COMPLEXITY ANALYSIS OF SECOND-ORDER LINE-SEARCH ALGORITHMS FOR SMOOTH NONCONVEX OPTIMIZATION

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Abstract. There has been much recent interest in finding unconstrained local minima of smooth functions, due in part of the prevalence of such problems in machine learning and robust statistics. A particular focus is algorithms with good complexity guarantees. Second-order Newton-type methods that make use of regularization and trust regions have been analyzed from such a perspective. More recent proposals, based chiefly on first-order methodology, have also been shown to enjoy optimal iteration complexity rates, while providing additional guarantees on computational cost.

In this paper, we present an algorithm with favorable complexity properties that differs in two significant ways from other recently proposed methods. First, it is based on line searches only: Each step involves computation of a search direction, followed by a backtracking line search along that direction. Second, its analysis is rather straightforward, relying for the most part on the standard technique for demonstrating sufficient decrease in the objective from backtracking. In the latter part of the paper, we consider inexact computation of the search directions, using iterative methods in linear algebra: the conjugate gradient and Lanczos methods. We derive modified convergence and complexity results for these more practical methods.

Key words. smooth nonconvex unconstrained optimization, line-search methods, second-order methods, second-order necessary conditions, iteration complexity.

AMS subject classifications. 49M05, 49M15, 90C06, 90C60.

1. Introduction. We consider the unconstrained optimization problem

\begin{equation}
\min f(x),
\end{equation}

where \( f : \mathbb{R}^n \to \mathbb{R} \) is a twice Lipschitz continuously differentiable function that is generally nonconvex. Some algorithms for this problem seek points that nearly satisfy the second-order necessary conditions for optimality, which are that \( \nabla f(x^*) = 0 \) and \( \nabla^2 f(x^*) \succeq 0 \). These iterative schemes terminate at an iterate \( x_k \) for which

\begin{equation}
\|\nabla f(x_k)\| \leq \epsilon_g \quad \text{and} \quad \lambda_{\text{min}}(\nabla^2 f(x_k)) \geq -\epsilon_H,
\end{equation}

where \( \epsilon_g, \epsilon_H \in (0, 1) \) are (typically small) prescribed tolerances. Numerous algorithms have been proposed in recent years for finding points that satisfy (2), each with a complexity guarantee, which is an upper bound on an index \( k \) that satisfies (2), in terms of \( \epsilon_g, \epsilon_H, \) and other quantities. We summarize below the main results.

Classical second-order convergent trust-region schemes [11] can be shown to satisfy (2) after at most \( \mathcal{O} \left( \max \{\epsilon_g^{-2}, \epsilon_H^{-3}\} \right) \) iterations [10]. Cubic regularization methods in their basic form [7] have better complexity bounds than trust-region schemes, requiring at most \( \mathcal{O} \left( \max \{\epsilon_g^{-2}, \epsilon_H^{-3}\} \right) \) iterations. The difference can be explained by the restriction enforced by the trust-region constraint on the norm of the...
steps. Recent work has shown that it is possible to improve the bound for trust-region algorithms using specific definitions of the trust-region radius [14]. The best known iteration bound for a second-order algorithm (that is, an algorithm relying on the use of second-order derivatives and Newton-type steps) is $O\left(\max\left\{\epsilon^{-3/2},\epsilon^{-3}\right\}\right)$. This bound was established originally (under the form of a global convergence rate) in [18], who considered cubic regularization of Newton’s method. The same result is achieved by the adaptive cubic regularization framework under suitable assumptions on the computed step [10]. Recent proposals have shown that the same bound can be attained by algorithms other than cubic regularization. A modified trust-region method [12], a variable-norm trust-region scheme [17], and a quadratic regularization algorithm with cubic descent condition [2] all achieve the same bound.

When $\epsilon_g = \epsilon_H = \epsilon$ for some $\epsilon \in (0, 1)$, all the bounds mentioned above reduce to $O(\epsilon^{-3})$. It has been established that this order is sharp for the class of second-order methods [10], and it can be proved for a wide range of algorithms that make use of second-order derivative information; see [13]. Setting $\epsilon_H = \epsilon^{1/2}$ and $\epsilon_g = \epsilon$ for some $\epsilon > 0$ yields bounds varying between $O(\epsilon^{-3})$ and $O(\epsilon^{-3/2})$, the latter being again optimal within the class of second-order algorithms [9].

