The Noncooperative Transportation Problem

Oliver Stein*       Nathan Sudermann-Merx†

May 23, 2017

Abstract

We extend the classical transportation problem from linear optimization and introduce several competing forwarders. This results in a noncooperative game which is commonly known as generalized Nash equilibrium problem. We show the existence of Nash equilibria and present numerical methods for their efficient computation. Furthermore, we discuss several equilibrium selection concepts that are applicable to this particular Nash game.

Keywords: Transportation problem with several forwarders, linear generalized Nash equilibrium problem, noncooperative game theory, subgradient method

AMS subject classifications: 91A06, 91A10, 90B06, 90C56


*Institute of Operations Research, Karlsruhe Institute of Technology (KIT), 76131 Karlsruhe, Germany, stein@kit.edu
†nathan.sudermann@gmail.com
1 Introduction

1.1 Motivation

Since its first mathematical formulation in the 18th century (cf. [23]), the transportation problem is one of the most famous problems in operations research. In the classical transportation problem we have one forwarder who transports a given good from manufacturers to producers while minimizing his transportation costs. In this paper we extend the transportation problem towards a more realistic scenario and introduce several forwarders. Each forwarder wants to minimize his transportation costs while sharing the supply and demand constraints with the remaining forwarders. This is exactly the situation which is addressed in noncooperative game theory. Therefore, in the following we will refer to the transportation problem with several forwarders as noncooperative transportation problem (NTP). The resulting setting is illustrated in Figure 1.

Nash equilibria in the NTP are configurations where no forwarder wants to deviate from his strategy given the decision of his competitors. This is not only of theoretical interest: Suppose, there is one owner of different factories who negotiates contracts with several competing forwarders. If the contract conditions are set such that the resulting configuration is a Nash equilibrium no forwarder has an incentive to deviate from the contract conditions. This yields a stable situation and is therefore preferable.
1.2 Literature review

In 1951, John Nash investigated the problem of finding equilibria in situations where several competing players try to optimize their objective functions over strategy sets that are independent of the decisions of the remaining players (cf. [24]). We will refer to this situation as the (classical) Nash equilibrium problem (NEP). Only a few years later, Kenneth Arrow and Gérard Debreu extended the setting towards a model with coupled strategy sets, that is, strategy sets that may depend on the decisions of the remaining players (cf. [1, 7]). Coupled strategy sets arise in a very natural way if, for instance, players share at least one constraint which could be a common budget or commonly used infrastructure. A Nash equilibrium problem with coupled constraints is called a generalized Nash equilibrium problem (GNEP). Despite its early introduction it took over 40 years until GNEPs attracted attention in the operations research community. During this time, that is, until the mid nineties, mainly existence results for (generalized) Nash equilibria where available and the numerical computation of equilibria was less developed (cf. [12]). However, the field of operations research had a deep impact on game theory and provided powerful numerical methods for the computation of Nash equilibria. Excellent overviews of theoretical and numerical results as well as numerous applications of GNEPs are given in [12] and [14]. Linear generalized Nash equilibrium problems (LGNEPs) are GNEPs under linearity assumptions (cf. [10], [11] and [27]). In particular, the noncooperative transportation problem can be reformulated as an LGNEP which was first mentioned as a small example in [27].

There exists some literature which examines the connection between noncooperative game theory and transportation. [13] is an early and well-known contribution to this field of research where the author examines a model of carriers competing for intercity passenger travel as an example of a Nash game. However, the resulting Nash game is discussed briefly and on a very general level without any exploitation of the underlying structure. In [29], the authors consider a generalized Nash equilibrium (GNEP) in transportation. However, the work is not related to the NTP but investigates an oligopolistic transit market with elastic demand. The authors of [18] present a comparative literature review of noncooperative games that describe transport problems in order to show their opportunities and risks. All reviewed articles in this context are categorized in games against a demon, games between travelers, games between travelers and authorities as well as games between authorities and it is possible to assign the NTP to the latter category. However, there exist only very few articles about games between authorities and
the authors state that

\[\ldots\text{the small number of such games is surprising, considering that noncooperative game theory seems a natural tool for analysing relations between authorities; }\ldots\]

In order to contribute to this this exciting and promising field of research this article provides a systematic examination of the noncooperative transportation problem. To the best of our knowledge, this paper yields the first systematic examination of the situation described in the noncooperative transportation problem.

1.3 Statement of contribution

We introduce a new game-theoretic model that describes a straightforward and realistic extension of the classical transportation problem. Furthermore, we establish a profound mathematical theory as well as numerical methods for the computation of Nash equilibria. Finally, we discuss the equilibrium selection problem in this setting and suggest some practical approaches to this problem which to the best of our knowledge has not been treated in the existing literature up to now.

1.4 Overview

In Section 2 we formulate the underlying mathematical model of the noncooperative transportation problem (NTP) with \(N\) forwarding agencies. The existence of Nash equilibria in the NTP is shown in Section 3. In particular, we construct an \((N-1)\)-dimensional set of Nash equilibria which can be computed efficiently. Further, a vertex enumeration algorithm and a projected subgradient method for the computation of Nash equilibria in the NTP are presented in Section 4. Since it is possible to compute many Nash equilibria efficiently, the question arises which equilibrium one should select in practical applications. This issue, the so-called Equilibrium Selection Problem, is addressed in Section 5 where we examine several criteria for the selection of specific Nash equilibria in terms of auxiliary optimization problems and show that for \(N = 2\) we even obtain closed form solutions for these optimization problems. Section 6 closes the article with some final remarks.


