on Y, then*

$$X \cap \mathrm{SEff}(Y \mid f) \subseteq \mathrm{SEff}(X \mid f) \subseteq [X \cap \mathrm{SEff}(Y \mid f)] \cup \operatorname{bd} X.$$

By Corollary 4.5 and under the assumption that the set X is open, we get the following result.

Corollary 4.6 *Let X be a nonempty open set and Y be a convex set such that $X \subseteq Y \subseteq \mathcal{V}$. Then, we have*

1°. *If $f : \mathcal{V} \to \mathbb{R}^m$ is componentwise semi-strictly quasi-convex on Y, then*

$$\begin{aligned} X \cap \mathrm{Eff}(Y \mid f) &= \mathrm{Eff}(X \mid f); \\ X \cap \mathrm{WEff}(Y \mid f) &= \mathrm{WEff}(X \mid f). \end{aligned}$$

2°. *If* $f : \mathcal{V} \to \mathbb{R}^m$ *is componentwise semi-strictly quasi-convex or quasi-convex on* Y*, then*

$$X \cap \mathrm{SEff}(Y \mid f) = \mathrm{SEff}(X \mid f).$$

4.3 The penalized multi-objective optimization problem $(\mathcal{P}_Y^{\oplus})$

In our approach, under assumption (4.1), we add an additional real-valued penalization function $f_{m+1} : \mathcal{V} \to \mathbb{R}$ to the original objective function f of the problem $(\mathcal{P}_Y)$ as a new component function. So, the new penalized multi-objective optimization problem can be stated as

$$f^{\oplus}(x) := \begin{pmatrix} f(x) \\ f_{m+1}(x) \end{pmatrix} = \begin{pmatrix} f_1(x) \\ \vdots \\ f_m(x) \\ f_{m+1}(x) \end{pmatrix} \to \underset{x \in Y}{\text{v-min}}\,. \qquad (\mathcal{P}_Y^{\oplus})$$

In the sequel, we will need in certain results some of the following assumptions concerning the lower-level sets / level lines of the function f_{m+1}:

$$\forall x^0 \in \mathrm{bd}\, X : \; L_{\leq}(Y, f_{m+1}, f_{m+1}(x^0)) = X, \qquad \text{(A1)}$$

$$\forall x^0 \in \mathrm{bd}\, X : \; L_{=}(Y, f_{m+1}, f_{m+1}(x^0)) = \mathrm{bd}\, X, \qquad \text{(A2)}$$

$$\forall x^0 \in X : \; L_{=}(Y, f_{m+1}, f_{m+1}(x^0)) = L_{\leq}(Y, f_{m+1}, f_{m+1}(x^0)) = X, \qquad \text{(A3)}$$

$$\forall x^0 \in X : \; L_{\leq}(Y, f_{m+1}, f_{m+1}(x^0)) \subseteq X, \qquad \text{(A4)}$$

$$L_{\leq}(Y, f_{m+1}, 0) = X. \qquad \text{(A5)}$$

Remark 4.7 *Notice, under both Assumptions* (A1) *and* (A2)*, we have*

$$\forall x^0 \in \mathrm{bd}\, X : \; L_{<}(Y, f_{m+1}, f_{m+1}(x^0)) = \mathrm{int}\, X,$$

while under Assumption (A3) *it holds*

$$\forall x^0 \in X : \; L_{<}(Y, f_{m+1}, f_{m+1}(x^0)) = \emptyset.$$

In addition, the following statements hold:

- *If* $\mathrm{int}\, X = \emptyset$*, then* (A1) $\wedge$ (A2) $\iff$ (A3)*.*
- (A3) $\implies$ (A1)*.*
- (A1) $\vee$ (A3) $\vee$ (A5) $\implies$ (A4)*.*

By taking a look on the literature in single- as well as multi-objective optimization theory, one can see that many authors (see, e.g., Apetrii *et al.* [2], Durea *et al.* [6], Ye [29], and references therein) use a penalization function $\phi : \mathcal{V} \to \mathbb{R} \cup \{+\infty\}$ (penalty term concerning X) which fulfils Assumption (A3) for $Y = \mathcal{V}$ (ϕ in the role of f_{m+1}). This means, for $x^0 \in Y = \mathcal{V}$, we have

$$x^0 \in X \iff \phi(x^0) = 0$$

and

$$x^0 \in \mathcal{V} \setminus X \iff \phi(x^0) > 0.$$

Such a penalization function ϕ will be given in Example 4.9 with $\phi = f_{m+1} = d_X$ (compare with *Clarke's Exact Penalty Principle* in optimization; see Ye [29] for more details).

Next, we present some examples for the penalization function f_{m+1}.

Example 4.8 *Let $X \subseteq \mathcal{V}$ be a closed convex set with $\tilde{x} \in \operatorname{int} X \neq \emptyset$ and $X \neq \mathcal{V}$. Let a gauge function $\mu : \mathcal{V} \to \mathbb{R}$ be given by*

$$\mu(x) := \inf\{\lambda > 0 \,|\, x \in \lambda \cdot (-\tilde{x} + X)\} \quad \text{for all } x \in \mathcal{V}.$$

Then, the function

$$f_{m+1}(\,\cdot\,) := \mu(\,\cdot\, - \tilde{x})$$

fulfils (A1) *and* (A2) *for $Y = \mathcal{V}$ (see Günther and Tammer [12]).*

Example 4.9 *Let X be a nonempty closed set in a normed space $(\mathcal{V}, ||\cdot||)$ and let the distance function $d_X : \mathcal{V} \to \mathbb{R}$ be given by*

$$d_X(x) := \inf\{||x - z|| \,|\, z \in X\} \quad \text{for all } x \in \mathcal{V}.$$

We recall some important properties of d_X (see, e.g., Mordukhovich and Nam [21], and references therein):

- *d_X is Lipschitz continuous on $\mathcal{V}$ with Lipschitz constant 1;*
- *d_X is convex on $\mathcal{V}$ if and only if X is convex in $\mathcal{V}$;*
- *$L_{\leq}(\mathcal{V}, d_X, 0) = L_{=}(\mathcal{V}, d_X, 0) = X$.*

Hence, the penalization function

$$f_{m+1} := d_X$$

fulfils Assumptions (A3) *and* (A5) *for $Y = \mathcal{V}$.*

Example 4.10 *Let X be a nonempty closed set in a normed space $(\mathcal{V}, ||\cdot||)$ with $X \neq \mathcal{V}$. Based on the distance function $d_X : \mathcal{V} \to \mathbb{R}$ (see Example 4.9) one can define a function $\triangle_X : \mathcal{V} \to \mathbb{R}$ by*

$$\triangle_X(x) := d_X(x) - d_{\mathcal{V}\setminus X}(x) = \begin{cases} d_X(x) & \text{for } x \in \mathcal{V} \setminus X, \\ -d_{\mathcal{V}\setminus X}(x) & \text{for } x \in X. \end{cases}$$

The function $\triangle_X$ was introduced by Hiriart-Urruty [13] and is known in the literature as signed distance function or Hiriart-Urruty function. Next, we recall some well-known properties of $\triangle_X$ (see Hiriart-Urruty [13], Liu et al. [18], Zaffaroni [30]):

- *$\triangle_X$ is Lipschitz continuous on $\mathcal{V}$ with Lipschitz constant 1;*
- *$\triangle_X$ is convex on $\mathcal{V}$ if and only if X is convex in $\mathcal{V}$;*
- *$L_{\leq}(\mathcal{V}, \triangle_X, 0) = X$ and $L_{=}(\mathcal{V}, \triangle_X, 0) = \operatorname{bd} X$.*

It follows that the penalization function

$$f_{m+1} := \triangle_X$$

fulfils Assumptions (A1), (A2) *and* (A5) *for* $Y = \mathcal{V}$.

Example 4.11 *In this example, we consider a nonlinear function introduced as a scalarizing tool in multi-objective optimization by Gerth and Weidner [9]. Let* $\mathcal{V}$ *be a normed space* $(\mathcal{V}, ||\cdot||)$. *Assume that* $C \subseteq \mathcal{V}$ *is a proper closed convex cone,* $k \in \operatorname{int} C \setminus (-C)$ *and* $X \subseteq \mathcal{V}$, $X \neq \mathcal{V}$, *is a nonempty closed set such that*

$$X - (C \setminus \{0\}) = \operatorname{int} X.$$

The function $\phi_{X,k} : \mathcal{V} \to \mathbb{R}$ *defined, for any* $x \in \mathcal{V}$, *by*

$$\phi_{X,k}(x) := \inf\{s \in \mathbb{R} \,|\, x \in sk + X\}$$

is finite-valued and fulfils the important properties (compare Khan et al. [16, Sec. 5.2]):

- $\phi_{X,k}$ *is Lipschitz continuous on* $\mathcal{V}$*;*
- $\phi_{X,k}$ *is convex on* $\mathcal{V}$ *if and only if* X *is convex in* $\mathcal{V}$*;*
- $L_{\leq}(\mathcal{V}, \phi_{X,k}, 0) = X$ *and* $L_{=}(\mathcal{V}, \phi_{X,k}, 0) = \operatorname{bd} X$.

This means that the penalization function

$$f_{m+1} := \phi_{X,k}$$

fulfils Assumptions (A1), (A2) *and* (A5) *for* $Y = \mathcal{V}$.

Examples 4.9, 4.10 and 4.11 show that a nonconvex set X can be considered in our approach. Let X be an arbitrarily nonempty closed set with $\emptyset \neq X \neq \mathcal{V}$. In a normed space $(\mathcal{V}, ||\cdot||)$ we know that the Hiriart-Urruty function $\triangle_X$ fulfils Assumptions (A1) and (A2), and moreover, the function d_X fulfils Assumption (A3). Hence, the results obtained in this paper extend and generalize the results given in the paper by Günther and Tammer [12], where X is supposed to be a convex set and a gauge function is used as penalization function (see Example 4.8).

5 Main results: Relationships between the multi-objective optimization problems $(\mathcal{P}_X)$, $(\mathcal{P}_Y)$ and $(\mathcal{P}_Y^{\oplus})$

In this section, under the assumptions given in (4.1), we study the relationships between the original multi-objective optimization problem $(\mathcal{P}_X)$ with a nonempty closed (not necessarily convex) feasible set $X \subseteq \mathcal{V}$, $X \neq \mathcal{V}$, and two related multi-objective optimization problems $(\mathcal{P}_Y)$ and $(\mathcal{P}_Y^{\oplus})$ with a convex feasible set $Y \subseteq \mathcal{V}$ for that $X \subseteq Y$.

We will generalize several results given by Günther and Tammer [12] that were derived under the assumptions

$$\begin{cases} \mathcal{V} = Y = \mathbb{R}^n; \\ X \text{ is a closed convex set in } \mathcal{V} \text{ with } \operatorname{int} X \neq \emptyset; \\ f_{m+1} \text{ is a special gauge function (see Example 4.8).} \end{cases} \tag{5.1}$$

Our new results presented in this section offer a way to solve nonconvex problems using algorithms for convex problems.

5.1 Sets of efficient solutions of $(\mathcal{P}_X)$, $(\mathcal{P}_Y)$ and $(\mathcal{P}_Y^{\oplus})$

In this section, we present relationships between the sets of efficient solutions of the problems $(\mathcal{P}_X)$, $(\mathcal{P}_Y)$ and $(\mathcal{P}_Y^{\oplus})$.

