On the Size of Integer Programs with Bounded Coefficients or Sparse Constraints

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Abstract

Integer programming formulations describe optimization problems over a set of integer points. A fundamental problem is to determine the minimal size of such formulations, in particular, if the size of the coefficients or sparsity of the constraints is bounded. This article considers lower and upper bounds on these sizes both in the original and in extended spaces, i.e., if additional variables are allowed. We show that every bounded (hole-free) integer set can be described by an extended integer formulation using at most three non-zero coefficients which are ±1. In the original space, we provide lower bounds on the size of integer formulations with bounded coefficients. For 0/1-problems, we also introduce a technique to compute a tight lower bound on the number of non-zeros of integer formulations in the original space. Moreover, we present statistics on these bounds in small dimensions. Finally, we consider conditions on the representability of integer optimization problems with arbitrary objective function as mixed-integer programs. All results are illustrated by examples.

1 Introduction

Let \( X \subseteq \mathbb{Z}^p \times \mathbb{R}^q \) be a set that should be described using a mixed-integer formulation in the original space, i.e.,

\[
X = \{ x \in \mathbb{Z}^p \times \mathbb{R}^q : Ax \leq b \},
\]

for \( A \in \mathbb{R}^{m \times (p+q)} \), \( b \in \mathbb{R}^m \). Clearly, a necessary condition for the existence of such a formulation is that \( X \) is hole-free, i.e., \( X = \text{conv}(X) \cap (\mathbb{Z}^p \times \mathbb{R}^q) \). More flexibility is given by considering extended formulations, that is, allowing additional variables and \( X \) to be the projection of the corresponding set. Whether \( X \) can be represented using an integer formulation has been completely characterized by Jeroslow and Lowe [18]. Recently, Basu et al. [2] derived an equivalent algebraic characterization. In particular, \( X \) can always be represented by an extended formulation if \( X \) is bounded.

In recent years, the investigation of extended formulations that describe the mixed-integer hull of \( X \) has received considerable attention, see, for example, Conforti et al. [5], Vanderbeck and Wolsey [27], and Kaibel [20] for an overview. Moreover, lower bounds for approximations of the mixed-integer hull have been considered as well, see, for example, Faenza and Sanità [11] and Braun et al. [3]. In this article, we consider arbitrary integer formulations that do not necessarily describe or approximate the mixed-integer hull. In this context, Kaibel and Weltge [22] prove exponential lower bounds on the number of
inequalities of such formulations for certain combinatorial (optimization) problems in the original space.

In this article, we extend the previous research on integer formulations by taking bounds on the size of the coefficients, the number of non-zeros in the constraints, and the objective function into account. We are interested in lower and upper bounds on the number of inequalities of such formulations. Since many interesting optimization problems have a feasible region which is a subset of \{0,1\}^n, we will concentrate in most of our results on techniques for sets \(X \subseteq \{0,1\}^n\). Moreover, we will demonstrate their applicability in many examples.

In Section 2, we formally define integer formulations in the original and extended space, and we recall some results from the literature that will be useful in our analysis. Afterwards, in Section 3 we focus on how to obtain mixed-integer formulations with bounded coefficients. We present in Section 3.1 a modeling technique to transform an arbitrary mixed-integer formulation into a formulation with bounded coefficients in an extended space. In Section 3.2, we concentrate on integer formulations with bounded coefficients in the original space, and we recall results from the literature that allow us to find the smallest integer formulation of an independence system with ternary left-hand side coefficients. Moreover, we derive a lower bound on the size of any integer formulation with bounded coefficients, and we present statistics on the size of integer formulations with bounded coefficients in small dimensions. Thereafter, in Section 4 we focus on sparsity, since sparse LP relaxations are often numerically more stable and faster to solve. In Section 4.1, we show how to sparsify any mixed-integer formulation in an extended space. Section 4.2 provides lower bounds on the sparsity of integer formulations in the original space as well as a technique to exactly compute the maximum sparsity of any integer formulation. In analogy to the results on integer formulations in small dimensions with bounded coefficients, we present in Section 4.3 statistics on the maximum sparsity of integer formulations in small dimensions. Finally, we discuss in Section 5 concepts of mixed-integer formulations concerning the incorporation of an objective function.

2 Basics of Mixed-Integer Formulations

To fix notation, we formally define the following.

**Definition 1.** A mixed-integer program (MIP) is an optimization problem

\[
\max \{g(x) : x \in Q \cap (\mathbb{Z}^p \times \mathbb{R}^q)\},
\]

specified by a polyhedron \(Q \subseteq \mathbb{R}^p \times \mathbb{R}^q\) and an affine function \(g : \mathbb{R}^p \times \mathbb{R}^q \rightarrow \mathbb{R}\). We will use the tuple \((Q,g)\) to denote a MIP.

The following presentation consists of two parts. The second part deals with questions concerning the objective function (Section 5). In the first part, we will exclusively deal with representations of the feasible set \(X\) using \(Q \cap (\mathbb{Z}^p \times \mathbb{R}^q)\). Thus, we define:

**Definition 2.** A mixed-integer formulation of a set \(X \subseteq \mathbb{Z}^p \times \mathbb{R}^q\) is a pair \((Q,\pi)\) with a polyhedron \(Q = \{(y,z) \in \mathbb{R}^{p'} \times \mathbb{R}^{q'} : Ay + Bz \leq b\}\) and an affine function \(\pi : \mathbb{R}^{p'} \times \mathbb{R}^{q'} \rightarrow \mathbb{R}^p \times \mathbb{R}^q\) such that

\[
X = \pi(Q \cap (\mathbb{Z}^{p'} \times \mathbb{R}^{q'})),
\]

where \(A \in \mathbb{R}^{m \times p'}, B \in \mathbb{R}^{m \times q'},\) and \(b \in \mathbb{R}^m\). If \(\pi\) is the identity, the formulation is in the original space, otherwise in an extended space (or an extended formulation for short). Furthermore, if \(q' = 0\), we call the the formulation an integer formulation.
Note that the original $x$-variables are allowed, but not required, to occur in $Q$ and $\pi$.

In preparation of the following, we refer to a result of Kaibel and Weltge [22] which shows that we can always find a polynomial size extended mixed-integer formulation of a set $X \subseteq \{0,1\}^n$ if the membership problem “Given $x \in \mathbb{R}^n$, is $x \in X$?” is in $\mathcal{NP}$. Thus, there always exists a mixed-integer formulation of polynomial size in an extended space under this complexity condition.

**Theorem 3** (Kaibel and Weltge [22]). Consider a 0/1-problem that defines $X^I \subseteq \{0,1\}^{n(I)}$ for each instance $I$, and let the membership problem for $X^I$ be in $\mathcal{NP}$. Then there exists a polynomial $p$ such that for each instance $I$, there is a system $Ax + By \leq b$ of at most $p(n(I))$ linear inequalities and $k(I) \leq p(n(I))$ auxiliary variables with

$$X^I = \{ x \in \{0,1\}^{n(I)} : Ax + By \leq b, y \in \mathbb{Z}^{k(I)} \}.$$  

If the membership problem is in $\mathcal{P}$, the integrality condition on $y$ can be dropped.

Moreover, analyzing the proof of Theorem 3 in [22] shows that $A$, $B$, and $b$ have coefficients in $\{0,\pm1\}$ only. The proof is constructive, but the resulting system might be very large and complicated. In Section 4.1 we will see that the mixed integer formulation of Theorem 3 can even be sparsified.

Furthermore, it is also possible to characterize cases in which no tractable mixed-integer formulation, i.e., a formulation that can be separated in polynomial time, exists in the original space.

**Proposition 4.** Consider some 0/1-problem that defines $X^I \subseteq \{0,1\}^{n(I)}$ for each instance $I$, and let the membership problem for $X^I$ be $\mathcal{NP}$-hard. Then, as long as $\mathcal{P} \neq \mathcal{NP}$, there exists no tractable mixed-integer formulation, in particular, no formulation of polynomial size, in the original space.

**Proof.** If $X^I$ admitted a tractable mixed-integer formulation or a mixed-integer formulation of polynomial size in the original space, this mixed-integer formulation would allow to answer the membership problem for $X^I$ in polynomial time. □

### 3 Bounded Coefficients

In this section, we focus on the questions of whether there exist tractable formulations or formulations of polynomial size such that the coefficients can be bounded by a constant. We start with the case of extended formulations and then consider formulations in the original space.

#### 3.1 Bounded Coefficients in Extended Formulations

Let us motivate the following with a classical example: Given a complete undirected graph $G = (V,E)$ and edge weights $w \in \mathbb{R}^E$, the (symmetric) traveling salesman problem (TSP) is to find a weight minimal Hamiltonian cycle in $G$. The feasible region of $X$ is the set of all incidence vectors $x \in \{0,1\}^E$ of Hamiltonian cycles. A standard integer formulation of $X$ with inequalities all of whose left-hand side coefficients are contained in $\{0,\pm1\}$ is the so-called subtour elimination formulation, see, e.g., Dantzig et al. [6]. Although this formulation contains exponentially many inequalities, it is tractable, since there is a polynomial time separation routine for its inequalities, see Hong [16].

This still leaves the question unanswered whether there exists an integer formulation of the TSP that uses at most a polynomial number of constraints. While the answer to
this question is negative in the original space, see Kaibel and Weltge [22], there exists an integer formulation (MTZ) by Miller et al. [25] in an extended space that uses $O(|E|)$ variables and $O(|E|)$ constraints (instead of exponentially many constraints in the subtour elimination formulation). Furthermore, all coefficients in this formulation are bounded by $|V|$. 

In the remainder of this section, we demonstrate how we can reformulate mixed-integer formulations such that all coefficients are contained in $\{0, \pm 1\}$ using additional variables. Moreover, the derived mixed-integer formulation is of polynomial size if the original mixed-integer formulation has polynomial size.

Assume we are given an extended mixed-integer formulation of a mixed-integer set $X \subseteq \mathbb{R}^n$ of the form

$$Q = \{(y, z) \in \mathbb{Z}^p \times \mathbb{R}^q : Ay + Bz \leq b, \ 0 \leq y \leq u, \ 0 \leq z \leq \tilde{u}\}, \quad (1)$$

where $A \in \mathbb{Z}^{m \times p}$, $B \in \mathbb{Z}^{m \times q}$, and $u \in \mathbb{Z}^p$, $\tilde{u} \in \mathbb{Z}^q$, together with its projection to the original space. Note that in particular, the entries in $y$ and $z$ are bounded by arbitrary integers. The aim of this section is to show that we can transform (1) into a mixed-integer formulation whose left-hand side coefficients are bounded by $\{0, \pm 1\}$. In the following, we use the notation $[n] := \{1, \ldots, n\}$.

**Lemma 5.** Let $X \subseteq \mathbb{R}^n$ be a mixed-integer set that admits a mixed-integer formulation (1). Then (1) can be expanded to an extended mixed-integer formulation of $X$ of size

$$O\left(\sum_{j=1}^p u_j \max\{|a_{ij}| : i \in [m]\} + \sum_{j=1}^q |\tilde{u}_j| \max\{|a_{ij}| : i \in [m]\}\right)$$

with left-hand side coefficients in $\{0, \pm 1\}$.

**Proof.** The main idea of this proof is to substitute the $y$-variables by variables in $\{0, 1\}$ and the $z$-variables by variables in $[0, 1]$. We distinguish whether a variable is integer or continuous.

**Integer variables:** Let $y_j$ be an integer variable with upper bound $u_j$, and define $\tilde{a}_{ij} := |a_{ij}|$ as well as $\tilde{A}_j := \max\{\tilde{a}_{ij} : i \in [m]\}$. We introduce $\tilde{A}_j u_j$ many binary variables $y_{j}^{k\ell}$, where $k \in [u_j]$ and $\ell \in [\tilde{A}_j]$. We then write

$$a_{ij} y_j = \text{sign}(a_{ij}) \sum_{k=1}^{u_j} \sum_{\ell=1}^{\tilde{a}_{ij}} y_{j}^{k\ell},$$

and we add the linking constraints $y_{j}^{k1} = y_j$, $\ell \in [\tilde{A}_j]$, $k \in [u_j]$. Thus, in any feasible solution with $y_{j}^{k1} = 1$, we have $\sum_{\ell=1}^{\tilde{a}_{ij}} y_{j}^{k\ell} = \tilde{a}_{ij}$.

**Continuous variables:** For continuous variables $z_j$ we can proceed similarly as for integer variables. As above, we define $\tilde{B}_j := \max\{|b_{ij}| : i \in [m]\}$ and introduce artificial variables $z_{j}^{k\ell}$ with $k \in [\tilde{u}_j]$, $\ell \in [\tilde{B}_j]$. We define the variables $z_{j}^{k\ell}$ to be contained in $[0, 1]$ and write

$$b_{ij} z_j = \text{sign}(b_{ij}) \sum_{k=1}^{\tilde{u}_j} \sum_{\ell=1}^{\tilde{B}_j} z_{j}^{k\ell}, \quad z_{j}^{k1} = \tilde{z}_j, \quad \ell \in [B_j], k \in [\tilde{u}_j].$$

To prove that the expanded mixed-integer formulation and (1) are equivalent, we have to show that each solution of (1) can be lifted to a feasible solution of the expanded system.
and vice versa. Let \((y^*, z^*)\) be a solution of (1). We can generate the value \(\bar{a}_{ij} y^*_j\) in constraint \(i\) by setting \(y^{k1}_j = 1\) for all \(k \in [y^*_j]\) and \(y^{k1}_j = 0\) for \(k > y^*_j\). We use the same assignment as in the integer case for \(z^*_j \in \mathbb{Z}\). Otherwise, if \(z^*\) is fractional, we can represent this value in the expanded formulation by setting \(z^{k1}_j = 1\) for all \(k \in \lfloor z^*_j \rfloor\),

\[
z^{[z^*_j]}_j = z^*_j - \lfloor z^*_j \rfloor,
\]

and \(z^{k1}_j = 0\) for \(k > \lfloor z^*_j \rfloor\). Hence, each solution of (1) can be lifted to a solution of the expanded system.

