Robust Utility Maximization with Drift and Volatility Uncertainty

Kerem Uğurlu

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Abstract

We give explicit solutions for utility maximization of terminal wealth problem $u(X_T)$ in the presence of Knightian uncertainty in continuous time $[0, T]$. We assume there is uncertainty on both drift and volatility of the underlying stocks, which induce nonequivalent measures on canonical space of continuous paths $\Omega$. We further take that the uncertainty set resides in compact sets that are time dependent, as well. In this framework, we solve the robust optimization problem with three classic utility functions explicitly, which are logarithmic, power and exponential utilities.

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1 Introduction

Starting with the pioneering works of [1, 2, 29, 4], the underlying risky assets are modelled as Markovian diffusions, where there exists a fixed underlying reference probability measure $\mathbb{P}$ that is retrieved from historical data of the price movements. However, it is mostly agreed that it is impossible to precisely identify $\mathbb{P}$. Hence, as a result, model ambiguity, also called Knightian uncertainty, in utility maximization is inevitably taken into consideration. Namely, the investor is diffident about the odds, and takes a robust approach to the utility maximization problem, where she minimizes over the priors, corresponding to different scenarios, and then maximizes over the investment strategies.

The literature on robust utility maximization in mathematical finance, (see e.g. [3, 7, 8, 10, 14] among others), mostly assumes that the set of priors is dominated by a reference
measure $\mathbb{P}$. Hence, it presumes a setting where volatility of risky assets are perfectly known, but drifts are uncertain. Namely, these approaches assume the equivalence of priors. In particular, they assume the equivalence of probability measures $P$ with a dominating reference prior $\mathbb{P}$. (see [11, 12, 8, 19, 22, 23, 20, 15, 16, 17, 18, 28, 33, 34, 35] among others.)

A more general direction is the case, where the uncertainty on both mean and volatility is taken into consideration. Here, the set of priors are nondominated, and there exists no dominating reference prior $\mathbb{P}$. This approach started with the seminal works of [36, 37] in an option pricing framework. Regarding utility maximization, [6] studied the case, where uncertainty in the volatility is due to an unobservable factor. [9] studied a similar setting in discrete time and has shown the existence of optimal portfolios; [25] works in a jump-diffusion context, with ambiguity on drift, volatility and jump intensity. [26] establishes a minimax result and the existence of a worst-case measure in a setup where prices have continuous paths and the utility function is bounded. [5] works in a diffusion context, where uncertainty is modelled by allowing drift and volatility to vary in two constant order intervals. Here, the optimization using power utility of the form $U(x) = x^\gamma$ for $0 < \gamma < 1$ is performed via a robust control (G-Brownian motion) technique, which requires the uncertain volatility matrix is restricted to be of diagonal type. We refer the reader to [21] for a detailed exposure on G-Brownian motion and its applications. [44] studies the utility maximization problem with power utility, where there is an ellipsoidal uncertainty for drift and volatility uncertainty that reside in a fixed compact set. [13] works in a continuous time setting, where the stock prices are allowed to be general discontinuous semi-martingales and strategies are required to be compact and studies power utility and give semi-explicit solutions, and the uncertainty is specified by a set of possible Levy triplets; that is, possible instantaneous drift, volatility and jump characteristics of the price process. [38] studied the mean variance optimization in a diffusion setting, where it is assumed that the drift of the stock is known with certainty, whereas the volatility is assumed to be in some compact set. Recently, [43] shows the existence of optimal strategy in the robust exponential utility maximization problem in discrete time.

On the other hand, we are studying a utility maximization problem in finite continuous time horizon in a diffusion setting, where there is time-dependent uncertainty on both drift and volatility residing in a compact set. Contrary to the usual stream that the compact set containing the differential characteristics is fixed throughout $[0, T]$, we assume that the set of priors is time dependent. There can be at least two arguments to support this construction. First, in an intraday movement of a stock, it is not reasonable to assume that the stock and drift and volatility uncertainty reside in a fixed compact set. Second, as time passes by, the drift and volatility of the stock can be learned (see e.g. [39]) and hence the corresponding
compact sets might change, as time proceeds. This more general approach entails additional technical problems. In particular, depending on the confidence set, the optimal value function might not be continuous in time, hence hindering the classical Hamilton-Jacobi-Bellman-Ishii (HJBI) approach, as well as the martingale optimality principle approach (see e.g. Theorem 1.1 [30]). The rest of the paper is as follows. In Section 2, we describe the model dynamics of the problem and state our general main problem and propose the solution methodology. In Section 3, we solve our utility maximization problem using logarithmic, power and exponential utility functions. In Section 4, we discuss our results and conclude the paper.

2 Model Dynamics and Investor’s Value Function

2.1 Framework for Model Uncertainty and Model Dynamics

We fix the dimension \( n \in \mathbb{N} \) and time horizon \( T \in (0, \infty) \). We let \( \Omega = C_0([0, T]) \) be the space of continuous paths \( \omega = (\omega_t)_{0 \leq t \leq T} \) starting at \( 0 \in \mathbb{R}^n \). We define the coordinate functional for \( \omega \in \Omega \) as \( W_t(\omega) := \omega_t \) and take the corresponding Borel \( \sigma \)-algebra by \( F_t := \sigma(W_s(\omega) : 0 \leq s \leq t) \). We denote \( \mathbb{P}_0 \) as the Wiener measure on \( \Omega \) such that \( W_t \) is the \( (\Omega, F_t) \) Wiener process and take \( \mathbb{P}_0 \) as the reference measure. We consider a market consisting of \( n \) risky assets \( S_t = (S^1_t, \ldots, S^n_t) \) and one riskless asset \( R_t \). We assume \( S_t \) and \( R_t \) satisfy the following dynamics

\[
\begin{align*}
dR_t &= rR_t dt \\
S_0 &= s_0 \\
dS_t &= \text{Diag}(S_t)(\mu_t dt + \sigma_t dW_t), \quad \mathbb{P}_0-\text{a.s.}
\end{align*}
\]

Here \( \text{Diag}(S_t) \) is an \( n \times n \) diagonal matrix with \( (S^1_t, \ldots, S^n_t) \) its diagonal entries. We take that \( \mu_t \) is progressively measurable \( \mathbb{R}^n \) valued mapping, whereas \( \sigma_t \) is \( n \times n \) matrix valued and progressively measurable. We further denote by \( \Sigma_t = \sigma_t \sigma_t^T \) that is assumed to be symmetric and positive definite.