A first main result of the paper is given in the next theorem where the penalization function f_{m+1} satisfies Assumptions (A1) and (A2). Notice that a special case of Theorem 5.1 is considered in Günther and Tammer [12, Th. 1] under the assumptions given in (5.1). Moreover, in the proof of the following theorem we are using ideas given in [12, Th. 1].

Theorem 5.1 *Let* (4.1) *be satisfied. Suppose that f_{m+1} fulfils Assumptions* (A1) *and* (A2)*. Then, the following statements are true:*

1°. *It holds that*

$$[X \cap \operatorname{Eff}(Y \mid f)] \cup \left[(\operatorname{bd} X) \cap \operatorname{Eff}(Y \mid f^{\oplus})\right] \subseteq \operatorname{Eff}(X \mid f).$$

2°. *In the case* $\operatorname{int} X \neq \emptyset$*, suppose additionally that $f : \mathcal{V} \to \mathbb{R}^m$ is componentwise semi-strictly quasi-convex on Y. Then, we have*

$$[X \cap \operatorname{Eff}(Y \mid f)] \cup \left[(\operatorname{bd} X) \cap \operatorname{Eff}(Y \mid f^{\oplus})\right] \supseteq \operatorname{Eff}(X \mid f).$$

Proof:

1°. The inclusion $X \cap \operatorname{Eff}(Y \mid f) \subseteq \operatorname{Eff}(X \mid f)$ follows by Lemma 4.4. Consider $x^0 \in (\operatorname{bd} X) \cap \operatorname{Eff}(Y \mid f^{\oplus})$. By Lemma 4.3 (applied for $(\mathcal{P}_Y^{\oplus})$ instead of $(\mathcal{P}_X)$) and Assumptions (A1) and (A2), it follows

$$\begin{aligned} S_{\leq}(X, f, x^0) &= S_{\leq}(Y, f, x^0) \cap X \\ &= S_{\leq}(Y, f, x^0) \cap L_{\leq}(Y, f_{m+1}, f_{m+1}(x^0)) \\ &\subseteq S_{=}(Y, f, x^0) \cap L_{=}(Y, f_{m+1}, f_{m+1}(x^0)) \\ &= S_{=}(Y, f, x^0) \cap \operatorname{bd} X \\ &\subseteq S_{=}(Y, f, x^0) \cap X \\ &= S_{=}(X, f, x^0). \end{aligned}$$

Hence, we get $x^0 \in \operatorname{Eff}(X \mid f)$ by Lemma 4.3 (applied for $(\mathcal{P}_X)$).

2°. Let $x^0 \in \operatorname{Eff}(X \mid f) \subseteq X$. In the case $x^0 \in X \cap \operatorname{Eff}(Y \mid f)$, the inclusion holds. So, we consider the case $x^0 \in X \setminus \operatorname{Eff}(Y \mid f)$.

If $\operatorname{int} X = \emptyset$, it holds $x^0 \in \operatorname{bd} X$.
If $\operatorname{int} X \neq \emptyset$, we get $x^0 \in \operatorname{bd} X$ from Corollary 4.5, taking into account the componentwise semi-strictly quasi-convexity of f on Y.
By Lemma 4.3 (applied for $(\mathcal{P}_X)$) and Assumption (A1), we can conclude

$$\begin{aligned}
S_{\leq}(Y, f, x^0) \cap L_{\leq}(Y, f_{m+1}, f_{m+1}(x^0)) &= S_{\leq}(Y, f, x^0) \cap X \\
&= S_{\leq}(X, f, x^0) \\
&\subseteq S_{=}(X, f, x^0) \\
&= S_{=}(Y, f, x^0) \cap X.
\end{aligned}$$

Now, we will prove the equation

$$S_{=}(Y, f, x^0) \cap X = S_{=}(Y, f, x^0) \cap \operatorname{bd} X. \tag{5.2}$$

In the case that $\operatorname{int} X = \emptyset$, (5.2) is obviously fulfilled. For the case $\operatorname{int} X \neq \emptyset$, it is sufficient to prove $S_{=}(Y, f, x^0) \cap \operatorname{int} X = \emptyset$ in order to get the validity of (5.2). Indeed, if we suppose that there exists $x^1 \in \operatorname{int} X$ with $x^1 \in S_{=}(Y, f, x^0)$, then we have to discuss following two cases:

Case 1: If $x^1 \in (\operatorname{int} X) \setminus \operatorname{Eff}(Y \mid f)$, then we get $x^1 \in (\operatorname{int} X) \setminus \operatorname{Eff}(X \mid f)$ by Lemma 4.4 under the assumption that f is componentwise semi-strictly quasi-convex on Y. This implies $x^0 \in X \setminus \operatorname{Eff}(X \mid f)$ because of $x^1 \in S_{=}(X, f, x^0)$, a contradiction to $x^0 \in \operatorname{Eff}(X \mid f)$.

Case 2: If $x^1 \in \operatorname{Eff}(Y \mid f)$, then we get $x^0 \in \operatorname{Eff}(Y \mid f)$ by $x^1 \in S_{=}(Y, f, x^0)$. This is a contradiction to $x^0 \in X \setminus \operatorname{Eff}(Y \mid f)$.

This means that (5.2) holds.

Furthermore, since $x^0 \in \operatorname{bd} X$ and (A2) holds, we have

$$S_{=}(Y, f, x^0) \cap \operatorname{bd} X = S_{=}(Y, f, x^0) \cap L_{=}(Y, f_{m+1}, f_{m+1}(x^0)).$$

From Lemma 4.3 (applied for $(\mathcal{P}_Y^{\oplus})$ instead of $(\mathcal{P}_X)$), we conclude $x^0 \in \operatorname{Eff}(Y \mid f^{\oplus})$. This means $x^0 \in (\operatorname{bd} X) \cap \operatorname{Eff}(Y \mid f^{\oplus})$ and 2° holds.

□

The semi-strict quasi-convexity assumption with respect to f in 2° of Theorem 5.1 can not be replaced by a quasi-convexity assumption (see Günther and Tammer [12, Ex. 1, Ex. 2, Ex. 3]). Moreover, the following inclusions do not hold under the assumptions supposed in Theorem 5.1 in general (see [12, Ex. 1, Ex. 5]):

$$\begin{aligned}
\operatorname{Eff}(X \mid f) &\subseteq X \cap \operatorname{Eff}(Y \mid f^{\oplus}); \\
\operatorname{bd} X \cap \operatorname{Eff}(Y \mid f) &\subseteq (\operatorname{bd} X) \cap \operatorname{Eff}(Y \mid f^{\oplus}); \\
\operatorname{Eff}(X \mid f) &\subseteq [(\operatorname{int} X) \cap \operatorname{Eff}(Y \mid f)] \cup \left[(\operatorname{bd} X) \cap \operatorname{Eff}(Y \mid f^{\oplus})\right].
\end{aligned}$$

In the next lemma, we present sufficient conditions for the fact that a solution $x^0 \in \operatorname{Eff}(X \mid f)$ is belonging to $\operatorname{Eff}(Y \mid f^{\oplus})$.

Lemma 5.2 *Let* (4.1) *be satisfied. Suppose that* f_{m+1} *fulfils Assumption* (A4). *If* $x^0 \in \mathrm{Eff}(X \mid f)$ *and*

$$S_=(X, f, x^0) \subseteq L_=(Y, f_{m+1}, f_{m+1}(x^0)),$$

then $x^0 \in X \cap \mathrm{Eff}(Y \mid f^{\oplus})$.

Proof: The proof is analogous to the proof given by Günther and Tammer [12, Lem. 8] for the case $Y = \mathcal{V} = \mathbb{R}^n$. $\square$

In the next theorem, we present a second main result that holds under the assumption that the penalization function f_{m+1} fulfils (A3).

Theorem 5.3 *Let* (4.1) *be satisfied. Suppose that* f_{m+1} *fulfils Assumption* (A3). *Then, the following statements are true:*

1°. *It holds*

$$[X \cap \mathrm{Eff}(Y \mid f)] \cup \left[(\mathrm{bd}\, X) \cap \mathrm{Eff}(Y \mid f^{\oplus})\right] \subseteq \mathrm{Eff}(X \mid f) = X \cap \mathrm{Eff}(Y \mid f^{\oplus}).$$

2°. *In the case* $\mathrm{int}\, X \neq \emptyset$, *suppose additionally that* $f : \mathcal{V} \to \mathbb{R}^m$ *is componentwise semi-strictly quasi-convex on* Y. *Then, we have*

$$[X \cap \mathrm{Eff}(Y \mid f)] \cup \left[(\mathrm{bd}\, X) \cap \mathrm{Eff}(Y \mid f^{\oplus})\right] \supseteq \mathrm{Eff}(X \mid f).$$

Proof:

1°. We are going to show $\mathrm{Eff}(X \mid f) = X \cap \mathrm{Eff}(Y \mid f^{\oplus})$.

Let us prove the inclusion "$\supseteq$". Consider $x^0 \in X \cap \mathrm{Eff}(Y \mid f^{\oplus})$. By Lemma 4.3 (applied for $(\mathcal{P}_Y^{\oplus})$ instead of $(\mathcal{P}_X)$) and Assumption (A3), it follows

$$\begin{aligned} S_\leq(X, f, x^0) &= S_\leq(Y, f, x^0) \cap X \\ &= S_\leq(Y, f, x^0) \cap L_\leq(Y, f_{m+1}, f_{m+1}(x^0)) \\ &\subseteq S_=(Y, f, x^0) \cap L_=(Y, f_{m+1}, f_{m+1}(x^0)) \\ &= S_=(Y, f, x^0) \cap X \\ &= S_=(X, f, x^0). \end{aligned}$$

By Lemma 4.3 (applied for $(\mathcal{P}_X)$), we get $x^0 \in \mathrm{Eff}(X \mid f)$.

Now, we prove the reverse inclusion "$\subseteq$". Let $x^0 \in \mathrm{Eff}(X \mid f)$. Due to

$$\begin{aligned} S_=(X, f, x^0) &= S_=(Y, f, x^0) \cap X \\ &= S_=(Y, f, x^0) \cap L_=(Y, f_{m+1}, f_{m+1}(x^0)) \\ &\subseteq L_=(Y, f_{m+1}, f_{m+1}(x^0)), \end{aligned}$$

it follows $x^0 \in X \cap \mathrm{Eff}(Y \mid f^{\oplus})$ by Lemma 5.2. Notice that (A3) implies (A4) by Remark 4.7.

Moreover, the inclusion $X \cap \mathrm{Eff}(Y \mid f) \subseteq \mathrm{Eff}(X \mid f)$ follows by Lemma 4.4 while $(\mathrm{bd}\, X) \cap \mathrm{Eff}(Y \mid f^{\oplus}) \subseteq \mathrm{Eff}(X \mid f)$ is a direct consequence of the equality $\mathrm{Eff}(X \mid f) = X \cap \mathrm{Eff}(Y \mid f^{\oplus})$.

2°. The proof is analogous to the first part of the proof of 2° in Theorem 5.1. Notice that we have

$$S_{=}(Y, f, x^0) \cap X = S_{=}(Y, f, x^0) \cap L_{=}(Y, f_{m+1}, f_{m+1}(x^0)).$$

□

5.2 Sets of weakly efficient solutions of $(\mathcal{P}_X)$, $(\mathcal{P}_Y)$ and $(\mathcal{P}_Y^{\oplus})$

In the first part of this section, we present some relationships between the sets of weakly efficient solutions of the problems $(\mathcal{P}_X)$, $(\mathcal{P}_Y)$ and $(\mathcal{P}_Y^{\oplus})$. The second part of this section is devoted to the study of the concept of *Pareto reducibility* for multi-objective optimizations problems that was introduced by Popovici [22, Def. 1].

