Robust Merton Problem with Time Varying Nondominated Priors

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Abstract

We give explicit solutions for utility optimization problems in the presence of Knightian uncertainty in continuous time with nondominated priors and finite time horizon in a diffusion model. We assume that the uncertainty set is compact and time dependent on \([0, T]\). We solve the robust optimization problem explicitly both when the investor’s utility is of CRRA and of CARA type and give an \(\alpha\)-maximin type expected type utility variation to the robust problem. To the best of our knowledge, this is the first work in deriving explicit solutions in continuous time with time varying uncertainty sets in a nondominated prior setting.

Mathematics Subject Classification: 91B28;93E20

Keywords: Knightian uncertainty; Robust optimization

1 Introduction

Starting with the pioneering works of [1, 8, 32, 4], the underlying risky assets are modelled as Markovian diffusions, where there exists a fixed underlying reference probability measure \(\mathbb{P}\) that is retrieved from historical data of the price movements. However, it is mostly agreed that it is impossible to precisely identify \(\mathbb{P}\). Hence, as a result, model ambiguity, also called Knightian uncertainty, in utility maximization is inevitably taken into consideration. Namely, the investor is diffident about the odds, and takes a robust approach to the utility maximization problem, where she minimizes over the priors, corresponding to different scenarios, and then maximizes over the investment strategies.

The literature on robust utility maximization in mathematical finance, (see e.g. [3, 7, 8, 12, 16] among others), mostly assumes that the set of priors is dominated by a reference
measure $\mathbb{P}$. Hence, it presumes a setting where volatility of risky assets are perfectly known, but drifts are uncertain. Namely, these approaches assume the equivalence of priors. In particular, they assume the equivalence of probability measures $\mathbb{P}$ with a dominating reference prior $\mathbb{P}$. (see [13, 14, 21, 24, 25, 22, 17, 18, 19, 20, 43, 44] among others.)

A more general direction is the case, where the uncertainty on both mean and volatility is taken into consideration. Here, the set of priors are nondominated, and there exists no dominating reference prior $\mathbb{P}$. This approach started with the seminal works of [39, 40]. Later, [6] studied the case, where uncertainty in the volatility is due to an unobservable factor. [9] studied a similar setting in discrete time and has shown the existence of optimal portfolios; [27, 7] works in a jump-diffusion context, with ambiguity on drift, volatility and jump intensity. [28] establishes a minimax result and the existence of a worst-case measure in a setup where prices have continuous paths and the utility function is bounded. [5] works in a diffusion context, where uncertainty is modelled by allowing drift and volatility to vary in two constant order intervals. The optimization using power utility of the from $U(x) = x^\gamma$ for $0 < \gamma < 1$ is then performed via a robust control (G-Brownian motion) technique, which requires the uncertain volatility matrix is restricted to be of diagonal type. We refer the reader to [23] for a detailed exposure on G-Brownian motion and its applications. In a more general setting [15] works in a continuous time setting, where the stock prices are allowed to be general discontinuous semi-martingales and strategies are required to be compact. Recently, [41] studied the mean variance optimization in in a diffusion setting, where it assumed that the drift of the stock is known with certainty, whereas the volatility is assumed to be in some compact set.

On the other hand, we are studying a utility maximization problem in finite time horizon in a diffusion setting, where there is uncertainty on both drift and volatility residing in a compact set at each time instant $t$. Contrary to the usual stream that the compact set containing the differential characteristics is fixed throughout $[0, T]$, we assume that the set of priors is time dependent. There can be at least two arguments to support this construction. First, in an intraday movement of a stock, it is not reasonable to assume that the stock and drift and volatility uncertainty reside in a fixed compact set. Secondly, as time passes by, the drift and volatility of the stock can be learned (see e.g. [42]) and hence the corresponding compact sets might change, as time proceeds. This more general approach entails additional technical problems. In particular, depending on the confidence set, the optimal value function might not be continuous in time, hence hindering the classical Hamilton-Jacobi-Bellman-Ishii (HJBI) approach, as well as the martingale optimality principle approach (see e.g. Theorem 1.1 [24]). But, we are able to solve the robust optimization problem via Sion’s minmax theorem explicitly in CRRA and CARA utility cases. Finally, we propose an adjustable
risk-attitude level of the investor to our problem based on our finding in the minmax setting.

The rest of the paper is as follows. In Section 2, we describe the model dynamics of the problem and state our general main problem. In Section 3, we solve our optimization problem using both with CRRA and CARA utility. In Section 4, we propose an adjustable level of robustness in volatility. Finally, in Section 5, we discuss our results and conclude the paper.

\section{Model Dynamics and Investor’s Value Function}

\subsection{Framework for Model Uncertainty and Model Dynamics}

We fix the dimension \( n \in \mathbb{N} \) and time horizon \( T \in (0, \infty) \). We let \( \Omega = \mathcal{C}_0([0,T]) \) be the space of continuous paths \( \omega = (\omega_t)_{0 \leq t \leq T} \) starting at \( 0 \in \mathbb{R}^n \) equipped with uniform topology. We define the coordinate functional for \( \omega \in \Omega \) as \( W_t(\omega) := \omega_t \) and take the corresponding Borel \( \sigma \)-algebra by \( F_t := \sigma(W_s(\omega) : 0 \leq s \leq t) \). We denote \( \mathbb{P}_0 \) as the Wiener measure on \( \Omega \) such that \( W_t \) is the \((\Omega, F_t)\) Wiener process and take \( \mathbb{P}_0 \) as the reference measure. We further denote by \( \mathcal{P}(\Omega) \) the Polish space of all probability measures on \( \Omega \) equipped with weak topology of probability measures on \( \Omega \). Similarly, we let \( \mathcal{P}(\Omega|\mathcal{F}_t) \) the Polish space of regular conditional probability measures (r.c.p.m.) on \( \Omega \) (see [34], Thm 5.1.9). We consider a market consisting of \( n \) risky assets \( S_t = (S^1_t, \ldots, S^n_t) \) and one riskless asset \( R_t \). We assume \( S_t \) and \( R_t \) satisfy the following dynamics

\[
\begin{align*}
    dR_t &= rR_t dt \\
    S_0 &= s_0 \\
    dS_t &= \text{Diag}(S_t)(\mu_t dt + \sigma_t dW_t), \quad \mathbb{P}_0 - \text{a.s.}
\end{align*}
\]

Here \( \text{Diag}(S_t) \) is an \( n \times n \) diagonal matrix with \( (S^1_t, \ldots, S^n_t) \) its diagonal entries, \( \mu_t \) is \( \mathbb{R}^n \)-valued process, whereas \( \sigma_t \) is \( n \times n \) lower triangular matrix with positive diagonal entries matrix values. We further denote by \( \Sigma_t = \sigma_t \sigma_t^T \) as the symmetric positive definite matrix.