**Assumption 2.1.** We assume that \( \|\mu_t\| \leq C^\mu_t \) and \( 0 < c^\Sigma_t,\min \leq \|\Sigma_t\| \leq C^\Sigma_t,\max \) for \( t_i \leq t < t_{i+1} \) and \( 0 = t_0 < t_1, \ldots, t_n < t_{n+1} = T \) \( \mathbb{P}_0 \)-a.s. We denote \( \Theta_{[t_i, t_{i+1}]} \subset \mathbb{R}^n \times \mathbb{S}_+^n \) the compact set containing the differential characteristics \( \theta_t \triangleq (\mu_t, \sigma_t) \) for \( t_i \leq t < t_{i+1} \). We further assume that there exists a strong solution to the Equation (2.1) for any given \( (\theta_t)_{0 \leq t \leq T} \) on \( (\Omega, F_T, \mathbb{P}_0) \). Namely, denoting \( C_0[0, T] = \Omega \) and \( S^\theta \) being as in Equation (2.1), we take that there exists an \( F_T \)-measurable mapping \( G : \Omega \to \Omega \) such that \( S^\theta(\cdot) \equiv G(x_{t_i}, W(\cdot)) \) solves Equation (2.1) on \( (\Omega, F_T, \mathbb{P}_0) \), as in Definition 10.9 in [24].
2.2 Alternative Models

Note that for different \((\theta_t)_{0 \leq t \leq T} \in (\Theta_t)_{0 \leq t \leq T}\), different probability measures are induced on \(\Omega\), which is defined as

\[
Q^\theta \triangleq \mathbb{P}_0 \circ (\log(S^\theta))^{-1},
\]

(2.2)

where \(S^\theta\) has the differential characteristics as in Equation (2.1) with \((\theta_t)_{0 \leq t \leq T} = (\mu_t, \sigma_t)_{0 \leq t \leq T}\). Further, different \(\sigma_1, \sigma_2\) induce nonequivalent probability measures. Indeed, for \(\theta_1 = (\mu_1, \sigma_1)\) and \(\theta_2 = (\mu_2, \sigma_2)\), where \((\mu_i, \sigma_i)_{i=1,2}\) are constants in \(\mathbb{R}^n\) and \(\mathbb{S}_n^+\), respectively

\[
Q^{\theta_1}(\log(S^{\theta_1})) = \sigma_1 \sigma_1^\top = 1
\]

\[
Q^{\theta_2}(\log(S^{\theta_2})) = \sigma_2 \sigma_2^\top = 1.
\]

Here \(\langle \cdot \rangle\) stands for the quadratic variation of \(\log(S^\theta)\). However, the dynamics of differential characteristics are given with respect to \(\mathbb{P}_0\), in particular, \textit{we look through the lenses of the Wiener measure} \(\mathbb{P}_0\). This is possible, since we consider only strong solutions in Equation (2.1).

2.3 Financial Scenario

We consider the problem of an agent investing in \(n\) risky assets \(S_t\) and a riskless asset \(R_t\). For a given initial endowment \(x_0 > 0\), the investor trades in a self financing way. We denote \(\pi_t\) as an \(n\)-dimensional progressively measurable stochastic process, which stands for the total amount of money invested in the \(n\) risky assets \(S_t\) at time \(t\), \(0 \leq t \leq T\). Then, we have for \(X_0 = x_0 > 0\)

\[
d\hat{X}_t^\pi = \hat{\pi}_t^\top S_t^{-1} \cdot dS_t + (\hat{X}_t^\pi - \hat{\pi}_t^\top 1) r dt
\]

\[
d\hat{X}_t^\pi = \hat{\pi}_t^\top \mu_t dt + \sigma_t \cdot dW_t + (\hat{X}_t^\pi - \hat{\pi}_t^\top 1) r dt \mathbb{P}_0-\text{a.s.}
\]

We further represent the amount of money invested in \(n\) risky assets as a fraction of current wealth via \(\hat{\pi}_t = \hat{X}_t^\pi \pi_t\) for \(0 \leq t \leq T\), where \(\pi_t\) stands for the corresponding fraction at time \(t\) and take the discounted wealth \(X_t^\pi = e^{-rt} \hat{X}_t^\pi \pi_t\). Hence, for \(X_0 = x_0\), the dynamics of wealth in this setting are given by

\[
dX_t^\pi = X_t^\pi \pi_t^\top ((\mu_t - r 1) dt + \sigma_t dW_t)
\]

\[
X_t^\pi = x_0 \exp \int_0^t \{\pi_u^\top (\mu_u - r 1) - \frac{1}{2} \pi_u^\top \Sigma_u \pi_u\} du + \int_0^t \pi_u^\top \sigma_u dW_u \mathbb{P}_0-\text{a.s.},
\]

(2.3)

where \(1\) stands for \(n\) dimensional vector \((1, \ldots, 1)\). We further denote \(X_{t_n}^{\pi, \theta_n}\) as the wealth process at time \(t_n\) with the dynamics \((\theta_t)_{0 \leq t \leq T} = (\mu_t, \sigma_t)_{0 \leq t \leq T}\) as in Equation (2.3) and \(\pi \in \Pi_{ad}\). Here \(\Pi_{ad}\) stands for the admissible portfolios that are defined below.
Definition 2.1. Let \( \{\pi_u\}_{0 \leq u \leq T} \) denote the \( \mathcal{B}([0,T]) \otimes \mathcal{F}_T \) progressively measurable process representing the cash-value allocated in \( n \) risky assets. We call \( (\pi_u)_{0 \leq u \leq T} \) admissible and denote it by \( \pi \in \Pi_{ad} \), if it satisfies
\[
X^\pi_t \geq 0, \ 0 \leq t \leq T, \ \mathbb{P}_0\text{-a.s.}
\]
Similarly, we denote by \( \Pi_{ad}^{[i,i+1)} \) the admissible cash value, if it satisfies
\[
X^\pi_t \geq 0 \ \forall \ i \leq t \leq i+1.
\]