2 Mathematical model

Consider $N$ competing forwarding agencies which want to transport one good from $R$ manufacturers to $T$ consumers. Manufacturer $r \in \{1, \ldots, R\}$ has a production capacity of $S_r \geq 0$ and consumer $t \in \{1, \ldots, T\}$ needs at least $D_t \geq 0$ units of this good. We assume $\sum_{r=1}^{R} S_r = \sum_{t=1}^{T} D_t$. The unitary transportation cost and the number of transported units from manufacturer $r$ to consumer $t$ by forwarder $\nu \in \{1, \ldots, N\}$ are denoted by $c_{rt}^\nu$ and $x_{rt}^\nu$, respectively. Let

$$x^\nu := (x_{11}^\nu, \ldots, x_{1T}^\nu, x_{21}^\nu, \ldots, x_{2T}^\nu, \ldots, x_{R1}^\nu, \ldots, x_{RT}^\nu)^\top \in \mathbb{R}^{R \cdot T}.$$ 

Given the decisions of the remaining agencies forwarder $\nu$ wants to minimize his total transportation costs

$$\min_{x^\nu} \sum_{r=1}^{R} \sum_{t=1}^{T} c_{rt}^\nu x_{rt}^\nu$$

subject to his constraints concerning the supply

$$\sum_{\ell=1}^{N} \sum_{t=1}^{T} x_{rt}^\ell = S_r, \ r \in \{1, \ldots, R\},$$

as well as his demand constraints

$$\sum_{\ell=1}^{N} \sum_{r=1}^{R} x_{rt}^\ell = D_t, \ t \in \{1, \ldots, T\},$$

and the nonnegativity condition $x^\nu \geq 0$.

A Nash equilibrium is a tuple $(\bar{x}^1, \ldots, \bar{x}^N)$, such that $\bar{x}^\nu$ is an optimal point of player $\nu$’s optimization problem for all $\nu \in \{1, \ldots, N\}$ given the decisions of the remaining players. Since the optimization problems of all players are linear and they share the same constraints, the resulting game is a so-called linear generalized Nash equilibrium problem (LGNEP) (cf. [10], [11] and [27]). In the following, we will refer to this specific LGNEP as noncooperative transportation problem or just NTP.

We pose the following assumption in order to avoid trivial NTPs.

**Assumption 2.1** We have $D_t > 0$ for at least one $t \in \{1, \ldots, T\}$. 

The following example will be used throughout this paper to illustrate our thoughts.

**Example 2.2** Consider an NTP with two manufacturers producing one good which is delivered by two forwarders to two consumers, that is, \( N = R = T = 2 \). The first manufacturer offers one unit of this good and the second one wants to sell two units, thus, we set \( S_1 = 1 \) and \( S_2 = 2 \). The demand of the first consumer is given by \( D_1 = 2 \) and consumer two needs one unit, that is, \( D_2 = 1 \).

The cost matrices \( C^\nu := (c^\nu_{rt}) \) are given by

\[
C^1 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \quad \text{and} \quad C^2 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}.
\]

In order to obtain a clear representation of player \( \nu \)'s optimization problem let \( \langle x, y \rangle := x^\top y \) denote the standard inner product and define

\[
e^\nu := (c^\nu_{11}, c^\nu_{1T}, c^\nu_{21}, \ldots, c^\nu_{R1}, \ldots, c^\nu_{RT})^\top \in \mathbb{R}^{RT},
\]

\[
b := (S_1, \ldots, S_R, D_1, \ldots, D_T)^\top \in \mathbb{R}^{R+T}
\]

as well as

\[
A := \begin{pmatrix} e^\top & 0 & \ldots & 0 \\ 0 & e^\top & \ddots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \ldots & e^\top \\ I_T & I_T & \cdots & I_T \end{pmatrix} \in \mathbb{R}^{(R+T) \times (R+T)}
\]

for all \( \nu \in \{1, \ldots, N\} \) with \( e := (1, \ldots, 1)^\top \in \mathbb{R}^T \) and the \( T \)-dimensional identity matrix \( I_T \).

Then we vectorize player \( \nu \)'s optimization problem and arrive at

\[
\min_{x^\nu} \langle e^\nu, x^\nu \rangle \quad \text{s.t.} \quad Ax^\nu + \sum_{\mu \neq \nu} A x^\mu = b, \quad x^\nu \geq 0
\]

where the decisions \( x^\mu, \mu \neq \nu \), of the remaining players are given.

Furthermore, let

\[
x^{-\nu} := (x^1, \ldots, x^{\nu-1}, x^{\nu+1}, \ldots, x^N)^\top \in \mathbb{R}^{(N-1) \cdot R \cdot T},
\]

be the vector which concatenates the decisions of the remaining players.
Example 2.3 In Example 2.2 we have
\[ c^1 = \begin{pmatrix} 1 \\ 2 \\ 2 \\ 1 \end{pmatrix}, \quad c^2 = \begin{pmatrix} 2 \\ 1 \\ 1 \\ 2 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \]
and
\[ A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}. \]
However, hereinafter, in order to improve the readability of the development of Example 2.2 we will prefer to work with its non-vectorized version using the cost matrices \( C^\nu \).

3 Existence of Nash equilibria

According to [9] one cannot expect to obtain an unique equilibrium in GNEPs since, generically, the set of Nash equilibria together with their KKT multipliers constitute a Lipschitz manifold whose dimension is connected to the number of players and the number of shared constraints. We prove the existence of an \((N - 1)\)-dimensional set of Nash equilibria in the NTP which can be computed by solving \( N \) linear optimization problems as we shall see in Theorem 3.1. Before that we need to introduce some notation.