5.2.1 Relationships between the sets of solutions

Some first relationships between the sets of weakly efficient solutions of the problems $(\mathcal{P}_X)$, $(\mathcal{P}_Y)$ and $(\mathcal{P}_Y^{\oplus})$ are given in the next theorem.

Theorem 5.4 *Let* (4.1) *and Assumption* (A4) *be satisfied. Then, we have*

$$X \cap \mathrm{WEff}(Y \mid f) \subseteq \mathrm{WEff}(X \mid f) \subseteq X \cap \mathrm{WEff}(Y \mid f^{\oplus}).$$

Proof: In view of Corollary 4.4, it follows $X \cap \mathrm{WEff}(Y \mid f) \subseteq \mathrm{WEff}(X \mid f)$. Now, let us prove the second inclusion $\mathrm{WEff}(X \mid f) \subseteq X \cap \mathrm{WEff}(Y \mid f^{\oplus})$.

Consider $x^0 \in \mathrm{WEff}(X \mid f) \subseteq X$. By Lemma 4.3 (applied for $(\mathcal{P}_X)$) and by Assumption (A4), we get

$$\begin{aligned}
\emptyset &= S_{<}(X, f, x^0) \\
&= S_{<}(Y, f, x^0) \cap X \\
&\supseteq S_{<}(Y, f, x^0) \cap L_{\leq}(Y, f_{m+1}, f_{m+1}(x^0)) \\
&\supseteq S_{<}(Y, f, x^0) \cap L_{<}(Y, f_{m+1}, f_{m+1}(x^0)).
\end{aligned}$$

In view of Lemma 4.3 (applied for $(\mathcal{P}_Y^{\oplus})$ instead of $(\mathcal{P}_X)$), it follows $x^0 \in X \cap \mathrm{WEff}(Y \mid f^{\oplus})$. □

Theorem 5.4 is a generalization of a result (under the assumptions given in (5.1)) by Günther and Tammer [12, Th. 3].

Remark 5.5 *Let* (4.1) *be satisfied and assume that* $\mathrm{int}\, X \neq \emptyset$. *Suppose that* f_{m+1} *is semi-strictly quasi-convex on* Y *and fulfils both Assumptions* (A1) *and* (A2). *Then,* *it follows*

$$\forall\, x^1 \in \operatorname{bd} X\, \forall\, \tilde{x} \in \operatorname{int} X:\ [\tilde{x}, x^1) \subseteq L_<(Y, f_{m+1}, f_{m+1}(x^1))$$

by Lemma 3.3. In particular, the function f_{m+1} satisifies

$$\forall\, x^1 \in \operatorname{bd} X\, \exists\, \tilde{x} \in \operatorname{int} X:\ [\tilde{x}, x^1) \subseteq L_<(Y, f_{m+1}, f_{m+1}(x^1)). \tag{A6}$$

Taking into account Remark 4.7, for any $x^0, x^1 \in \operatorname{bd} X$, we have

$$L_<(Y, f_{m+1}, f_{m+1}(x^0)) = L_<(Y, f_{m+1}, f_{m+1}(x^1)) = \operatorname{int} X.$$

The result given in Theorem 5.6 presents important relationships between the sets of weakly efficient solutions of the problems $(\mathcal{P}_X)$, $(\mathcal{P}_Y)$ and $(\mathcal{P}_Y^{\oplus})$. In Günther and Tammer [12, Th. 4], a special case (see the assumptions given in (5.1)) of Theorem 5.6 is considered.

Theorem 5.6 *Let* (4.1) *be satisfied. The following statements are true:*

1°. *Assume that* $\operatorname{int} X \neq \emptyset$. *Let $f : \mathcal{V} \to \mathbb{R}^m$ be componentwise upper semi-continuous along line segments on Y. Furthermore, we suppose that f_{m+1} fulfils Assumptions* (A1), (A2) *and* (A6). *Then, we have*

$$[X \cap \operatorname{WEff}(Y \mid f)] \cup \left[(\operatorname{bd} X) \cap \operatorname{WEff}(Y \mid f^{\oplus})\right] \subseteq \operatorname{WEff}(X \mid f).$$

2°. *Let Assumption* (A4) *be fulfilled. In the case* $\operatorname{int} X \neq \emptyset$, *suppose additionally that $f : \mathcal{V} \to \mathbb{R}^m$ is componentwise semi-strictly quasi-convex on Y. Then, we have*

$$[X \cap \operatorname{WEff}(Y \mid f)] \cup \left[(\operatorname{bd} X) \cap \operatorname{WEff}(Y \mid f^{\oplus})\right] \supseteq \operatorname{WEff}(X \mid f).$$

Proof: Consider $i \in I_m$. Notice that the following statements are equivalent (see Section 3):

- f_i is upper semi-continuous on line segments in Y.
- $L_{\geq}([0,1], (f_i \circ l_{a,b}), s)$ is closed for all $s \in \mathbb{R}$ and all $a, b \in Y$.
- $L_<([0,1], (f_i \circ l_{a,b}), s) \cup (\mathbb{R} \setminus [0,1])$ is open for all $s \in \mathbb{R}$ and all $a, b \in Y$.

Now, we are going to prove both statements 1° and 2°:

1°. In view of Corollary 4.4, it follows $X \cap \operatorname{WEff}(Y \mid f) \subseteq \operatorname{WEff}(X \mid f)$. Now, let us consider $x^0 \in (\operatorname{bd} X) \cap \operatorname{WEff}(Y \mid f^{\oplus})$. By Lemma 4.3 (applied for $(\mathcal{P}_Y^{\oplus})$ instead of $(\mathcal{P}_X)$), it follows

$$\emptyset = S_<(Y, f, x^0) \cap L_<(Y, f_{m+1}, f_{m+1}(x^0)). \tag{5.3}$$

Now, we will prove that

$$S_<(Y, f, x^0) \cap L_<(Y, f_{m+1}, f_{m+1}(x^0)) = S_<(Y, f, x^0) \cap X. \tag{5.4}$$

Then, by (5.3) and (5.4), we get $S_<(X, f, x^0) = \emptyset$, hence $x^0 \in \mathrm{WEff}(X \mid f)$ by Lemma 4.3 (applied for $(\mathcal{P}_X)$).

By Assumption (A1), the inclusion "$\subseteq$" in (5.4) follows directly.

Let us prove the reverse inclusion "$\supseteq$" in (5.4). Assume the contrary holds, i.e., it exists $x^1 \in S_<(Y, f, x^0) \cap X$ such that $x^1 \notin L_<(Y, f_{m+1}, f_{m+1}(x^0))$. So, we have $x^1 \in L_=(Y, f_{m+1}, f_{m+1}(x^0)) = \mathrm{bd}\, X$ by Assumption (A1). By Assumption (A6) (see Remark 5.5), for the given points $x^0, x^1 \in \mathrm{bd}\, X$, there exists $\tilde{x} \in \mathrm{int}\, X$ such that

$$l_{x^1,\tilde{x}}(\lambda) \in L_<(Y, f_{m+1}, f_{m+1}(x^1)) = L_<(Y, f_{m+1}, f_{m+1}(x^0))$$

for all $\lambda \in (0, 1]$.

Consider $i \in I_m$. Since $x^1 \in L_<(Y, f_i, f_i(x^0))$, we get

$$0 \in L_<([0, 1], (f_i \circ l_{x^1,\tilde{x}}), f_i(x^0)) \cup (\mathbb{R} \setminus [0, 1]).$$

The openness of the set $L_<([0, 1], (f_i \circ l_{x^1,\tilde{x}}), f_i(x^0)) \cup (\mathbb{R} \setminus [0, 1])$ implies that there exists a $\overline{\lambda}_i \in \mathbb{R}$ with $0 < \overline{\lambda}_i < 1$ such that $f_i(l_{x^1,\tilde{x}}(\lambda)) < f_i(x^0)$ for all $\lambda \in (0, \overline{\lambda}_i]$.

Now, the point $x^2 := l_{x^1,\tilde{x}}(\min\{\overline{\lambda}_i \mid i \in I_m\})$ fulfils

$$x^2 \in S_<(Y, f, x^0) \cap L_<(Y, f_{m+1}, f_{m+1}(x^0))$$

in contradiction to (5.3).

Consequently, we infer that (5.4) holds.

2°. Consider $x^0 \in \mathrm{WEff}(X \mid f) \subseteq X$. Of course, we can have $x^0 \in \mathrm{WEff}(Y \mid f)$ and therefore $x^0 \in X \cap \mathrm{WEff}(Y \mid f)$. Let us assume that $x^0 \in X \setminus \mathrm{WEff}(Y \mid f)$. In view of Theorem 5.4, we know that $x^0 \in \mathrm{WEff}(X \mid f)$ implies $x^0 \in X \cap \mathrm{WEff}(Y \mid f^\oplus)$.

Now, consider two cases:

Case 1: Let $\mathrm{int}\, X \neq \emptyset$. By Corollary 4.5, we get $x^0 \in \mathrm{bd}\, X$ by the componentwise semi-strictly quasi-convexity of f on Y.

Case 2: Let $\mathrm{int}\, X = \emptyset$. Obviously, we have $x^0 \in X = \mathrm{bd}\, X$.

Finally, we get $x^0 \in (\mathrm{bd}\, X) \cap \mathrm{WEff}(Y \mid f^\oplus)$.

□

The semi-strict quasi-convexity assumption with respect to f in 2° of Theorem 5.6 can not be replaced by a quasi-convexity assumption (see Günther and Tammer [12, Ex. 2, Ex. 3]). Due to Theorem 5.4, the set $X \cap \mathrm{WEff}(Y \mid f)$ can be replaced by $(\mathrm{int}\, X) \cap \mathrm{WEff}(Y \mid f)$ in Theorem 5.6.

Theorem 5.7 *Let* (4.1) *be satisfied. Suppose that f_{m+1} fulfils Assumption* (A3). *Then, we have*

$$X \cap \mathrm{WEff}(Y \mid f^\oplus) = X.$$

Proof: The inclusion "$\subseteq$" is obvious. Let us prove the reverse inclusion "$\supseteq$".

Let $x^0 \in X$. By Assumption (A3), it follows $L_<(Y, f_{m+1}, f_{m+1}(x^0)) = \emptyset$. So, we get

$$S_<(Y, f, x^0) \cap L_<(Y, f_{m+1}, f_{m+1}(x^0)) = \emptyset,$$

hence we infer $x^0 \in X \cap \mathrm{WEff}(Y \mid f^{\oplus})$ by Lemma 4.3 (applied for $(\mathcal{P}_Y^{\oplus})$ instead of $(\mathcal{P}_X)$). $\square$

Remark 5.8 *Assume that f_{m+1} fulfils Assumption* (A3). *By* 2° *of Theorem 5.6 and by Theorem 5.7, we get*

$$\begin{aligned}\mathrm{WEff}(X \mid f) &\subseteq [X \cap \mathrm{WEff}(Y \mid f)] \cup \left[(\mathrm{bd}\, X) \cap \mathrm{WEff}(Y \mid f^{\oplus})\right] \\ &= [(\mathrm{int}\, X) \cap \mathrm{WEff}(Y \mid f)] \cup \mathrm{bd}\, X.\end{aligned}$$

However, the reverse inclusion

$$\mathrm{WEff}(X \mid f) \supseteq [(\mathrm{int}\, X) \cap \mathrm{WEff}(Y \mid f)] \cup \mathrm{bd}\, X.$$

does not hold in general, since $\mathrm{bd}\, X \subseteq \mathrm{WEff}(X \mid f)$ *is not true in general (see, e.g., Günther and Tammer [12, Ex. 5]). Hence, it seems to be more appropriate to work with a penalization function f_{m+1} that fulfils Assumptions* (A1) *and* (A2) *instead of Assumption* (A3) *in order to compute the set* $\mathrm{WEff}(X \mid f)$.