2.4 Investor’s Problem

The investor utilizes the classical Merton problem, but she is also diffident about the underlying dynamics of the stocks both in terms of drift \( \mu_t \) and covariance matrix \( \Sigma_t \). She assumes that \( \theta_t \equiv (\mu_t, \sigma_t) \) is in some compact set \( (\Theta_t)_{0 \leq t \leq T} \) standing for the priors on the underlying dynamics. The investor reevaluates its priors \( (\Theta_t)_{0 \leq t \leq T} \) on some prespecified times \( 0 = t_0 < t_1 < \ldots < t_n < T \). At time \( t_n \), we write the optimization problem of the investor for \( X^\pi_{t_n} = x_{t_n} \) as
\[
V(t_n, x_{t_n}) \equiv \sup_{\pi \in \Pi_{ad}^{[t_n,t_{n+1})}} \inf_{\theta \in \Theta^{[t_n,t_{n+1})}} \mathbb{E}^P_0[t_n \circ u(X^\pi_{T})],
\]
where \( \mathbb{E}^P_0[\cdot] \equiv \mathbb{E}^P[\cdot | \mathcal{F}_{t_n}] \). Hence, at time \( t_0 \) the optimization problem reads backwardly as
\[
V(t_0, x_0) \equiv \sup_{\pi \in \Pi_{ad}^{[t_0,t_1)}} \inf_{\theta \in \Theta^{[t_0,t_1)}} \mathbb{E}^P_0 \left[ \sup_{\pi' \in \Pi_{ad}^{[t_1,t_2)}} \inf_{\theta' \in \Theta^{[t_1,t_2)}} \mathbb{E}^P_{t_1} \left[ \sup_{\pi'' \in \Pi_{ad}^{[t_2,t_3)}} \inf_{\theta'' \in \Theta^{[t_2,t_3)}} \mathbb{E}^P_{t_2} \left[ \ldots \sup_{\pi^n \in \Pi_{ad}^{[t_{n-1},t_n)}} \inf_{\theta^n \in \Theta^{[t_{n-1},t_n)}} \mathbb{E}^P_{t_{n-1}} \left[ u(X^\pi_{T}) \right] \ldots \right] \right] \right] \right] \right] \right].
\]

We continue with the following variant of so called Martingale Optimality Principle (See also Theorem 1.1 of [30]).

Theorem 2.1. Suppose that

1. there exists a function \( v : [t_n, T] \times \mathbb{R}^+ \rightarrow \mathbb{R} \), that is \( C^{1,2} \) on \( (t_n, T) \times \mathbb{R}^+ \) with \( v(T, \cdot) = u(\cdot) \),
2. for any \( \pi \in \Pi_{ad}^{[t_n,t_{n+1})} \), there exists an optimal solution \( \theta \in \Theta^{[t_n,t_{n+1})} \) of
\[
\inf_{\theta \in \Theta^{[t_n,t_{n+1})}} \mathbb{E}^P_{t_n}[u(X^\pi_{T})]
\]
such that

\[ Y^\pi_{tn} \triangleq v(t_n, X^\pi_{tn}) \]  \hfill (2.4)

satisfies for \( t_n \leq s \leq t \leq T \)

\[ \mathbb{E}_s[Y^\pi_t] \leq Y^\pi_s, \mathbb{P}_0-\text{a.s.} \]

3. there exists some \( \bar{\pi} \in \Pi_{[t_n,t_{n+1}]}^{ad} \) such that for \( t_n \leq s \leq t \leq T \)

\[ \mathbb{E}_s[Y^\pi_t] = Y^\pi_s, \mathbb{P}_0-\text{a.s.} \]

Then, \( \bar{\pi} \in \Pi_{ad} \) is optimal for the problem

\[
V(t_n, x_{tn}) = \sup_{\pi \in \Pi_{[t_n,t_{n+1}]}^{ad}} \inf_{\theta[t_n,t_{n+1}] \in \Theta(t_n,t_{n+1})} \mathbb{E}^{\mathbb{P}_0}_{t_n}[u(X^\pi_{T})] \leq Y^\pi_{tn}, \mathbb{P}_0-\text{a.s.}
\]

\[ Y^\pi_{tn} = v(t_n, x) \]

Proof. By Assumption 2. and by \( V(T, \cdot) = u(\cdot) \), we have

\[ \mathbb{E}^{\mathbb{P}_0}_{tn}[Y^\pi_{T}] = \mathbb{E}^{\mathbb{P}_0}_{tn}[u(X^\pi_{T})] \leq Y^\pi_{tn}, \mathbb{P}_0-\text{a.s.} \]

Taking supremum over \( \Pi_{[t_n,t_{n+1}]}^{ad} \), we get

\[ V(t_n, x_{tn}) = \sup_{\pi \in \Pi_{[t_n,t_{n+1}]}^{ad}} \mathbb{E}^{\mathbb{P}_0}_{tn}[u(X^\pi_{T})] \leq v(t_n, x_{tn}). \]

By assumption, for some \( \bar{\pi} \in \Pi_{[t_n,t_{n+1}]}^{ad} \), we have \( \mathbb{E}_s[Y^\pi_t] = \bar{Y}^\pi_s, \mathbb{P}_0-\text{a.s.} \) for \( t_n \leq s \leq t \leq T \). Then,

\[ V(t_n, x_{tn}) = \mathbb{E}^{\mathbb{P}_0}_{tn}[ar{Y}^\pi_t] = \bar{Y}^\pi_{tn} = v(t_n, x_{tn}). \]

Hence, we conclude the proof. \( \square \)

Applying Ito lemma for \( t \geq t_n \) to \( Y^\pi \) as in (2.4), we have

\[
dY^\pi_t = \left( v_t + v_x X^\pi_t \left( \pi_t^\top (\mu_t - r 1) \right) + (X^\pi_t \Sigma_t \pi_t)^2 \frac{1}{2} v_x \Sigma_t \pi_t \right) dt + X^\pi_t v_x \pi_t \sigma dW_t, \ \mathbb{P}_0-\text{a.s.}
\]
By Theorem 2.1, we need to have
\[
\sup_{\pi \in \Pi^\text{ad}_{[t_n, t_{n+1}]} \theta_{[t_n, t_{n+1}]}} \inf_{\Theta_{[t_n, t_{n+1}]}} \left[ v_t + v_x (\pi_t^T (\mu_t - r 1)) + \frac{1}{2} v_{xx} \pi_t^T \Sigma_t \pi_t \right]
\]
(2.6)
\[
= v_t + \sup_{\pi \in \Pi^\text{ad}_{[t_n, t_{n+1}]}} \inf_{\theta \in \Theta_{[t_n, t_{n+1}]}} \left[ \pi_t^T (\mu_t - r 1) + \frac{1}{2} v_{xx} \pi_t^T \Sigma_t \pi_t \right]
\]
\[
= 0
\]

**Lemma 2.1.** The value function \( V(t_n, x_{t_n}) \) as defined in Equation (2.5) is increasing and concave in \( x_{t_n} \).