Let \( y^\nu \in \mathbb{R}^{R^T} \) be an optimal point of player \( \nu \)'s classical transportation problem, that is, for \( x^{\nu} = 0 \). This optimal point exists since the classical transportation problem is solvable. Further, define the vectors
\[ \hat{y}^1 := \begin{pmatrix} y^1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \hat{y}^2 := \begin{pmatrix} 0 \\ y^2 \\ \vdots \\ 0 \end{pmatrix}, \quad \ldots, \quad \hat{y}^N := \begin{pmatrix} 0 \\ \vdots \\ 0 \\ y^N \end{pmatrix} \in \mathbb{R}^{R^T \cdot N} \]
and the set
\[ Y := \text{conv}(\hat{y}^1, \ldots, \hat{y}^N), \]
that is, \( Y \) is the convex hull of the vectors \( \hat{y}^1, \ldots, \hat{y}^N \). Further, we define the dimension of a convex set by the dimension of its affine hull, that is, the smallest affine subspace that contains \( Y \).
In Theorem 3.1 we see that each element of $Y$ is a Nash equilibrium and, therefore, we obtain a large set of Nash equilibria which can be computed very efficiently.

**Theorem 3.1** Let Assumption 2.1 be valid. Then $Y$ is an $(N-1)$-dimensional set and each element of $Y$ is a Nash equilibrium in the NTP.

**Proof.** The affine hull of $Y$ is spanned by the $N-1$ vectors $\tilde{y}_1 - \tilde{y}_2, \ldots, \tilde{y}_1 - \tilde{y}_N$ which are linearly independent due to Assumption 2.1. Therefore, $Y$ is an $(N-1)$-dimensional set.

The concatenated KKT systems of all players are given by

$$
\begin{align*}
\mathbf{c}^\nu + A^\nu \mathbf{\mu}^\nu - \lambda^\nu &= 0 \quad (1) \\
A x^\nu + \sum_{\mu \neq \nu} A x^\mu &= b \quad (2) \\
x^\nu &\geq 0 \\
\lambda^\nu &\geq 0 \\
(\lambda^\nu)^T x^\nu &= 0
\end{align*}
$$

for all $\nu \in \{1, \ldots, N\}$. Now take an arbitrary $y \in Y$, that is, there exist nonnegative scalars $\sigma_1, \ldots, \sigma_N \geq 0$ with $\sum_{i=1}^N \sigma_i = 1$ and

$$y = \sum_{i=1}^N \sigma_i \tilde{y}_i = \begin{pmatrix} \sigma_1 \tilde{y}_1 \\ \vdots \\ \sigma_N \tilde{y}_N \end{pmatrix}.$$

Note that $y$ is a Nash equilibrium if and only if $\sigma_\nu y^\nu$ solves (1)-(5) for all $\nu \in \{1, \ldots, N\}$. Since $y^\nu$ is an optimal point of the classical transportation problem, there exist vectors $\bar{\lambda}^\nu \geq 0$ and $\bar{\mu}^\nu$ with

$$
\begin{align*}
\mathbf{c}^\nu + A^\nu \bar{\mu}^\nu - \bar{\lambda}^\nu &= 0 \\
A y^\nu &= b \\
y^\nu &\geq 0 \\
\bar{\lambda}^\nu &\geq 0 \\
(\bar{\lambda}^\nu)^T y^\nu &= 0
\end{align*}
$$

for all $\nu \in \{1, \ldots, N\}$. Now, we take the vector $\sigma_\nu y^\nu$ together with the dual variables $\bar{\lambda}^\nu$ and $\bar{\mu}^\nu$ of $y^\nu$ and plug them into (1)-(5) for all $\nu \in \{1, \ldots, N\}$. 

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It is easy to see that \((\sigma_\nu y^\nu, \bar{\lambda}^\nu, \bar{\mu}^\nu)\) satisfies (1), (3), (4), and (5) for all \(\nu \in \{1, \ldots, N\}\). Equation (2) is also valid for \((\sigma_\nu y^\nu, \bar{\lambda}^\nu, \bar{\mu}^\nu)\) since we have

\[
A(\sigma_\nu y^\nu) + \sum_{\mu \neq \nu} A(\sigma_\mu y^\mu) = \sum_{\nu=1}^{N} \sigma_\nu Ay^\nu = b \sum_{\nu=1}^{N} \sigma_\nu = b.
\]

Altogether, we have shown that each element of \(Y\) is a Nash equilibrium in the NTP.

\section*{Example 3.2}

In Example 2.2, the optimization problems of the two forwarders have the unique optimal transportation plans

\[
y^1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \text{and} \quad y^2 = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix}.
\]

Hence, we have

\[
Y = \left\{ \begin{pmatrix} \lambda_1 & 0 \\ \lambda_1 & \lambda_1 \\ 0 & \lambda_2 \\ 2\lambda_2 & 0 \end{pmatrix} \in \mathbb{R}^{4 \times 2} : \lambda_1 + \lambda_2 = 1, \lambda_1, \lambda_2 \geq 0 \right\}.
\]

\section{4 Computation of Nash equilibria}

\subsection{4.1 Sign bingo algorithm}

It is possible to compute the whole set of all Nash equilibria of an NTP applying the Sign Bingo algorithm from [10] which is a vertex enumeration algorithm.

\section*{Example 4.1}

We apply the Sign Bingo algorithm to Example 2.2 and compute the set of all Nash equilibria in the NTP which is given by
Especially, recalling Example 3.2 we see that the set $Y$ does not contain all Nash equilibria but is a strict subset of the latter.

The Sign Bingo algorithm is suitable for the computation of all Nash equilibria in small and medium sized NTPs. However, due to its exponential complexity, which is intrinsic to all vertex enumeration algorithms, the computation of all Nash equilibria fails for large NTPs, such that we restrict ourselves in the remainder of this article to the computation of special Nash equilibria instead of computing all Nash equilibria.