5.2.2 Pareto reducibility for multi-objective optimizations problems

According to Popovici [22], the multi-objective optimization problem $(\mathcal{P}_X)$ is called *Pareto reducible* if the set of weakly efficient solutions of $(\mathcal{P}_X)$ can be represented as the union of the sets of efficient solutions of its subproblems.

Considering the objective function

$$f_I = (f_{i_1}, \cdots, f_{i_k}) : \mathcal{V} \to \mathbb{R}^k,$$

for a selection of indices $I = \{i_1, \ldots, i_k\} \subseteq I_{m+1}$, $i_1 < \cdots < i_k$, with cardinality $\mathrm{card}(I) = k \geq 1$, we define the problem

$$f_I(x) \to \operatorname*{v-min}_{x \in X}. \tag{$\mathcal{P}_X^I$}$$

The problem $(\mathcal{P}_X^I)$ is a single-objective optimization problem when I is a singleton set, otherwise being a multi-objective optimization problem. Notice that $f_{I_m} = f$ and $f_{I_{m+1}} = f^{\oplus}$. If $\emptyset \neq I \subseteq I_m$, then the problem $(\mathcal{P}_X^I)$ can be seen as a subproblem of the original problem $(\mathcal{P}_X)$.

For any index set I with $\emptyset \neq I \subseteq I_m$, we define the following subproblem of the penalized problem $(\mathcal{P}_Y^{\oplus})$:

$$f_I^{\oplus}(x) := \begin{pmatrix} f_I(x) \\ f_{m+1}(x) \end{pmatrix} \to \operatorname*{v-min}_{x \in Y}.$$

Next, we recall sufficient conditions for Pareto reducibility (see the papers by Popovici [22, Prop. 4] and [24, Cor. 4.5]):

Proposition 5.9 ([22, 24]) *Consider the space $\mathcal{V}$ given in* (4.1) *and assume that X is a nonempty convex set in $\mathcal{V}$. If f is componentwise semi-strictly quasi-convex and upper semi-continuous along line segments on X, then*

$$\mathrm{WEff}(X \mid f) = \bigcup_{\emptyset \neq I \subseteq I_m} \mathrm{Eff}(X \mid f_I).$$

In addition, if $\mathcal{V}$ is the n-dimensional Euclidean space $\mathbb{R}^n$ and f is componentwise lower semi-continuous along line segments on X, then

$$\mathrm{WEff}(X \mid f) = \bigcup_{\substack{\emptyset \neq I \subseteq I_m; \\ \operatorname{card} I \leq n+1}} \mathrm{Eff}(X \mid f_I).$$

In the next theorem, we present a Pareto reducibility type result for multi-objective optimization problems.

Theorem 5.10 *Let* (4.1) *be satisfied and let* $\operatorname{int} X \neq \emptyset$. *Suppose that f_{m+1} fulfils Assumptions* (A1) *and* (A2). *Moreover, assume that $f^{\oplus}$ is componentwise semi-strictly quasi-convex as well as upper semi-continuous along line segments on Y. Then, we have*

$$\mathrm{WEff}(X \mid f) = \left[X \cap \bigcup_{\emptyset \neq I \subseteq I_m} \mathrm{Eff}(Y \mid f_I) \right] \cup \left[(\operatorname{bd} X) \cap \bigcup_{\emptyset \neq I \subseteq I_{m+1}} \mathrm{Eff}(Y \mid f_I) \right].$$

In addition, if $\mathcal{V}$ is the n-dimensional Euclidean space $\mathbb{R}^n$ and $f^{\oplus}$ is componentwise lower semi-continuous along line segments on Y, then

$$\mathrm{WEff}(X \mid f) = \left[X \cap \bigcup_{\substack{\emptyset \neq I \subseteq I_m; \\ \operatorname{card} I \leq n+1}} \mathrm{Eff}(Y \mid f_I) \right] \cup \left[(\operatorname{bd} X) \cap \bigcup_{\substack{\emptyset \neq I \subseteq I_{m+1}; \\ \operatorname{card} I \leq n+1}} \mathrm{Eff}(Y \mid f_I) \right].$$

Proof: By Theorem 5.6 and Remark 5.5, we have

$$\mathrm{WEff}(X \mid f) = [X \cap \mathrm{WEff}(Y \mid f)] \cup \left[(\operatorname{bd} X) \cap \mathrm{WEff}(Y \mid f^{\oplus}) \right].$$

Applying Proposition 5.9 for both problems $(\mathcal{P}_Y)$ and $(\mathcal{P}_Y^{\oplus})$, we get the desired equalities given in this theorem. $\square$

Under the assumption $Y = \mathcal{V}$, Theorem 5.10 provides a representation for the set of weakly efficient solutions of the constrained problem $(\mathcal{P}_X)$ using the sets of efficient solutions of families of unconstrained (free) optimization problems. In Lemma 6.16, we will see that the set X given in Theorem 5.10 is a convex set if f_{m+1} is semi-strictly quasi-convex on X and satisfies the Assumption (A5) (i.e., $X = L_{\leq}(Y, f_{m+1}, 0)$).

Theorem 5.11 *Let* (4.1) *be satisfied and let X be convex. Suppose that f_{m+1} fulfils Assumption* (A3). *Moreover, assume that f is componentwise semi-strictly quasi-*

convex as well as upper semi-continuous along line segments on Y. Then, we have

$$\mathrm{WEff}(X \mid f) = X \cap \bigcup_{\substack{\{m+1\} \subseteq I \subseteq I_{m+1};\\ \operatorname{card} I \geq 2}} \mathrm{Eff}(Y \mid f_I).$$

Proof: Due to Theorem 5.3, we have

$$X \cap \mathrm{Eff}(Y \mid f_I^{\oplus}) = \mathrm{Eff}(X \mid f_I) \quad \text{for all } \emptyset \neq I \subseteq I_m. \tag{5.5}$$

By Proposition 5.9, it follows

$$\begin{aligned}
\mathrm{WEff}(X \mid f) &= \bigcup_{\emptyset \neq I \subseteq I_m} \mathrm{Eff}(X \mid f_I) \\
&\overset{(5.5)}{=} X \cap \bigcup_{\emptyset \neq I \subseteq I_m} \mathrm{Eff}(Y \mid f_I^{\oplus}) \\
&= X \cap \bigcup_{\substack{\{m+1\} \subseteq I \subseteq I_{m+1};\\ \operatorname{card} I \geq 2}} \mathrm{Eff}(Y \mid f_I).
\end{aligned}$$

$\square$

Notice that the condition $\operatorname{card} I \geq 2$ for the index set I considered in Theorem 5.11 is essential, since, for any $x^0 \in X$, we have

$$\mathrm{Eff}(Y \mid f_{m+1}) = L_=(Y, f_{m+1}, f_{m+1}(x^0)) = X$$

under the Assumption (A3). This means that

$$X \cap \bigcup_{\{m+1\} \subseteq I \subseteq I_{m+1}} \mathrm{Eff}(Y \mid f_I) = X.$$

5.3 Sets of strictly efficient solutions of $(\mathcal{P}_X)$, $(\mathcal{P}_Y)$ and $(\mathcal{P}_Y^{\oplus})$

We present some first relationships between the sets of strictly efficient solutions of the problems $(\mathcal{P}_X)$, $(\mathcal{P}_Y)$ and $(\mathcal{P}_Y^{\oplus})$.

Theorem 5.12 *Let* (4.1) *and Assumption* (A4) *be satisfied. Then, we have*

$$X \cap \mathrm{SEff}(Y \mid f) \subseteq \mathrm{SEff}(X \mid f) \subseteq X \cap \mathrm{SEff}(Y \mid f^{\oplus}).$$

Proof: By Corollary 4.4, we get $X \cap \mathrm{SEff}(Y \mid f) \subseteq \mathrm{SEff}(X \mid f)$. We now show the second inclusion.

Consider $x^0 \in \mathrm{SEff}(X \mid f) \subseteq X$. In view of Lemma 4.3 (applied for $(\mathcal{P}_X)$) and the assumption (A4), we get

$$\begin{aligned}
S_{\leq}(Y, f, x^0) \cap L_{\leq}(Y, f_{m+1}, f_{m+1}(x^0)) &\subseteq S_{\leq}(Y, f, x^0) \cap X \\
&= S_{\leq}(X, f, x^0) = \{x^0\},
\end{aligned}$$

Therefore, it follows $x^0 \in X \cap \mathrm{SEff}(Y \mid f^{\oplus})$ by Lemma 4.3 (applied for $(\mathcal{P}_Y^{\oplus})$ instead of $(\mathcal{P}_X)$). $\square$

Notice that a result by Günther and Tammer [12, Th. 3] (under the assumptions given in (5.1)) is a special case of Theorem 5.12.

The following Theorem 5.13 presents important relationships between the sets of strictly efficient solutions of the problems $(\mathcal{P}_X)$, $(\mathcal{P}_Y)$ and $(\mathcal{P}_Y^{\oplus})$. It should be noted that Günther and Tammer [12, Th. 4] consider a special case of Theorem 5.13 (see the assumptions given in (5.1)).

Theorem 5.13 *Let* (4.1) *be satisfied. The following statements are true:*

1°. *If Assumption* (A1) *holds, then we have*

$$[X \cap \mathrm{SEff}(Y \mid f)] \cup \big[(\mathrm{bd}\, X) \cap \mathrm{SEff}(Y \mid f^{\oplus})\big] \subseteq \mathrm{SEff}(X \mid f).$$

2°. *Assume that Assumption* (A4) *holds. In the case* $\mathrm{int}\, X \neq \emptyset$*, suppose additionally that* $f : \mathcal{V} \to \mathbb{R}^m$ *is componentwise semi-strictly quasi-convex or quasi-convex on* Y*. Then, we have*

$$[X \cap \mathrm{SEff}(Y \mid f)] \cup \big[(\mathrm{bd}\, X) \cap \mathrm{SEff}(Y \mid f^{\oplus})\big] \supseteq \mathrm{SEff}(X \mid f).$$

Proof:

1°. By Corollary 4.4, we have $X \cap \mathrm{SEff}(Y \mid f) \subseteq \mathrm{SEff}(X \mid f)$. Consider $x^0 \in (\mathrm{bd}\, X) \cap \mathrm{SEff}(Y \mid f^{\oplus})$. In view of Lemma 4.3 (applied for $(\mathcal{P}_Y^{\oplus})$ instead of $(\mathcal{P}_X)$) and Assumption (A1), we have

$$\begin{aligned} S_{\leq}(X, f, x^0) &= S_{\leq}(Y, f, x^0) \cap X \\ &= S_{\leq}(Y, f, x^0) \cap L_{\leq}(Y, f_{m+1}, f_{m+1}(x^0)) = \{x^0\}. \end{aligned}$$

From Lemma 4.3 (applied for $(\mathcal{P}_X)$), we get $x^0 \in \mathrm{SEff}(X \mid f)$.

2°. Consider $x^0 \in \mathrm{SEff}(X \mid f) \subseteq X$. If we have $x^0 \in \mathrm{SEff}(Y \mid f)$, then $x^0 \in X \cap \mathrm{SEff}(Y \mid f)$. We now suppose that $x^0 \in X \setminus \mathrm{SEff}(Y \mid f)$. By Theorem 5.12, we immediately get $x^0 \in X \cap \mathrm{SEff}(Y \mid f^{\oplus})$.