\textbf{Assumption 2.1.} We assume that \( \|\mu(t,\omega)\| \leq C_t^\mu \) and \( 0 < c_t^\Sigma_{\min} \leq \|\Sigma(t,\omega)\| \leq C_t^\Sigma_{\max} \) for all \( (t,\omega) \in [0,T] \times \Omega \). We denote by \( \Theta_t \subset \mathbb{R}^n \times S^+_n \) the compact set containing the differential characteristics \( \mu(t,\omega) \) and \( \Sigma(t,\omega) \). More precisely, we assume that \( \Theta : [0,T] \to \mathbb{R}^n \times S^+_n \) is a continuous set valued mapping (see [35]), whose image is a compact set of \( \mathbb{R}^n \times S^+_n \) at each time \( t \). We further denote the corresponding set of \( (\mu(t,\omega),\sigma(t,\omega)) \) as \( \Gamma_t \) and \( (\mu_u,\sigma_u)_{t\leq u\leq T} \) as \( \Gamma_{[t,T]} \).
2.2 Alternative Models

At each fixed time $t \in [0, T]$, we consider the class of alternative models denoted by $Q_{[t,T]}$. These are the set of r.c.p.m.’s on $\Omega$ that are induced by the process $S(\cdot)$ as in Equation (2.1). Namely,

$$Q_{[t,T]} = \{ P \in \mathfrak{P}(\Omega|\mathcal{F}_t) | P \text{ is the r.c.p.m. induced by } S \text{ satisfying } dS_u = S_u(\mu_u du + \sigma_u dW_u), \mathbb{P}_0 - \text{a.s.} \text{ for } u \in [t,T] \text{ given } \{S_s(\omega)_{0 \leq s \leq t}\} \text{ and } (\mu_u,\Sigma_u) \in \Gamma_u \text{ for } t \leq u \leq T \}. $$

We assume there exists a strong solution to the Equation (2.1) for any given $\gamma = (\mu_t, \sigma_t)_{0 \leq t \leq T} \in \Gamma_{[0,T]}$ on $(\Omega, \mathcal{F}_T, \mathbb{P}_0)$. Namely, denoting $C_0[0,T] = \Omega$, there exists an $\mathcal{F}_T$ measurable mapping $G : \Omega \rightarrow \Omega$ such that $S^\gamma(\cdot) \equiv G(x_t, W(\cdot))$ solves Equation (2.1) on $(\Omega, \mathcal{F}_T, \mathbb{P}_0)$, as in Definition 10.9 in [26]. We further note that there is a one-to-one correspondence between the set of r.c.p.m.’s $Q_{[t,T]}$ and $(\mu_u, \Sigma_u)_{u \in [t,T]} \in \Gamma_{[t,T]}$. Namely, $\gamma_u = (\mu_u, \sigma_u) \in \Gamma_u$ for $t \leq u \leq T$ uniquely defines $S^\gamma$ on $[t,T]$ $\mathbb{P}_0$-a.s. We denote the r.c.p.m. induced by $(S^\gamma_u)_{t \leq u \leq T}$ for fixed $\gamma_u \in \Gamma_u$ and $A \in \mathcal{F}_T$ as $Q^\gamma_t(A)$ with

$$Q^\gamma_t(A) \doteq \mathbb{P}_0 \left( \omega \in \Omega : \{\log(S^\gamma_u(\omega))_{0 \leq u \leq T}\} \in A ; \text{ given } \{\log(S^\gamma_s(\omega))_{0 \leq s \leq t}\} \right).$$

We further take the closed convex hull of $Q_{[t,T]}$ and denote it by $\bar{Q}_{[t,T]}$. Here, closure is taken with respect to weak convergence of probability measures on $\Omega$ (see Definition 3.1).

2.3 Financial Scenario

We consider the problem of an agent investing in $n$ risky assets $S_i$ and a riskless asset $R_t$. For a given initial endowment $x_0 > 0$, the investor trades in a self financing way. Namely, denoting $\hat{\pi}_t$ as an $\mathcal{F}_t$ adapted $n$-dimensional stochastic process, which stands for the total amount of money invested in the $n$ risky assets $S_i$ at time $t$, $0 \leq t \leq T$, we have, for $X_0 = x_0 > 0$,

$$dX^\hat{\pi}_t = \hat{\pi}_t^S S_t^{-1} \cdot dS_t + (X^\hat{\pi}_t - \hat{\pi}_t^1) \mu dt $$

$$dX^\pi_t = \hat{\pi}_t^\pi (\mu dt + \sigma \cdot dW_t) + (X^\pi_t - \hat{\pi}_t^1) \mu dt. $$

We further represent the amount of money invested in $n$ risky assets as a fraction of current wealth via $\hat{\pi}_t = X^\hat{\pi}_t/\pi_t$ for $0 \leq t \leq T$, where $\pi_t$ stands for the corresponding fraction
at time $t$. Hence, for $X_0 = x_0$, the dynamics of wealth in this setting are given by

$$dX_t^\pi = X_t^\pi \pi_t^T (\mu dt + \sigma dW_t) + rX_t^\pi (1 - \pi_t \cdot 1) dt$$

(2.2)

$$X_t^\pi = x_0 \exp \int_0^t \left\{ \pi_s^T \mu + r(1 - \pi_s \cdot 1) - \frac{1}{2} \pi_u^T \Sigma_u \pi_u \right\} ds + \int_0^t \pi_s \sigma dW_s,$$

where $1$ stands for $n$ dimensional vector $(1, \ldots, 1)$.

**Definition 2.1.** Let $\pi_t$ denote the $\mathcal{B}([0,t]) \otimes \mathcal{F}_t$ progressively measurable process representing the cash-value allocated in the risky asset at time $t$. We call $(\pi_t)_{0 \leq t \leq T}$ admissible and denote it by $\pi \in \Pi_{ad}$, if

$$X_t^\pi > 0 \mathbb{P}_0 - \text{a.s. for all } 0 \leq t \leq T,$$

and $\mathbb{E}_0 \int_0^T [||\pi_t||^4 dt] < \infty$. We denote the set of all admissible $\pi$ as $\Pi_{ad}$. We note that $\Pi_{ad}$ is nonempty, since $\pi \equiv 0$ is in $\Pi_{ad}$. We further say that $\pi_n \to \pi$ in $L^4(\Omega, [t,T])$, if

$$\mathbb{E}_0 \int_t^T ||\pi(u) - \pi_n(u)||^4 dt \to 0, \mathbb{P}_0 \text{ a.s.}$$