### 4.2 Projected subgradient method

As we have seen in Example 4.1, not all Nash equilibria are part of the set $Y$. Therefore, besides the computation of the set $Y$ there is a need to apply some additional methods which are able to compute arbitrary Nash equilibria and not just equilibria in $Y$. As we shall see in Remark 4.5, this can be accomplished applying the projected subgradient method from [11] to an auxiliary optimization problem. Let us briefly recall the required notation and results from [11].
For given \(x^{-\nu}\) player \(\nu\)'s dual problem is given by
\[
\max_{\lambda^\nu \in \mathbb{R}^{R+T}} \langle b - \sum_{\mu \neq \nu} Ax^\mu, \lambda^\nu \rangle \quad \text{s.t.} \quad A^T \lambda^\nu \leq c^\nu.
\]
We define player \(\nu\)'s optimal value function by
\[
\varphi^\nu(x^{-\nu}) := \max_{\lambda^\nu \in Z^\nu} \langle b - \sum_{\mu \neq \nu} Ax^\mu, \lambda^\nu \rangle
\]
with
\[
Z^\nu := \{ \lambda^\nu \in \mathbb{R}^{R+T} : A^T \lambda^\nu \leq c^\nu \}.
\]
Furthermore, we define the gap function
\[
V(x) := \sum_{\nu=1}^N \langle c^\nu, x^\nu \rangle - \varphi^\nu(x^{-\nu})
\]
which does not have to be real-valued outside of
\[
W := \{ x \in \mathbb{R}^{N \times R^T} : Ax = b, x \geq 0 \}.
\]
Nash equilibria are exactly the global minimal points of the possibly non-smooth and nonconvex optimization problem
\[
P : \quad \min V(x) \quad \text{s.t.} \quad x \in W
\]
with optimal value zero (cf. [17]). Since, in general, \(V\) is a nonsmooth and nonconvex function, we use a projected subgradient method with subgradients in the sense of Clarke (cf. [6]) in order to solve \(P\).

Furthermore, let \(O^\nu\) denote the set of vertices of \(Z^\nu\). According to [19] Ex. 6.1.9], the set \(O^\nu\) is nonempty, that is \(Z^\nu\) possesses at least one vertex. Then due to the vertex theorem of linear programming we may extend \(\varphi^\nu\) to the real-valued function
\[
\hat{\varphi}^\nu(x^{-\nu}) := \max_{\lambda^\nu \in O^\nu} \langle b - \sum_{\mu \neq \nu} Ax^\mu, \lambda^\nu \rangle
\]
and define
\[
O^\nu(x^{-\nu}) := \{ \lambda^\nu \in O^\nu : \langle b - \sum_{\mu \neq \nu} Ax^\mu, \lambda^\nu \rangle = \hat{\varphi}^\nu(x^{-\nu}) \}.
\]
This allows us to define the real-valued extension $\hat{V}$ of the gap function $\hat{V}$ by setting

$$\hat{V}(x) := \sum_{\nu=1}^{N} \langle c^\nu, x^\nu \rangle - \hat{\varphi}_\nu(x^{-\nu}).$$

Let us recall the definition of a generalized gradient in the sense of Clarke (cf. [6]) to which we will just refer as Clarke subgradient throughout this work.

**Definition 4.2** Let $f$ be a Lipschitz continuous function and denote the set of its nondifferentiability points by $ND$. Then we define the Clarke subdifferential of $f$ at $x$ by

$$\partial f(x) := \text{conv} \left( \lim_{x_i \to x, x_i \notin ND} \nabla f(x_i) \right)$$

and each element $s \in \partial f(x)$ is called a Clarke subgradient of $f$ at $x$.

In the following, for two sets $A$ and $B$ we will use their (Minkowski) sum $A + B = \{a + b : a \in A, b \in B\}$.

**Theorem 4.3** In the NTP we have

$$\partial \hat{V}(x) = \left\{ \begin{array}{l} \begin{pmatrix} c^1 \\ A^T \lambda^1 \\ \vdots \\ A^T \lambda^1 \end{pmatrix} : \lambda^1 \in \text{conv} \left( O_1(x^{-1}) \right) \\ + \begin{pmatrix} c^2 \\ A^T \lambda^2 \\ \vdots \\ A^T \lambda^2 \end{pmatrix} : \lambda^2 \in \text{conv} \left( O_2(x^{-2}) \right) \\ + \ldots + \begin{pmatrix} c^N \\ A^T \lambda^N \end{pmatrix} : \lambda^N \in \text{conv} \left( O_N(x^{-N}) \right) \end{array} \right\}$$

**Proof.** Recall that we have

$$\hat{\varphi}_\nu(x^{-\nu}) = \max_{\lambda^\nu \in O_\nu} \langle \lambda^\nu, B^\nu x^{-\nu} - b^\nu \rangle.$$
The Clarke subdifferential is homogeneous with respect to also negative scalars. Furthermore, convex functions are subdifferentiable regular (cf. [6]) which implies the validity of a sum rule. Altogether, we have

$$\partial \hat{V}(x) = \partial \left( \sum_{\nu=1}^{N} (\langle c^\nu, x^\nu \rangle - \hat{\varphi}_\nu(x^\nu)) \right)$$

$$= \partial \left( - \sum_{\nu=1}^{N} (\hat{\varphi}_\nu(x^{-\nu}) - \langle c^\nu, x^\nu \rangle) \right)$$

$$= - \partial \left( \sum_{\nu=1}^{N} \hat{\varphi}_\nu(x^{-\nu}) - \langle c^\nu, x^\nu \rangle \right)$$

$$= - \left( \sum_{\nu=1}^{N} \partial \hat{\varphi}_\nu(x^{-\nu}) - \nabla_x \langle c^\nu, x^\nu \rangle \right)$$

and the assertion follows from a standard formula for the subdifferential of convex functions which have a representation as pointwise maximum of finitely many smooth functions (cf. [2]).