Let us consider two cases:

Case 1: If $\mathrm{int}\, X \neq \emptyset$, then we conclude $x^0 \in \mathrm{bd}\, X$ because of Corollary 4.5.

Case 2: If $\mathrm{int}\, X = \emptyset$, then clearly it follows $x^0 \in \mathrm{bd}\, X$.

So, we infer that $x^0 \in (\mathrm{bd}\, X) \cap \mathrm{SEff}(Y \mid f^{\oplus})$.

$\square$

The proof of Theorem 5.13 uses ideas given in the paper by Günther and Tammer [12, Th. 8]. In contrast to 1° in Theorem 5.1 (Theorem 5.3) as well as 1° in Theorem 5.6, we only need the Assumption (A1) concerning the level sets of the function f_{m+1} in 1° of Theorem 5.13. In accordance to 2° in Theorem 5.6, only Assumption (A4) concerning the level sets of f_{m+1} must be fulfilled in 2° of Theorem 5.13. In 2° of

Theorem 5.1 (Theorem 5.3) Assumptions (A1) and (A2) (Assumption (A3)) must be fulfilled. In view of Theorem 5.12, the set $X \cap \mathrm{SEff}(Y \mid f)$ can be replaced by the set $(\operatorname{int} X) \cap \mathrm{SEff}(Y \mid f)$ in Theorem 5.13.

Next, we present a corresponding result to the equality given in 1° of Theorem 5.3 for the concept of strict efficiency that holds under the assumption that the penalization function f_{m+1} fulfils (A3).

Theorem 5.14 *Let* (4.1) *be satisfied. Suppose that f_{m+1} fulfils Assumption* (A3). *Then, we have*

$$\mathrm{SEff}(X \mid f) = X \cap \mathrm{SEff}(Y \mid f^{\oplus}).$$

Proof: First, we show the inclusion "$\supseteq$", therefore consider $x^0 \in X \cap \mathrm{SEff}(Y \mid f^{\oplus})$. Because of Lemma 4.3 (applied for $(\mathcal{P}_Y^{\oplus})$ instead of $(\mathcal{P}_X)$) and Assumption (A3) it follows

$$\begin{aligned} S_{\leq}(X, f, x^0) &= S_{\leq}(Y, f, x^0) \cap X \\ &= S_{\leq}(Y, f, x^0) \cap L_{\leq}(Y, f_{m+1}, f_{m+1}(x^0)) = \{x^0\}. \end{aligned}$$

By Lemma 4.3 (applied for $(\mathcal{P}_X)$), we have $x^0 \in \mathrm{SEff}(X \mid f)$.

In view of Theorem 5.12, we get immediately the reverse inclusion "$\subseteq$". Notice that (A3) implies (A4) by Remark 4.7. □

Under the assumption that f is componentwise semi-strictly quasi-convex or quasi-convex on Y and that f_{m+1} fulfils Assumption (A3), we get

$$(\operatorname{int} X) \cap \mathrm{SEff}(Y \mid f) = (\operatorname{int} X) \cap \mathrm{SEff}(Y \mid f^{\oplus})$$

by Theorems 5.13 and 5.14.

6 Sufficient conditions for the validity of the Assumptions (A1) and (A2) based on (local) generalized-convexity concepts

As we have seen in Section 5, we need some additional assumptions concerning the level sets / level lines of the penalization function f_{m+1} in order to obtain the main results of the paper. In particular, the Assumptions (A1) and (A2) play an important role in our penalization approach. We already know that under certain assumptions the gauge function given in Example 4.8, the Hiriart-Urruty function given in Example 4.10 and the nonlinear scalarizing function given in Example 4.11 fulfil these Assumptions (A1) and (A2). In this section, our aim is to identify further classes of functions that satisfy both Assumptions (A1) and (A2).

In the first part of the section, we introduce local concepts of generalized-convexity while in the second part we will use these concepts in order to derive sufficient conditions for the validity of (A1) and (A2).

6.1 Local versions of generalized-convexity

In the next definition, for any normed space $\mathcal{V}$ equipped with the norm $||\cdot||$ (we write $(\mathcal{V}, ||\cdot||)$), we introduce local versions of semi-strict quasi-convexity and quasi-convexity for a real-valued function $h : \mathcal{V} \to \mathbb{R}$.

Definition 6.1 *Let $(\mathcal{V}, ||\cdot||)$ be a normed space and let $X \subseteq \mathcal{V}$ be open. A real-valued function $h : \mathcal{V} \to \mathbb{R}$ is called*

- *locally semi-strictly quasi-convex (locally quasi-convex) at a point $x^0 \in X$ if there exists $\varepsilon > 0$ such that h is semi-strictly quasi-convex (quasi-convex) on $B_{||\cdot||}(x^0, \varepsilon)$.*
- *locally explicitly quasi-convex at $x^0 \in X$ if it is both locally semi-strictly quasi-convex and locally quasi-convex at $x^0 \in X$.*

Notice that the open ball $B_{||\cdot||}(x^0, \varepsilon)$ is a open and convex set in a normed space $(\mathcal{V}, ||\cdot||)$. We defined local semi-strict quasi-convexity and local quasi-convexity on open but not necessarily convex sets. Clearly, if h is locally semi-strictly quasi-convex at $x^0 \in X$ and lower semi-continuous on $B_{||\cdot||}(x^0, \varepsilon)$, then h is locally quasi-convex at $x^0 \in X$.

The local concepts of generalized-convexity given in Definition 6.1 will be used in Lemma 6.10, Lemma 6.11 and Theorem 6.12

In the following lemma, we present relationships between global and corresponding local versions of generalized-convexity.

Lemma 6.2 *Let $(\mathcal{V}, ||\cdot||)$ be a normed space and let $X \subseteq \mathcal{V}$ be an open convex set. A function $h : \mathcal{V} \to \mathbb{R}$, which is semi-strictly quasi-convex (quasi-convex) on the set X, is locally semi-strictly quasi-convex (locally quasi-convex) at every point $x^0 \in X$.*

The reverse implications are not true, as shown in the next example.

Example 6.3 *For the function $h = h_{\max} : \mathbb{R} \to \mathbb{R}$ considered in Example 3.8 we know that h is not semi-strictly quasi-convex on $X := \mathbb{R}$. However, h is semi-strictly quasi-convex on $B_{||\cdot||}(x, \varepsilon) = (x - \varepsilon, x + \varepsilon)$ for every $x \in X$ and for $\varepsilon \in (0, 1)$. Moreover, the function $h : \mathbb{R} \to \mathbb{R}$ defined by*

$$h(x) := \begin{cases} x + 1 & \text{for all } x < -1, \\ 0 & \text{for all } x \in [-1, 1], \\ 1 - x & \text{for all } x > 1 \end{cases}$$

is not quasi-convex on $X := \mathbb{R}$, but quasi-convex on $B_{||\cdot||}(x, \varepsilon) = (x - \varepsilon, x + \varepsilon)$ for every $x \in X$ and for $\varepsilon \in (0, 1)$.

A further relationship between global and corresponding local versions of generalized-convexity is given in the next lemma.

Lemma 6.4 *Let $(\mathcal{V}, ||\cdot||)$ be a normed space, let $X \subseteq \mathcal{V}$ be an open convex set, and let $h : \mathcal{V} \to \mathbb{R}$ be upper semi-continuous along line segments on X. The function h is semi-strictly quasi-convex on the set X if the following statements are fulfilled:*

1°. *h is locally explicitly quasi-convex at each point $x^0 \in X$.*

2°. *Every local minimum of h is also global for each restriction on a line segment in X.*

Proof: Assume that h is not semi-strictly quasi-convex on X. By Lemma 3.3, there exist $s \in \mathbb{R}$, $x^0 \in L_=(X,h,s)$ and $x^1 \in L_<(X,h,s)$ such that $x^\lambda := l_{x^0,x^1}(\lambda) \in L_\geq(X,h,s)$ for some $\lambda \in (0,1)$. Since $h(x^\lambda) \geq h(x^0) > h(x^1)$ and $h \circ l_{x^0,x^1} : [0,1] \to \mathbb{R}$ is upper semi-continuous on $[0,1]$, we can choose

$$\lambda_{\max} \in \left\{ \overline{\lambda} \in (0,1) \,|\, h(l_{x^0,x^1}(\overline{\lambda})) = \max_{\lambda \in [0,1]} h(l_{x^0,x^1}(\lambda)) \right\}$$

by a well-known Weierstrass type existence theorem (see, e.g., Aliprantis and Border [1, Th. 2.43]). Now, put $x^2 := x^{\lambda_{\max}}$. Consider $\varepsilon > 0$ such that h is explicitly quasi-convex on $B_\varepsilon := B_{||\cdot||}(x^2, \varepsilon)$. Now, by Lemma 2.2, we get that $B_\delta := [x^2 - \delta v, x^2 + \delta v] \subseteq B_\varepsilon$ holds for $v := \frac{x^1 - x^0}{||x^1 - x^0||}$ (note that $x^1 \neq x^0$) and $\delta \in (0, \varepsilon)$. Define

$$\overline{\delta} := \min(\delta, ||x^2 - x^0||, ||x^2 - x^1||) \in (0, \delta]$$

and

$$B_{\overline{\delta}} := [x^2 - \overline{\delta} v, x^2 + \overline{\delta} v].$$

Consider $\delta', \delta'' > 0$. We know that $x^2 + \delta' v = x^1$ implies $\delta' = ||x^2 - x^1||$ as well as $x^2 - \delta'' v = x^0$ implies $\delta'' = ||x^2 - x^0||$. Hence, we get $B_{\overline{\delta}} \subseteq [x^0, x^1]$ and $B_{\overline{\delta}} \subseteq B_\delta \subseteq B_\varepsilon$. Since $h(x^1) < s \leq h(x^2)$, we know by our assumptions that x^2 can not be a local minimum point of h on the line segment $[x^0, x^1]$. Hence, there exists $x^3 \in B_{\overline{\delta}} \setminus \{x^2\}$ with $h(x^3) < h(x^2)$. For the point $x^4 := x^2 + (x^2 - x^3) \in B_{\overline{\delta}}$ we have $h(x^4) \leq h(x^2)$. Now, we consider three cases:

Case 1: If $h(x^3) = h(x^4)$, then $x^2 \in (x^3, x^4) \subseteq L_\leq(B_\varepsilon, h, h(x^3))$ by the quasi-convexity of h on B_ε, a contradiction to $h(x^3) < h(x^2)$.

Case 2: If $h(x^3) < h(x^4)$, then $x^2 \in (x^3, x^4) \subseteq L_<(B_\varepsilon, h, h(x^4))$ by the semi-strict quasi-convexity of h on B_ε, a contradiction to $h(x^4) \leq h(x^2)$.

Case 3: If $h(x^4) < h(x^3)$, then $x^2 \in (x^3, x^4) \subseteq L_<(B_\varepsilon, h, h(x^3))$ by the semi-strict quasi-convexity of h on B_ε, a contradiction to $h(x^3) < h(x^2)$.

We get that h is semi-strictly quasi-convex on the set X. □

In the next theorem, we present a new characterization of semi-strictly quasi-convex functions.