### 2.4 Investor’s Problem

Our main problem is as follows. The investor is risk-averse and maximizes her minimal expected utility over her set of alternative models $Q \in \mathcal{Q}_{[t,T]}$. We write the optimization problem of the investor as

$$\sup_{\pi \in \Pi_{ad}} \inf_{(\mu, \sigma) \in \Gamma_{[t,T]}} \{ \mathbb{E}_0^\pi \left[ u(X_T^{\pi,\mu,\sigma}) \right] \},$$

which can be written as

$$\sup_{\pi \in \Pi_{ad}} \inf_{Q \in \mathcal{Q}_{[t,T]}} \{ \mathbb{E}_0^Q \left[ u(X_T) \right] \},$$

(2.3)

where $u(\cdot)$ is the utility function of the investor that is assumed to be concave and in $C^2(0, \infty)$, whereas $\mathbb{E}_0^\pi$ and $\mathbb{E}_0^Q[\cdot]$ stands for the conditional expectation with respect to $\mathbb{P}_0$ and $Q \in \mathcal{Q}_{[t,T]}$, respectively and $X_T^{\mu,\sigma,\pi}$ has the dynamics as in Equation (2.2) with the corresponding $(\mu, \sigma) \in \Gamma$ and $\pi \in \Pi_{ad}$.

### 3 Explicit Solutions with Specific Utility Functions

We will be working with the logarithmic, power and exponential utility functions, that are of the form $\log(x)$, $x^\gamma$ with $0 < \gamma < 1$ and exponential, $-\gamma e^{-\gamma x}$ for $x > 0$ with $0 < \gamma < 1$, respectively and give explicit solutions in our robust setting.
3.1 Logarithmic Utility Case

First, we are going to solve the robust optimization problem in Equation (2.3) with logarithmic utility $U(x) = \log(x)$ and $x > 0$

$$\sup_{\pi \in \Pi_{ad}} \inf_{Q \in \mathcal{Q}_{[t,T]}} E^Q_t[\log(X^\pi_T)].$$

To attack problem (3.4), first we solve the classical expected utility optimization, namely for a fixed $(\mu_u, \sigma_u)_{t \leq u \leq T} \in \Gamma_{[t,T]}$ with $S_t$ having the corresponding differential characteristics, the optimization problem reads as

$$\sup_{\pi \in \Pi_{ad}} E^F_0[\log(X^\pi_T)].$$

By Ito lemma, we have

$$\sup_{\pi \in \Pi_{ad}} E^F_0[\log(X^\pi_T)] = \log(x_t) + \sup_{\pi \in \Pi_{ad}} E^F_0 \left[ \int_t^T (\pi_u^T \mu_u + r(\pi_u \cdot 1))du \right] - \frac{1}{2} \pi_u^T \Sigma_u \pi_u ds$$

Hence, by concavity on $\pi$ inside the integral, we conclude that checking first order condition inside the expectation on $\pi$ is sufficient and get that

$$(\mu_u - r1) - \Sigma \pi_u = 0.$$  

Hence, we have

$$\pi_u^* = \Sigma_u^{-1}(\mu_u - r1)$$

for all $t \leq u \leq T \mathbb{P}_0$ a.s., and the optimal value function reads as

$$E^F_0[\log(X^\pi_T)] = \log(x_t) + \int_t^T \left( r + \frac{1}{2}(\mu_u - r1)^T \Sigma_u^{-1}(\mu_u - r1) du \right).$$

We state next our main result in this section.

**Theorem 3.1.** Let $\pi \in \Pi_{ad}$ be as defined in Definition 2.1 and $X^\pi_t$ have the dynamics as in Equation (2.2). Then, we have

$$\sup_{\pi \in \Pi_{ad}} \inf_{Q \in \mathcal{Q}_{[t,T]}} E^Q_t[\log(X^\pi_T)] = \inf_{Q \in \mathcal{Q}_{[t,T]}} \sup_{\pi \in \Pi_{ad}} E^Q_t[\log(X^\pi_T)]$$
Theorem 3.1 is an application of Sion’s minimax theorem, which we recall here for convenience.

**Theorem 3.2.** [29] Let $X$ be a compact convex subset of a linear topological space and $Y$ a convex subset of a linear topological space. Let $f$ be a real-valued function on $X \times Y$ such that

- $f(x, \cdot)$ is upper semi continuous and quasi-concave on $Y$ for each $x \in X$.
- $f(\cdot, y)$ is lower semi continuous and quasi-convex on $X$ for each $y \in Y$.

Then

$$\min_{x \in X} \sup_{y \in Y} f(x, y) = \sup_{y \in Y} \min_{x \in X} f(x, y).$$

We define a suitable topology to work with the mapping

$$Q \to E^Q[U(X^T_T)] \quad \text{for} \quad Q \in \tilde{Q}_{[t,T]}.$$ 

**Definition 3.1.** Let $\mathcal{S}$ be a family of probability measures on $\Omega$. $\mathcal{S}$ is called relatively compact, if for every sequence $\{Q_n\}$ in $\mathcal{S}$, there exists a subsequence $\{Q_m\}$ of $\{Q_n\}$ and a probability measure on $\Omega$ (not necessarily in $\mathcal{S}$) such that $\{Q_m\}$ converges weakly to $Q$. That is

$$E^{Q_m}[g] \to E^Q[g]$$

for every $g : \Omega \to \mathbb{R}$ with $g \in C_b(\Omega)$, where $C_b(\Omega)$ is the space of continuous bounded functions on $\Omega$. We say in this case that $Q_m$ converges weakly to $Q$, and denote it as $Q_m \rightharpoonup Q$.

**Lemma 3.1.** $\tilde{Q}_{[t,T]}$ is compact with respect to topology defined in Definition 3.1.

**Proof.** We know by construction $\{\log(S_t(\omega))\}_{0 \leq t \leq T} \in C([0,T])$. We have for any $N > 0$

$$N \mathbb{P}_0(\|\log(S_u(\omega))\|_{\sup_{0 \leq u \leq T}} \geq N | \mathcal{F}_t) \leq \log(x_t) + \sup_{(\mu, \sigma) \in \Gamma_{[t,T]}} \mathbb{E}^0_t\left[\int_t^T |\mu_u - \frac{1}{2} \Sigma_u| du + \sup_{t \leq r \leq T} |\int_t^r \sigma_u dW_u|\right]$$

\[
\leq \log(x_t) + C(T - t) + \sup_{\sigma \in \Gamma_{[t,T]}} \mathbb{E}^0_t\left[\sup_{t \leq r \leq T} |\int_t^r \sigma_u dW_u|\right] \\
\leq \log(x_t) + C(T - t) + C \sup_{\sigma \in \Gamma_{[t,T]}} (\mathbb{E}^0_t[(\sup_{t \leq r \leq T} |\int_t^r \sigma_u dW_u|)^2])^{1/2} \\
\leq \log(x_t) + C(T - t) + C \sup_{\sigma \in \Gamma_{[t,T]}} (\mathbb{E}^0_t[\int_t^T \Sigma_u du])^{1/2} \\
\leq \log(x_t) + C(T - t) + (T - t)^{1/2},
\]
where we use the boundedness of $\mu(t, \omega)$ and $\sigma(t, \omega)$ for all $(t, \omega) \in [0, T] \times \Omega$ by Assumption 2.1. Similarly, we have

$$
\mathbb{P}_0 \left( \sup_{t_1 \leq u \leq t_2} \left| \int_{t_1}^{t_2} \mu_u - \frac{1}{2} \Sigma_u du + \int_{t_1}^{t_2} \sigma_u dW_u \right| \right) \leq C(t_2 - t_1) + C(t_2 - t_1)^{1/2}
$$