For the reverse direction, assume we are given a solution \((Y,Z)\) of the expanded mixed-integer formulation, where \(Y := (y^{k1})_{j\ell}\) and \(Z := (z^{k1})_{j\ell}\). This solution can be projected to a solution of (1) by

\[
(y,z) := \pi(Y,Z) = \left( \left( \sum_{k=1}^{u_j} y^{k1}_j \right)_j, \left( \sum_{k=1}^{z^{k1}_j} \right)_j \right)
\]

because of the linking constraints: Let \(i \in [m]\) be the index of an expanded constraint. Then we can rearrange the \(i\)-th expanded constraint via the following steps

\[
\sum_{j \in [p]} \text{sign}(a_{ij}) \sum_{k=1}^{u_j} a_{ij} y^{k1}_j + \sum_{j \in [q]} \text{sign}(b_{ij}) \sum_{k=1}^{\bar{u}_j} b_{ij} \bar{z}^{k1}_j \leq b_i
\]

\[
\Leftrightarrow \sum_{j \in [p]} \text{sign}(a_{ij}) \bar{a}_{ij} \sum_{k=1}^{u_j} y^{k1}_j + \sum_{j \in [q]} \text{sign}(b_{ij}) \bar{b}_{ij} \sum_{k=1}^{\bar{u}_j} \bar{z}^{k1}_j \leq b_i
\]

\[
\Leftrightarrow \sum_{j \in [p]} a_{ij} \pi(Y)_j + \sum_{j \in [q]} b_{ij} \pi(Z)_j \leq b_i.
\]

Hence, \(\pi(Y,Z)\) is feasible for (1). For this reason, both mixed-integer formulations are equivalent.

As a consequence, we can replace arbitrary integer coefficients of any variable by introducing \(O(\sum_{j \in [p]} \tilde{A}_j u_j + \sum_{j \in [q]} \tilde{B}_j [\bar{A}_j])\) new variables and constraints. \(\square\)

**Theorem 6.** Let \(X \subseteq \mathbb{R}^n\) be a mixed-integer set that admits an extended mixed-integer formulation (1).

1. If the number of constraints and variables as well as the coefficients in (1) are bounded by a polynomial in \(n\), \(X\) has an extended mixed-integer formulation of polynomial size in \(n\) with left-hand side coefficients in \(\{0, \pm 1\}\).

2. Let the following conditions hold:
   - all entries of \(A, B, u,\) and \(\bar{u}\) are bounded by a polynomial in \(n\),
   - the mixed-integer formulation (1) is tractable, and
   - the number of variables in (1) is bounded by a polynomial in \(n\).

Then \(X\) has a tractable extended mixed-integer formulation with coefficients in \(\{0, \pm 1\}\).

**Proof.** The first part of the theorem is a direct consequence of Lemma 5. To see that the second part holds, we show that the mixed-integer formulation that is obtained by applying Lemma 5 is tractable.

W.l.o.g. we assume that \(q = 0\), i.e., the mixed-integer formulation (1) contains only integer variables. Then the expanded mixed-integer formulation from Lemma 5 contains only binary variables \(y^{k1}_j\).
Let $Y := (y_{jk}^k)_{jkl}$ be the variable vector in the space of the expanded mixed-integer formulation. Because the expanded mixed-integer formulation contains at most polynomially many linking and box constraints, these constraints can be separated in polynomial time. Thus, we can assume in the following that $Y$ fulfills all linking and box constraints, and it suffices to check whether the expanded constraints from (2) can be separated efficiently.

Since $Y$ fulfills the linking constraints of Lemma 5, the $i$-th expanded constraint is violated by this solution if and only if $\pi(Y)$ violates the corresponding constraint in (1), cf. the argumentation in the proof of Lemma 5. Because (1) is tractable, finding an inequality of (1) that is violated by $\pi(Y)$ (if it exists) is possible in polynomial time. Since the transformation and the projection can be calculated in polynomial time, (2) can be separated in polynomial time.

In particular, this means that the TSP has an extended mixed integer formulation of polynomial size all of whose left-hand side coefficients are contained in $\{0, \pm 1\}$ by applying Theorem 6 to MTZ.

### 3.2 Bounded Coefficients in the Original Space

The aim of this section is to discuss integer formulations with bounded left-hand side coefficients in the original space. We first investigate integer formulations of sets $X \subseteq \{0, 1\}^n$ with left-hand side coefficients in $\{0, \pm 1\}$. Afterwards, we discuss lower bounds on integer formulations with bounded coefficients in Section 3.2.2.

#### 3.2.1 Constructing Integer Formulations with $\{0, \pm 1\}$-Coefficients

Let $X \subseteq \{0, 1\}^n$. Then, $X$ always has an integer formulation with coefficients in $\{0, \pm 1\}$ via infeasibility cuts: Given a set $X \subseteq \{0, 1\}^n$ and a point $\tilde{x} \in \{0, 1\}^n \setminus X$, the infeasibility or no-good cut w.r.t. $\tilde{x}$ is given by

$$\sum_{i: \tilde{x}_i = 0} x_i + \sum_{i: \tilde{x}_i = 1} (1 - x_i) \geq 1. \quad (3)$$

The only binary point that violates the infeasibility cut (3) is $\tilde{x}$. Thus, if we take the box constraints as well as the infeasibility cuts for all infeasible binary points, we obtain an integer formulation of $X$ with coefficients in $\{0, \pm 1\}$. Obviously, the tractability of these inequalities depends on the way $X$ is given. For example, if $|\{0, 1\}^n \setminus X|$ is small and can be enumerated explicitly, (3) yields a polynomial size integer formulation.

**Theorem 7.** Consider some $0/1$-problem that defines $X^I \subseteq \{0, 1\}^{n(I)}$ for each instance $I$, and let the membership problem for $X^I$ be solvable in polynomial time in $n$. Then the integer formulation given by (3) for each $\tilde{x} \in \{0, 1\}^{n(I)} \setminus X^I$ and the bounds $0 \leq x \leq 1$ is tractable in $n$.

**Proof.** Since the box constraints $0 \leq x \leq 1$ can be separated in linear time, it suffices to show that the separation problem for (3) and $x^* \in [0, 1]^n$ can be solved in polynomial time.

Note that the left-hand side of (3) is $\|\tilde{x} - x^*\|_1$, where $\|x\|_1$ denotes the 1-norm. To solve the separation problem of infeasibility cuts for $X$, we solve the auxiliary problem

$$\min_{\tilde{x} \in \{0, 1\}^n} \|\tilde{x} - x^*\|_1$$

first. Obviously,

$$\tilde{x}_i = \begin{cases} 0 & \text{if } x_i^* \leq \frac{1}{2}, \\ 1 & \text{otherwise} \end{cases}$$
is a solution of this problem and it can be computed in linear time.

If \( \|\tilde{x} - x^*\|_1 \geq 1 \), the point \( x^* \) cannot violate (3) for any \( \tilde{x} \in \{0, 1\}^n \). Hence, \( x^* \) lies inside the integer formulation of \( X \) via infeasibility cuts. Otherwise, \( \|\tilde{x} - x^*\|_1 < 1 \), and we are done if \( \tilde{x} \in \{0, 1\}^n \setminus X \), because the infeasibility cut for \( \tilde{x} \) is violated by \( x^* \). Thus, we can assume in the following that \( \tilde{x} \in X \).

Assume now that \( \|\tilde{x} - \hat{x}\|_1 \geq 2 \). Then plugging \( x^* \) into (3) yields

\[
\sum_{i: \hat{x}_i = 0} x^*_i + \sum_{i: \hat{x}_i = 1} (1-x^*_i) = \|\tilde{x} - x^*\|_1 = \|\hat{x} - \hat{x}\|_1 + \|\hat{x} - \tilde{x}\|_1 - \|\hat{x} - x^*\|_1 > 1.
\]

Thus, (3) cannot be violated in this case. It therefore suffices to check at most \( n + 1 \) points \( \tilde{x} \in \{0, 1\}^n \) with \( \|\tilde{x} - \tilde{x}\|_1 \leq 1 \). We therefore call the membership problem for each of these points and check whether (3) is violated by \( x^* \) if \( \tilde{x} \) is infeasible.

Unfortunately, the number of constraints in the integer formulation via infeasibility cuts may become very large. But for independence systems, we will see below that the number of inequalities in an integer formulation with ternary left-hand side coefficients can be much smaller than the bound via infeasibility cuts. In particular, we show that the size of any integer formulation with ternary coefficients is defined by so-called hypercliques. At the end of this section, we describe how these results can be used for general sets \( X \subseteq \{0, 1\}^n \).

A (finite) independence system is a set system \((E, \mathcal{I})\) such that \( E \) is a finite set, \( \mathcal{I} \subseteq 2^E \), \( \emptyset \in \mathcal{I} \), and if \( S \subseteq T \in \mathcal{I} \), then \( S \in \mathcal{I} \). We encode an independence system as the set \( X(\mathcal{I}) := \{\chi^I : I \in \mathcal{I}\} \subseteq \{0, 1\}^E \), where \( \chi^I \) is the characteristic vector of \( I \). In the following, we do not distinguish between \((E, \mathcal{I})\) and \( X(\mathcal{I}) \); we refer to both as independence systems. A well-known characterization of \( X = X(\mathcal{I}) \) uses the concept of circuits: Given an independence system \( X \subseteq \{0, 1\}^n \), a circuit of \( X \) is a set \( C \subseteq [n] \) such that \( \chi^C \not\in X \) but \( \chi^C - e_k \in X \) for each \( k \in C \). The circuit inequality associated with \( C \) is

\[
\sum_{i \in C} x_i \leq |C| - 1. \tag{4}
\]

A classical result in the literature is that the circuit inequalities suffice to characterize the points in \( X \), cf. Balas and Jeroslow [1], i.e.,

\[
\{x \in [0, 1]^n : x fulfills (4) for all circuits C of X\} \tag{5}
\]

is an integer formulation of \( X \).

Thus, there is always an integer formulation of \( X \) with coefficients in \( \{0, \pm 1\} \) which is given by the circuit inequalities. Observe that there may be exponentially many circuits of \( X \), but that this formulation may be smaller than the formulation via infeasibility cuts, see Example 11 below. In the following, we show how to strengthen this integer formulation, and we show that there is always a minimum size integer formulation of box constraints and so-called hyperclique inequalities. The latter inequalities are a natural generalization of clique inequalities for the stable set problem to arbitrary independence systems.

**Definition 8.** Let \( X \subseteq \{0, 1\}^n \) be an independence system. A set \( \mathcal{H} \subseteq [n] \) is called a \( k \)-hyperclique of \( X \) if for each \( C \subseteq \mathcal{H} \) with \( |C| = k \) the vector \( \chi^C \) is not contained in \( X \). The \( k \)-hyperclique inequality of a \( k \)-hyperclique \( \mathcal{H} \) is the inequality

\[
\sum_{i \in \mathcal{H}} x_i \leq k - 1. \tag{6}
\]
As will be shown in Example 12, cliques in an undirected graph $G$ can be seen as 2-hypercliques for the stable set problem in $G$.

Note that $k$-hyperclique inequalities are valid inequalities for independence systems, see, e.g., Easton et al. [10]. Moreover, if we are given two $k$-hypercliques $H$ and $H'$ with $H \subseteq H'$, the hyperclique inequality for $H'$ dominates the hyperclique inequality for $H$. Consequently, since each circuit of an independence system $X$ is a hyperclique, an integer formulation of $X$ is given by the hyperclique inequalities for maximal hypercliques and box constraints. In particular, if there are at most polynomially many maximal hypercliques, there exists a formulation of polynomial size with bounded coefficients. Moreover, it turns out that each non-trivial inequality of an integer formulation all of whose coefficients are contained in $\{0, \pm 1\}$ can be dominated by hypercliques inequalities.

Proposition 9. Let $X \subseteq \{0, 1\}^n$ be an independence system and let $a^\top x \leq \beta$ be valid for $X$ with $a \in \{0, \pm 1\}^n$ such that it cuts off a circuit of $X$. Then $a^\top x \leq \beta$ is dominated by a $([\beta] + 1)$-hyperclique inequality.

Proof. If the inequality $a^\top x \leq \beta$ is valid and it cuts off a circuit $C$ of $X$, we have
\[
a^\top \chi^C = \sum_{i \in C} a_i > \beta
\]
for the characteristic vector $\chi^C$ of $C$. Since $a^\top x \leq \beta$ is valid, it follows that $\beta \geq 0$, because the zero vector is feasible. Moreover, for each $k \in C$, we obtain
\[
\sum_{i \in C \setminus \{k\}} a_i \leq \beta,
\]
because $C$ is a circuit. This implies $a_k > 0$, i.e., $a_k = 1$, for each $k \in C$. Furthermore, it follows that $[\beta] = |C| - 1$. Thus,
\[
a^\top x \leq \beta \iff \sum_{i \in C} x_i + \sum_{i \notin C} a_i x_i \leq |C| - 1.
\]

Observe that $H := C \cup \{i \in [n] \setminus C : a_i = 1\}$ is a $([\beta] + 1)$-hyperclique, because otherwise, there would exist $H \subseteq H$, $|H| = |C| = [\beta] + 1$, with $\chi^H \in X$ that is cut off by $a^\top x \leq \beta$. Thus, $a^\top x \leq \beta$ is a hyperclique inequality or it can be dominated by the hyperclique inequality for $H$.

By Proposition 9, the concept of hyperclique inequalities enables us to derive minimum size integer formulations with left-hand side coefficients in $\{0, \pm 1\}$.