After having evaluated the gap function $\hat{V}$ at a given point $x$ the vector

$$s := \begin{pmatrix} c^1 \\ A^T \bar{\lambda}^1 \\ \vdots \\ A^T \bar{\lambda}^1 \end{pmatrix} + \begin{pmatrix} A^T \bar{\lambda}^2 \\ c^2 \\ \vdots \\ A^T \bar{\lambda}^2 \end{pmatrix} + \cdots + \begin{pmatrix} A^T \bar{\lambda}^{N-1} \\ \vdots \\ c^{N-1} \\ A^T \bar{\lambda}^{N-1} \end{pmatrix} + \begin{pmatrix} A^T \bar{\lambda}^N \\ \vdots \\ c^N \end{pmatrix}$$

is a Clarke subgradient of $\hat{V}$ at $x$ in the NTP. Furthermore, obviously, we have

$$\nabla V(x) = \begin{pmatrix} c^1 \\ \vdots \\ c^N \end{pmatrix} - \begin{pmatrix} A^T \bar{\lambda}^1 \\ \vdots \\ A^T \bar{\lambda}^N \end{pmatrix} + \sum_{\nu=1}^{N} \begin{pmatrix} A^T \bar{\lambda}^\nu \\ \vdots \\ A^T \bar{\lambda}^\nu \end{pmatrix}$$

for all $x$ where $V$ is smooth.

Having introduced all necessary objects we state the projected subgradient method (PSM) in Algorithm 1.
Algorithm 1: Projected Subgradient Method (PSM)

(S.0): Choose $x^0 \in W, \varepsilon \geq 0$, and set $k := 0$.
(S.1): If $V(x^k) \leq \varepsilon$: STOP.
(S.2): Compute a subgradient $s^k \in \partial V(x^k)$.
(S.3): Set $\alpha_k := \frac{1}{\sqrt{k+1}}$.
(S.4): Compute $x^{k+1} = P_W[x^k - \alpha_k s^k]$.
(S.5): Set $k := k + 1$ and go to (S.1).

Remark 4.4 One major drawback of general nonsmooth methods is that an ordinary ‘pointwise’ subdifferential does not yield a numerical stopping criterion. One possibility to overcome this difficulty is the use of epsilon subdifferentials that contain information on subderivatives from a neighborhood of the point of interest as it is done, e.g., in bundle methods or the robust gradient sampling method (RGS) from [3]. Usually, the approximated epsilon subdifferential is used to compute a search direction or to check a stopping criterion at the cost of an quadratic optimization problem in each iteration (cf. [3, 22]). In particular for high dimensional problem data this computation becomes numerically expensive. However, in contrast to general nonsmooth optimization problems, we can exploit that our optimal function value is known to be zero, which provides the handy stopping criterion $V(x^k) \leq \varepsilon$, and we do not have to construct a numerically expensive epsilon subdifferential.

Remark 4.5 It is, trivially, possible to compute each Nash equilibrium of an arbitrary LGNEP by the projected subgradient method. In order to see this recall that all generalized Nash equilibria are global minimal points of the concave function $V$ over the polyhedron $W$ and, therefore, the Nash equilibria of LGNEPs lie at the boundary of the polyhedron $W$. Hence, initializing PSM with a starting point in the normal cone of the desired Nash equilibrium yields a direct solution in one iteration. However, of course, in practice the desired Nash equilibrium is not known in advance.

We generate random test examples for the NTP for different combinations of $N$, $R$ and $T$ and apply PSM to these test instances using a Matlab® implementation. We choose the origin as starting point, $V(x) < 10^{-6}$ as stopping criterion and report the needed number of iterations $iter$, the final function value of the gap function $Vterm$ as well as the computation time $time$. We see that PSM did not terminate in 6 test examples. This happened since PSM started cycling and we suspect that PSM got stuck in a local
minimal point of $V$ since we were able to overcome this difficulty by choosing different starting points. However, note that PSM was able to compute an equilibrium for $N = 50$, $R = 10$ and $T = 50$ in about ten minutes which is a nonsmooth nonconvex optimization problem with 25000 variables. The numerical results are presented in Table 1.

5 Selection of Nash equilibria

Recall that in the transportation problem with $N$ forwarders it is possible to compute the $(N - 1)$-dimensional set $Y$ of Nash equilibria by solving $N$ linear optimization problems which can be done very efficiently. Further, we may have computed additional equilibria by applying the projected subgradient method to the NTP. Thus, the question arises which equilibrium one should select. In the economics literature this problem is known as Equilibrium Selection or Nash Selection Problem (see [15, 16, 21] for possible entry points in this field of research from an economic perspective). In this chapter, we will concentrate on selection techniques that are based on the idea of minimizing some objective function $f$ over the set $Y$, that is, we consider the optimization problem

$$P_{sel}: \min_y f(y) \quad \text{s.t.} \quad y \in Y$$

with optimal point $y^*$. In the following sections, we shall discuss several choices of objective functions $f$ that may be reasonable.