Hence, choosing $|t_2 - t_1|$ small enough and $N$ large enough, we have that there exists $\tilde{\Omega} \subset \Omega$ that is compact by Arzela-Ascoli theorem such that $Q(\Omega \setminus \tilde{\Omega}) \leq \epsilon$ for all $Q \in \bar{Q}_{[t, T]}$. Hence, by Prokhorov theorem ([34] Theorem 3.9.2) $\bar{Q}_{[t, T]}$ is relatively compact, and since it is closed it is compact with respect to weak convergence. Hence, we conclude the proof.

Lemma 3.2. Let $Q_1, Q_2, Q \in \bar{Q}_{[t, T]}$ and let $g : \Omega \to \mathbb{R}$ be an $\mathcal{F}_T$ measurable mapping with $\mathbb{E}^Q[g] < \infty$. Then, for any $\alpha \in \mathbb{R}$, we have

$$
\mathbb{E}^{\alpha Q_1 + (1-\alpha) Q_2}[g] = \alpha \mathbb{E}^{Q_1}[g] + (1-\alpha) \mathbb{E}^{Q_2}[g].
$$

In particular, the mapping $Q \to \mathbb{E}^Q[\log(X_T^\pi)]$ is quasi-convex for any $Q \in \bar{Q}_{[t, T]}$.

Proof. The result follows by the observation

$$
\mathbb{E}^Q[\log(X_T^\pi)] \leq \max_{(\mu, \sigma) \in \Gamma_{[t, T]}} \mathbb{E}^{P_0}_t[\log(X_T^\pi)],
$$

for any $Q \in \bar{Q}_{[t, T]}$ and via the approximation of Lebesgue integration argument. \hfill \Box

Lemma 3.3. Let $\pi$ be in $\Pi_{ad}$. Let $Q \in \bar{Q}_{[t, T]}$ be fixed. Then, the mapping

$$
\pi \to \mathbb{E}^Q_t[\log(X_T^\pi)]
$$

quasi-concave in $\pi$.

Proof. By Ito lemma, we have

$$
\mathbb{E}^{P_0}_t[\log(X_T^\pi)] = \mathbb{E}^{P_0}_t \left( \int_t^T \pi_u^\pi (\mu - r \cdot 1) - \frac{1}{2} \pi_u^\pi \Sigma_u \pi_u du \right),
$$

from which it is easy to see that $\pi \to \mathbb{E}^{P_0}_t[\log(X_T^\pi)]$ is concave, hence quasi-concave in $\pi$. Hence, we conclude the result. \hfill \Box

Next, we continue with the following lemma.
Lemma 3.4. Let $\pi \in \Pi_{\text{ad}}$ be fixed and $(Q_n)_{n \geq 1} \in \bar{Q}^c_{[t,T]}$. Then, the mapping

$$Q_n \rightarrow \mathbb{E}^Q_0[\log(X^n_T)]$$

is lower continuous. Namely,

$$\liminf_{n \rightarrow \infty} \mathbb{E}^Q_n[\log(X^n_T)] \geq \mathbb{E}^Q_0[\log(X^n_T)],$$

as $Q_n \rightharpoonup Q^*$ for some $Q \in \bar{Q}^c_{[t,T]}$ in the sense of Definition 3.1.

Proof. By truncating our utility function $\log(x)$ as

$$V_k(x) = \begin{cases} k & \text{if } \log(x) \geq k \\ \log(x) & \text{if } |\log(x)| \leq k \\ -k & \text{if } \log(x) \leq -k \end{cases}$$

for $x > 0$ and for $k > 0$, we note that

$$\mathbb{E}^P_0[\log(X^n_T)] + \epsilon(k) \geq \mathbb{E}^P_0[V_k(X^n_T)],$$

uniformly for all $(\mu, \sigma) \in \Gamma_{[t,T]}$ for some $\epsilon(k)$ depending on $k$ only with $\epsilon(k) \downarrow 0$ as $k \rightarrow \infty$. Indeed, we have for any $Q \in \bar{Q}^c_{[t,T]}$

$$\mathbb{E}^Q_0[\log(X^n_T)] \leq \mathbb{E}^P_0[\log(X^n_T)I_{\{ |\log(X^n_T)| > k \}}] < \epsilon,$$

for $k$ large enough. Furthermore, since $\mathbb{E}^Q_0[\log(X^n_T)]$ is integrable with

$$V_k(x) \leq |\log(x)| \text{ for any } k \geq 0$$

(3.5)

$$\mathbb{E}^Q_0[V_k(X^n_T)] \leq \mathbb{E}^Q_0[|\log(X^n_T)|] < \infty \text{ for any } Q \in \bar{Q}^c_{[t,T]}.$$

Hence, we have

$$\liminf_{n \rightarrow \infty} \mathbb{E}^Q_n[\log(X^n_T)] \geq \liminf_{n \rightarrow \infty} \mathbb{E}^Q_n[V_k(X^n_T)] - \epsilon(k)$$

$$= \mathbb{E}^Q_{x_0}[V_k(X^n_T)] - \epsilon(k),$$

where the last equality is due to convergence $Q_n \rightharpoonup Q^*$. Finally, by letting $k \rightarrow \infty$ with $\epsilon(k) \downarrow 0$ and via Lebesgue dominated convergence theorem by Equation (3.5), we have

$$\liminf_{n \rightarrow \infty} \mathbb{E}^Q_n[\log(X^n_T)] \geq \mathbb{E}^Q_{x_0}[\log(X^n_T)].$$

Hence, we conclude the proof. $\square$
Proof of Theorem 3.3. We have that \( \Pi_{ad} \) is a convex set by Definition 2.1. Similarly, \( \bar{Q}_{c}^{[t,T]} \) is a convex topological subset of \( \mathcal{P}(\Omega|\mathcal{F}_t) \). Further \( \mathbb{E}_t^Q[\log(X_T^\pi)] \) is real valued for any fixed \( \pi \in \Pi_{ad} \) and any \( Q \in \bar{Q}_{c}^{[t,T]} \). By Lemma 3.1, 3.2, 3.3 and 3.4, the conditions for Theorem 3.3 are satisfied. Hence, we conclude the result via Theorem 3.1. \( \square \)