Corollary 10. Let $X$ be an independence system. An integer formulation of $X$ with coefficients in $\{0, \pm 1\}$ is given by
\[
\{x \in [0, 1]^n : x fulfills all maximal hyperclique inequalities of X\}.
\]
In particular, there is an integer formulation that uses only inequalities with coefficients in $\{0, \pm 1\}$ of minimum size such that each inequality that cuts off a circuit of $X$ is a maximal hyperclique inequality.

The following example illustrates the concepts that were discussed in this section.
Example 11. Consider the independence system

\[ X := \left\{ x \in \{0,1\}^{2n} : \sum_{i=1}^{n} x_i + 2 \sum_{i=n+1}^{2n} x_i \leq n \right\}, \]

where \( n \) is an even positive integer. A binary point \( x \) is not contained in \( X \) if and only if \( x = \chi^A + \chi^B \), where \( A \subseteq [n], |A| = k \), and \( B \subseteq \{n+1, \ldots, 2n\}, |B| > \lfloor \frac{n-k}{2} \rfloor \). For this reason, the number of infeasibility cuts in the integer formulation given by (3) is

\[ \sum_{k=0}^{n} \binom{n}{k} \sum_{\ell=\lfloor \frac{n-k}{2} \rfloor+1}^{n} \binom{n}{\ell}. \]

By a simple case analysis, we can show that each circuit \( C \) of this independence system can be written as \( C = A \cup B \), where \( A \subseteq [n], |A| = k \) is odd, and \( B \subseteq \{n+1, \ldots, 2n\}, |B| = \lfloor \frac{n-k}{2} \rfloor + 1 \). Thus, \( X \) has exactly

\[ \sum_{k=0}^{n} \binom{n}{k} \left( \binom{n}{\lfloor \frac{n-k}{2} \rfloor + 1} \right) \quad (7) \]

circuits, and the integer formulation via circuit inequalities is much smaller than the integer formulation via infeasibility cuts.

Furthermore, one can show that it suffices to take the non-negativity constraints and

\[ \sum_{k=0}^{n} \binom{n}{k} = 2^{n-1} \]

different hyperclique inequalities to obtain an integer formulation for \( X \) which is much smaller than (7), see Section A in the appendix.

Example 12. Let \( G = (V, E) \) be an undirected simple graph and consider the set \( X \) of incidence vectors of all stable sets in \( G \). Then \( X \) is an independence system and can be represented via the integer formulation

\[ X = \{ x \in [0,1]^V : x_u + x_v \leq 1, \ \{u,v\} \in E \}. \]

It is well-known that this formulation can be strengthened by replacing the edge inequalities \( x_u + x_v \leq 1 \) with maximal clique inequalities \( \sum_{v \in C} x_v \leq 1 \), for all maximal cliques \( C \) in \( G \). Clearly, maximal clique inequalities are 2-hyperclique inequalities. Thus, hyperclique inequalities generalize clique inequalities for the stable set polytope to arbitrary independence systems.

Moreover, Corollary 10 implies that \( X \) does not admit a small integer formulation in general, because there are graphs with exponentially many maximal cliques, e.g., complete multipartite graphs. Nevertheless, there are graph classes with a polynomial number of maximal cliques. For instance, Jing-Ho et al. [19] proved that threshold graphs contain at most linearly many maximal cliques. Furthermore, we can describe \( X \) for threshold graphs with non-negativity constraints and one additional inequality, see Chvátal and Hammer [4], if we allow arbitrary coefficients.
In the remainder of this section, we discuss how we can transfer the previous results for independence systems to arbitrary sets $X \subseteq \{0,1\}^n$. Let
\begin{equation}
\left\{ x \in \{0,1\}^n : \sum_{j=1}^n a_{ij} x_j \leq b_i, \ i \in [m] \right\}
\end{equation}
be an integer formulation of $X$ with $m$ constraints. This integer formulation can be interpreted as the intersection of integer formulations for the sets $X^i := \left\{ x \in \{0,1\}^n : \sum_{j=1}^n a_{ij} x_j \leq b_i \right\}$. Thus, if $a^\top x \leq \beta$ is the $i$-th inequality of $Ax \leq b$, it suffices to show how to construct an integer formulation for $X^i$ with ternary coefficients. Note that the integer formulation for $X$ obtained by intersecting the formulations for $X^i$ need not be minimal.

If all coefficients of $a^\top x \leq \beta$ are non-negative, the set $X^i$ is an independence system. Hence, the results on hyperclique inequalities can be applied directly to this constraint to obtain an integer formulation with left-hand side coefficients in $\{0,\pm 1\}$ for $X^i$. If there is a negative coefficient $a_{ij}, j \in [n]$, we can complement the variable $x_j \mapsto (1 - x_j)$ to modify the coefficient of $x_j$ to $-a_{ij}$ (the right-hand side of the constraint has to be adjusted accordingly). Performing these steps for each variable with a negative coefficient leads to a system allowing for the techniques above. Complementing the same variables of the resulting system again leads to an integer formulation of $X^i$ with left-hand side coefficients in $\{0,\pm 1\}$.

### 3.2.2 A Lower Bound on the Size of Integer Formulations with Bounded Coefficients

Kaibel and Weltge [22] developed a criterion to determine lower bounds on the size of integer formulations of a hole-free set $X \subseteq \mathbb{Z}^n$ based on so-called hiding sets. A set $H \subseteq (\text{aff}(X) \cap \mathbb{Z}^n) \setminus \text{conv}(X)$ is a hiding set if for any two distinct points $x, y \in H$ the relation $\text{conv}(\{x, y\}) \cap \text{conv}(X) \neq \emptyset$ holds, where $\text{aff}(\cdot)$ denotes the affine hull.

**Proposition 13** (Kaibel and Weltge [22]). Let $X \subseteq \mathbb{Z}^n$ be a hole-free set, and let $H \subseteq (\text{aff}(X) \cap \mathbb{Z}^n) \setminus X$ be a hiding set for $X$. Then any integer formulation of $X$ in the original space contains at least $|H|$ inequalities.

By finding large hiding sets, Kaibel and Weltge [22] proved that several examples do not admit a small integer formulation: the TSP, the set of incidence vectors of connected subgraphs of a graph, or the permutahedron. But Proposition 13 does not characterize the size of coefficients in integer formulations. For example, a knapsack polytope can be described by box constraints and one inequality, and thus, admits a small integer formulation. But in practice, this integer formulation may be impractical if the coefficients of the additional inequality are very large.

For this reason, one is often interested in integer formulations for which the size of the coefficients is bounded and that the inequalities in the formulation belong to a family of inequalities that can be separated efficiently. The aim of this section is to present a criterion that provides lower bounds on the size of integer formulations that use only inequalities fulfilling some specific requirements. In particular, we will see that the new criterion gives the same lower bound as the hiding set bound of Proposition 13 if we allow arbitrary inequalities.

**Definition 14.** Let $X \subseteq \mathbb{Z}^n$ be a hole-free set and let $\mathcal{F}$ be a family of valid inequalities for $X$. A set $H(\mathcal{F}) \subseteq (\text{aff}(X) \cap \mathbb{Z}^n) \setminus X$ is called an $\mathcal{F}$-hiding set if for all $\bar{y} \in H(\mathcal{F})$ and for each $(a^\top x \leq b) \in \mathcal{F}$ with $a^\top \bar{y} > b$, we have $a^\top y \leq b$ for all $y \in H(\mathcal{F}) \setminus \{\bar{y}\}$.
Geometrically, an \( \mathcal{F} \)-hiding set \( H(\mathcal{F}) \) of \( X \) is a set of points such that each hyperplane induced by an inequality in \( \mathcal{F} \) separates at most one point in \( H(\mathcal{F}) \) from \( \text{conv}(X) \). In the following, we call an integer formulation that consists only of inequalities in \( \mathcal{F} \) an \( \mathcal{F} \)-integer formulation.

**Proposition 15.** Let \( X \subseteq \mathbb{Z}^n \) be a hole-free set and let \( \mathcal{F} \) be a family of valid inequalities for \( X \) such that \( X \) admits an \( \mathcal{F} \)-integer formulation. If \( H(\mathcal{F}) \subseteq (\text{aff}(X) \cap \mathbb{Z}^n) \setminus X \) is an \( \mathcal{F} \)-hiding set of \( X \), any \( \mathcal{F} \)-integer formulation of \( X \) contains at least \(|H(\mathcal{F})|\) inequalities.

**Proof.** Let \( H(\mathcal{F}) \) be an \( \mathcal{F} \)-hiding set of \( X \), and assume there is an \( \mathcal{F} \)-integer formulation of \( X \) with at most \(|H(\mathcal{F})| - 1\) constraints. Then there are distinct \( y_1, y_2 \in H(\mathcal{F}) \) as well as an inequality \( (a^\top x \leq b) \in \mathcal{F} \) such that \( a^\top x \leq b \) separates both \( y_1 \) and \( y_2 \) from \( X \). But this contradicts the fact that \( H(\mathcal{F}) \) is an \( \mathcal{F} \)-hiding set. Hence, each \( \mathcal{F} \)-integer formulation contains at least \(|H(\mathcal{F})|\) inequalities. \( \square \)

Furthermore, the concept of \( \mathcal{F} \)-hiding sets is a generalization of the concept of hiding sets in the sense of Kaibel and Weltge [22].

**Proposition 16.** Let \( X \subseteq \mathbb{Z}^n \) be a hole-free set and let \( \mathcal{F} \) be the set of all valid inequalities for \( X \). Then \( H \) is an \( \mathcal{F} \)-hiding set of \( X \) if and only if \( H \) is a hiding set of \( X \).

**Proof.** On the one hand, let \( H \) be a hiding set of \( X \) w.r.t. the definition of Kaibel and Weltge [22]. By the proof of Proposition 13 in [22], each valid inequality for \( X \) can separate at most one point in \( H \) from \( X \). Hence, \( H \) is an \( \mathcal{F} \)-hiding set.

On the other hand, let \( H \) be an \( \mathcal{F} \)-hiding set. If \( H \) was not a hiding set, there would exist distinct \( y_1, y_2 \in H \) with \( \text{conv}({y_1, y_2}) \cap \text{conv}(X) = \emptyset \) by definition. But then there exists an inequality valid for \( X \) that separates both \( y_1 \) and \( y_2 \) simultaneously from \( X \), contradicting that \( H \) is an \( \mathcal{F} \)-hiding set. \( \square \)

In the remainder of this section, we give an example in which an \( \mathcal{F} \)-hiding set produces a tight lower bound on the number of inequalities in an \( \mathcal{F} \)-integer formulation.

**Example 17.** Consider the knapsack polytope

\[
O_m := \text{conv} \left( \{ (x, y) \in \{0,1\}^m \times \{0,1\}^m : \sum_{i=1}^m 2^{m-i}(x_i + y_i) \leq 2^m - 1 \} \right),
\]

which is the knapsack polytope associated to the so-called orbisack, see Kaibel and Loos [21] and Loos [24]. Since \( O_m \) is a knapsack polytope, it admits a small integer formulation. But in practice, the integer formulation given in the definition of \( O_m \) is impractical due to the large coefficients. For this reason, we are interested in integer formulations of \( O_m \) with small coefficients, and we construct large \( \mathcal{F} \)-hiding sets for particular choices of \( \mathcal{F} \). For the rest of this section, we denote by \( \mathcal{F}^c \) the set of valid inequalities for \( O_m \) whose left-hand side coefficients are contained in \( \{-c, \ldots, c\} \). By \( \mathcal{F}^c_j \) we denote the set of all inequalities in \( \mathcal{F}^c \) that define facets of \( O_m \). The arguments to obtain the following results are omitted, but can be found in Section B of the appendix of this paper.

By analyzing the structure of maximal hypercliques of \( O_m \), one can show that there exists an \( \mathcal{F}^1 \)-hiding set for \( O_m \) of size \( 2^{m-1} - 1 \), i.e., each \( \mathcal{F}^1 \)-integer formulation of \( O_m \) needs at least \( 2^{m-1} - 1 \) inequalities. Observe that this bound is asymptotically tight: By a simple analysis of the circuits of \( O_m \), one can prove that there exist \( 2^{m-1} \) distinct maximal hypercliques for the orbisack \( O_m \). Moreover, there are \( 4m \) box constraints for \( O_m \) since \( O_m \subseteq \mathbb{R}^{2m} \). Hence, there exists an \( \mathcal{F}^1 \)-integer formulation of \( O_m \) containing \( 2^{m-1} + 4m \) inequalities by Corollary 10.
Furthermore, one can construct an $\mathcal{F}_I^2$-integer formulation of $O_m$ that consists of $4m + 2^{m-2}$ inequalities. This $\mathcal{F}_I^2$-integer formulation is asymptotically minimal, since one can construct an $\mathcal{F}_I^2$-hiding set of size $2^{m-3} - 1$ by exploiting the structure of $\mathcal{F}_I^2$-inequalities. Hence, the minimum size of an $\mathcal{F}_I^2$-integer formulation of $O_m$ contains $\Theta(2^{m-3})$ inequalities.

Observe the qualitative difference between the upper and lower bounds for $\mathcal{F}_I^1$-integer and $\mathcal{F}_I^2$-integer formulations: While both bounds differ in the case of $\mathcal{F}_I^1$ by a linear term, we only know the minimum size of an $\mathcal{F}_I^2$-integer formulations up to a factor of 2.

### 3.2.3 Statistics on the Size of Integer Formulations with Bounded Coefficients

In Section 3.2.1, we have discussed techniques to derive integer formulations all of whose inequalities have normal vectors in $\{0, \pm 1\}^n$. Furthermore, Proposition 15 allows to construct lower bounds on the number of inequalities in an integer formulation with bounded left-hand side coefficients. The aim of this section is to quantify the number of nonempty subsets $X \subseteq \{0, 1\}^n$ that admit an integer formulation with bounded coefficients and whose number of inequalities does not exceed a given threshold. In particular, we focus on the cases for which the left-hand side coefficients are contained in $\{0, \pm 1\}$ and $\{-2, \ldots, 2\}$.