Recall that we have

$$Y = \left\{ y \in \mathbb{R}^{RTN} : \exists \lambda \in \Delta_N \text{ with } y = \sum_{i=1}^{N} \lambda_i \hat{y}_i \right\}$$

where $\Delta_N \in \mathbb{R}^N$ denotes the $(N - 1)$-dimensional standard simplex, that is, we have

$$\Delta_N := \{ \lambda \in \mathbb{R}^N : \lambda \geq 0, \sum_{i=1}^{N} \lambda_i = 1 \}.$$ 

Therefore, it is possible to transform $P_{sel}$ into an optimization problem over the standard simplex (i.e., in barycentric coordinates)

$$P_{sel} : \min_{\lambda} f\left( \sum_{i=1}^{N} \lambda_i \hat{y}_i \right) \quad \text{s.t.} \quad \lambda \in \Delta_N.$$ 

Note that this is an optimization problem with $N - 1$ degrees of freedom since it is possible to eliminate one variable from the representation above.
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Table 1: Numerical results with PSM for the extended transportation problem
Example 5.1 We will apply the subsequent results to our running example in order to illustrate our thoughts. In Example 2.2 the set \( Y \) was given by

\[
Y = \left\{ y : \exists \lambda \in [0, 1] \text{ with } y = \lambda \cdot \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} + (1 - \lambda) \cdot \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 2 & 0 \end{pmatrix} \right\}
\]

which is a one-dimensional set.

Remark 5.2 Of course it is also possible to consider known equilibria that are not elements of the set \( Y \) in the selection problem \( P_{sel} \). Therefore, in the following, we will also evaluate the different objective functions on the equilibrium

\[
\bar{x} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{pmatrix}
\]

which is an equilibrium in Example 3.2 that is not part of the set \( Y \).

5.1 Minimal total costs

In order to select the ‘best’ Nash equilibrium from the set \( Y \), perhaps the most obvious thought would be to minimize the sum of the cost functions of all forwarders over \( Y \), that is, to set

\[
f_1(y) := \langle c, y \rangle
\]

with \( c = (c^1, \ldots, c^N)^T \) and to solve

\[
P_1: \min_y \langle c, y \rangle \quad \text{s.t.} \quad y \in Y
\]

which is a linear optimization problem. There is a very efficient way to solve \( P_1 \) without employing a numerical method of linear programming since the set of vertices of \( Y \) possesses only \( N \) elements which are given by

\[
\text{vert } Y = \{ \hat{y}^1, \ldots, \hat{y}^N \}.
\]

Therefore, due to the vertex theorem of linear programming, we can solve \( P_1 \) by enumeration of the \( N \) vertices of \( Y \) which can be done very efficiently even for a large number of forwarders.
Example 5.3 The vertices of $Y$ in Example 2.2 are given by

\[
\hat{y}^1 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \hat{y}^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 2 & 0 \end{pmatrix}.
\]

An evaluation of the sum of both cost functions $f$ at these vertices yields $f_1(\hat{y}^1) = 4$ and $f_1(\hat{y}^2) = 3$, such that we would select the Nash equilibrium $\hat{y}^2$ since it is the equilibrium in $Y$ that minimizes the sum of both cost functions. Further, we have $f_1(\bar{x}) = 3$, that is $\bar{x}$ is also an optimal choice.

Remark 5.4 Note that all Nash equilibria that solve $P_1$ are also Pareto optimal since they maximize the social welfare. In general, all Pareto optimal Nash equilibria in the set $Y$ can be computed by solving the optimization problem

\[
P_\eta : \min_y \sum_{\nu=1}^{N} \eta_\nu (c^{\nu})^\top y^{\nu} \quad \text{s.t.} \quad y \in Y
\]

with positive weights $\eta_\nu$, $\nu = 1, \ldots, N$.

The advantage of this approach is that it yields a decision criterion which is easy to interpret and very efficiently computable. One disadvantage in this approach is that we will always arrive at an equilibrium that is a vertex of $Y$ because these equilibria are exactly the points in which all goods are delivered by only one forwarder, such that the remaining forwarders transport nothing. Therefore, in Section 5.2 we shall consider an approach that aims at obtaining a Nash equilibrium which possesses a uniform distribution of all goods within the forwarders.

5.2 Uniform distribution of goods

Another approach of choosing a Nash equilibrium from the set $Y$ may be to distribute the good from manufacturers to the consumers in a way that the maximal delivery size is minimal. This has at least two advantages: First, this approach is very risk averse and may be preferable if the costs of transportation errors is very high. Second, it enforces a uniform transportation over all forwarders and possible combinations. This is also the main difference to the approach in Section 5.1 where all units were delivered by only
one forwarder. Thus, we set
\[ f_2(y) := \max_{r,t,\nu} y_{r,t}^{\nu} \]
and obtain the optimization problem
\[ P_2 : \min \max_y \max_{r,t,\nu} y_{r,t}^{\nu} \quad \text{s.t.} \quad y \in Y \]
which, using the epigraph reformulation, can be reformulated, such that \( P_2 \) is equivalent to the linear optimization problem
\[ \min_{(y,\alpha)} \quad \text{s.t.} \quad y \in Y, \]
\[ y_{r,t}^{\nu} \leq \alpha, \quad r = \{1, \ldots, R\}, \quad t = \{1, \ldots, T\}, \quad \nu = \{1, \ldots, N\}. \]
Note that the polyhedron \( Y \) is given in the so-called \( V \)-representation of polyhedral sets (cf. [30]), that is, as convex hull of its vertices. However, standard solvers of linear programming require the so-called \( H \)-representation of polyhedral feasible sets, that is, a representation as intersection of finitely many half spaces. Fortunately, as mentioned in the beginning of Section 5, we can overcome this difficulty by transforming the linear optimization problem into barycentric coordinates and arrive at
\[ \min_{(\lambda,\alpha)} \quad \lambda \geq 0, \quad \text{s.t.} \quad \lambda \cdot 1 = (1 - \lambda) \cdot 2 \]
which is an optimization problem whose representation is accessible to standard solvers in linear optimization.