**Proof.** Recall that by assumption, the utility function \( u(\cdot) \) is increasing and concave and
\[
X_{T}^{\pi, \theta} = x_{t_n} \exp \left[ \int_{t_n}^{T} (\pi_u^T (\mu_u - r 1) - \frac{1}{2} \pi_u^T \Sigma_u \pi_u) du + \int_{t_n}^{T} \pi_u^T \sigma_u dW_u \right], \mathbb{P}_0 - \text{a.s.}
\]
In particular, for any \( x_{t_n}^1 \leq x_{t_n}^2 \) with fixed \( \Pi^\text{ad}_{[t_n, t_{n+1}]} \) and fixed \( \theta \in \Theta_{[t_n, t_{n+1}]} \), by monotonicity of \( u(\cdot) \), we have
\[
\mathbb{E}^\mathbb{P}_0[u(X_{T}^{\pi, \theta})] \leq \mathbb{E}^\mathbb{P}_0[u(X_{T}^{\pi_1, \theta_1})].
\]
Since this holds for any \( \Pi^\text{ad}_{[t_n, t_{n+1}]} \) and \( \theta \in \Theta_{[t_n, t_{n+1}]} \), taking first infimum over \( \theta \in \Theta_{[t_n, t_{n+1}]} \) for fixed \( \Pi^\text{ad}_{[t_n, t_{n+1}]} \) and then supremum over \( \Pi^\text{ad}_{[t_n, t_{n+1}]} \), we have
\[
V(t_n, x_{t_n}^1) \leq V(t_n, x_{t_n}^2).
\]
Next, we show concavity of \( V(t_n, x) \). Let \( 0 < \alpha < 1 \) with
\[
\alpha x_{t_n}^1 + (1 - \alpha) x_{t_n}^2 = x_{t_n}^3.
\]
Then, we have
\[
V(t_n, x_{t_n}^3) \geq \sup_{\pi_1 \in \Pi^\text{ad}_{[t_n, t_{n+1}]}, \pi_2 \in \Pi^\text{ad}_{[t_n, t_{n+1}]}} \inf_{\theta \in \Theta_{[t_n, t_{n+1}]}} \mathbb{E}^\mathbb{P}_0[u(\alpha X_{T}^{\pi_1, \theta_1} + (1 - \alpha) X_{T}^{\pi_2, \theta_2})]
\]
Since \( u \) is concave by assumption, we have
\[
V(t_n, x_{t_n}^3) \geq \sup_{\pi_1 \in \Pi^\text{ad}_{[t_n, t_{n+1}]}, \pi_2 \in \Pi^\text{ad}_{[t_n, t_{n+1}]}} \left\{ \alpha \inf_{\theta \in \Theta_{t_n}} \mathbb{E}^\mathbb{P}_0[u(X_{T}^{\pi_1, \theta})] + (1 - \alpha) \inf_{\theta \in \Theta_{t_n}} \mathbb{E}^\mathbb{P}_0[u(X_{T}^{\pi_2, \theta})] \right\}.
\]
Since the last expression is sum of two suprema, we conclude that
\[ V(t_n, x_{t_n}^3) \geq \alpha V(t_n, x_{t_n}^1) + (1 - \alpha) V(t_n, x_{t_n}^2). \]
Hence, we conclude the proof. \(\square\)

### 3 Explicit Solutions with Specific Utility Functions

We will be working with the logarithmic, power and exponential utility functions. These are of the form \(\log(x)\), \(x^\gamma\) for \(0 < \gamma < 1\), \(-\beta e^{-\beta x}\) with \(\beta > 0\) for \(x > 0\), respectively, and give explicit solutions in our robust setting. First, we give the following lemma.

**Lemma 3.1.** Let \(v(t_n, x_{t_n})\) on \([t_n, T] \times \mathbb{R}^+\) be a strictly increasing and strictly concave \(C^{1,2}\) function on \(x\) such that \(\frac{v_x}{v_{xx}} = c\) for some positive \(c\). Then, for \(t_n \leq t \leq T\), the supremum and infimum in the HJBI equation in (2.6) are attained for

\[
\begin{align*}
\mu_t^* &= \arg\min_{\mu \in \Theta(t_n, t_{n+1})} (\|\mu - r\|) \quad (3.7) \\
\Sigma_t^* &= \arg\max_{\Sigma \in \Theta(t_n, t_{n+1})} (\|\pi^\top \Sigma \pi\|) = C_{t_n} * I_{n \times n},
\end{align*}
\]

where \(I_{n \times n}\) stands for the \(n\)-dimensional identity matrix.

**Proof.**

\[
\sup_{\pi \in \Pi_{ad}[t_n, t_{n+1}]} \inf_{\theta(t_n, t_{n+1}) \in \Theta(t_n, t_{n+1})} \left[ v_t + v_x (x \pi^\top t (\mu_t - r) - v_{xx}) \right] = v_t + \sup_{\pi \in \Pi_{ad}[t_n, t_{n+1}]} \inf_{\theta(t_n, t_{n+1}) \in \Theta(t_n, t_{n+1})} \left[ x v_x \pi^\top t (\mu_t - r) + 2 x^2 \frac{1}{2} \pi^\top t \Sigma_t \pi v_{xx} \right]
\]

Since \(v\) is \(C^{1,2}\) and strictly increasing and strictly concave, the result follows by inner minimization for any fixed \(\pi \in \Pi_{ad}\). Since \(\Sigma_t\) is positive definite and \(v\) is concave, to minimize \(\frac{1}{2} \pi^\top u \Sigma_u \pi_u\), we need to choose \(\Sigma_t^* = C_{t_n} * I_{n \times n}\) for any fixed \(\pi\).