To obtain statistics, we modeled the problem “Is there an integer formulation of $X \subseteq \{0, 1\}^n$ with at most $k$ inequalities all of whose left-hand side coefficients are contained in $\{-c, \ldots, c\}$?” as the problem to decide whether the following system is feasible. Let $B_r := \{-r, \ldots, r + 1\}^n$ be the box that adds $r$ integer layers around $\{0, 1\}^n$.

\[
\begin{align*}
\sum_{j=1}^n A_{ij} x_j &\leq b_i, & i \in [k], x \in X, \quad (9a) \\
\sum_{j=1}^n A_{ij} y_j &\geq b_i + 1 - (1 - z^i_y) M, & i \in [k], y \in B_r \setminus X, \quad (9b) \\
\sum_{i=1}^k z^i_y &\geq 1, & y \in B_r \setminus X, \quad (9c) \\
A_{ij} &\in \{-c, \ldots, c\}, & i \in [k], j \in [n], \\
b_i &\in \{-nc, \ldots, nc\}, & i \in [k], \\
 z^i_y &\in \{0, 1\}, & i \in [k], y \in B_r \setminus X,
\end{align*}
\]

where $M$ is a sufficiently large constant, e.g., $ncr + 1$. In this model, $A_i$ is the normal vector of a linear inequality and $b_i$ is its right-hand side coefficient. Inequality (9a) ensures that $A_i x \leq b_i$ is valid for $X$. If $z^i_y = 1$ for $y \in B_r \setminus X$, Inequality (9b) implies that $A_i x \leq b_i$ separates $y$ from $X$. Finally, Inequality (9c) enforces that each point in $B_r \setminus X$ is separated by at least one inequality. Observe that it suffices to consider $b_i \in \{-nc, \ldots, nc\}$, because the absolute value of a left-hand side coefficient in $Ax \leq b$ is bounded by $c$ and all feasible points are contained in $\{0, 1\}^n$. Thus, if this system has a solution, $Ax \leq b$ is an integer formulation of $X$ for all points in $B_r$ with $k$ inequalities all of whose left-hand side coefficients are contained in $\{-c, \ldots, c\}$. Otherwise, no such integer formulation of $X$ exists.

Observe that this problem is a relaxation of the infinite dimensional condition that $y \in \mathbb{Z}^n \setminus X$ is cut off by some inequality. Hence, the feasible region of the relaxed problem (9) is too large in general, i.e., we cannot guarantee that all points in $\mathbb{Z}^n \setminus X$ are separated by an inequality $A_i x \leq b_i$. Thus, we have to check whether a found solution is feasible. This can be done, for example, by checking whether the feasible region of the found integer formulation is bounded and by counting the number of integer points contained in the
Table 1: Statistics on the number of nonempty sets $X \subseteq \{0, 1\}^n$ with an integer formulation with bounded absolute values of left-hand side coefficients ($c = 1$ and $c = 2$, respectively) such that the given number of inequalities is both necessary and sufficient for an integer formulation in comparison with lower bounds on the size of any integer formulation

<table>
<thead>
<tr>
<th>$c = 1$:</th>
<th>number of inequalities</th>
</tr>
</thead>
<tbody>
<tr>
<td>dimension</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>12</td>
</tr>
<tr>
<td>3</td>
<td>164</td>
</tr>
<tr>
<td>4</td>
<td>8388</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$c = 2$:</th>
<th>number of inequalities</th>
</tr>
</thead>
<tbody>
<tr>
<td>dimension</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>14</td>
</tr>
<tr>
<td>3</td>
<td>242</td>
</tr>
<tr>
<td>4</td>
<td>57248</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>lower bound $\ell(X)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>dimension</td>
</tr>
<tr>
<td>2</td>
</tr>
<tr>
<td>3</td>
</tr>
<tr>
<td>4</td>
</tr>
</tbody>
</table>

feasible region, since no feasible point in $X$ is cut off by construction. If the found solution is not feasible, we increase the bound $r$ to cut off more invalid points. In our experiments a maximum bound of $r = 4$ was sufficient to find a valid integer formulation for each $X$ or to prove that no integer formulation with bounded coefficients exists.

Furthermore, we are interested in lower bounds on the size of any integer formulation for nonempty $X \subseteq \{0, 1\}^n$, to be able to decide whether small coefficients suffice to describe $X$ with few inequalities. To this end, we computed for each set $X$ a maximum hiding set in $\{0, 1\}^n$. Note that the restriction to binary hiding sets is due to practical reasons, since we need a bounded search space. A generalization to hiding sets in other bounded regions would of course be possible.

To compute binary hiding sets, we used the following procedure. Given a nonempty set $X \subseteq \{0, 1\}^n$, we build a graph $G = (V, E)$ with $V = \{0, 1\}^n \setminus X$. The edge set $E$ of $G$ is defined as

$$E = \{\{x, y\} : x, y \in V, \text{conv}\{\{x, y\}\} \cap \text{conv}(X) \neq \emptyset\}.$$ 

Thus, any clique in $G$ is a hiding set by definition. By computing the size of a maximum clique in $G$, we get the maximum size of any hiding set $H \subseteq \{0, 1\}^n$ for $X$.

Note that the hiding set bound can be very weak, e.g., if $|X| = 2^n - 1$, a maximum binary hiding set has size 1. To improve this bound, observe that any nonempty $X \subseteq \{0, 1\}^n$ needs at least $n + 1$ inequalities in any integer formulation. We denote this lower bound, i.e., the maximum of $n + 1$ and the size of a maximum binary hiding set, by $\ell(X)$. Table 1 summarizes our experiments.

The results show that increasing the size of the coefficients can decrease the size of an integer formulation even in small dimensions. In each dimension the maximum number of inequalities decreases by at least one if we allow left-hand side coefficients in $\{-2, \ldots, 2\}$, i.e., $c = 2$ instead of $c = 1$. Furthermore, there are many point sets that can be described
by few inequalities if we allow larger coefficients. For example, about 95% of all non-trivial subsets of \( \{0,1\}^3 \) can be described by four inequalities and about 87% of all non-trivial subsets of \( \{0,1\}^4 \) can be described by five inequalities if we allow \( c = 2 \). For \( c = 1 \), however, these numbers are only 65% and 13%.

Moreover, the lower bounds in Table 1 show that we cannot benefit from increasing \( c \) to a value larger than 2 for the given dimensions: In dimension 2, the lower bound of 3 inequalities is met by all non-trivial sets \( X \), i.e., all sets admit an integer formulation with 3 inequalities if \( c = 2 \). In dimensions 3 and 4, there are two sets which need at least 5 and 8 inequalities, respectively. If we allow \( c = 2 \), we can describe both sets with 5 and 8 inequalities, respectively. Thus, increasing \( c \) cannot reduce the maximum number of inequalities needed in the worst case any further.

Finally, we remark that, at least in theory, the above approach can be used to obtain statistics in higher dimensions. But since there exist \( 2^{2^n} - 2 \) non-trivial subsets of \( \{0,1\}^n \), we would have to solve at least \( 4.29 \cdot 10^9 \) MIPs for \( n = 5 \), which is impractical. Of course, one can reduce the number of treated subsets \( X \) by exploiting symmetries between different point configurations. But it is likely that the number of considered sets is still too large.

4 Sparse Formulations

The set of feasible points of a 0/1-knapsack problem for some \( a \in \mathbb{R}_{\geq 0}^n \) is

\[
X = \{ x \in \{0,1\}^n : a^\top x \leq \beta \}.
\]

This formulation seems to be easy to handle, since it consists of a single constraint. But is there also a sparse description of \( X \)? In this section, we analyze the existence of formulations with sparse constraints in an extended space as well as in the original space. We call an inequality of the form \( a^\top x \leq \beta \) \( s \)-sparse if at most \( s \) entries of \( a \) are non-zero. Thus, the sparser an inequality is, the smaller \( s \). In particular, the maximum sparsity of a mixed-integer set \( X \) is \( s \) if \( X \) admits a mixed-integer formulation by \( s \)-sparse inequalities but not by \( (s-1) \)-sparse inequalities.

In the literature, sparsity was discussed, e.g., by Dey et al. [8] who studied the approximation of polyhedra by sparse cutting planes, and the same authors investigated also the impact of sparse cutting planes for sparse integer programs, see Dey et al. [9].

4.1 Sparsity in Extended Spaces

Given a set \( X \subseteq \{0,1\}^n \), we can always find a mixed-integer formulation of \( X \) in an extended space such that each inequality in the mixed-integer formulation contains at most three non-zero coefficients.

Proposition 18. Let \( X \subseteq \mathbb{R}^n \) be a hole-free set and \( Ax \leq b \) be a mixed-integer formulation of \( X \) with \( m \) inequalities in the original space. Then there exists a mixed-integer formulation of \( X \) all of whose inequalities are 3-sparse with \( O(mn) \) variables and constraints.

Proof. For each inequality \( a^\top x \leq \beta \) in this formulation with \( s \geq 4 \) non-zero coefficients, we introduce \( s - 2 \) artificial variables and the following \( s - 1 \) additional constraints: W.l.o.g. we can assume that the first \( s \) coefficients are the non-zero coefficients of \( a^\top x \leq b \).
Define \( y_1 := a_1 x_1 + a_2 x_2 \) and \( y_k = y_{k-1} + a_{k+1} x_{k+1} \) for each \( k \in \{2, \ldots, s - 2\} \), as well as \( y_{s-2} + a_s x_s \leq \beta \). Then the additional constraints have at most three non-zero coefficients, and a point \((x,y)\) is feasible for this set of constraints if and only if \( x \) fulfills \( a^\top x \leq \beta \). \( \square \)
Remark 19. If \( X \) has an extended mixed-integer formulation of polynomial size, the sparsified mixed-integer formulation of Proposition 18 has polynomial size as well. This shows that there exists a polynomial size formulation with sparse constraints in an extended space if the membership problem for \( X \) is in \( 
abla P \), see Theorem 3 and its discussion in Section 2.

The question arises whether it is possible to further decrease the number of non-zero coefficients in an extended formulation. If only one non-zero coefficient is allowed, the only possible constraints are of the form \( x_i \leq q \) or \( -x_i \geq q \) for some \( q \in \mathbb{R} \), since the coefficient of \( x_i \) can be scaled to \( \pm 1 \). This means that the only polyhedra with a description using only one non-zero coefficient are cuboids that may be translated or unbounded in some directions. Thus, any polytope that can be written as a projection of a cube has an extended formulation using only one non-zero coefficient. Polytopes with this property are called zonotopes, see, for example, Grünbaum [14].

If we allow an additional non-zero coefficient, i.e., 2-sparse inequalities, an easy example of an integer set that admits such an integer formulation is the vertex set of the standard simplex.

Example 20. Let \( S^n \) be the vertices of the standard simplex, i.e., \( S^n = \{0, e_1, \ldots, e_n\} \). Then \( S^n \) can be described by

\[
S^n := \{ x \in \{0,1\}^n : x_i + x_j \leq 1, 1 \leq i < j \leq n \}.
\]

Since the convex hull of any \( X \subseteq \{0,1\}^n \), can be written as the projection of the \( |X| \)-dimensional simplex, \( X \) has an integer extended formulation using only two non-zero coefficients. Note that, in general, an exponential number of variables and inequalities is needed since the number of points in \( X \) may be exponential in \( n \), whereas Proposition 18 implies a polynomial formulation if the original formulation is of polynomial size.

4.2 Sparsity in the Original Space

A sparse description in the original space seems to be more complicated. The positive examples are mentioned in the previous section: the standard simplex and cube. Unfortunately, not every set \( X \) has small sparse formulations like these examples. For this reason, our next goal is to derive lower bounds on the number of non-zero coefficients needed in an integer formulation of a given set \( X \subseteq \{0,1\}^n \). Let \( \bar{x} \in \{0,1\}^n \) and let \( N(\bar{x}) \) be the neighbors of \( \bar{x} \) in the 0/1-cube, i.e.,

\[
N(\bar{x}) := \{ x \in \{0,1\}^n : \|x - \bar{x}\|_1 = 1 \},
\]

that is, the neighbors of \( \bar{x} \) are those 0/1-points that differ from \( \bar{x} \) in exactly one coordinate.

Lemma 21. Let \( X \subseteq \{0,1\}^n \) be nonempty and \( \bar{x} \in \{0,1\}^n \setminus X \). Then there exists no integer formulation for \( X \) in the original space with at most \( |N(\bar{x}) \cap X| - 1 \) non-zeros on the left-hand side of each defining inequality.

Proof. Define \( s = |N(\bar{x}) \cap X| \) and assume that \( s > 1 \), since otherwise the statement is trivial. Let \( X = P \cap \mathbb{Z}^n \) with \( P = \{ x : Ax \leq b \} \), where each row of \( A \) has at most \( s - 1 \) non-zeros. Since \( \bar{x} \notin X \), there exists some inequality \( a^T x \leq \beta \) of \( Ax \leq b \) such that \( a^T \bar{x} > \beta \), but \( a^T \bar{x} \leq \beta \) for all \( \bar{x} \in N(\bar{x}) \cap X \).

Assume w.l.o.g. that \( a_1 = a_2 = \cdots = a_{n-s+1} = 0 \). Thus, the remaining \( s - 1 \) entries \( a_{n-s+2}, \ldots, a_n \) might have nonzero coefficients. Since \( |N(\bar{x}) \cap X| = s \), there exists \( \bar{x} \in N(\bar{x}) \cap X \) such that \( a_i^T \bar{x} = \beta \) for some \( i = 1, \ldots, n - s + 1 \) with \( \bar{x}_i \neq \bar{x}_i \). But then \( \beta < a^T \bar{x} = a_i^T \bar{x} \leq \beta \), a contradiction. \( \square \)
Next, we demonstrate applications of Lemma 21.