**Example 5.5** In Example 2.2, direct inspections show that finding the equilibrium in \( Y \) with the smallest component is equivalent to determining \( \lambda \geq 0 \), such that
\[ \lambda \cdot 1 = (1 - \lambda) \cdot 2 \]
since 1 and 2 are the largest components of \( \hat{y}^1 \) and \( \hat{y}^2 \), respectively, and the order is not influenced by scaling with \( \lambda \). Therefore, we arrive at \( \lambda = \frac{2}{3} \), that is, we would select the Nash equilibrium
\[ y^* = \begin{pmatrix} \frac{2}{3} & 0 & 0 \\ 0 & \frac{1}{3} & -\frac{1}{3} \\ \frac{2}{3} & 0 & 0 \end{pmatrix} \]
from the set $Y$ which, indeed, possesses a rather uniform distribution of the good over all forwarders and possible combinations. Further, we have

$$f_2(\bar{x}) = 1 > \frac{2}{3} = f_2(y^*),$$

that is, in this situation we would not choose the equilibrium $\bar{x}$.

## 5.3 Minimizing the sum of squares

A more technical approach addresses the fact that one may not expect to obtain one unique equilibrium by the decision rules in the Sections 5.1 and 5.2 since they result from solving linear optimization problems that may have nonisolated solutions. A standard approach to tackle this issue is to choose norm minimal solution (cf. e.g. [20]), that is, solutions with minimal Euclidean norm. These solutions are unique since they can be obtained as optimal points of an unconstrained strictly convex quadratic optimization problem. Thus, we set

$$f_3(y) := \|y\|_2^2$$

and obtain the convex quadratic optimization problem

$$P_3 : \min_y y^T y \quad \text{s.t.} \quad y \in Y,$$

which possesses a unique solution.

**Example 5.6** In Example 2.2 solving $P_3$ is equivalent to computing a scalar $\lambda \in [0, 1]$, such that

$$\begin{bmatrix} \lambda \cdot \begin{pmatrix} 1 & 0 \\ 1 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} + (1 - \lambda) \cdot \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 2 & 0 \end{pmatrix} \end{bmatrix}_F^2 = \begin{bmatrix} \lambda & 0 \\ 0 & 1 - \lambda \\ 0 & 0 \end{bmatrix}_F^2 = 3\lambda^2 + 5(1 - \lambda)^2$$

is minimal where we used the Frobenius norm

$$\|A\|_F := \sum_{i,j} a_{ij}^2.$$

Thus, we compute $\lambda = \frac{5}{8}$ and choose the Nash equilibrium

$$y^* = \begin{pmatrix} 5 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$
from the set \( Y \) with \( f_3(y^*) = \frac{15}{3} \). Further, we have \( f_3(\bar{x}) = 3 \), that is, the equilibrium \( \bar{x} \) is not an optimal choice.

Note that the obtained norm minimal solution is not sparse at all. However, in the following section we will see that it is possible to compute the sparsest equilibria in the set \( Y \) very efficiently.

5.4 Sparse equilibria

One may also think of situations where it is advantageous to select a sparse equilibrium, that is, an equilibrium with as little non-zero components as possible. This could be preferable in situations where each non-zero entry \( y^\nu_{rt} \) is associated with high fixed costs that occur for ‘activating’ the combination \( x^\nu_{rt} \) which for instance might be the costs for constructing new infrastructure or legal fees for designing new contracts.

Selecting the sparsest equilibrium is equivalent to minimizing the so-called zero norm of a given vector \( y \in \mathbb{R}^n \) which is defined by

\[
\|y\|_0 := \sum_{i=1}^{n} |y_i|^0 = |\{1 \leq i \leq n : y_i \neq 0\}|
\]

where we defined \( 0^0 := 0 \), that is, we set

\[
f_4(y) := \|y\|_0
\]

and obtain the \( \ell_0 \)-minimization problem

\[
P_4 : \min_y \|y\|_0 \quad \text{s.t.} \quad y \in Y
\]

In general, the \( \ell_0 \)-minimization problem is computationally intractable and is therefore replaced by minimizing the \( \ell_1 \)-norm of \( y \) (cf., e.g., [4, 5, 8]).

However, in the NTP, the situation differs greatly from the general case. First, the \( \ell_1 \)-minimization does not make any sense for the NTP since each equilibrium of \( Y \) possesses the same \( \ell_1 \)-norm which follows from the following, more general, result.

**Proposition 5.7** There exists a scalar \( D > 0 \) with

\[
\|x\|_1 = D
\]

for all \( x \in W \), that is, each \( x \in W \) possesses the same \( \ell_1 \)-norm.
Proof. Let \( x \in W \). Then we have

\[
\|x\|_1 = \sum_{\nu=1}^{N} \sum_{t=1}^{T} \sum_{r=1}^{R} x_{rt}^\nu
\]

\[
= \sum_{t=1}^{T} \sum_{\nu=1}^{N} \sum_{r=1}^{R} x_{rt}^\nu
\]

\[
= \sum_{t=1}^{T} D_t,
\]

that is, the assertion is shown with \( D := \sum_{t=1}^{T} D_t \) and Assumption 2.1.

Second, for the extended transportation problem the \( \ell_0 \)-minimization is tractable since it can be reduced to selecting the sparsest vertex \( \hat{y}^\nu, \nu \in \{1, \ldots, N\}, \) of the set \( Y \). This can be seen easily, since the number of zeros of each element in \( Y \) cannot be higher than the number of zeros of a vertex \( \hat{y}^\nu, \nu \in \{1, \ldots, N\} \) of \( Y \) because the creation of a ‘new zero’ as convex combination of nonnegative elements is not possible.