3.1.1 The Optimal Solution with Logarithmic Utility

We have the following series of equations

\[
\inf_{Q \in \bar{Q}_{c}^{[t,T]}} \sup_{\pi \in \Pi_{ad}} \mathbb{E}_t^Q[\log(X_T^\pi)] = \inf_{Q \in \bar{Q}_{c}^{[t,T]}} \sup_{\pi \in \Pi_{ad}} \mathbb{E}_t^Q[\log(X_T^\pi)] \\
= \sup_{\pi \in \Pi_{ad}} \inf_{Q \in \bar{Q}_{c}^{[t,T]}} \mathbb{E}_t^Q[\log(X_T^\pi)] \\
\leq \sup_{\pi \in \Pi_{ad}} \inf_{Q \in \bar{Q}_{c}^{[t,T]}} \mathbb{E}_t^Q[\log(X_T^\pi)] \\
\leq \inf_{Q \in \bar{Q}_{c}^{[t,T]}} \sup_{\pi \in \Pi_{ad}} \mathbb{E}_t^Q[\log(X_T^\pi)]
\]

where the first equality is by Lemma 3.2, the second equality is by Theorem 3.1. The first inequality is by \( \bar{Q}_{c}^{[t,T]} \supset Q_{c}^{[t,T]} \) and second inequality is by classical minimax inequality. Thus, we have

\[
\inf_{Q \in \bar{Q}_{c}^{[t,T]}} \sup_{\pi \in \Pi_{ad}} \mathbb{E}_t^Q[\log(X_T^\pi)] \\
= \inf_{(\mu, \sigma) \in \Gamma_{[t,T]}} \left( \log(x_0) + (r(T - t) + \frac{1}{2} \int_t^T (\mu_u - r_1)^\top \Sigma_u^{-1} (\mu_u - r_1) du \right)
\]

By uniform ellipticity on \( \Sigma(t, \omega) \) and the bounded assumption on \( \mu(t, \omega) \) as in Assumption 2.1, we conclude that

\[
\inf_{Q \in \bar{Q}_{c}^{[t,T]}} \sup_{\pi \in \Pi_{ad}} \mathbb{E}_t^Q[\log(X_T^\pi)] \\
= \log(x_t) + r(T - t) + \frac{1}{2} \int_t^T C_{t, \Sigma, \max} \arg \min_{\|\mu_u\| \leq C_{u}^\mu} \|\mu_u - r_1\|^2 du
\]

Denoting

\[
\mu_t^* = \arg \min_{\|\mu_t\| \leq C_t^\mu} \|\mu_t - r_1\|,
\]

the optimal controls at each time \( t \) reads as

\[
\pi_t^* = \frac{1}{C_{t, \Sigma, \max}} (\mu_t^* - r \cdot 1) \\
\Sigma_t^* = C_{t, \max} \cdot I_{n \times n},
\]

where \( C_{t, \max} \) is as in Assumption 2.1.
3.2 Power Utility Case

In this section, we will work with utility function that are of the form

\[ u(x) = x^\gamma \text{ for } 0 < \gamma < 1 \text{ and } x > 0. \]

Hence, our optimization problem reads as

\[
\sup_{\pi \in \Pi_{ad}} \inf_{Q \in \bar{Q}_{[t,T]}} \mathbb{E}_t^{Q} [ (X_T^\pi)^\gamma ] \tag{3.6}
\]

First, we will restrict to deterministic \((\mu, \sigma)\) and deterministic policies \(\pi \in \Pi_{ad}\). Namely, we consider only \((\mu(t), \sigma(t)) \in \Gamma_{[t,T]}\) that do not depend on \(\omega\). Similarly, we take deterministic policies of \(\Pi_{ad}\) into consideration and denote it by \(\Pi_{\text{det}}^{ad}\). Then, we show that allowing adapted processes \((\mu(t, \omega), \sigma(t, \omega)) \in \Gamma_{[t,T]}\) and policies \(\pi(t, \omega) \in \Pi_{ad}\) does not change the optimal value and the optimal policies of the corresponding problem.

In this setting, our robust utility maximization problem reads as

\[
\sup_{\pi \in \Pi_{\text{det}}^{ad}} \inf_{(\mu(t), \sigma(t)) \in \Gamma_{[t,T]}} \mathbb{E}_t^{P_0} [ (X_T^\pi)^\gamma ]. \tag{3.7}
\]

As in log-utility case, via similar steps (see e.g. [31], Chapter 11 ), the classical expected utility maximization problem for a fixed \((\mu_u, \sigma_u)_{t \leq u \leq T} \in \Gamma_{[t,T]}\) is

\[
\sup_{\pi \in \Pi_{\text{det}}^{ad}} \mathbb{E}_t^{P_0} [ (X_T^\pi)^\gamma ],
\]

which has the solution

\[
\sup_{\pi \in \Pi_{ad}} \mathbb{E}_t^{P_0} [ (X_T^\pi)^\gamma ] = x_t^\gamma \exp \left( \gamma \int_t^T \left( r + \frac{1}{2} \frac{1}{1-\gamma} (\mu_u - r \cdot 1)^\top \Sigma_u^{-1} (\mu_u - r \cdot 1) \right) du \right),
\]

with the optimal policy

\[
\pi_u^* = \frac{1}{1 - \gamma} \Sigma_u^{-1} (\mu_u - r \cdot 1), \ t \leq u \leq T.
\]

**Theorem 3.3.** Let \(\pi \in \Pi_{\text{det}}^{ad}\) and \((\mu(u), \sigma(u))_{t \leq u \leq T} \in \Gamma_{[t,T]}\) be as defined in Definition 2.1 and \(X_t^\pi\) have the dynamics as in Equation (2.2). Then, we have

\[
\sup_{\pi \in \Pi_{\text{det}}^{ad}} \inf_{(\mu(t), \sigma(t)) \in \Gamma_{[t,T]}} \mathbb{E}_t^{P_0} [ (X_T^\pi)^\gamma ] = \inf_{(\mu(t), \sigma(t)) \in \Gamma_{[t,T]}} \sup_{\pi \in \Pi_{\text{det}}^{ad}} \mathbb{E}_t^{P_0} [ (X_T^\pi)^\gamma ].
\]

The solution of Problem (3.7) is via Sion’s minimax theorem, and the steps are identical except the quasi-concavity and continuity of the mapping \(\pi \rightarrow \mathbb{E}_t^{P_0} [ (X_T^\pi)^\gamma ]\). Hence, we give this result below.
Lemma 3.5. The mapping

\[ \pi \rightarrow \mathbb{E}^Q[(X_T^\pi)^\gamma] \]

is continuous and quasi-concave for a fixed probability measure \( Q \in \bar{\mathcal{M}}_c \).