**Example 22.** The parity polytopes $P_e$ and $P_o$ are the convex hulls of binary points that contain an even and odd number of 1-entries, respectively. Observe that each neighbor of a vertex of $P_e$ is contained in $P_o$ and vice versa. Hence, Lemma 21 implies that both $P_e$ and $P_o$ need completely dense inequalities in any integer formulation in the original space. Jeroslow [17] has shown that there does not exist a integer formulation for the vertices of Lemma 25.

The next example will also show that the sparsity bound derived in Lemma 21 can be tight for every sparsity level.

**Example 23.** Consider the set $X_s := \{x \in \{0,1\}^n : 1^\top x \leq s\}$, where $s \in [n - 1]$. The set $X_s$ is the feasible region of a 0/1-knapsack problem, and we claim that $X_s$ can be described by $(s + 1)$-sparse inequalities but not by $s$-sparse inequalities.

Let $\tilde{x} \in \{0,1\}^n$ such that $1^\top \tilde{x} = s + 1$. Then $\tilde{x} \notin X$ and there are $s + 1$ neighbors of $\tilde{x}$ in $X$. Hence, Lemma 21 implies that $X$ cannot be represented by an integer formulation all of whose inequalities have at most $s$ non-zero coefficients on the left-hand side. But $X_s$ admits an $(s + 1)$-sparse integer formulation via box constraints and

$$\sum_{i \in I} x_i \leq s, \ I \in \binom{[n]}{s+1},$$

where $\binom{[n]}{s+1}$ is the set of all $(s + 1)$-element subsets of $[n]$.

Lemma 21 allows to derive bounds on the sparsity of any integer formulation of a set $X \subseteq \{0,1\}^n$ by a simple neighborhood argument. But unfortunately, its bound need not be tight in every case. To be able to compute the maximum sparsity of any integer formulation, we introduce the concept of infeasible face coverings.

**Definition 24.** Let $X \subseteq \{0,1\}^n$. A face $F$ of $[0,1]^n$ is called infeasible w.r.t. $X$ if every integer point $x \in F$ is not contained in $X$. A collection $\mathcal{F}$ of infeasible faces of $[0,1]^n$ w.r.t. $X$ is called an infeasible face collection of $X$. Moreover, if $\mathcal{F}$ is an infeasible face collection and for every $x \in \{0,1\}^n \setminus X$ there exists $F \in \mathcal{F}$ such that $x \notin F$, then $\mathcal{F}$ is called an infeasible face covering of $X$. An infeasible face covering $\mathcal{F}$ of $X$ is called maximal if for each face $F \in \mathcal{F}$ there is no infeasible face $F'$ of $[0,1]^n$ with $F \subsetneq F'$. An infeasible face covering $\mathcal{F}$ is called irredundant if for each $F \in \mathcal{F}$ there is no further maximal infeasible face covering of $X$. Assume there exists a maximal infeasible face covering $\mathcal{F}' \neq \mathcal{F}$. Then $\mathcal{F}'$ contains a face $F' \notin \mathcal{F}$ which is not a maximal infeasible face, contradicting the maximality or irredundancy of $\mathcal{F}'$.

**Lemma 25.** Every $X \subseteq \{0,1\}^n$ has a unique maximal irredundant infeasible face covering.

**Proof.** Let $\mathcal{F}$ be the collection of all infeasible faces of $[0,1]^n$ w.r.t. $X$, i.e., if $F \in \mathcal{F}$, then each integer point in $F$ is not contained in $X$. The collection $\mathcal{F}$ turns into a poset if we order the faces in $\mathcal{F}$ w.r.t. inclusion. Obviously, the $\subseteq$-maximal elements in $\mathcal{F}$ form a maximal irredundant infeasible face covering of $X$. Furthermore, there is no further maximal irredundant infeasible face covering of $X$. Assume there exists a maximal infeasible face covering $\mathcal{F}' \neq \mathcal{F}$. Then $\mathcal{F}'$ contains a face $F' \notin \mathcal{F}$ which is not a maximal infeasible face, contradicting the maximality or irredundancy of $\mathcal{F}'$. \hfill $\square$

We call the smallest dimension of a face in the unique maximal irredundant infeasible face covering of $X \subseteq \{0,1\}^n$ the maximal irredundant covering bound of $X$, abbreviated as micb($X$). In the following, we show that micb($X$) completely characterizes the sparsity of any irredundant integer formulation of $X$. Here, we say that an integer formulation
is irredundant if each inequality in the integer formulation separates at least one infeasible binary point from \( \text{conv}(X) \). Moreover, if \( F \) and \( F' \) are faces of \([0,1]^n\), we denote by \( F^\text{min}(F,F') \) the smallest face of \([0,1]^n\) that contains both \( F \) and \( F' \). If \( F' \) is vertex \( x \), we write \( F^\text{min}(F,x) \). Note, that \( F^\text{min}(F,F') \) can also be written as \( F \lor F' \), the join of these two faces in the face lattice of \([0,1]^n\).

To show the main result characterizing the maximum sparsity in Theorem 28, we first need some technical lemmata.

**Lemma 26.** Let \( \bar{x} \in \{0,1\}^n \) and let \( F \) be a face of \([0,1]^n\). Moreover, let \( a^\top x \leq \beta \) be an inequality valid for \([0,1]^n\) with \( a^\top \bar{x} = \beta \). If all binary points in \( \{\bar{x}\} \cup F \) violate the inequality \( a^\top x \leq \beta', \beta' \in \mathbb{R} \), then all binary points in \( F^\text{min}(F,\bar{x}) \) violate \( a^\top x \leq \beta' \).

**Proof.** The proof proceeds via induction on \( n \). If \( n = 1 \), the statement is trivially fulfilled. In the induction step we distinguish two cases. First, if \( F \) and \( \bar{x} \) are contained in a facet of \([0,1]^n\), the induction hypothesis can be used to show the statement, since each facet is an \((n-1)\)-dimensional cube. In the second case, \( F \) and \( \bar{x} \) are not contained in a facet of \([0,1]^n\). This implies the existence of a \( y \in F \cap \{0,1\}^n \) such that \( \bar{x} \) and \( y \) lie on a diagonal of \([0,1]^n\). W.l.o.g. we can assume (by complementing entries) that \( \bar{x} = 1 \) and \( y = 0 \).

Since \( a^\top \bar{x} = \beta \) by assumption, it follows that \( a^\top \bar{x} = a^\top 1 = \sum_{i=1}^n a_i = \beta \). Thus, \( a_i \geq 0 \) holds for all \( i \in [n] \), because \( a^\top x \leq \beta \) is valid for \([0,1]^n\). This means that \( a^\top y \leq a^\top x' \leq a^\top \bar{x} \) for all \( x' \in \{0,1\}^n \). Therefore, the statement follows by \( \beta' < a^\top y \leq a^\top x' \leq a^\top \bar{x} \) for all \( x' \in \{0,1\}^n \).

Now we are able to derive bounds on the sparsity of an integer formulation for a set \( X \subseteq \{0,1\}^n \).

**Lemma 27.** Let \( F \) be a face of \([0,1]^n\) and let \( a^\top x \leq \beta \) be an inequality with \( a^\top \bar{x} > \beta \) for all \( \bar{x} \in F \). If \( a^\top x \leq \beta \) does not completely cut off a face \( F' \) of \([0,1]^n\) with \( F \subseteq F' \), then \( a \) has at least \( n - \text{dim}(F) \) non-zero entries. Furthermore, there is an inequality \( \bar{a}^\top x \leq \bar{\beta} \) such that the binary points that violate this inequality are exactly the binary points in \( F \) and such that \( \bar{a} \) has exactly \( n - \text{dim}(F) \) non-zero entries.

**Proof.** For a face \( \bar{F} \) of \([0,1]^n\)

\[
C_F = \{ w \in \mathbb{R}^n : w \text{ induces the face } \bar{F} \text{ of } [0,1]^n \} \\
= \{ w \in \mathbb{R}^n : w_i > 0 \text{ if } \bar{x}_i = 1 \text{ for all } \bar{x} \in \bar{F}, \ \\
w_i < 0 \text{ if } \bar{x}_i = 0 \text{ for all } \bar{x} \in \bar{F}, \ \\
w_i = 0 \text{ otherwise} \}
\]

denotes the (open) normal cone of \( \bar{F} \). Let \( a^\top x \leq \beta \) be violated by all points in \( F \) and assume that it does not completely cut off a face \( F' \) of \([0,1]^n\) with \( F \subseteq F' \). Then \( a \in C_F := \bigcup_{\bar{F} \subseteq F} C_{\bar{F}} \). Otherwise, if \( a \notin C_F \), then \( a^\top x \leq \beta \) has to cut off a binary point \( \bar{x} \notin F \). By Lemma 26, inequality \( a^\top x \leq \beta \) has to cut off \( F^\text{min}(F,\bar{x}) \), which contradicts the assumption. Because \( a \in C_F \), it has at least \( n - \text{dim}(F) \) non-zero entries due to the structure of \( C_F \).

To prove the second part of the lemma, let \( \bar{a} \) be any vector in \( C_F \) and define \( \bar{\beta} = \max\{\bar{a}^\top x : x \in [0,1]^n\} \) as well as \( \beta' = \max\{\bar{a}^\top x : x \in \{0,1\}^n \setminus F\} \). Since \( \bar{a} \in C_F \), \( \arg\max\{\bar{a}^\top x : x \in [0,1]^n\} = F \) and thus \( \bar{\beta} > \beta' \). Consequently, \( \bar{a}^\top x \leq \beta' \) is an inequality which is violated by exactly those binary points that are contained in \( F \). Furthermore, \( \bar{a} \) has exactly \( n - \text{dim}(F) \) non-zero entries by the structure of the normal cone \( C_F \). \( \square \)

This allows us to completely characterize the maximum sparsity of an integer formulation of any non-trivial \( X \subseteq \{0,1\}^n \).
Theorem 28. The maximum sparsity of an integer formulation of a non-trivial set $X \subseteq \{0, 1\}^n$ is $n - \text{micb}(X)$.

Proof. To prove the assertion, we first show that each integer formulation of $X$ contains an inequality with at least $n - \text{micb}(X)$ non-zero entries. Afterwards, we construct an integer formulation all of whose inequalities are $(n - \text{micb}(X))$-sparse to prove that this bound is tight.

To show the first part, let $a^\top x \leq \beta$ be an inequality valid for conv($X$) that cuts off at least one point $\bar{x} \in \{0, 1\}^n \setminus X$. Moreover, denote with $s_a$ the number of non-zero entries of $a$. Consider the face

$$F_a := \{x \in [0, 1]^n : x_i = 1 \text{ if } a_i > 0 \text{ and } x_i = 0 \text{ if } a_i < 0, \ i \in [n]\}$$

of $[0, 1]^n$, which coincides with $\text{argmax}\{a^\top x : x \in [0, 1]^n\}$. Observe that $s_a = n - \dim(F_a)$.

Since $a^\top x \leq \beta$ separates $\bar{x}$ from conv($X$), it separates at least one point in $F_a$. Hence, all points in $F_a$ are cut off. Furthermore, let $F$ be a maximal face of $[0, 1]^n$ which is completely cut off by $a^\top x \leq \beta$. On the one hand, if $F_a \subseteq F$, then

$$\dim(F) \geq \dim(F_a). \quad (10)$$

On the other hand, if $F_a \not\subseteq F$, then $a^\top x \leq \beta$ cuts off $F_{\min}(F, F_a)$ by applying Lemma 26 to all points of $F_a$. Thus, all points in $F_{\min}(F, F_a)$ have to be infeasible, which is a contradiction to the maximality of $F$. Consequently, all maximal infeasible faces of $[0, 1]^n$ contain $F_a$ and fulfill (10).

Let $F^*$ denote a maximal infeasible face of $[0, 1]^n$ of minimal dimension. Then,

$$\max_a s_a = \max_a (n - \dim(F_a)) \overset{(10)}{=} \max_a (n - \dim(F^*)) = n - \text{micb}(X),$$

which proves the proposed lower bound.

To prove the theorem, it suffices to construct an integer formulation all of whose inequalities are $(n - \text{micb}(X))$-sparse. Let $F$ be the maximal irredundant infeasible face covering of $X$. To construct the desired integer formulation, we take the box constraints $x \in [0, 1]^n$ to guarantee that each feasible point is contained in the hypercube. Note that each such constraint is 1-sparse. Now it suffices to cut off the infeasible faces of $[0, 1]^n$ w.r.t. $X$. By Lemma 27, each face $F \in F$ can be cut off by an inequality that has exactly $n - \dim(F)$ non-zero left-hand side coefficients. If we take these inequalities for all faces in $F$ we have ensured that we cut off each point in $[0, 1]^n \setminus X$ since $F$ is a maximal infeasible face covering. These inequalities are $(n - \text{micb}(X))$-sparse, which proves the assertion. \qed

Remark 29. Since the normal cone of any face of $[0, 1]^n$ contains a vector in $\{0, \pm 1\}^n$, there always exists a sparsest integer formulation of any non-trivial set $X \subseteq \{0, 1\}^n$ all of whose left-hand side coefficients are contained in $\{0, \pm 1\}$.

In the remainder of this section, we apply Theorem 28 to show that there is no sparse integer formulation of the vertices of the TSP polytope.

Theorem 30. Let $K_n = (V, E)$ be the complete undirected graph with $n \geq 5$ nodes and let $X \subseteq \{0, 1\}^E$ be the set of incidence vectors of Hamiltonian cycles of $K_n$. Then the maximum sparsity of an integer formulation of $X$ is $n - 2$.