Theorem 6.5 *Let $(\mathcal{V}, ||\cdot||)$ be a normed space, let $X \subseteq \mathcal{V}$ be open and convex, and let $h : \mathcal{V} \to \mathbb{R}$ be continuous along line segments on X. Then, h is semi-strictly quasi-convex on X if and only if both of the following statements hold:*

1°. *h is locally semi-strictly quasi-convex at each point $x^0 \in X$.*

2°. *Every local minimum of h is also global for each restriction on a line segment in X.*

Proof: Statement 1° together with the lower semi-continuity along line segments of h on X imply the local explicit quasi-convexity of h at every point $x^0 \in X$. In view of Lemma 6.4, we infer that both statements 1° and 2° imply the semi-strict quasi-convexity of h on X, taking into account the upper semi-continuity along line segments of h on X.

Now, we prove the reverse implication. The validity of statement 1° follows by Lemma 6.2 and the semi-strict quasi-convexity of h on X ensures that 2° holds. □

An analogous statement as given in Theorem 6.5 holds for the concept of (local) explicit quasi-convexity.

6.2 Sufficient conditions for the validity of the Assumptions (A1) and (A2)

Now, we are going to identify further classes of functions that satisfy both Assumptions (A1) and (A2).

At the beginning of this section, we present two preliminary lemmata.

Lemma 6.6 *Let* (4.1) *be satisfied. Then, we have*

1°. f_{m+1} *fulfils* (A1) *and* (A2) *for* $Y = \mathcal{V}$ *if and only if* f_{m+1} *fulfils* (A1) *and* (A2) *for each set* Y *with* $X \subseteq Y \subseteq \mathcal{V}$.

2°. f_{m+1} *fulfils* (A1) *and* (A2) *if and only if* $\tilde{f}_{m+1} := h \circ f_{m+1} : Y \to \mathbb{R}$ *fulfils* (A1) *and* (A2) *(with* $\tilde{f}_{m+1}$ *in the role of* f_{m+1}*), where* $h : \mathbb{R} \to \mathbb{R}$ *is a strictly increasing function on the image set* $f_{m+1}[Y]$.

3°. f_{m+1} *fulfils* (A1) *and* (A2) *if and only if* $\tilde{f}_{m+1} := f_{m+1} - f_{m+1}(x^0)$, $x^0 \in \operatorname{bd} X$, *fulfils* (A1), (A2) *and* (A5) *(with* $\tilde{f}_{m+1}$ *in the role of* f_{m+1}*).*

Lemma 6.7 *Let* (4.1) *be satisfied and let* Y *be open. Suppose that* f_{m+1} *is upper semi-continuous on* $\mathcal{V}$ *and fulfils Assumption* (A5)*. Assume that* $L_<(Y, f_{m+1}, 0) \neq \emptyset$. *Then,* X *has a nonempty interior, since*

$$\emptyset \neq L_<(Y, f_{m+1}, 0) \subseteq \operatorname{int} X. \tag{6.1}$$

Proof: In view of (A5), we have

$$\emptyset \neq L_<(Y, f_{m+1}, 0) = L_<(\mathcal{V}, f_{m+1}, 0) \cap Y \subseteq L_\leq(\mathcal{V}, f_{m+1}, 0) \cap Y = X.$$

By the upper semi-continuity of f_{m+1} on $\mathcal{V}$ and by Lemma 3.5, the set $L_<(\mathcal{V}, f_{m+1}, 0)$ is open. Hence, the intersection of $L_<(\mathcal{V}, f_{m+1}, 0)$ with the open set Y is open too. So, we conclude (6.1). □

In the formulation of Lemma 6.7, the openness assumption concerning the set Y is essential, as to see in the next example.

Example 6.8 *We consider the function* $f_{m+1} := ||\cdot||_\infty - 1$, *where the maximum norm* $||\cdot||_\infty : \mathbb{R}^2 \to \mathbb{R}$ *is defined by* $||x||_\infty := \max(|x_1|, |x_2|)$ *for all* $x = (x_1, x_2) \in \mathbb{R}^2$. *Notice that* f_{m+1} *is convex on* $\mathbb{R}^2$, *hence explicitly quasi-convex as well as continuous on* $\mathbb{R}^2$. *Moreover, put* $x^1 := (0, 0)$, $x^2 := (1, 0)$ *and* $Y := \overline{B}_{||\cdot||_\infty}(x^2, 1)$.

First, we have

$$\begin{aligned} L_{\leq}(Y, f_{m+1}, 0) &= L_{\leq}(\mathbb{R}^2, ||\cdot||_\infty, 1) \cap Y \\ &= \overline{B}_{||\cdot||_\infty}(x^1, 1) \cap \overline{B}_{||\cdot||_\infty}(x^2, 1) \\ &= [0,1] \times [-1,1] \\ &=: X. \end{aligned}$$

Moreover, it can easily be seen that

$$\begin{aligned} L_{<}(Y, f_{m+1}, 0) &= L_{<}(\mathbb{R}^2, ||\cdot||_\infty, 1) \cap Y \\ &= B_{||\cdot||_\infty}(x^1, 1) \cap \overline{B}_{||\cdot||_\infty}(x^2, 1) \\ &= [(0,1) \times (-1,1)] \cup [\{0\} \times (-1,1)] \\ &\supsetneq (0,1) \times (-1,1) \\ &= \operatorname{int} X, \end{aligned}$$

which shows that the inclusion given in (6.1) *of Lemma 6.7 does not hold. Hence, the openness assumption of* Y *in Lemma 6.7 can not be removed.*

In the following, we are looking for conditions such that Assumptions (A1) and (A2) are fulfilled for the penalization function f_{m+1}.

Lemma 6.9 *Let* (4.1) *be satisfied and let* Y *be open. Suppose that* f_{m+1} *is upper semi-continuous on* $\mathcal{V}$ *and fulfils Assumption* (A5)*. Assume that* $L_{<}(Y, f_{m+1}, 0) \neq \emptyset$*. Then, Assumption* (A1) *is fulfilled, and moreover, for every* $x^0 \in \operatorname{bd} X$*, we have* $f_{m+1}(x^0) = 0$*.*

Proof: Let $x^0 \in \operatorname{bd} X$. We are going to show that $f_{m+1}(x^0) = 0$, hence Assumption (A1) follows by the fact that $X = L_{\leq}(Y, f_{m+1}, 0)$.

Assume the contrary, i.e., $f_{m+1}(x^0) < 0$. By Lemma 6.7 we get

$$x^0 \in L_{<}(Y, f_{m+1}, 0) \subseteq \operatorname{int} X,$$

in contradiction to $x^0 \in \operatorname{bd} X$. □

The next lemma uses the definitions of local explicit quasi-convexity of the function f_{m+1} (see Definition 6.1) and presents sufficient condition for the validity of the Assumptions (A1) and (A2).

Lemma 6.10 *Let* (4.1) *be satisfied and let* Y *be open. Suppose that* $(\mathcal{V}, ||\cdot||)$ *is a normed space. Assume that* f_{m+1} *is upper semi-continuous on* $\mathcal{V}$ *and fulfils Assumption* (A5)*. For every* $x^0 \in (\operatorname{int} X) \cap L_{=}(Y, f_{m+1}, 0)$*, we suppose that there exists* $\varepsilon > 0$ *such that* f_{m+1} *is explicitly quasi-convex on* $B_{||\cdot||}(x^0, \varepsilon)$*, and there is*

$$x^1 \in B_{||\cdot||}(x^0, \varepsilon) \cap L_{<}(Y, f_{m+1}, 0). \tag{6.2}$$

Then, Assumptions (A1) *and* (A2) *are fulfilled.*

Proof: The validity of Assumption (A1) follows by Lemma 6.9. We are going to prove that Assumption (A2) holds.

For $x^0 \in \operatorname{bd} X$, we know that $f_{m+1}(x^0) = 0$ by Lemma 6.9, hence wit follows $x^0 \in L_=(Y, f_{m+1}, 0)$. This shows $\operatorname{bd} X \subseteq L_=(Y, f_{m+1}, 0)$.

Let us prove the inclusion $\operatorname{bd} X \supseteq L_=(Y, f_{m+1}, 0)$. Consider $x^0 \in L_=(Y, f_{m+1}, 0) \subseteq X$. Assume the contrary, i.e., $x^0 \in \operatorname{int} X$. Since $x^0 \in \operatorname{int} X$, there exists $\overline{\varepsilon} > 0$ such that $B_{||\cdot||}(x^0, \overline{\varepsilon}) \subseteq X$. Obviously, we have $B_{\tilde{\varepsilon}} := B_{||\cdot||}(x^0, \tilde{\varepsilon}) \subseteq X$ for $\tilde{\varepsilon} := \min(\overline{\varepsilon}, \varepsilon)$. By Lemma 2.2, we know that

$$B_\delta := [x^0 - \delta v, x^0 + \delta v] \subseteq B_{\tilde{\varepsilon}}$$

for $\delta \in (0, \tilde{\varepsilon})$ and $v := \frac{x^1 - x^0}{||x^1 - x^0||}$ (note that $x^1 \neq x^0$). Due to the semi-strict quasi-convexity of f_{m+1} on $B_\varepsilon := B_{||\cdot||}(x^0, \varepsilon)$ and the fact that $x^0 \in L_=(Y, f_{m+1}, 0)$ and $x^1 \in L_<(Y, f_{m+1}, 0)$, we can choose $x^2 \in B_\delta \cap (x^0, x^1]$ with $x^2 \in L_<(Y, f_{m+1}, 0)$. For $x^3 := x^0 + (x^0 - x^2)$, we have $x^3 \in B_\delta \subseteq B_{\tilde{\varepsilon}} \subseteq X$ and $x^0 \in (x^2, x^3)$.

Now, since we have $x^3 \in X = L_\leq(Y, f_{m+1}, 0)$, we can consider two cases:

Case 1: Let $x^3 \in L_=(Y, f_{m+1}, 0)$. Under the semi-strict quasi-convexity of f_{m+1} on B_ε, we get $x^0 \in (x^2, x^3) \subseteq L_<(B_\varepsilon, f_{m+1}, 0)$. Since $(x^2, x^3) \subseteq B_\delta$, it follows $x^0 \in L_<(B_\delta, f_{m+1}, 0) \subseteq L_<(Y, f_{m+1}, 0)$, a contradiction to $x^0 \in L_=(Y, f_{m+1}, 0)$.

Case 2: Let $x^3 \in L_<(Y, f_{m+1}, 0)$. Since $x^2, x^3 \in L_<(B_\varepsilon, f_{m+1}, 0)$, it follows $x^0 \in (x^2, x^3) \subseteq L_<(B_\varepsilon, f_{m+1}, 0)$ by the quasi-convexity of f_{m+1} on B_ε. Because of $(x^2, x^3) \subseteq B_\delta$, we have $x^0 \in L_<(B_\delta, f_{m+1}, 0) \subseteq L_<(Y, f_{m+1}, 0)$, again a contradiction to $x^0 \in L_=(Y, f_{m+1}, 0)$.

Consequently, we get that $x^0 \in \operatorname{bd} X$. □

The next lemma gives sufficient conditions for the validity of (6.2).

Lemma 6.11 *Let* (4.1) *be satisfied and let Y be open. Suppose that $(\mathcal{V}, ||\cdot||)$ is a normed space. Assume that f_{m+1} is upper semi-continuous on $\mathcal{V}$ and fulfils Assumption* (A5). *Consider two points $\tilde{x} \in L_<(Y, f_{m+1}, 0)$ and $x^0 \in (\operatorname{int} X) \cap L_=(Y, f_{m+1}, 0)$. Let f_{m+1} be explicitly quasi-convex on $B_{||\cdot||}(x^0, \varepsilon)$ for some $\varepsilon > 0$. Then, there exists $x^1 \in \operatorname{int} X$ such that condition* (6.2) *holds if one of the following statements is true:*

1°. *Every local minimum point of f_{m+1} on* $\operatorname{int} X$ *is also global.*

2°. *Assume that X is convex. Every local minimum of f_{m+1} is also global for each restriction on a line segment in* $\operatorname{int} X$.