Next, to find the optimal \(\mu_t^*\) for any fixed \(\pi \in \Pi_{ad}\), we proceed as follows. We choose specifically \(\tilde{\pi}_t\) as

\[
\tilde{\pi}_t = \frac{1}{C_{t_n}} \frac{v_x (\mu_t^* - r) - v_{xx}}{-v_{xx}}, \quad \text{for } t_n \leq t < t_{n+1},
\]

where \(\mu_t^*\) is as in Equation (3.14). Note that \(\tilde{\pi}_t\) is constant, deterministic and is an element of \(\Pi_{ad}[t_n, t_{n+1}]\). Since \(v\) is increasing with \(v_x > 0\), for that \(\tilde{\pi}_t\) to minimize the expression

\[
v_x \tilde{\pi}_t^\top (\mu_t - r) \quad \text{for } t_n \leq t < t_{n+1},
\]

...
over $\mu_t$, we must choose
\[ \mu_t^* = \arg\min_{\mu \in \Theta(t_n, t_{n+1})} \|\mu - r1\|, \text{ for } t_n \leq t < t_{n+1}. \]

Furthermore, we have
\[ \sup_{\Pi_{ad}(t_n, t_{n+1})} \inf_{\theta(t_n, t_{n+1})} \left[ v_x \pi_t^T (\mu_t - r1) + \frac{1}{2} v_x^2 \pi_t^T \Sigma_t \pi_t v_x \right] \leq \inf_{\theta(t_n, t_{n+1})} \sup_{\pi \in \Pi_{ad}} \left[ v_x \pi_t^T (\mu_t - r1) + \frac{1}{2} v_x^2 \pi_t^T \Sigma_t \pi_t v_x \right]. \] (3.8)

Next, for the right hand side of the inequality above, for a fixed $\theta(t_n, t_{n+1})$ with $v_x < 0$ and $\Sigma_t$ being positive definite, we must have
\[ \arg\max_{\pi \in \Pi_{ad}} \left[ v_x \pi_t^T (\mu_t - r1) + \frac{1}{2} v_x^2 \pi_t^T \Sigma_t \pi_t v_x \right] = (\Sigma_t)^{-1} v_x (\mu_t - r1)^{-1} \]
and plugging that $\pi$ to the right hand side of the inequality (3.8), we have for $t_n \leq t < t_{n+1}$
\[ \arg\min_{(\mu, \sigma) \in \Theta(t_n, t_{n+1})} \left( \frac{1}{2} (\Sigma_t)^{-1} v_x \|\mu_t - r1\|^2 \right) = \left( \arg\min_{\mu \in \Theta(t_n, t_{n+1})} \frac{\|\mu_t - r1\|}{v_x^2} \right), C_{t_n} * I_{n \times n}. \]

But these are the values that we have plugged in and found for the left hand side of Equation (3.8). Hence, again by inequality (3.8), we conclude that the HJBI equation are attained for the values as in (3.14). Hence, we conclude the proof. \[ \square \]

Based on Lemma 3.1, our solution methodology is, as follows. We assume first that $V(t_n, x_{t_n})$ is $C^{1,2}$. Then, by Lemma 3.1, we plug in the corresponding $\theta(t_n)$ parameters and solve the classical Merton problem. Next, we verify that the resulting value function $V(t_n, x_{t_n})$ is indeed $C^{1,2}$. Hence, we will have solved the problem for $[t_n, T]$. Then, we will solve the problem for $(t_{n-1}, t_n]$ and we proceed backwards up to $[0, t_1]$ via the same methodology. We emphasize here that the resulting value function $V : \mathbb{R}^+ \times [0, T] \to \mathbb{R}$ is not necessarily $C^{1,2}$ or even continuous, but it is a concatenation of $C^{1,2}$ functions on $[0, t_1], [t_1, t_2], \ldots, [t_n, T]$.

### 3.1 Logarithmic Utility Case

First, we are going to solve the robust optimization problem with logarithmic utility $\log(x_{t_n})$ and $x_{t_n} > 0$.
\[ V(t_n, x_{t_n}) = \sup_{\pi \in \Pi_{ad}(t_n, t_{n+1})} \inf_{\theta(t_n, t_{n+1}) \in \Theta(t_n, t_{n+1})} \mathbb{E}_{t_n}^\Theta \left[ \log(X_T^\pi, \theta) \right]. \]
We assume that $V(t_n, x_{t_n})$ is $C^{1,2}$ and by Lemma 3.1, we let

$$\mu_{t_n}^* = \arg \min_{\mu \in \Theta_{t_n, t_{n+1}}} \| \mu_t - r \|$$

$$\Sigma_t^* = C_t \ast I_{n \times n}$$

and let

$$dX_t^\pi = X_t^\pi \pi_t^T (((\mu_{t_n}^* - r 1)dt + \sigma_{t_n}^* dW_t), \ for \ t_n < t \leq T.$$ 

The optimization problem reads as

$$\sup_{\pi \in \Pi_{ad}} E_{t_n}^{\mathbb{P}_0} [ \log(X_T^\pi, \theta) ].$$

By Ito lemma, we have

$$\sup_{\pi \in \Pi_{ad}} E_{t_n}^{\mathbb{P}_0} [ \log(X_T^\pi, \theta) ]$$

$$= \log(x_{t_n}) + \sup_{\pi \in \Pi_{ad}} E_{t_n}^{\mathbb{P}_0} \left[ \int_{t_n}^T (\pi_u^T (\mu_u - r \cdot 1)$$

$$\left. - \frac{1}{2} \pi_u^T \Sigma_u \pi_u \right) du \right].$$

Hence, by concavity on $\pi$ inside the integral, we conclude that checking first order condition inside the expectation on $\pi$ is sufficient and get that