Proof. We have

\[ \mathbb{E}^P_0[(X_T^\pi)^\gamma] = x_0^\gamma \mathbb{E}^P_0 \left[ Z_\pi \exp \left( \int_t^T (r + \sum_{i=1}^n (\mu_i - r \pi_u^i) - \frac{1}{2} (1 - \gamma) \pi_u^\top \Sigma u \pi_u) du \right) \right], \]

where

\[ Z_\pi = \exp \left( \int_t^T \gamma \pi_u^\top \sigma dW_u - \frac{1}{2} \gamma^2 \int_t^T \pi_u^\top \Sigma u \pi_u du \right) \]

and note that \( \mathbb{E}^P_0[Z_\pi] = 1 \) with \( Z_\pi \) being a martingale satisfying the Novikov condition

\[ \exp \left( \int_t^T \frac{1}{2} \pi_u^\top \Sigma u \pi_u dt \right) < \infty \]

by \((\mu_t, \Sigma_t)_{0 \leq t \leq T}\) being bounded. We further denote

\[ F(\pi) \triangleq \exp \left( \int_t^T \gamma (r + \pi_u^\top (\mu_u - r \cdot 1) - \frac{1}{2} (1 - \gamma) \pi_u^\top \Sigma u \pi_u) du \right) \quad (3.8) \]

Hence, we have

\[ \mathbb{E}^P_0[Z_\pi F(\pi)] = F(\pi), \]

where we note that \( F(\pi_n) \to F(\pi) \) as \( \pi_n \to \pi \) in \( L^4([0, T]) \) as \( n \to \infty \). Indeed,

\[ \left| \int_0^T \gamma (r + \pi_u^\top (\mu_u - r \cdot 1) - \frac{1}{2} (1 - \gamma) \pi_u^\top \Sigma u \pi_u) ds \right. \]

\[ \left. - \int_0^T \gamma (r + \pi_u^n^\top (\mu_u - r \cdot 1) - \frac{1}{2} (1 - \gamma) \pi_u^n^\top \Sigma u \pi_u) ds \right| \]

\[ \leq C \int_0^T \| \pi_t - \pi_t^n \|^4 dt + C \int_0^T \| \pi_t - \pi_t^n \|^4 dt \]

\[ \leq C \int_0^T \| \pi_t - \pi_t^n \|^4 dt \to 0, \]

where we use norm boundedness assumption on \((\mu, \Sigma)\), generalized Holder inequality in the first inequality and the fact that convergence in \( L^4([0, T]) \) implies convergence in \( L^1([0, T]) \) in the second inequality. Hence, the convergence is preserved as \( x \to e^x \) being a continuous function. We next note that

\[ \int_t^T \gamma (r + \pi_u^\top (\mu_u - r \cdot 1) - \frac{1}{2} (1 - \gamma) \pi_u^\top \Sigma u \pi_u) du \]

is concave and hence quasi-concave. Exponentiating the expression as in Equation (3.8) preserves the quasi-concavity. Hence, we conclude the proof. \( \square \)
3.2.1 The Optimal Solution with Power Utility

The optimal solution in the robust setting with deterministic dynamics reads as

$$\inf_{\mu(t) \in \Gamma_{[t,T]}} \sup_{\pi \in \Pi_{\text{ad}}^\text{det}} E^P_0 [f(T)^\gamma] = \inf_{\|\mu(t)\| \leq C(t)} x^\gamma T \exp \left( \gamma \int_t^T \left( r + \frac{1}{2} \frac{1}{1 - \gamma} \frac{1}{C_u^\max} \|\mu_u - r \cdot 1\|^2 \right) du \right)$$

Denoting

$$\mu^*_t = \arg \min_{\mu(t) \in \Gamma_{[t,T]}} \|\mu(t) - r \cdot 1\|$$

the optimal policy at $t$ reads as

$$\pi^*_t = \frac{1}{1 - \gamma} \frac{1}{C_u^\max} (\mu^*_u - r \cdot 1)$$

**Lemma 3.6.** Considering $\Pi_{\text{ad}}$ instead of $\Pi_{\text{ad}}^\text{det}$ and $(\mu(t, \omega), \sigma(t, \omega)) \in \Gamma$ that also depend on $\omega$ does not change the utility maximization problem (3.6).

**Proof.** We denote by $(\mu(t), \sigma(t))$ below as the deterministic $\mathcal{F}_t$ adapted deterministic processes, whereas $(\mu(t, w), \sigma(t, \omega))$ are $\mathcal{F}_t$ adapted processes depending on $\omega$, as well. We have

$$\inf_{(\mu(t, \omega), \sigma(t, \omega)) \in \Gamma_{[t,T]}} \sup_{\pi \in \Pi_{\text{ad}}^\text{det}} E^P_0 [f(T)^\gamma] = \inf_{(\mu(t, \omega), \sigma(t, \omega)) \in \Gamma_{[t,T]}} \sup_{\pi \in \Pi_{\text{ad}}^\text{det}} E^P_0 [f(T)^\gamma]$$

Hence, we have

$$\inf_{(\mu(t, \omega), \sigma(t, \omega)) \in \Gamma_{[t,T]}} \sup_{\pi \in \Pi_{\text{ad}}^\text{det}} E^P_0 [f(T)^\gamma] = \inf_{(\mu(t), \sigma(t)) \in \Gamma_{[t,T]}} \sup_{\pi \in \Pi_{\text{ad}}^\text{det}} E^P_0 [f(T)^\gamma]$$

By Theorem 3.3, we have

$$\inf_{(\mu(t), \sigma(t)) \in \Gamma_{[t,T]}} \sup_{\pi \in \Pi_{\text{ad}}^\text{det}} E^P_0 [f(T)^\gamma] = \sup_{\pi \in \Pi_{\text{ad}}^\text{det}} \inf_{(\mu(t), \sigma(t)) \in \Gamma_{[t,T]}} E^P_0 [f(T)^\gamma]$$

We further have for

$$\sup_{\pi \in \Pi_{\text{ad}}^\text{det}} \inf_{(\mu(t, \omega), \sigma(t, \omega)) \in \Gamma_{[t,T]}} x^\gamma T \exp \left( \gamma \int_t^T \left( r + \frac{1}{2} \frac{1}{1 - \gamma} \frac{1}{C_u^\max} \|\mu_u - r \cdot 1\|^2 \right) du \right)$$
the following

\[
\arg \min_{(\mu(t,\omega),\sigma(t,\omega)) \in \Gamma} \left( \gamma (r + \pi_u^T (\mu_u - r)) - \frac{1}{2} (1 - \gamma) \pi_u^T \Sigma_u \pi_u \right)
\]

\[
= \arg \min_{(\mu(t,\omega),\sigma(t,\omega)) \in \Gamma} \left( \gamma (r + \pi_u^T (\mu_u - r)) - \frac{1}{2} (1 - \gamma) \pi_u^T \Sigma_u \pi_u \right)
\]