Proof: Let 1° be fulfilled. Assume that there is no $x^1 \in \operatorname{int} X$ such that (6.2) holds. Then, x^0 is a local minimum of f_{m+1} on $\operatorname{int} X$, hence under 1° also global on $\operatorname{int} X$. This is a contradiction because we have $\tilde{x} \in L_<(Y, f_{m+1}, 0) \subseteq \operatorname{int} X$ (see Lemma 6.7) and $f_{m+1}(\tilde{x}) < 0 = f_{m+1}(x^0)$.

Now, let 2° be fulfilled. By Lemma 2.1, we have $x^0 \in \operatorname{cor} X$. For $v := x^0 - \tilde{x} \neq 0$ there exists $\delta > 0$ such that $x^0 + [0, \delta] \cdot v \subseteq X$. Define $x^2 := x^0 + \delta v$. By $\tilde{x} \in L_<(Y, f_{m+1}, 0) \subseteq \operatorname{int} X$ (see Lemma 6.7) and the convexity of X, we know that $x^0 \in (\tilde{x}, x^2) \subseteq \operatorname{int} X$ (see, e.g., Zălinescu [31, Th. 1.1.2]). Choose $x^3 \in (x^0, x^2)$. Assume that there is no $x^1 \in \operatorname{int} X$ such that (6.2) holds, hence x^0 is a local minimum of f_{m+1} on $\operatorname{int} X$. Then, $x^0 \in (\tilde{x}, x^3)$ is also a local minimum of f_{m+1} on the line segment $[\tilde{x}, x^3] \subseteq \operatorname{int} X$. By 2° of this lemma, we infer that x^0 is also global minimum of f_{m+1} on the line segment $[\tilde{x}, x^3]$, in contradiction to $f_{m+1}(\tilde{x}) < 0 = f_{m+1}(x^0)$. □

In the following theorem, we identify a further class of functions that fulfils the Assumptions (A1) and (A2).

Theorem 6.12 *Let* (4.1) *be satisfied and let* Y *be open. Suppose that* $(\mathcal{V}, ||\cdot||)$ *is a normed space. Assume that* f_{m+1} *is upper semi-continuous on* $\mathcal{V}$ *and fulfils Assumption* (A5). *Let* $L_<(Y, f_{m+1}, 0) \neq \emptyset$ *(i.e.,* $\operatorname{int} X \neq \emptyset$*). The function* f_{m+1} *fulfils the Assumptions* (A1) *and* (A2) *if both of the following statements hold:*

1°. *Every local minimum point of* f_{m+1} *on* $\operatorname{int} X$ *is also global.*

2°. f_{m+1} *is locally explicitly quasi-convex on* $\operatorname{int} X$.

Proof: Directly follows by Lemmata 6.10 and 6.11. □

Notice that every local minimum point of a semi-strictly quasi-convex function on a convex set is also global (see, e.g., Bagdasar and Popovici [3]).

Theorem 6.13 *Let* (4.1) *be satisfied and let* Y *be open. Assume that* f_{m+1} *is upper semi-continuous on* $\mathcal{V}$ *and fulfils Assumption* (A5). *Let* $L_<(Y, f_{m+1}, 0) \neq \emptyset$. *If* f_{m+1} *is explicitly quasi-convex on* Y*, then Assumptions* (A1) *and* (A2) *hold.*

Proof: If, in addition, $\mathcal{V}$ is normed, then we get the statement of this corollary by Theorem 6.12. Now, let us assume that $\mathcal{V}$ is not necessarily normed.

By Lemma 6.9, we know that Assumption (A1) is fulfilled. We are going to prove that Assumption (A2) holds. In view of the proof of Lemma 6.10 we know $\operatorname{bd} X \subseteq L_=(Y, f_{m+1}, 0)$. Now, we show $L_=(Y, f_{m+1}, 0) \cap \operatorname{int} X = \emptyset$.

Assume the contrary, i.e., we have $x^0 \in L_=(Y, f_{m+1}, 0) \cap \operatorname{int} X$. Consider $\tilde{x} \in L_<(Y, f_{m+1}, 0)$. By Lemma 2.1, it follows $x^0 \in \operatorname{cor} X$. Hence, there exists $\delta > 0$ such that $x^1 := x^0 + \delta(x^0 - \tilde{x}) \in X$. Notice that $x^0 \in (\tilde{x}, x^1)$ and $x^1 \in X = L_\leq(Y, f_{m+1}, 0)$.

Now, we look at two cases:

Case 1: Let $f_{m+1}(x^1) < 0$. Then, the quasi-convexity of f_{m+1} on Y implies $x^0 \in (\tilde{x}, x^1) \subseteq L_<(Y, f_{m+1}, 0)$, a contradiction to $f_{m+1}(x^0) = 0$.

Case 2: Let $f_{m+1}(x^1) = 0$. By the semi-strict quasi-convexity of f_{m+1} on Y, we get $x^0 \in (\tilde{x}, x^1) \subseteq L_<(Y, f_{m+1}, 0)$, again a contradiction to $f_{m+1}(x^0) = 0$. □

The openness assumption concerning the set Y can not be removed in Lemma 6.9 and Theorems 6.12 and 6.13, as shown in the next example.

Example 6.14 *If we assume that Assumption* (A1) *holds for the problem considered in Example 6.8, then we have* $x^2 \notin L_\leq(Y, f_{m+1}, f_{m+1}(x^1)) = X$ *for the point* $x^2 \in \operatorname{bd} X$ *since* $f_{m+1}(x^1) = -1 < 0 = f_{m+1}(x^2)$*, a contradiction.*

Suppose that Assumption (A2) *is fulfilled for the convex problem considered in Example 6.8. Then, we have* $L_=(Y, f_{m+1}, f_{m+1}(x^2)) = \operatorname{bd} X$ *for the point* $x^2 \in \operatorname{bd} X$. *So, due to* $f_{m+1}(x^2) = 0$*, we must have* $f_{m+1}(x) = 0$ *for all* $x \in \operatorname{bd} X$. *However, it is easily seen that* $f_{m+1}(x^1) = -1$ *for the point* $x^1 \in \operatorname{bd} X$*, a contradiction.*

Consequently, the Assumptions (A1) *and* (A2) *do not hold for the problem given in Example 6.8. This means that the openness assumption concerning the set Y can not be removed in Lemma 6.9 and Theorems 6.12 and 6.13.*

Corollary 6.15 *Let* (4.1) *be satisfied and let $Y = \mathcal{V}$. Assume that f_{m+1} is semi-strictly quasi-convex as well as continuous on $\mathcal{V}$ and fulfils Assumption* (A5)*. Let $L_<(\mathcal{V}, f_{m+1}, 0) \neq \emptyset$. Then, Assumptions* (A1) *and* (A2) *hold.*

The next lemma shows that under the Assumption (A5) a (semi-strictly) quasi-convex function f_{m+1} on Y ensures that the set X is convex. Hence, in order to describe a nonconvex feasible set X using the level set $L_\leq(Y, f_{m+1}, 0)$, it is necessary that f_{m+1} is not a (semi-strictly) quasi-convex function on Y.

Lemma 6.16 *Let* (4.1) *be satisfied. Assume that f_{m+1} is quasi-convex or semi-strictly quasi-convex on the convex set Y and fulfils Assumption* (A5)*. Then, X is a convex set in $\mathcal{V}$.*

Proof: Since $X = L_\leq(Y, f_{m+1}, 0)$ by Assumption (A5), we know that the quasi-convexity of f_{m+1} on Y implies convexity of X.

Let f_{m+1} be semi-strictly quasi-convex on Y. Assume the contrary, i.e., there exist $x^1, x^2 \in X$, $\lambda \in (0,1)$ such that $x^3 := l_{x^1,x^2}(\lambda) \notin X$. Consider the complement of $X = L_\leq(Y, f_{m+1}, 0)$, i.e, the set

$$X^c := \mathcal{V} \setminus X = L_>(Y, f_{m+1}, 0) \cup (\mathcal{V} \setminus Y). \tag{6.3}$$

The convexity of Y ensures $x^3 \in (x^1, x^2) \subseteq Y$, and therefore,

$$x^3 \in L_>(Y, f_{m+1}, 0). \tag{6.4}$$

Since X is closed, the set X^c is open, hence by (6.3), (6.4) and Lemma 2.1 we get $x^3 \in \operatorname{cor} X^c$. Therefore, for $v := x^1 - x^3 \neq 0$, it exists $\delta > 0$ such that $x^3 + [0,\delta] \cdot v \subseteq X^c$. Moreover, we have $x^3 + [0,1] \cdot v = [x^3, x^1] \subseteq Y$. Hence, by (6.3), it follows

$$x^3 + [0,\overline{\delta}] \cdot v \subseteq L_>(Y, f_{m+1}, 0) \cap (x^1, x^2) \tag{6.5}$$

for $\overline{\delta} := \min(\delta, 0.5) > 0$. By Assumption (A5) and by $x^1, x^2 \in X$, we have

$$\max(f_{m+1}(x^1), f_{m+1}(x^2)) \leq 0. \tag{6.6}$$

In view of (6.5) and (6.6), we get

$$\begin{aligned} x^3 + [0,\overline{\delta}] \cdot v &\subseteq L_> \left((0,1), (f_{m+1} \circ l_{x^1,x^2}), 0\right) \\ &\subseteq L_> \left((0,1), (f_{m+1} \circ l_{x^1,x^2}), \max(f_{m+1}(x^1), f_{m+1}(x^2))\right). \end{aligned}$$

Notice that $\operatorname{card}\left(x^3 + [0,\overline{\delta}] \cdot v\right) > 1$. However, in view of Lemma 3.4, under the semi-strict quasi-convexity of f_{m+1} on the convex set Y, it follows

$$\operatorname{card}\left(L_>((0,1), (f_{m+1} \circ l_{x^1,x^2}), \max(f_{m+1}(x^1), f_{m+1}(x^2)))\right) \leq 1,$$

a contradiction. This completes the proof. $\square$

7 Problems involving constraints given by a system of inequalities

In the previous sections, the feasible set $X \subseteq \mathcal{V}$ was always represented by certain level sets of a penalization function $f_{m+1} : \mathcal{V} \to \mathbb{R}$ (see the Assumptions (A1), (A3) and (A5)). However, in many cases the feasible set X is given by a system of inequalities, i.e., we have

$$X := \{x \in Y \mid g_1(x) \leq 0, \dots, g_q(x) \leq 0\} = \bigcap_{i \in I_q} L_{\leq}(Y, g_i, 0)$$

for some constraint functions $g_1, \dots, g_q : \mathcal{V} \to \mathbb{R}$, $q \in \mathbb{N}$, and a convex set $Y \subseteq \mathcal{V}$. For notational convenience, let us consider $g := (g_1, \dots, g_q) : \mathcal{V} \to \mathbb{R}^q$ as the vector-valued constraint function.