Proof. To be able to apply Theorem 28, we have to analyze the binary points that are not contained in $X$. To this end, we construct for each infeasible binary point $x$ a certificate, i.e., fixings of variables that ensure infeasibility of $x$, of minimum size. The following situations cover all possible cases of infeasibility: The graph $G'$ induced by edges $e \in E$ with $x_e = 1$.
contains a subgraph isomorphic to $K_{1,3}$,
contains an induced subgraph which is a path,
is the empty graph,
contains an induced subgraph which is a cycle of length less than $|V|$,

In the first case, a certificate of infeasibility is given by fixing the three edges of $K_{1,3}$ to 1, because this implies that there is a node in a solution with degree 3. In the second and third case, there is a node $v$ of $G'$ whose degree is at most 1. Thus, there are $n-2$ pairwise different edges $e \in E$ incident to $v$ (in $K_n$) such that $x_e = 0$. By fixing these $n-2$ edges to 0, we obtain a certificate of infeasibility of size $n-2$, since this enforces that $v$ has at most degree 1 in any solution. Finally, in the fourth case, every node has degree 2 and if the solution is infeasible, a subtour contains at most $n-3$ nodes. Fixing the corresponding $n-3$ edges to 1 generates a certificate of infeasibility.

Thus, for each infeasible $x \in \{0,1\}^E$, there exists a certificate of infeasibility of size at most $n-2$. Therefore, each infeasible point is contained in an infeasible face of $[0,1]^n$ of dimension at least $|E| - n + 2$. To be able to apply Theorem 28, we have to show that there is indeed an infeasible binary point for which no certificate of size less than $n-2$ exists. To see this, consider the infeasible point $x = 0$. If at most $n-3$ variables are fixed to 0, a Hamiltonian cycle exists on the remaining edges by Ore’s Theorem [26], which guarantees the existence of a Hamiltonian cycle if for every pair of distinct non-adjacent nodes the sum of their degrees is at least the number of nodes. Consequently, a minimum size certificate of infeasibility for the null vector has size $n-2$, and Theorem 28 implies the assertion.

Note that the maximum sparsity of the TSP for $n = 4$ is 3, because the certificate for a $K_{1,3}$ subgraph has size 3, which is larger than the variable bound $n-2$. For $n = 3$, however, Theorem 30 holds, since $K_3$ does not contain a $K_{1,3}$ subgraph.

Finally, we show that it is hard to compute the maximum sparsity of an integer formulation or to separate inequalities of such formulations.

**Theorem 31.** It is $\mathcal{NP}$-hard to determine the maximum sparsity of an integer formulation of a set $X \subseteq \{0,1\}^n$, even if $X$ corresponds to the independent sets of a graphic matroid.

**Proof.** Let $G = (V,E)$ be an undirected simple graph and let $X \subseteq \{0,1\}^E$ correspond to the independent sets of a graphic matroid of $G$, i.e., $X$ contains the incidence vectors of cycle free edge sets in $G$. Let $\bar{x} \in \{0,1\}^E \setminus X$, and let $F$ be an infeasible face of $[0,1]^E$. If $\bar{x}$ is the incidence vector of an induced cycle $C$ of $G$, 

$$F_C := \text{conv}\{\{x \in \{0,1\}^E : x_e = 1 \text{ for all } e \in E \text{ with } \bar{x}_e = 1\}\}$$

is a face of $[0,1]^E$ that contains only infeasible binary points including $\bar{x}$. Furthermore, we cannot drop any condition in the definition of $F_C$, since this relaxed face would contain the incidence vector of a tree. Thus, each maximal infeasible face of $[0,1]^E$ that contains the incidence vector of an induced cycle $C$ has dimension $|E| - |C|$. Moreover, if $\bar{x}$ is not the incidence vector of an induced cycle but of an arbitrary set containing a cycle, $\bar{x}$ is contained in $F_C$ for each induced cycle $C$ of the graph induced by $\bar{x}$. Hence, the dimension of each maximal infeasible face of $[0,1]^E$ that contains $\bar{x}$ is at least the size of the longest induced cycle encoded by $\bar{x}$. For this reason, the faces $F_C$, where $C$ is an induced cycle of $G$, form the maximal irredundant infeasible face covering of $X$. Hence, computing the maximum sparsity of an integer formulation of $X$ is equivalent to computing the length of a longest induced cycle in $G$ by Theorem 28 and the definition of $F_C$.

Computing the maximum length of an induced cycle is $\mathcal{NP}$-hard, see Garey and Johnson [12, Problem GT23]. Thus, the assertion follows.

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Table 2: Number of non-trivial subsets of \( \{0,1\}^n \) that admit an integer formulation with a given maximum sparsity

<table>
<thead>
<tr>
<th>dimension</th>
<th>maximum sparsity</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>8 6</td>
</tr>
<tr>
<td>3</td>
<td>26 138 90</td>
</tr>
<tr>
<td>4</td>
<td>80 4088 38 976 22 390</td>
</tr>
</tbody>
</table>

In general, formulations with maximum sparsity are \( \mathcal{NP} \)-hard to separate as the following example of a knapsack polytope shows.

**Example 32.** Let \( X \) be the feasible region of a binary knapsack polytope. Similar as in the proof of Theorem 31, one can show that the faces

\[
F_C := \text{conv}\left( \{ x \in \{0,1\}^n : x_i = 1 \text{ for all } i \in C \} \right),
\]

where \( C \) is a circuit of \( X \), form the maximal irredundant infeasible face covering of \( X \). Since the sparsity of the circuit inequality for a circuit \( C \) equals \( n - \dim(F_C) \), an integer formulation with maximum sparsity of \( X \) is given by box constraints and all circuit inequalities for \( X \), cf. (5). Separating this integer formulation is \( \mathcal{NP} \)-hard, see Klabjan et al. [23].

### 4.3 Statistics on the Maximum Sparsity of Integer Formulations

The aim of this section is to give an impression on the distribution of maximum sparsity of integer formulations for non-trivial sets \( X \subseteq \{0,1\}^n \) in small dimensions. To be able to determine the maximum sparsity of an integer formulation for a given \( X \), we computed by a brute force method its maximal irredundant infeasible face covering, which characterizes the maximum sparsity by Theorem 28. Table 2 summarizes our results.

Since we were only able to compute the statistics for very small dimensions, an interpretation of these numbers has to be treated carefully. Our results indicate that there are only few non-trivial subsets of \( \{0,1\}^n \) that admit a very sparse integer formulation. From a combinatorial point of view, this is reasonable because the sets \( X \subseteq \{0,1\}^n \) that can be represented by 1-sparse inequalities are exactly the \( 3^n - 1 \) non-trivial faces of \( [0,1]^n \). In contrast to this, it is much more likely that an integer formulation of \( X \) has to be dense, since this depends only on local properties of infeasible points. For example, the sets \( X \) that need the densest integer formulations are those for which there exists \( \bar{x} \in \{0,1\}^n \setminus X \) which has only feasible neighbors in \( [0,1]^n \). Consequently, only a small amount of sets \( X \subseteq \{0,1\}^n \) admits a sparse integer formulation.

### 5 Discussion of Concepts of Mixed-Integer Formulations Including an Objective Function

Most of the results in the preceding sections or in the literature do not mention the objective function. For example, the geometric characterization of Jeroslow and Lowe [18] of subsets of \( \mathbb{R}^n \) that can be represented as the projection of mixed-integer points in a polyhedron is independent from an objective function. To address this issue, the main topic of this section is the objective function.
Let $\Pi$ be an optimization problem of the form
\[
\max \{ f(x) : x \in X \},
\] (11)
where $X \subseteq \mathbb{Z}^p \times \mathbb{R}^q$ is hole-free and $f : \mathbb{Z}^p \times \mathbb{R}^q \rightarrow \mathbb{R}$. The aim of this section is to represent $\Pi$ via a mixed-integer programming formulation. Observe that since we allow general objective functions, we cannot use the definition from Section 2, which required the objective function to be linear. We first have to adapt the definition of a mixed-integer formulation.

**Definition 33.** A mixed-integer programming formulation (MIP formulation) for an optimization problem with objective function $f$ is a triple $(Q, g, \tau)$ with a polyhedron $Q \subseteq \mathbb{R}^p \times \mathbb{R}^q$ and an affine function $g : \mathbb{R}^p \times \mathbb{R}^q \rightarrow \mathbb{R}$ such that
\begin{enumerate}[(a)]  
  \item $(Q, g)$ is a MIP,
  \item $\tau : \mathbb{R}^p \times \mathbb{R}^q \rightarrow \mathbb{R}^p \times \mathbb{R}^q$, and
  \item $\tau(\arg\max \{ g(y) : y \in Q \cap (\mathbb{Z}^p \times \mathbb{R}^q) \}) \subseteq \arg\max \{ f(x) : x \in X \}$. \end{enumerate}

Assume we are given an optimization problem $\Pi$ as in (11). If there exists a MIP formulation of an instance $I$, we can solve $I$ by solving the maximization problem of $g$ over the mixed integer set $Q\cap(\mathbb{Z}^p \times \mathbb{R}^q)$ obtaining a solution $y$. By applying the transformation $\tau$ on $y$, we obtain a solution $\tau(y) \in X$, which by Definition 33 c) is an optimal solution of $I$. Therefore, if there exists a MIP formulation of (11), the optimization problem $\Pi$ can be solved by a MIP.

If $\tau$ is the identity and we ignore the objective functions $f$ and $g$, the polyhedron $Q$ together with the identity form a mixed-integer formulation in the original space as defined in Definition 2. Moreover, if $\tau$ is an affine projection and we ignore Part c) of Definition 33, the concept of an extended mixed-integer formulation is generalized by Definition 33.

One problem of Definition 33 is that such a formulation always exists, but might not be useful. It is clear that $Q$, $g$, and $\tau$ can be chosen easily once an optimal value of (11) is known. For example, let an optimum of (11) be attained at $x^*$. In this case, setting $g$ to be constant and $\tau \equiv x^*$, $Q$ can be any nonempty polyhedron, in particular it can be chosen as a single point. Since solving (11) in order to obtain $x^*$ might be $\mathcal{NP}$-hard, we cannot expect that such a MIP formulation can be computed efficiently. Thus, the definition of a MIP formulation has to be refined.

**Definition 34.** Let $I$ be an instance of an optimization problem with objective $f$. A MIP formulation $(Q, g, \tau)$ is called efficient if
\begin{enumerate}[(1)]  
  \item $Q$ has polynomial size in $\langle I \rangle$,
  \item $g$ can be computed and $g(x)$ can be evaluated in polynomial time in $\langle I \rangle$,
  \item $\tau(y)$ can be computed in polynomial time in $\langle I \rangle$ for all $y \in Q \cap (\mathbb{Z}^p \times \mathbb{R}^q)$,
\end{enumerate}
where $\langle \cdot \rangle$ denotes the encoding length (see for example Grötschel et al. [13]).

Note that Theorem 3 shows that an efficient integer programming formulation always exists if $f$ is linear, $X \subseteq \{0,1\}^n$, and the membership problem is in $\mathcal{NP}$. Furthermore, note that it is necessary that $g$ can be computed efficiently, since once an optimal solution is known, $g$ easily be chosen as to be only optimal in that particular solution.

Moreover, it can be shown that efficient formulations do not exist under some circumstances.

**Lemma 35.** Let $I$ be an instance of an $\mathcal{NP}$-hard problem $\Pi$, and let $(Q, g, \tau)$ be a MIP formulation of $I$ (either in the original or an extended space) such that
\[
\max \{ g^\top x : x \in Q \cap (\mathbb{Z}^p \times \mathbb{R}^q) \}
\]

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can be solved in polynomial time in \( \langle I \rangle \). As long as \( P \neq NP \), the MIP formulation \((Q, g, \tau)\) is not efficient.

**Proof.** The assumptions guarantee that the formulation can be derived in polynomial time and that the computation of an optimal solution of \( I \) can be achieved in polynomial time as well.

**Example 36.** Let \( G = (V, E) \) be an edge or node weighted bipartite graph. The maximum biclique problem on \( G \) is to find an induced complete bipartite subgraph \( G' \) of \( G \) whose (node or edge) weight is maximal. The node version of this problem is solvable in polynomial time, see Yannakakis [28, Theorem 4], and there exists a MIP formulation \((Q, g)\) using only node variables, see Hochbaum [15].

The edge version, however, is \( NP \)-hard, see Dawande et al. [7]. It can be modeled by variables representing whether an edge or node is contained in \( G' \), see Hochbaum [15]. Thus, the question arises whether there also exists a MIP formulation using \( Q \), since \( G' \) can be represented by specifying only its nodes. Lemma 35 implies that no efficient MIP formulation for the edge version exists over \( Q \).

From a practical point of view, there are additional requirements that seem useful.

**Remark 37.**

1. If a MIP formulation is known and \( f \) is changed, it would be preferable if \( Q \) and \( \tau \) can be chosen as before and only \( g \) has to be changed, that is, \( Q \) and \( \tau \) are independent of \( f \).
2. \( Q \) should be tractable. Otherwise, using this MIP formulation may not be helpful.
3. Part c) in Definition 33 can be changed to equality, i.e., all optimal solutions can be generated.

The first part can be guaranteed in the case that \( f \) and \( \tau \) are affine, since if \( f(x) = f^T x \) and \( \tau(y) = \tau y \), for \( f \in \mathbb{R}^{p+q} \), \( \tau \in \mathbb{R}^{(p'+q') \times (p+q)} \), we obtain:

\[
\max_{x \in X} f(x) = \max_{y \in Y} f(\tau(y)) = \max_{y \in Y} f^T \tau y = \max_{y \in Y} (\tau^T f)^T y
\]

\[
= \max_{y \in Y} (\tau^T f)(y) =: \max_{y \in Y} g(y),
\]

where \( Y := Q \cap (\mathbb{Z}^p \times \mathbb{R}^q) \). The first equality holds because of Definition 33 Condition c) and the others because of linearity.