$$(\mu_{t_n}^* - r 1) - \Sigma^* \pi_t = 0,$$

Thus,

$$\pi_t^* = (\Sigma_{t_n}^*)^{-1} (\mu_{t_n}^* - r 1)$$

$$= \frac{1}{C_{t_n}} (\mu_{t_n}^* - r 1)$$

for $t_n \leq t < T$, and the optimal value function reads as

$$V(t_n, x_{t_n}) = E_{t_n}^{\mathbb{P}_0} [ \log(X_T^\pi) ]$$

$$= \log(x_{t_n}) + \frac{1}{2C_{t_n}} \| \mu_{t_n}^* - r \cdot 1 \|^2 (T - t_n).$$ (3.9)
Hence, we verify that $V(t_n, x_{t_n})$ is indeed $C^{1,2}$ and the corresponding optimal $\Pi_{[t_n, t_{n+1}]}^{ad}$ and $\theta \in \Theta_{[t_n, t_{n+1}]}$ for $t_n \leq t < T$ are

$$
\mu^*_t = \arg \min_{\mu_t \in \Theta_{[t_n, t_{n+1}]}^n} (||\mu_t - r 1||)
$$

$$
\Sigma^*_t = \arg \max_{\Sigma_t \in \Theta_{[t_n, t_{n+1}]}^n} (||\pi^*_t \Sigma_t \pi_t||) = C_{t_n} \ast I_{n \times n}
$$

$$
\pi^*_t = \frac{1}{C_{t_n}} (\mu^*_t - r 1)
$$

Next, we go one time step backwards and examine the following optimization problem

$$
V(t_{n-1}, x_{t_{n-1}}) = \sup_{\Pi_{[t_{n-1}, t_{n+1}]}^{ad}} \inf_{\theta \in \Theta_{[t_{n-1}, t_{n+1}]}^n} \mathbb{E}^{F}_{t_{n-1}}[u(X_T)]
$$

$$
= \log(x_{t_{n-1}}) + \sup_{\Pi_{[t_{n-1}, t_{n+1}]}^{ad}} \inf_{\theta \in \Theta_{[t_{n-1}, t_{n+1}]}^n} \mathbb{E}^{F}_{t_{n-1}}\left[\int_{t_{n-1}}^{T} (\pi^*_t \mu_t - r \cdot 1) dt\right] - \frac{1}{2} \pi^*_t \Sigma_t \pi_t dt
$$

$$
= \log(x_{t_{n-1}}) + \sup_{\Pi_{[t_{n-1}, t_{n+1}]}^{ad}} \inf_{\theta \in \Theta_{[t_{n-1}, t_{n+1}]}^n} \mathbb{E}^{F}_{t_{n-1}}\left[\int_{t_{n-1}}^{t_n} (\pi^*_t \mu_t - r \cdot 1) dt\right] - \frac{1}{2} \pi^*_t \Sigma_t \pi_t dt + \sup_{\pi \in \Pi_{[t_{n-1}, t_{n+1}]}^{ad}} \inf_{\theta \in \Theta_{[t_{n-1}, t_{n+1}]}^n} \mathbb{E}^{F}_{t_{n}}\left[\int_{t_n}^{T} (\pi^*_t \mu_t - r \cdot 1) dt\right] - \frac{1}{2} \pi^*_t \Sigma_t \pi_t dt
$$

By Equation (3.9) for $[t_n, T]$, we have

$$
V(t_{n-1}, x_{t_{n-1}}) = \sup_{\Pi_{[t_{n-1}, t_{n+1}]}^{ad}} \inf_{\theta \in \Theta_{[t_{n-1}, t_{n+1}]}^n} \mathbb{E}^{F}_{t_{n-1}}[u(X_T)]
$$

$$
= \log(x_{t_{n-1}}) + \inf_{\pi \in \Pi_{[t_{n-1}, t_{n+1}]}^{ad}} \mathbb{E}^{F}_{t_{n-1}}\left[\int_{t_{n-1}}^{t_n} (\pi^*_t \mu_t - r \cdot 1) dt\right] - \frac{1}{2} \pi^*_t \Sigma_t \pi_t dt + \left(\frac{1}{2} ||\mu^*_t - r \cdot 1||^2\right) (T - t_n)
$$

$$
= \left(\frac{1}{2} ||\mu^*_t - r \cdot 1||^2\right) (T - t_n) + \sup_{\pi \in \Pi_{[t_{n-1}, t_{n+1}]}^{ad}} \inf_{\theta \in \Theta_{[t_{n-1}, t_{n+1}]}^n} \mathbb{E}^{F}_{t_{n-1}}\left[\int_{t_{n-1}}^{t_n} (\pi^*_t \mu_t - r \cdot 1) dt\right] - \frac{1}{2} \pi^*_t \Sigma_t \pi_t dt
$$

Going backwards, on the interval $[t_{n-1}, t_n]$, we apply Lemma 3.1 and Theorem 2.1

$$
\sup_{\pi \in \Pi_{[t_{n-1}, t_{n+1}]}^{ad}} \inf_{\theta \in \Theta_{[t_{n-1}, t_{n+1}]}^n} \mathbb{E}^{F}_{t_{n-1}}\left[\int_{t_{n-1}}^{t_n} (\pi^*_t \mu_t - r \cdot 1) dt - \frac{1}{2} \pi^*_t \Sigma_t \pi_t dt\right].
$$
We find that for $t_{n-1} \leq t < t_n$

$$
\mu_t^* = \arg \min_{\mu_t \in \Theta_{t_{n-1}}} (||\mu_t - r1||)
$$

$$
\Sigma_t^* = \arg \max_{\Sigma_t \in \Theta_{t_{n-1}}} (||\pi_t^T \Sigma_t \pi_t||) = C_{t_{n-1}} \cdot I_{n \times n}
$$

$$
\pi_t^* = \frac{1}{C_{t_{n-1}}} (\mu_{t_{n-1}}^* - r1)
$$

Thus,

$$
V(t_{n-1}, x_{t_{n-1}}) = \log(x_{t_{n-1}}) + \frac{1}{2}(\mu_{t_{n-1}}^* - r1)^T (\Sigma_{t_{n-1}}^*)^{-1} (\mu_{t_{n-1}}^* - r1) (t_{n} - t_{n-1})
$$

$$
+ \frac{1}{2}(\mu_{t_{n}}^* - r1)^T (\Sigma_{t_{n}}^*)^{-1} (\mu_{t_{n}}^* - r1) (T - t_{n})
$$