In order to apply results from Sections 5 and 6, for the penalization function f_{m+1} considered in $(\mathcal{P}_Y^{\oplus})$, we put

$$f_{m+1} := \max(g_1, \dots, g_q).$$

Then, Assumption (A5) is satisfied, i.e., we have

$$X = \bigcap_{i \in I_q} L_{\leq}(Y, g_i, 0) = L_{\leq}(Y, f_{m+1}, 0). \tag{7.1}$$

For the special approach considered in this section, the standard assumption (4.1) reads as

$$\begin{cases} \text{Let } \mathcal{V} \text{ be a real topological linear space;} \\ \text{let } X = L_{\leq}(Y, f_{m+1}, 0) \text{ be nonempty and closed;} \\ \text{let } Y \subseteq \mathcal{V} \text{ be convex.} \end{cases} \tag{7.2}$$

Notice that under the assumptions that Y is closed and f_{m+1} is lower semi-continuous on Y, the set X is closed too. In addition, due to Lemmata 3.5, 3.7 and 6.16, under the assumptions in (7.2) we get the following implications:

- If g is componentwise convex (quasi-convex) on Y, then f_{m+1} is convex (quasi-convex) on Y.
- If f_{m+1} is quasi-convex or semi-strictly quasi-convex on Y, then the set X is convex.
- Assume that Y is closed. If g is componentwise lower semi-continuous on Y, then f_{m+1} is lower semi-continuous on Y.

In some results, we need the well-known *Slater condition* that is given by

$$\bigcap_{i \in I_q} L_{<}(Y, g_i, 0) = L_{<}(Y, f_{m+1}, 0) \neq \emptyset. \tag{7.3}$$

In order to force the validity of the Assumptions (A1) and (A2), we can use Lemmata 6.9 and 6.10, Theorems 6.12 and 6.13, and Corollary 6.15.

Next, we present relationships between the original problem $(\mathcal{P}_X)$ with constraint set X given by a system of inequalities and the objective function

$$f = (f_1, \dots, f_m),$$

and two related problems $(\mathcal{P}_Y)$ and $(\mathcal{P}_Y^{\oplus})$ with a convex feasible set Y and the objective functions

$$f = (f_1, \dots, f_m),$$

and

$$f^{\oplus} = (f_1, \dots, f_m, f_{m+1}) = (f_1, \dots, f_m, \max(g_1, \dots, g_q)),$$

respectively.

Theorem 7.1 *Let* (7.2) *be satisfied and let* Y *be open. Suppose that* f_{m+1} *is upper semi-continuous on* $\mathcal{V}$*. Assume that Slater's condition* (7.3) *holds.*

1°. *Let the Assumption* (A2) *be fulfilled. If* f *is componentwise semi-strictly quasi-convex on* Y*, then*

$$\mathrm{Eff}(X \mid f) = [X \cap \mathrm{Eff}(Y \mid f)] \cup \left[(\mathrm{bd}\, X) \cap \mathrm{Eff}(Y \mid f^{\oplus})\right]. \tag{7.4}$$

2°. *Assume that Assumptions* (A2) *and* (A6) *hold. If* f *is componentwise semi-strictly quasi-convex as well as upper semi-continuous along line segments on* Y*, then*

$$\mathrm{WEff}(X \mid f) = [(\mathrm{int}\, X) \cap \mathrm{WEff}(Y \mid f)] \cup \left[(\mathrm{bd}\, X) \cap \mathrm{WEff}(Y \mid f^{\oplus})\right]. \tag{7.5}$$

3°. *If* f *is componentwise semi-strictly quasi-convex or quasi-convex on* Y*, then*

$$\mathrm{SEff}(X \mid f) = [(\mathrm{int}\, X) \cap \mathrm{SEff}(Y \mid f)] \cup \left[(\mathrm{bd}\, X) \cap \mathrm{SEff}(Y \mid f^{\oplus})\right]. \tag{7.6}$$

Proof: The validity of Assumption (A1) follows by Slater's condition (7.3) and Lemma 6.9. In addition, we have $\mathrm{int}\, X \neq \emptyset$ by Lemma 6.7. Hence, this theorem follows directly by Theorems 5.1, 5.6 and 5.13. Notice that (7.1) (i.e., (A5) holds) implies (A4) by Remark 4.7. □

Under the assumption that f_{m+1} is explicitly quasi-convex on Y, we directly get the following result by Theorem 7.1.

Corollary 7.2 *Let* (7.2) *be satisfied and let* Y *be open. Suppose that* f_{m+1} *is upper semi-continuous on* $\mathcal{V}$ *and explicitly quasi-convex on* Y*. Assume that Slater's condition* (7.3) *holds.*

1°. *If* f *is componentwise semi-strictly quasi-convex on* Y*, then* (7.4) *holds.*

2°. *If* f *is componentwise semi-strictly quasi-convex as well as upper semi-continuous along line segments on* Y*, then* (7.5) *holds.*

3°. *If* f *is componentwise semi-strictly quasi-convex or quasi-convex on* Y*, then* (7.6) *holds.*

Proof: Follows directly by Theorem 7.1. Notice that Assumptions (A1) and (A2) are fulfilled by Theorem 6.13. Moreover, due to the semi-strict quasi-convexity of f_{m+1} on Y, the Assumption (A6) is satisfied. $\square$

For the case $Y = \mathcal{V}$, we conclude the following result by Corollary 7.2.

Corollary 7.3 *Let* (7.2) *be satisfied and let* $Y = \mathcal{V}$. *Suppose that* f_{m+1} *is semi-strictly quasi-convex and continuous on* $\mathcal{V}$. *Assume that Slater's condition* (7.3) *holds. Then, assertions* 1°, 2° *and* 3° *of Corollary 7.2 are fulfilled.*

8 Concluding remarks

In this paper, we derived a new approach for solving generalized-convex multi-objective optimization problems involving not necessarily convex constraints. These results extend and generalize the results given by Günther and Tammer [12]. We showed that the set of (strictly, weakly) efficient solutions (in an arbitrarily real topological linear space) of a multi-objective optimization problem involving a nonempty closed (not necessarily convex) feasible set, can be computed completely using at most two corresponding multi-objective optimization problems with a new feasible set that is an convex upper set of the original feasible set. Our approach relies on the fact that the original feasible set can be described using level sets of a certain scalar function (see Assumptions (A1), (A2) and (A3)). We applied our approach to problems where the feasible set is given by a system of inequalities with a finite number of constraint functions. For deriving our new results, we assumed that the well-known Slater constraint condition is fulfilled.

In a forthcoming paper, we apply our results to special types of nonconvex multi-objective optimization problems. It is interesting to study problems where the nonconvex feasible set is given by a union of convex sets, as well as problems involving multiple forbidden regions. Such problems can be motivated by several models in location theory.

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References

[1] C.D. Aliprantis and K. C. Border, *Infinite dimensional analysis*, 2nd edn. Springer, Berlin, Heidelberg, 2006.

[2] M. Apetrii, M. Durea and R. Strugariu, *A new penalization tool in scalar and vector optimization*, Nonlinear Anal. **107** (2014), 22–33.

[3] O. Bagdasar, N. Popovici, *Local maximum points of explicitly quasiconvex functions*, Optimization Letters. **9(4)** (2015), 769–777.

[4] V. Barbu and T. Precupanu, *Convexity and Optimization in Banach Spaces*, Springer Netherlands, 2012.

[5] A. Cambini and L. Martein, *Generalized Convexity and Optimization: Theory and Applications*, Springer, Berlin, Heidelberg, 2009.

[6] M. Durea, R. Strugariu and C. Tammer, *On Some Methods to Derive Necessary and Sufficient Optimality Conditions in Vector Optimization*, J. Optim. Theory Appl. doi:10.1007/s10957-016-1059-y.

[7] M. Ehrgott, *Multicriteria Optimization*, 2nd edn. Springer, Berlin, Heidelberg, 2005.

[8] G. Eichfelder, *Variable Ordering Structures in Vector Optimization*, Springer, Berlin, Heidelberg, 2014.

[9] C. Gerth (Tammer) and P. Weidner, *Nonconvex separation theorems and some applications in vector optimization*, J. Optim. Theory Appl. **67** (1990), 297–320.

[10] G. Giorgi, A. Guerraggio and J. Thierfelder, *Mathematics of Optimization - Smooth and Nonsmooth Case*, Elsevier, Amsterdam, 2004.

[11] A. Göpfert, H. Riahi, C. Tammer and C. Zălinescu, *Variational Methods in Partially Ordered Spaces*, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, 17, Springer-Verlag, New York, 2003.

[12] C. Günther and C. Tammer, *Relationships between constrained and unconstrained multi-objective optimization and application in location theory*, Math Meth Oper Res. **84(2)** (2016), 359–387.

[13] J.-B. Hiriart-Urruty, *New concepts in nondifferentiable programming*, Bull. Soc. Math. France **60** (1979), 57–85.

[14] J.-B. Hiriart-Urruty and C. Lemaréchal, *Convex Analysis and Minimization Algorithms I*, Springer, Berlin Heidelberg, 1993.

[15] J. Jahn, *Vector Optimization - Theory, Applications, and Extensions*, 2nd edn. Springer, Berlin Heidelberg, 2011.

[16] A. A. Kahn, C. Tammer and C. Zălinescu, *Set-valued Optimization: An Introduction with Applications*, Springer, Berlin, Heidelberg, 2015.

[17] K. Klamroth and J. Tind, *Constrained Optimization Using Multiple Objective Programming*, J Global Optim. **37(3)** (2007), 325–355.

[18] C. G. Liu, K. F. Ng and W. H. Yang, *Merit functions in vector optimization*, Math. Program. Ser. A. **119** (2009), 215–237.

[19] M. M. Mäkelä, V.-P. Eronen and N. Karmitsa, *On Nonsmooth Optimality Conditions with Generalized Convexities*, Optimization in Science and Engineering. (2014), 333–357.

[20] C. Malivert and N. Boissard, *Structure of efficient sets for strictly quasi-convex objectives*, J. Convex Anal. **1** (1994), 143–150.

[21] B. S. Mordukhovich and N. M. Nam, *Subgradients of distance functions at out-of-set points*, Taiwanese Journal of Mathematics. **10(2)** (2006), 299–326.

[22] N. Popovici, *Pareto reducible multicriteria optimization problems*, Optimization. **54** (2005), 253–263.

[23] N. Popovici, *Structure of efficient sets in lexicographic quasiconvex multicriteria optimizatio*, Operations Research Letters. **34** (2006), 142-–148.

[24] N. Popovici, *Involving the Helly number in Pareto reducibility*, Operations Research Letters. **36** (2008), 173–176.

[25] N. Popovici, *The role of generalized convexity in vector optimization and related variational problems*, Habilitation thesis, Babes-Bolyai University Cluj-Napoca, 2016.

[26] J. Puerto and A.M. Rodríguez-Chía, *Quasiconvex constrained multicriteria continuous location problems: Structure of nondominated solution sets*, Comput Oper Res. **35(3)** (2008), 750–765.

[27] S. M. Robinson, *A quadratically convergent algorithm for general nonlinear programming problems*, Mathematical Programming (1972), 145-–156.

[28] R. T. Rockafellar, *Convex Analysis*, Princeton University Press, Princeton, 1970.

[29] J. J. Ye, *The exact penalty principle*, Nonlinear Anal. **75(3)** (2012), 1642—1654.

[30] A. Zaffaroni, *Degrees of efficiency and degrees of minimality*, SIAM J. Control Optim. **42** (2003), 1071–1086.

[31] C. Zălinescu, *Convex analysis in general vector spaces*, River Edge, N.J.: World Scientific, 2002.

[32] E. Zeidler, *Nonlinear Functional Analysis and its Applications. Part III: Variational Methods and Optimization, Springer, New York, 1986.*