To expand on the last question in Remark 37, we consider the following example:

**Example 38.** Given a graph \( G = (V, E) \), edge weights \( w \), and a positive integer \( k \), the aim of the graph partitioning problem is to find a partition of the nodes \( V \) into at most \( k \) subsets such that the sum of the weights of edges between two different partitions is maximized. This can be formulated using binary variables \( x_{vi} \) modeling whether node \( v \) is in partition \( i \) by the following quadratic optimization problem

\[
\max \quad \sum_{\{u,v\} \in E} w_{uv} \left( 1 - \sum_{i=1}^{k} x_{ui} x_{vi} \right)
\]

s.t.

\[
\sum_{i=1}^{k} x_{vi} = 1, \quad v \in V, \quad (12)
\]

\[
x_{vi} \in \{0, 1\}, \quad v \in V, \ i \in [k].
\]

In the following, we refer to this problem as the quadratic graph partitioning formulation.
The first observation is, that Lemma 35 can be used for showing the non-existence of an efficient formulation. Let $P_k(G)$ be the polytope defined as the convex hull of the points fulfilling (12) and box constraints.

**Corollary 39.** There does not exist an efficient MIP formulation for the graph partitioning problem in the original space of the $x$-variables using only (12) and box constraints.

**Proof.** Observe that $P_k(G)$ is integral and that any linear objective can be maximized in polynomial time over $P_k(G)$. Thus, the statement follows from Lemma 35 since the graph partitioning problem is $\mathcal{NP}$-hard, see Garey and Johnson [12, Problem ND14].

One problem with the formulation of the graph partitioning problem is the quadratic objective function. It could be linearized by introducing a binary variable $y_{uv}$ modeling whether the endpoints $u$ and $v$ of edge $\{u,v\} \in E$ lie in different partitions. An integer linear program is then given by

$$\max \sum_{\{u,v\} \in E} w_{uv} y_{uv}$$

s.t. $\sum_{i=1}^{k} x_{vi} = 1$, $v \in V$;

$$x_{ui} - x_{vi} \leq y_{uv}, \quad \{u,v\} \in E, i \in [k],$$

$$x_{vi} - x_{ui} \leq y_{uv}, \quad \{u,v\} \in E, i \in [k],$$

$$x_{ui} + x_{vi} + y_{uv} \leq 2, \quad \{u,v\} \in E, i \in [k],$$

$$x_{vi} \in \{0,1\}, \quad v \in V, i \in [k],$$

$$y_{uv} \in \{0,1\}, \quad \{u,v\} \in E.$$  \hfill (13)

Formulation (13) is an example of an efficient MIP formulation. Note that this formulation is given in an extended space, if we consider the original space to be given by the $x$-variables alone. This shows that there is a MIP formulation of the graph partitioning problem in an extended space such that condition c) of Definition 33 holds with equality. A question that arises is whether this equality can also be achieved in the original space. Intuitively this should not be the case, because of the objective function. In the following, we show that this indeed is not possible.

**Lemma 40.** There does not exist a linear objective $w \in \mathbb{R}^{V \times [k]}$ for $P_k(G)$ such that the maximizers of $w$ over $P_k(G)$ are exactly the maximizers of the quadratic graph partitioning problem if $k \geq 2$ and not all solutions of the quadratic problem are optimal.

**Proof.** For the sake of contradiction, assume there is a non-trivial face $F$ of $P_k(G)$ that contains exactly the maximizers of the quadratic graph partitioning problem. Let $a^T x \leq \beta$ be an inequality that induces $F$. Since exactly the points in $F$ maximize $a^T x \leq \beta$, we have $a^T x = \beta$ for all $x \in F$, and $a^T x < \beta$ for all $x \in P_k(G) \setminus F$.

Observe that $x \in \{0,1\}^{V \times [k]}$ is a vertex of $P_k(G)$ if and only if for every $v \in V$ there is exactly one $i(v) \in [k]$ with $x_{vi(v)} = 1$. Hence, a vertex is contained in $F$ if and only if $i(v) \in A^v := \arg\max\{a_{vi} : i \in [k]\}$ for every $v \in V$.

Let $x$ be a vertex of $P_k(G)$ that is contained in $F$. If there exists $\tilde{v} \in V$ such that $A^{\tilde{v}} \neq [k]$, there exist an index $j \in A^{\tilde{v}}$ and an index $\tilde{j} \in [k] \setminus A^{\tilde{v}}$. By exchanging the entries of $x$ in columns $j$ and $\tilde{j}$, we obtain another vertex $\tilde{x}$ of $P_k(G)$ that is optimal for the quadratic graph partitioning problem, because changing the labels of assigned partitions (that is,
exchanging columns) does not affect the objective value. But \( \tilde{x} \) cannot be contained in \( F \), because for every \( v \in V \setminus \{ \tilde{v} \} \)
\[
\sum_{i=1}^{k} a_{vi} \tilde{x}_{vi} \leq \sum_{i=1}^{k} a_{vi} x_{vi} \quad \text{and} \quad \sum_{i=1}^{k} a_{\tilde{v}i} \tilde{x}_{\tilde{v}i} = a_{\tilde{v}j} < a_{\tilde{v}j} = \sum_{i=1}^{k} a_{\tilde{v}i} x_{\tilde{v}i}.
\]
Consequently, \( a^\top \tilde{x} < a^\top x \). For this reason, \( A^v = [k] \) for all \( v \in V \).

But if \( A^v = [k] \) for every \( v \in V \), every vertex of \( P_k(G) \) maximizes \( a^\top x \leq \beta \) due to (12). This contradicts the assumption that not all vertices of \( P_k(G) \) are optimal for the quadratic problem.

Note that this lemma does not depend on the graph partitioning problem, but only on assigning nodes to partitions with the requirement that interchanging two partitions does not change the objective value. Thus, in the original space of the \( x \)-variables there does not exist a linear objective function that allows for finding all optimal solutions.

6 Conclusion

In this paper, we have seen several techniques to decide whether a set \( X \subseteq \{0, 1\}^n \) admits a mixed-integer formulation that is of polynomial size or tractable (and has left-hand side coefficients in \( \{0, \pm 1\} \)), see Theorem 3 and Theorem 6, or that is sparse, see Section 4.1. In particular, Proposition 13 and Proposition 15 can be used to show that \( X \) does not admit a polynomial size integer formulation (with bounded coefficients), whereas Theorem 28 allows to decide whether \( X \) cannot be represented by a sparse integer formulation. Finally, Proposition 4 allows to detect whether \( X \) does not admit any tractable mixed-integer formulation. Summarizing, most of our results for mixed-integer formulations in extended spaces are constructive, whereas the results in the original space can be used to prove non-existence of mixed-integer formulations with the mentioned properties.

Nevertheless, there remain some interesting open questions for further research. For example, it is not known if it is possible to decide in polynomial time whether a set \( X \subseteq \{0, 1\}^n \) admits an \( F \)-hiding set of specific size. In particular, the complexity of deciding whether a given set is a hiding set (in the sense of Kaibel and Weltge [22]) of \( X \) is not known.

Moreover, there are further interesting aspects of MIP formulations involving the objective function, which are not covered by this paper. In Definition 33 c), we required that optimal points of a MIP formulation have to be transformable to optimal points of the original problem. But in the graph partitioning example we have seen that not all of the original optimal points can be attained by any MIP formulation. Thus, the questions remains under which conditions all of the original optimal points are guaranteed to be found. Furthermore, we have already mentioned some preferable properties of MIP formulation, e.g., tractability or invariance under objective changes (compare Remark 37). Deriving criteria that ensure these properties are out of scope of this paper, but form a possible direction of future research.

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References


A Details in Example 11

In this section, we present missing arguments for Example 11.

Example 41 (Example 11 revisited). Consider the independence system

$$X := \left\{ x \in \{0, 1\}^{2n} : \sum_{i=1}^{n} x_i + 2 \sum_{i=n+1}^{2n} x_i \leq n \right\},$$

where $n$ is an even positive integer. A binary point $x$ is not contained in $X$ if and only if $x = \chi^A + \chi^B$, where $A \subseteq [n]$, $|A| = k$, and $B \subseteq \{n+1, \ldots, 2n\}$, $|B| > \lfloor \frac{n-k}{2} \rfloor$. For this reason, the integer formulation given by (3) contains

$$\sum_{k=0}^{n} \binom{n}{k} \sum_{\ell=\lceil \frac{n-k}{2} \rceil + 1}^{n} \binom{n}{\ell}$$

many infeasibility cuts. By a simple case analysis, we can show that each circuit $C$ of this independence system can be written as $C = A \cup B$, where $A \subseteq [n]$, $|A| = k$ is odd, and $B \subseteq [2n] \setminus [n]$, $|B| = \lceil \frac{n-k}{2} \rceil + 1 =: k'$. Thus, $X$ has exactly

$$\sum_{k=0}^{n} \binom{n}{k} \binom{\lfloor \frac{n}{2} \rfloor}{|\frac{n-k}{2}| + 1}$$

circuits, and the integer formulation via circuit inequalities is much smaller than the integer formulation via infeasibility cuts.
To analyze the integer formulation via maximal hyperclique inequalities, let $C$ be a circuit of $X$ that contains exactly $k$ elements from $[n]$ and exactly $k'$ elements from $[2n] \setminus [n]$. Then there is a hyperclique $\mathcal{H} = \mathcal{H}_1 \cup \mathcal{H}_2$, $\mathcal{H}_1 \subseteq [n]$, $\mathcal{H}_2 \subseteq [2n] \setminus [n]$, whose hyperclique inequality is of the form
\[
\sum_{i \in \mathcal{H}_1} x_i + \sum_{i \in \mathcal{H}_2} x_i \leq k + k' - 1
\]
and that dominates (or coincides with) the circuit inequality associated with $C$. Furthermore, the characteristic vector of $C$ violates the defining constraint of $X$ by
\[
\sum_{i=1}^n \chi_i^C + 2 \sum_{i=n+1}^{2n} \chi_i^C - n = k + 2k' - n = k + 2\left(\left\lfloor \frac{n-k}{2} \right\rfloor + 1\right) - n = 1,
\]
because $k$ is odd and $n$ is even.

On the one hand, if we delete an element with coefficient 2 from the circuit and add an element $i \in [n]$ which is not already contained in the circuit, we obtain a set $C'$, which has the same cardinality as $C$, but which does not violate the knapsack constraint. Hence, $i$ cannot be contained in the hyperclique, since otherwise, the corresponding hyperclique inequality would cut off $\chi_i^{C'}$. Consequently, the coefficient of all variables with an index in $[n] \setminus C$ is 0 in the hyperclique inequality.

On the other hand, if we set $x_{i'} = 1$ for some $i' \in \{n+1, \ldots, 2n\} \setminus C$, at most $k + k' - 2$ variables with an index in $C$ can be set to 1 to generate a feasible point $x$. For this reason, in any maximal hyperclique inequality that cuts off $\chi^C$, the coefficient of $i'$ is 1.

Combining both observations shows that the strongest hyperclique inequality that cuts off $\chi^C$ is of the form
\[
\sum_{i \in C \cap [n]} x_i + \sum_{i=n+1}^{2n} x_i \leq k + k' - 1.
\]

For this reason, the integer formulation of $X$ that consists of the strongest valid hyperclique inequalities contains
\[
\sum_{k=0}^{n} \binom{n}{k} = 2^{n-1}
\]
different hyperclique inequalities.

**B Omitted Proofs of Section 3.2.2**

In this section, we provide the missing arguments to show that the concept of $F$-hiding sets can produce lower bounds which are asymptotically tight. To this end, recall from Section 3.2.2 the definition of the knapsack polytope corresponding to the orbisack

\[
O_m := \text{conv}\left(\{(x, y) \in \{0, 1\}^m \times \{0, 1\}^m : \sum_{i=1}^m 2^{m-i}(x_i + y_i) \leq 2^m - 1\}\right).
\]

Observe that the vertices of $O_m$ are binary $m \times 2$-matrices, where we denote the first column of such a vertex by $x$ and the second column by $y$. Thus, $O_m \subseteq \mathbb{R}^{[m] \times [2]}$.

To prove the claims of Section 3.2.2, we first present a complete characterization of hypercliques of $O_m$. For this characterization, we have to analyze circuits of $O_m$. Note that $O_m$ is a knapsack polytope, and thus, one would typically refer to circuits as *minimal covers* in the literature. But to be consistent with our terminology in Section 3.2, we use the name circuit instead.
Lemma 42. A set $C \subseteq [m] \times [2]$ is a circuit of $O_m$ if and only if there exists $i \in [m]$ such that

1. $C \subseteq [i] \times [2]$,
2. $\{i\} \times [2] \subseteq C$, and
3. either $(j, 1) \in C$ or $(j, 2) \in C$ for all $j \in \{1, \ldots, i-1\}$.

Furthermore, any maximal hyperclique $\mathcal{H}$ of $O_m$ can be written as $\mathcal{H} = C \cup (\{1\} \times [2])$, where $C$ is a circuit of $O_m$.

Proof. Let

$$a^\top x + a^\top y = \sum_{j=1}^{m} 2^{m-j}(x_j + y_j) \leq \beta$$

be the knapsack inequality of $O_m$, and let $C \subseteq [m] \times [2]$. Moreover, we denote the first and second column of $O_m^C$ by $x^C$ and $y^C$, respectively.

First, we show that $C$ is a circuit of $O_m$ if it fulfills Conditions 1–3 above. To this end, we observe

$$(a, a)^\top \chi^C \leq \sum_{j=1}^{i} a_j (x_j^C + y_j^C) \quad \text{and} \quad \sum_{j=1}^{i-1} a_j = 2^{m-i+1} + \sum_{j=1}^{i-1} 2^{m-j}$$

$$= 2^{m-i+1} + 2^{m} \sum_{j=1}^{i-1} \left(\frac{1}{2}\right)^{j} = 2^{m-i+1} + 2^{m}(1 - 2^{1-i}) = 2^{m} = \beta + 1.$$  

Thus, $\chi^C \notin O_m$. But if we remove any element from $C$ to obtain $C'$, we have

$$(a, a)^\top \chi^{C'} \leq (a, a)^\top \chi^C - 1 = \beta$$

because $a_i \geq 1$ for all $i \in [m]$. Consequently, the characteristic vector of each proper subset of $C$ is contained in $O_m$, and thus, $C$ is a circuit of $O_m$.