Going backwards at $t_0 = 0$, we have

$$
V(t_0, x_0) = \log(x_0) + \sum_{t_i} \frac{1}{2C_{t_i}} ||\mu_{t_i}^* - r1||^2 (t_{i+1} - t_i)
$$

and the corresponding optimal parameters for $t \in [t_i, t_{i+1})$ and the admissible policy $(\pi_t)_{t_i \leq t < t_{i+1}}$ are

$$
\mu_t^* = \arg \min_{\mu_t \in \Theta_{[t_i, t_{i+1})}} (||\mu_t - r1||)
$$

$$
\Sigma_t^* = \arg \max_{\Sigma_t \in \Theta_{[t_i, t_{i+1})}} (||\pi_t^T \Sigma_t \pi_t||) = C_{t_i} \cdot I_{n \times n}
$$

$$
\pi_t^* = \frac{1}{C_{t_i}} (\mu_{t_i}^* - r1)
$$

### 3.2 Power Utility Case

We proceed to solve the robust optimization problem in power utility case. As in logarithmic utility function, following the above recipe, we assume that $V(t_n, x)$ is $C^{1,2}$ and pick the corresponding $\theta_{t_n}^* \in \Theta_{t_n}$. We let for $t_n \leq t < T$

$$
\mu_t^* = \arg \min_{\mu_t \in \Theta_{[t_n, T)}} ||\mu_t - r1||
$$

$$
\Sigma_t^* = C_{t_n} \cdot I_{n \times n}.
$$

and solve the classical nonrobust problem

$$
V(t_n, x_{t_n}) = \sup_{\pi \in \Pi_{[t_n, T)}} \mathbb{E}_{t_n}^\pi [(\hat{X}_T^\pi, \hat{\theta}_T^\pi) \gamma],
$$

(3.10)
for $0 < \gamma < 1$. The equation for (3.10) on $[t_n, T)$ retrieved from Lemma 3.1 reads as

$$
V_t + \sup_{\pi} \left \{ x\pi^T(\mu - r1)V_x + x^2 \frac{1}{2C_{t_n}} \pi^T \pi V_{xx} \right \} = 0 \quad (3.11)
$$

$$
V(T, x) = x^\gamma
$$

We make the Ansatz to the Equation (3.11) on $[t_n, T)$ for $V(t_n, x_t)$ along with the optimal policy

$$
V(t_n, x_t) = x_{t_n}^\gamma \exp \left( \frac{\gamma ||\mu_{t_n} - r1||^2}{1 - \gamma} (T - t_n) \right) \quad (3.12)
$$

for $t_n \leq t < T$, which is $C^{1,2}$ on $[t_n, T)$ and satisfies the condition in Lemma 3.1 and fulfills Equation (3.12). Hence, going backwards in time with $t_n \to T$ and $t_{n-1} \to t_n$ and making the corresponding Ansatz for Equation (3.11) on $[t_{n-1}, t_n)$ and iterating up to $t_0 = 0$, we conclude that

$$
V(t_0, x_0) = x_0^\gamma \mathbb{E}_0 \left[ \exp \left( \sum_{t_i} \frac{\gamma ||\mu_{t_i}^* - r1||^2}{2(1 - \gamma)C_{t_i}} (t_{i+1} - t_i) \right) \right]
$$

for $t_i \leq t < t_{i+1}$ with $i = 0, \ldots, n - 1$

$$
\mu_{t_i}^* = \arg\min_{\mu \in \Theta_{t_i}} (||\mu - r1||)
$$

$$
\Sigma_{t_i}^* = \arg\max_{\Sigma_{t_i} \in \Theta_{t_i}} (||\pi_t^T \Sigma_t \pi_t||) = C_{t_i} * I_{n \times n}
$$

$$
\pi_t^* = \frac{\mu_{t_i} - r1}{C_{t_i}(1 - \gamma)}.
$$

### 3.3 Exponential Utility Case

We next analyze the robust utility optimization problem for the exponential utility, also called Constant Absolute Risk Aversion (CARA) utility case

$$
u(x) = -x \beta e^{-\beta x}
$$

for $x > 0$ and $\beta > 0$. We take $\hat{\pi}_t = X_t^{\pi, \theta^*}$ in Equation (2.3) such that

$$
d\hat{X}_T^{\pi, \theta} = \hat{\pi}_t^T (\mu_t - r1) dt + \hat{\pi}_t^T \sigma_t dW_t, \mathbb{P}_0-a.s. \quad (3.13)
$$

We take the optimization problem analogous to previous two subsections as

$$
V(t_n, x_{t_n}) = \sup_{\pi \in \Pi_{ad}} \inf_{\theta \in \Theta_{t_n, t_{n+1}}} \mathbb{E}_{t_n}^{P_0} \left[ -\beta e^{-\beta \hat{X}_T^{\pi, \theta}} \right]
$$
Based on Equation (3.13), the HJBI reads as

\[ v_t + \sup_{\pi \in \Pi_{\left(t_n, t_{n+1}\right)}} \inf_{\theta \in \Theta_{\left(t_n, t_{n+1}\right)}} \left[ v_x \pi_t^\top (\mu_t - r \mathbf{1}) + \frac{1}{2} v_{xx} \pi_t^\top \Sigma_t \pi_t \right] = 0 \]

Lemma 3.2. Let \( v(t_n, x_t) \) on \([t_n, T] \times \mathbb{R}^+\) be a strictly increasing and strictly concave \( C^{1,2} \) function on \( x \) such that \( \frac{\partial \pi}{\partial x} = c \) for some positive \( c \). Then, for \( t_n \leq t < t_{n+1} \), the supremum and infimum in the HJBI equation in (2.6) are attained for

\[ \mu^*_t = \arg \min_{\mu \in \Theta_{\left(t_n, t_{n+1}\right)}} (\| \mu_t - r \mathbf{1} \|) \]

\[ \Sigma^*_t = \arg \max_{\Sigma \in \Theta_{\left(t_n, t_{n+1}\right)}} (\| \pi_t^\top \Sigma_t \pi_t \|) = C_t \ast I_{n \times n}, \]

where \( I_{n \times n} \) stands for the \( n \)-dimensional identity matrix.