Hence, we have

\[
\sup_{\pi \in \Pi_{ad}} \inf_{(\mu(t,\omega),\sigma(t,\omega)) \in \Gamma_{[t,T]}^u} \mathbb{E}_{t}^P \left[ (X_T^\pi)^\gamma \right] = \sup_{\pi \in \Pi_{ad}} \inf_{(\mu(t,\omega),\sigma(t,\omega)) \in \Gamma_{[t,T]}^u} \mathbb{E}_{t}^P \left[ (X_T^\pi)^\gamma \right]
\]

But, we also have

\[
\sup_{\pi \in \Pi_{ad}} \inf_{(\mu(t,\omega),\sigma(t,\omega)) \in \Gamma_{[t,T]}^u} \mathbb{E}_{t}^P \left[ (X_T^\pi)^\gamma \right] \leq \sup_{\pi \in \Pi_{ad}} \inf_{(\mu(t,\omega),\sigma(t,\omega)) \in \Gamma_{[t,T]}^u} \mathbb{E}_{t}^P \left[ (X_T^\pi)^\gamma \right]
\]

But, by min max inequality, we also have

\[
\sup_{\pi \in \Pi_{ad}} \inf_{(\mu(t,\omega),\sigma(t,\omega)) \in \Gamma_{[t,T]}^u} \mathbb{E}_{t}^P \left[ (X_T^\pi)^\gamma \right] \leq \inf_{(\mu(t,\omega),\sigma(t,\omega)) \in \Gamma_{[t,T]}^u} \sup_{\pi \in \Pi_{ad}} \mathbb{E}_{t}^P \left[ (X_T^\pi)^\gamma \right]
\]

Hence, we have

\[
\sup_{\pi \in \Pi_{ad}} \inf_{(\mu(t,\omega),\sigma(t,\omega)) \in \Gamma_{[t,T]}^u} \mathbb{E}_{t}^P \left[ (X_T^\pi)^\gamma \right] = \inf_{(\mu(t,\omega),\sigma(t,\omega)) \in \Gamma_{[t,T]}^u} \sup_{\pi \in \Pi_{ad}} \mathbb{E}_{t}^P \left[ (X_T^\pi)^\gamma \right]
\]

Hence, we conclude the proof. \[ \square \]

### 3.3 Exponential Utility Case

We next analyze the robust utility optimization problem for the exponential utility, also called Constant Absolute Risk Aversion (CARA) utility case

\[
U(x) = -\gamma e^{-\gamma x},
\]

for \( x > 0 \). For this utility function, we slightly change our problem formulation and notation. The wealth dynamics are

\[
\frac{dX_t}{X_t} = \pi_t^T \text{Diag}(S_t)^{-1} dS_t + (X_t - \pi_t^T 1) r dt.
\] (3.9)

We consider the discounted wealth \( \hat{X}_t \) and discounted amount invested in assets \( \hat{\pi}_t \). Namely,

\[
\hat{\pi}_t = e^{-rt} \pi_t
\]

\[
\hat{X}_t^\pi = e^{-rt} X_t^\pi.
\]
Hence, then by Equation (3.9), we have
\[ d\hat{X}_t^\pi = \hat{\pi}_t^\top (\mu - r \cdot 1)dt + \hat{\pi}_t^\top \sigma dW_t, \]
\[ \hat{X}_t^\pi = x_t + \int_0^T \hat{\pi}_u^\top (\mu - r \cdot 1)du + \int_0^T \hat{\pi}_u^\top \sigma dW_u. \]

We take the optimization problem analogous to previous cases as
\[ \sup_{\hat{\pi} \in \Pi_{\text{ad}}} \inf_{Q \in \mathcal{Q}_{[t,T]}} E^Q_t \left[ -\gamma e^{-\gamma \hat{X}_T^\pi} \right] \quad (3.10) \]

First, we restrict to deterministic \((\mu,\sigma) \in \Gamma_{[t,T]}\) and \(\hat{\pi} \in \Pi_{\text{ad}}\) as in power utility case and solve the deterministic problem.

\[ \sup_{\hat{\pi} \in \Pi_{\text{det}}} \inf_{(\mu(t),\sigma(t)) \in \Gamma_{[t,T]}} E^P_0 \left[ -\gamma e^{-\gamma \hat{X}_T^\pi} \right] \quad (3.11) \]

Next, we state our analogous minimax relation for the exponential utility case for Equation (3.11).

**Theorem 3.4.** Equation (3.11) satisfies the following minimax relation
\[ \sup_{\hat{\pi} \in \Pi_{\text{det}}} \inf_{(\mu(t),\sigma(t)) \in \Gamma_{[t,T]}} E^P_0 \left[ U(\hat{X}_T^\pi) \right] = \inf_{(\mu(t),\sigma(t)) \in \Gamma_{[t,T]}} \sup_{\hat{\pi} \in \Pi_{\text{det}}} E^P_0 \left[ U(\hat{X}_T^\pi) \right] \]

The proof of Theorem 3.4 is, as in power and log utility case, a direct application of Sion’s min max theorem. Other than the continuity and quasiconcavity of the mapping
\[ \hat{\pi} \rightarrow E^P_0 \left[ -e^{-\gamma \hat{X}_T^\pi} \right], \]
verification of its conditions are identical. Hence, we give the proof of that fact below.

**Lemma 3.7.** Let \(\hat{\pi}\) and \(\hat{\pi}_n\) be in \(\hat{\Pi}_{\text{ad}}\) with \(\hat{\pi}_n \rightarrow \hat{\pi}\) in \(L^4([0,T])\) as \(n \rightarrow \infty\). Then, for a fixed \((\mu(t),\sigma(t)) \in \Gamma_{[t,T]}\), the mapping
\[ \hat{\pi} \rightarrow E^P_0 \left[ -e^{-\gamma \hat{X}_T^\pi} \right] \]
is continuous. Namely,
\[ E^P_0 \left[ -e^{-\gamma \hat{X}_T^\pi} \right] \rightarrow E^P_0 \left[ -e^{-\gamma \hat{X}_T^\pi} \right], \]
as \(\hat{\pi}_n \rightarrow \hat{\pi}\) in \(L^4[0,T]\).