For the reverse direction, we distinguish three cases. In the first case, assume that Condition 2 does not hold. Then, we can estimate for the characteristic vector $\chi^C$ that

$$(a, a)^\top \chi^C \leq \sum_{j=1}^{m} a_j = 2^{m} - 1 = \beta.$$  

Hence, $C$ is not a circuit of $O_m$. Consequently, we can assume in the following that there exists $i \in [m]$ such that $\{i\} \times [2]$ is contained in $C$. Let $i'$ be the minimal index $i$ for which Condition 2 holds.

In the second case, consider that Condition 2 holds but Condition 3 is violated for $i'$, i.e., there exists $j \in [i' - 1]$ such that $\{j\} \times [2] \cap C = \emptyset$. Let $j'$ be the minimal of all such indices $j$. Then,

$$(a, a)^\top \chi^C \geq \sum_{j=1}^{j'-1} a_j + \sum_{j=j'+1}^{m} a_j (x_j^C + y_j^C) \leq \sum_{j=1}^{j'-1} a_j + 2 \sum_{j=j'+1}^{m} a_j$$

$$= \sum_{j=1}^{j'-1} a_j + \sum_{j=j'}^{m-1} a_j = \sum_{j=1}^{m-1} 2^{m-j} = 2^{m} - 2 < \beta.$$  

Note that the first equation holds because $i'$ and $j'$ are chosen minimally, and thus, Condition 3 holds for all $j < j'$. For this reason, $C$ is not a circuit of $O_m$.  

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In the last case, assume that Condition 2 is valid and that Condition 3 holds for \( i' \) but Condition 1 does not hold for \( i' \). Define \( C' := C \cap ([i'] \times [2]) \). By the beginning of the proof we have that \( C' \) is a circuit of \( O_m \). For this reason, \( C \) cannot be a circuit if Condition 1 does not hold, since this implies \( C' \subseteq C \).

Finally, we prove the characterization of hypercliques. To this end, let \( A := \{(1, 1), (1, 2)\} \) and define for any circuit \( C \) of \( O_m \), the set \( C^A := C \cup A \). If we take any \( |C| \)-subset \( C' \) of \( C^A \), either exactly one element from \( A \) or both elements from \( A \) are contained in \( C' \).

In the first case, \( C' \) either coincides with \( C \) or we replaced the unique index \( C \cap A \) by the other index in \( A \). Hence, \( C' \) is a circuit by the above characterization. In the second case, \( \chi^{C'} \) cannot be contained in \( O_m \), because \( A \) is a circuit and \( A \subseteq C' \). Consequently, \( C^A \) is a \(|C|\)-hyperclique.

To prove that, in fact, each maximal hyperclique can be written in this way, we proceed as follows. Let \( H \) be a \( k \)-hyperclique of \( O_m \). Then by definition there exists a circuit \( C \subseteq H \) with \(|C| = k \). We distinguish two cases.

First, assume \( C = \{(1, 1), (1, 2)\} \), i.e., \( k = 2 \). If \( H \) contained an index \( (i, j) \in \{2, \ldots, m\} \times [2] \), then \( C' := \{(1, 1), (i, j)\} \) would be infeasible by definition of a hyperclique. But \( C' \) cannot be infeasible for \( O_m \) because it does not contain a circuit. Hence, the only maximal 2-hyperclique is \( \{(1, 1), (1, 2)\} \).

Second, assume \( C \subseteq [i] \times [2] \) with \( \{i\} \times [2] \subseteq C \) for some \( i \in \{2, \ldots, m\} \). Since \( C \) is a circuit, we know by the above characterization that either \( (1, 1) \) or \( (1, 2) \) is contained in \( C \). If \( H \) contained an index \( (i, j) \in \{2, \ldots, m\} \times [2] \setminus C \), then \( C' := (C \setminus \{(1, 1), (1, 2)\}) \cup \{(i, j)\} \) has cardinality \( k \), and thus would be infeasible since \( H \) is a \( k \)-hyperclique. But because \( C' \) neither contains \( (1, 1) \) nor \( (1, 2) \), it cannot contain a circuit, and as a consequence \( \chi^{C'} \in O_m \).

For this reason, any maximal \( k \)-hyperclique is contained in \( C \cup \{(1, 1), (1, 2)\} \). By the initial arguments, we know that \( C \cup \{(1, 1), (1, 2)\} \) is, in fact, such a hyperclique, which proves the assertion.

Due to the structure of circuits and hypercliques described in Lemma 42, there is exactly one maximal hyperclique that is contained in \( \{1\} \times [2] \), and there are exactly \( 2^{i-1} \) maximal hypercliques that contain \( \{i\} \times [2] \), \( i \in \{2, \ldots, n\} \). Consequently, there are \( 2^{m-1} \) distinct maximal hypercliques for \( O_m \). Moreover, there are \( 4m \) box constraints for \( O_m \), since it consists of \( 2m \) variables. Hence, there exists an \( F^1 \)-integer formulation of \( O_m \) containing \( 2^{m-1} + 4m \) inequalities by Corollary 10.

To prove that the concept of \( F^1 \)-hiding sets can produce a lower bound on any \( F^1 \)-integer formulation for \( O_m \) that is asymptotically tight, we construct a large \( F^1 \)-hiding set: Consider the matrices in \( \mathcal{X}^1 = \bigcup_{i=2}^{m} \chi^{1}_i \), where \( \chi^{1}_i \) is the set of circuits of \( O_m \) which contain \( \{i\} \times [2] \). Since all matrices in \( \chi^{1}_i \) are circuits of \( O_m \), they are not contained in \( O_m \). Furthermore, the combinatorial structure of hypercliques of Lemma 42 shows that each hyperclique inequality of \( O_m \) cuts off at most one point in \( \chi^{1}_i \). By Proposition 9, any inequality in \( F^1 \) cuts off at most one point in \( \chi^{1}_i \), since, otherwise, the inequality cannot be dominated by a hyperclique inequality. Consequently, Proposition 15 implies that any \( F^1 \)-integer formulation of \( O_m \) contains at least \(|\chi^{1}_i| = 2^{m-1} - 1\) inequalities. Hence, the lower bound provided by \( F^1 \)-hiding sets is asymptotically tight.

Finally, we consider \( F^2 \)-integer formulations of \( O_m \), where \( F^2 \) is the restriction of \( F^2 \) to facet defining inequalities of \( O_m \). A motivation for the restriction to \( F^2 \) may be that one is interested in a formulation that contains the “strongest” inequalities which fulfill the coefficient bound, or because one knows that such inequalities can be separated efficiently. For example, for \( O_m \) inequalities in \( F^2 \) can be separated in \( O(n^2) \) time, which can be seen by the structure of such facets described in the proof of Lemma 44 below.
Lemma 43. An $F^2$-integer formulation of the orbisack $O_m$, $m \geq 3$, is given by box constraints, $x_1 + y_1 \leq 1$, $x_1 + x_2 + y_1 + y_2 \leq 2$, and all inequalities

$$2x_1 + 2y_1 + \sum_{i=2}^{m} (a_{i-1}x_i + b_{i-1}y_i) \leq 2 + \beta,$$

where $a^\top x + b^\top y \leq \beta$ is a maximal hyperclique inequality of $O_{m-1}$, i.e., $a, b \in \{0,1\}^{m-1}$.

Proof. Kaibel and Loos [21] proved that the proposed inequalities define facets of $O_m$, and thus are valid for $O_m$. Furthermore, the box constraints ensure that all feasible integer points of this inequality system are binary. Consequently, it suffices to check that each circuit of $O_m$ is cut off by at least one of the given inequalities.

By Lemma 42, we know an iterative scheme that generates all circuits of $O_m$. First, we observe that the only circuit that is contained in $[1] \times [2]$ is $\{(1,1),(1,2)\}$. This circuit is cut off by $x_1 + y_1 \leq 1$. Second, both remaining circuits contained in $[2] \times [2]$ are separated from $O_m$ by $x_1 + x_2 + y_1 + y_2 \leq 2$.

Finally, let $i \geq 3$, and let $C$ be a circuit that is contained in $[i] \times [2]$. Then $C' := C \cap \{2,\ldots,m\} \times [2]$ is a circuit of $O_{m-1}$ by Lemma 42, and there exists a maximal hyperclique inequality $a^\top x + b^\top y \leq \beta$ for $O_{m-1}$ that separates $\chi^{C'}$ and $O_{m-1}$. Since either $(1,1)$ or $(1,2)$ is contained in $C'$,

$$2x_1 + 2y_1 + \sum_{i=2}^{m} (a_{i-1}x_i + b_{i-1}y_i) \leq 2 + \beta$$

defines a separating hyperplane for $\chi^C$ and $O_m$. □

Thus, there exists an $F^2$-integer formulation of $O_m$ that consists of $4m + 2 + 2^{m-2}$ inequalities. Observe that the inequalities $x_1 \leq 1$ and $y_1 \leq 1$ are redundant and that the remaining inequalities define facets of $O_m$, see Kaibel and Loos [21]. Consequently, if we remove both redundant constraints, the remaining system is even an $F^2$-integer formulation of $O_m$. To see that this integer formulation is asymptotically minimal if we consider only facet defining inequalities of $O_m$ that are contained in $F^2$, we construct an $F^2$-hiding set.

Lemma 44. Let $m \geq 4$ and let $\chi^2 \in \{0,1\}^{m \times 2}$, $i \in \{4,\ldots,m\}$, be the set of all matrices $X \in \{0,1\}^{m \times 2}$ with

- $(X_{11},X_{12}) = (X_{21},X_{22}) = (1,0),$
- $(X_{11},X_{12}) = (1,1),$
- $(X_{j1},X_{j2}) = (0,0)$ for all $j \in \{i+1,\ldots,m\},$
- $X_{j1} + X_{j2} = 1$ for all $j \in \{3,\ldots,i-1\}$, and
- $\sum_{j=3}^{m-1} X_{j1}$ is even.

Then $\chi^2 := \bigcup_{i=4}^{m} \chi^2_i$, is an $F^2$-hiding set of $O_m$.

Proof. Due to the argumentation after Lemma 42 and since $\chi^2 \subseteq \chi^1$, where $\chi^1$ is defined as above, each facet inequality with left-hand side coefficients in $\{0,\pm 1\}$ cuts off at most one point in $\chi^2$. Hence, it suffices to show that each facet inequality of $O_m$ whose left-hand side coefficients are contained in $\{-2,\ldots,2\}$ but not in $\{0,\pm 1\}$ cuts off at most one point in $\chi^2$.

Based on Kaibel and Loos [21, Thm. 5.1], an inequality $(a,b)^\top(x,y) \leq \beta$ with left-hand side coefficients in $\{-2,\ldots,2\}$ such that $|a_i| = 2$ or $|b_i| = 2$ for some $i \in [m]$ defines a facet of $O_m$ if and only if there exists $i, i' \in \{2,\ldots,m\}$, $i' < i$, such that
The left-hand side of the facet inequality evaluates for the constructed point to

Thus, we have only one degree of freedom in such a maximizer. But since exactly one of

and consequently, this facet inequality simplifies to

Due to the structure of $a$, $b$, and $\beta$, this simplified inequality coincides with a circuit

inequality of $O_{m-2}$. Thus, at most one point $X \in \mathcal{X}^2$ can be cut off by any such inequality, since rows 3 to $m$ of $X$ form the incidence vector of a circuit of $O_{m-2}$.

Second, we consider the case $i' = 3$. Since the restriction of $X \in \mathcal{X}^2$ to rows 3 to $m$ is the incidence vector of a circuit $C$ of $O_{m-2}$, it suffices to show that any possible inequality can be simplified to a maximal hyperclique inequality for $O_{m-2}$, because any such inequality cuts off at most one point in $\mathcal{X}^2$. W.l.o.g. we can assume that $a_2 = 2$ and $b_2 = 0$, since the other case follows by symmetry. Then we have

and consequently, this facet inequality simplifies to

Because the simplified inequality is a maximal hyperclique inequality for $O_{m-2}$, the assertion follows.

Finally, let $i' \geq 4$. To prove that a facet in $\mathcal{F}^2$ separates at most one point in $\mathcal{X}^2$ from $O_m$, we construct for each left-hand side of a facet a maximizer in $\mathcal{X}^2$ and we show that this maximizer is cut off by a unique facet. Clearly, any maximizer $X \in \mathcal{X}^2$ of the left-hand side of a facet fulfills

Thus, we have only one degree of freedom in such a maximizer. But since exactly one of both possibilities fulfills $\sum_{j=3}^{m-1} X_{j1}$ being even, only one is contained in $\mathcal{X}^2$ by assumption. The left-hand side of the facet inequality evaluates for the constructed point to $i + i'$, and thus, violates the facet inequality by 1. Hence, all other points in $\mathcal{X}^2$ cannot violate the facet inequality.
Since the first column of a matrix $X \in \mathcal{X}^2$ determines $X$, an easy calculation using the conditions in Lemma 44 shows that there is an $\mathcal{F}_2^2$-hiding set containing

$$\left| \bigcup_{i=4}^{m} \mathcal{X}_i^2 \right| = \sum_{i=4}^{m} \sum_{k=0}^{i-3} \binom{i-3}{k} = \sum_{i=4}^{m} 2^{i-4} = 2^{m-3} - 1$$

matrices. Hence, the minimum size of an $\mathcal{F}_2^2$-integer formulation of $O_m$ contains $\Theta(2^{m-2})$ inequalities.

Unfortunately, we are not aware of a lower bound on the size of an $\mathcal{F}_2^2$-integer formulation of $O_m$, because we do not know whether the matrices in $\mathcal{X}^2$ form an $\mathcal{F}_2^2$-hiding set. In particular, we cannot use the above argumentation to prove or disprove that $\mathcal{X}^2$ is an $\mathcal{F}_2^2$-hiding set, because the structure of general valid inequalities in $\mathcal{F}_2^2$ for $O_m$ is unknown.