Proof. The proof is a simple modification of Lemma 3.1.

Based on Lemma 3.1 and Theorem 2.1, we proceed to solve

\[ V(t_n, x_t) = \sup_{\hat{\pi} \in \Pi_{\left[t_n, t_{n+1}\right]}} \mathbb{E}^{\mathcal{F}_t}_{t_n} \left[ -\beta e^{-\beta \hat{X}_T^{\pi, \theta}} \right] \]

\[ = -\beta \inf_{\hat{\pi} \in \Pi_{\left[t_n, t_{n+1}\right]}} \mathbb{E}^{\mathcal{F}_t}_{t_n} \left[ e^{-\beta \hat{X}_T^{\pi, \theta}} \right], \]

with \( \hat{X}_T^{\pi, \theta} \) having the dynamics Equation (3.13) for \( t_n \leq t < T \)

\[ \mu^*_t = \arg \min_{\mu \in \Theta_{\left(t_n, t_{n+1}\right)}} (\| \mu_t - r \mathbf{1} \|) \]

\[ \Sigma^*_t = \arg \max_{\Sigma \in \Theta_{\left(t_n, t_{n+1}\right)}} (\| \pi_t^\top \Sigma_t \pi_t \|) = C_t \ast I_{n \times n}. \]

As in the previous two cases, we find \( V(t_n, x_t) \) and verify that it is in \( C^{1,2} \) as follows. Let

\[ V(t_n, x_t) = -\beta e^{-\beta x_{t_n}} \inf_{\hat{\pi} \in \Pi_{\left[t_n, t_{n+1}\right]}} \mathbb{E}^{\mathcal{F}_t}_{t_n} \left[ \exp \left( \int_{t_n}^{T} -\beta \hat{\pi}_u^\top (\mu^*_{t_n} - r \cdot \mathbf{1}) du + \int_{0}^{T} \hat{\pi}_u^\top \Sigma^*_{t_n} \pi_u dW_u \right) \right] \]

\[ = -\beta e^{-\beta x_{t_n}} \inf_{\hat{\pi} \in \Pi_{\left[t_n, t_{n+1}\right]}} \mathbb{E}^{\mathcal{F}_t}_{t_n} \left[ \exp \left( \int_{t_n}^{T} -\beta \hat{\pi}_u^\top (\mu^*_{t_n} - r \cdot \mathbf{1}) + \frac{1}{2} \beta^2 \hat{\pi}_u^\top \Sigma^*_{t_n} \pi_u du \right) \right] \]

We note that

\[ -\beta \hat{\pi}_u^\top (\mu^*_{t_n} - r \cdot \mathbf{1}) + \frac{1}{2} \beta^2 \hat{\pi}_u^\top \Sigma^*_{t_n} \pi_u \]
is convex in \( \hat{\pi} \). Hence, by pointwise minimisation, we get that for \( t_n \leq t < T \)

\[
\hat{\pi}_t^* = \frac{1}{C_{t_n} \beta}(\mu^*_{t_n} - r1)
\]

\[
V(t_n, x_{t_n}) = -\beta e^{-\beta x_{t_n}} \exp \left( -\frac{\|\mu^*_{t_n} - r \cdot 1\|^2}{2} (T - t_n) \right)
\]

We see that \( V(t_n, x_{t_n}) \) is \( C^{1,2} \). Hence, the verificaton is complete. Going backwards by repeating the above verification procedure for \( [t_n-1, t_n), [t_n-2, t_n-1), \ldots [0, t_1) \), we conclude that the optimal parameters for \( t \in [t_i, t_{i+1}) \) are

\[
\hat{\pi}_t^* = \frac{1}{C_{t_i} \beta}(\mu^*_t - r1)
\]

\[
\mu^*_t = \arg \min_{\mu^*_t \in \Theta_{t_i, t_{i+1}}} (\|\mu^*_t - r \cdot 1\|)
\]

\[
\Sigma^*_t = \arg \max_{\Sigma^*_t \in \Theta_{t_i, t_{i+1}}} (\|\pi^*_t \Sigma^*_t \pi^*_t\|) = C_{t_i} * I_{n \times n}
\]

\[
V(t_0, x_0) = -\beta e^{-\beta x_0} \exp \left( -\sum_{t_i} \frac{\|\mu^*_t - r \cdot 1\|^2}{2C_{t_i}} (t_{i+1} - t_i) \right)
\]

### 4 Concluding Remarks

We see that the robust approach in three classical utility functions necessitates to choose the volatility of the largest magnitude with \( \Sigma^*_t = C_{t_i} * I_{n \times n} \) for \( t_i \leq t < t_{i+1} \), whereas the drift term is to be chosen closest to the risk free interest rate with \( \mu^*_t = \arg \min_{\mu^*_t \in \Theta_{t_i, t_{i+1}}} (\|\mu^*_t - r \cdot 1\|) \) for \( t_i \leq t < t_{i+1} \). The optimal portion to be invested in risky assets decreases proportional to the uncertainty of \( \Sigma^*_t \). A limiting argument of the uncertainty sets is also immediate by our framework. In particular, given that the uncertainty interval changes at each time \( t \in [0, T] \) rather than at prespecified times \( 0 = t_0 < t_1 < \ldots < t_n < T \), letting \( t_{i+1} - t_i \to 0 \), we have the optimal parameters along with the value function in exponential utility case for \( t_i \leq t < t_{i+1} \)

\[
\mu^*_t = \arg \min_{\mu^*_t \in \Theta_{t_i}} (\|\mu^*_t - r \cdot 1\|)
\]

\[
\hat{\pi}_t^* = \frac{1}{C_{t_i} \beta}(\mu^*_t - r1)
\]

\[
\Sigma^*_t = \arg \max_{\Sigma^*_t \in \Theta_{t_i}} (\|\pi^*_t \Sigma^*_t \pi^*_t\|) = C_{t_i} * I_{n \times n}
\]

\[
V(t_0, x_0) = -\beta e^{-\beta x_0} \exp \left( -\int_0^T \frac{\|\mu^*_t - r \cdot 1\|^2}{2C_{t_i}} dt \right)
\]
The power and utility cases have the analogous optimal parameters and optimal values, accordingly.

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