**Proof.** The proof follows from the observation
\[ -E^P_0 e^{\gamma t} \left[ e^{(\int_t^T -\gamma \hat{\pi}_u^\top (\mu_u - r \cdot 1)du + \int_t^T \hat{\pi}_u^\top \sigma dW_u)} \right] \]
\[ = -e^{\gamma t} \exp \left( -\gamma \int_t^T \hat{\pi}_u^\top (\mu_u - r \cdot 1) + \frac{1}{2} \gamma^2 \hat{\pi}_u^\top \Sigma \hat{\pi}_u dt \right). \]
We see that
\[ -e^{x_t} \exp \left( \int_t^T -\gamma \hat{\pi}_u^\top (\mu_u - r \cdot 1) + \frac{1}{2} \gamma^2 \hat{\pi}_u^\top \Sigma \hat{\pi}_u du \right) \]
is continuous in \( \pi_n \) as \( \pi_n \to \pi \) in \( L^4([0, T]) \). Indeed,
\[
\left| \int_t^T -\gamma \hat{\pi}_u^\top (\mu_u - r \cdot 1) + \frac{1}{2} \gamma^2 \hat{\pi}_u^\top \Sigma \hat{\pi}_u du - \int_t^T -\gamma \hat{\pi}_u^n (\mu_u - r \cdot 1) + \frac{1}{2} \gamma^2 \hat{\pi}_u^n \Sigma \hat{\pi}_u^n du \right|
\leq C \int_t^T \| \pi_u - \pi_u^n \| du + C \int_t^T \| \pi_u - \pi_u^n \|^4 du \to 0,
\]
where we use norm boundedness assumption on \((\mu, \sigma) \in \Gamma_{[t,T]}\), generalized Holder inequality and convergence in \( L^4[t, T] \) implying convergence in \( L^1[t, T] \) in the last line. Hence, the convergence is preserved as \( x \to -e^x \) being a continuous function.

To see quasi-concavity, we note that the expression
\[
\int_t^T -\gamma \hat{\pi}_u^\top (\mu_u - r \cdot 1) + \frac{1}{2} \gamma^2 \hat{\pi}_u^\top \Sigma \hat{\pi}_u du
\]
is convex in \( \pi \), hence exponentiating (3.12) preserves convexity and negating the expression gives concavity and hence quasi-concavity, hence we conclude the proof.

\[ \square \]

3.3.1 The Optimal Solution with Exponential Utility

Via derivations analogous to the power utility case, for a fixed \((\mu(t), \sigma(t)) \in \Gamma_{[t,T]}\), the optimal discounted cash flow and the optimal value to classical expected utility maximization problem reads as
\[
\hat{V}(x_t) = -\exp (-\gamma x_t) \exp (-\int_t^T (\mu_u - r \cdot 1) \Sigma_u^{-1} (\mu_u - r \cdot 1)) du
\]
\[
\hat{\pi}_t^* = \frac{1}{\gamma \Sigma_t^{-1}} (\mu_t - r 1)
\]

Hence, by appealing to Lemma 3.6 the corresponding controls and the value of the robust optimization problem Equation (3.10) are
\[
\hat{\pi}_t^* = \frac{1}{\gamma C_{t,\text{max}}} (\mu_t - r \cdot 1)
\]
\[
\mu_t^* = \arg \min_{\mu(t) \in \Gamma_t} \| \mu(t) - r \cdot 1 \|
\]
\[
\hat{V}(x_t) = -\exp (-\gamma x_t) \exp (-\int_t^T \frac{1}{C_{t,\text{max}}} \| \mu_u^* - r \cdot 1 \|^2) du
\]
4 An Adjustable Level of Risk Aversion

It is clear that the minimax, i.e., robust approach exhibits extremely strong ambiguity aversion by considering the worst-case scenario by choosing $\mu^*_t = \arg \min_{\mu_t \in \Gamma_t} \|\mu_t - r\|$, and $\Sigma^*_t = C^{\Sigma,\max}_t I_{n \times n}$. However, no empirical experiments have supported such extreme pessimistic ambiguity attitude of investors (see [33], Section 2.33 for further discussion in this issue). In fact, in [36], the authors have shown that investor’s reaction can be ambiguity loving rather than ambiguity averse, if they feel knowledgeable and experienced about the market. Based on that, the concept of $\alpha$-maxmin expected utility has been introduced with the corresponding optimization problem

$$
\rho_t(u(X^\pi_T)) \triangleq \alpha_t \inf_{Q \in \mathcal{Q}} \mathbb{E}^P_0[u(X^\pi_T)] + (1 - \alpha_t) \sup_{Q \in \mathcal{Q}} \mathbb{E}^P_0[u(X^\pi_T)]
$$

(see e.g. [37, 38] among others) Our minmax approach gives the lower and upper bound of $\rho_t(u(X^\pi_T))$ explicitly

$$
\inf_{Q \in \mathcal{Q}} \sup_{\pi \in \Pi_{ad}} \mathbb{E}^P_0[u(X^\pi_T)] = \sup_{\pi \in \Pi_{ad}} \inf_{Q \in \mathcal{Q}} \mathbb{E}^P_0[u(X^\pi_T)] \\
\leq \rho_t(u(X^\pi_T)) \\
\leq \sup_{\pi \in \Pi_{ad}} \sup_{Q \in \mathcal{Q}_{[t,T]}} \mathbb{E}^P_0[u(X^\pi_T)] \\
= \sup_{Q \in \mathcal{Q}_{[t,T]}} \sup_{\pi \in \Pi_{ad}} \mathbb{E}^P_0[u(X^\pi_T)]
$$

Hence, we propose the following dynamics with the corresponding optimal policy

$$
\tilde{\mu}_t = \alpha_t \arg \min_{\mu_t \in \Gamma_t} \|\mu_t - r\| + (1 - \alpha_t) \arg \max_{\mu_t \in \Gamma_t} \|\mu_t - r\| \\
\tilde{\Sigma}_t^{-1} = \alpha_t C^{\Sigma,\min}_t I_{n \times n} + (1 - \alpha_t) C^{\Sigma,\max}_t I_{n \times n} \\
\tilde{\pi}_t = \tilde{\Sigma}_t^{-1}(\mu_t - r)
$$

Here $0 \leq \alpha_t \leq 1$ for all $0 \leq t \leq T$, determining the risk attitude of the investor. We remark that

$$
\inf_{Q \in \mathcal{Q}} \sup_{\pi \in \Pi_{ad}} \mathbb{E}^P_0[u(X^\pi_T)] \leq \mathbb{E}^P_0[u(X^\pi_T)] \leq \sup_{\pi \in \Pi_{ad}} \sup_{Q \in \mathcal{Q}_{[t,T]}} \mathbb{E}^P_0[u(X^\pi_T)]
$$

as $\rho_t(u(X))$ is valued between the most risk seeking $\alpha_t = 0$ and most risk averse $\alpha_t = 1$. 


References