Example 5.8 In Example 2.2, we have \( \|y\|_1 = 3 = D_1 + D_2 \) for all \( y \in Y \). Further, \( \hat{y}^2 \) is an optimal point of \( P_4 \) since it belongs to the sparsest player solution. The equilibrium \( \bar{x} \) is not as sparse at \( y^* = \hat{y}^2 \) and therefore not preferable.

5.5 Closed form solutions for two player games

Two player games form an important field of research in the existing literature on game theory (cf. [25, 28]). Also for NTPs, the case \( N = 2 \) deserves some special attention since for \( N = 2 \) the set of efficiently computable Nash equilibria \( Y \) is a one-dimensional line segment which implies that the selection problem \( P_{sel} \) is a one-dimensional optimization problem.

Remark 5.9 Note that \( N = 2 \) is the only restriction in this section. Particularly, each player may possess a large number of decision variables and constraints.

In Sections 5.1 and 5.4 we selected the desired equilibrium out of a set of \( N \) equilibria which just shrinks to a set of two equilibria in the case \( N = \)
2. However, as described in Sections 5.2 and 5.3, the computation of the equilibrium that possesses a uniform distribution of goods or minimizes the sum of squares, respectively, requires the solution of a linear or a quadratic optimization problem, respectively. In the case $N = 2$ we can avoid the numerical solution of these optimization problems and obtain closed form solutions instead as stated in the following propositions.

As mentioned at the beginning of Section 5, the problem of selecting a Nash equilibrium $y$ out of the set $Y$ can be transformed into determining a suitable weight $\lambda \in \Delta_N$. In the following, we will denote the desired Nash equilibrium by $y^* \in \mathbb{R}^{2T-R}$ and the corresponding weight by $\lambda^* \in \mathbb{R}$.

**Proposition 5.10** Let $\hat{y}_1^{\text{max}}, \hat{y}_2^{\text{max}} \in \mathbb{R}$ be the largest entries of the vectors $\hat{y}^1$ and $\hat{y}^2$, respectively, and define

$$
\lambda^* := \frac{\hat{y}_2^{\text{max}}}{\hat{y}_1^{\text{max}} + \hat{y}_2^{\text{max}}}.
$$

Then the vector

$$
y^* = \lambda^* \hat{y}^1 + (1 - \lambda^*) \hat{y}^2
$$

is the Nash equilibrium that solves $P_2$, that is, the equilibrium which enforces a uniform distribution of all goods.

**Proof.** It is straightforward to see that in order to solve $P_2$ we have to choose $\lambda$, such that we arrive at $\lambda \hat{y}_1^{\text{max}} = (1 - \lambda) \hat{y}_2^{\text{max}}$ since the position of the maximal component is not influenced by the multiplication of the corresponding vectors with the scalar $\lambda \geq 0$. Further, Assumption 2.1 implies $\hat{y}_1^{\text{max}} + \hat{y}_2^{\text{max}} \neq 0$, such that we have

$$
\lambda^* := \frac{\hat{y}_2^{\text{max}}}{\hat{y}_1^{\text{max}} + \hat{y}_2^{\text{max}}}.
$$


**Proposition 5.11** Let us define

$$
\lambda^* := \frac{\|\hat{y}^2\|_F^2}{\|\hat{y}^1\|_F^2 + \|\hat{y}^2\|_F^2}.
$$

Then the vector

$$
y^* = \lambda^* \hat{y}^1 + (1 - \lambda^*) \hat{y}^2
$$

is the Nash equilibrium that solves $P_3$, that is, the equilibrium with minimal Euclidean norm.
Proof. It is easy to see that the vectors $\hat{y}^1$ and $\hat{y}^2$ are orthogonal. Therefore, we have
\[
f(\lambda) := \|\lambda \hat{y}^1 + (1 - \lambda) \hat{y}^2\|_F^2 = \lambda^2 \|\hat{y}^1\|_F^2 + (1 - \lambda)^2 \|\hat{y}^2\|_F^2 \]
and differentiating $f$ with respect to $\lambda$ yields
\[
f'(\lambda) = 2\lambda \|\hat{y}^1\|_F^2 - 2(1 - \lambda) \|\hat{y}^2\|_F^2.
\]
Then we have
\[
f'(\lambda) = 0 \iff 2\lambda \|\hat{y}^1\|_F^2 = 2(1 - \lambda) \|\hat{y}^2\|_F^2 \\
\iff \lambda(\|\hat{y}^1\|_F^2 + \|\hat{y}^2\|_F^2) = \|\hat{y}^2\|_F^2 \\
\iff \lambda^* = \frac{\|\hat{y}^2\|_F^2}{\|\hat{y}^1\|_F^2 + \|\hat{y}^2\|_F^2}
\]
where
\[
\|\hat{y}^1\|_F^2 + \|\hat{y}^2\|_F^2 \neq 0
\]
is ensured by Assumption 2.1. Since $f$ is a strongly convex function, we have shown that $\lambda^*$ is the unconstrained global minimal point of $f$ on $\mathbb{R}$. In particular, we also have $\lambda^* \in [0, 1]$ which implies that $\lambda^*$ is also an optimal point of the constrained optimization problem $P_3$ which proves the desired assertion.

6 Final remarks

Using the results of this article, it is possible to establish contract conditions that yield stable situations in transportation problems with many forwarders. Therefore, we have provided guidelines for the algorithmic treatment of highly complex coupled transportation problems. While our proposals are based on game theoretic foundations, we have also shown their applicability within real world frameworks since the equilibria are computable in an efficient way.

The generalization of our approach to even more complex problems like transportation on graphs is left for future research.

Acknowledgments

The authors are grateful to Ralf Werner for fruitful discussions on the subject of this manuscript.
References


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