A new trend in complexity analyses has emerged recently, that focuses on measuring not just the number of iterations to achieve (2) but also the computational cost of the iterations. Two independent proposals, respectively based on adapting accelerated gradient to the nonconvex setting [5] and approximately solving the cubic subproblem [1], require $O(\log(\frac{1}{\epsilon})\epsilon^{-7/4})$ operations (with high probability) to find a point $x_k$ that satisfies

$$\|\nabla f(x_k)\| \leq \epsilon \quad \text{and} \quad \lambda_{\min}(\nabla^2 f(x_k)) \geq -\sqrt{L_g}\epsilon,$$

with $L_g$ being a Lipschitz constant of the gradient. (These complexity bounds show only dependence on $\epsilon$.) The difference factor of $\epsilon^{-1/4}$ by comparison with the complexities of the previous paragraph is due to the cost of computing a negative eigenvalue of $\nabla^2 f(x_k)$ and/or the cost of solving the linear system. A later proposal [4] focuses on solving cubic subproblems via gradient descent, together with an inexact eigenvalue computation: It satisfies (3) in at most $O(\log(\frac{1}{\epsilon})\epsilon^{-2})$ with high probability. Another technique [15] requires only gradient computations, with noise being added to some iterates. It reaches with high probability a point satisfying (3) in at most $O(\log^4(\frac{1}{\epsilon})\epsilon^{-2})$ iterations. Up to the logarithmic factor, this bound is characteristic of gradient-type methods, but classical work establishes only first-order guarantees [6].

Although this setting is not explicitly addressed in the cited papers, it appears that to reach an iterate satisfying (2) with $\epsilon_g = \epsilon_H = \epsilon$, the methods studied in [1, 5] would require $O\left(\log^{3/2}(\frac{1}{\epsilon})\epsilon^{-7/4}\right)$ iterations, while the methods described in [4] and [15] could require $O\left(\log^3(\frac{1}{\epsilon})\epsilon^{-3}\right)$ and $O\left(\log^4(\frac{1}{\epsilon})\epsilon^{-3}\right)$ iterations, respectively. Although these bounds look worse than those of classical nonlinear optimization schemes, they are more informative, in that they not only account for the number of outer iterations of the algorithm, but also for the number of inner iterations of auxiliary procedures. We note, however, that unlike the classical complexity results, the newer procedures make use of randomization, so the bounds typically hold only with high probability.

Our goal in this paper is to describe an algorithm that achieves optimal complexity, whether measured by the number of iterations required to satisfy the condition (2) or by an estimate of the number of fundamental operations required (gradient evaluations or Hessian-vector multiplications). Each iteration of our algorithm takes the form of a step calculation followed by a backtracking line search. (To our knowledge,
ours is the first line-search algorithm that is endowed with a second-order complexity analysis.) The “reference” version of our algorithm is presented in Section 2, along with its complexity analysis. In this version, we assume that two key operations — solution of the linear equations to obtain Newton-like steps and calculation of the most negative eigenvalue of a Hessian — are performed exactly. In Section 3, we refine our study by introducing inexactness into these operations, and adjusting the complexity bounds appropriately. Finally, we discuss the established results and their practical connections in Section 4.

2. A Line-Search Algorithm Based on Exact Step Computations. We now describe an algorithm based on exact computation of search directions, in particular, the Newton-like search directions and the eigenvector that corresponds to the most negative eigenvalue of the Hessian.

2.1. Outline. We use a standard line-search framework [19, Chapter 3]. Starting from an initial iterate \( x_0 \), we apply an iterative scheme of the form \( x_{k+1} = x_k + \alpha_k d_k \), where \( d_k \) is a chosen search direction and \( \alpha_k \) is a step length computed by a backtracking line-search procedure.

Algorithm 1 defines our method. Each iteration begins by evaluating the gradient, together with the curvature of the function along the gradient direction. This information determines whether the negative gradient direction is a suitable choice for search direction \( d_k \). If not, we compute the minimum eigenvalue of the Hessian. The corresponding eigenvector is used as the search direction whenever the eigenvalue is sufficiently negative. Otherwise, we compute a Newton-like search direction, adding a regularization term if needed to ensure sufficient positive definiteness of the coefficient matrix. There are a total of five possible choices for the search direction \( d_k \) (including two different scalings of the negative gradient). Table 1 summarizes the various steps that can be performed and the conditions under which those steps are chosen.

<table>
<thead>
<tr>
<th>Context</th>
<th>Direction</th>
<th>Decrease</th>
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<tbody>
<tr>
<td>( |g_k| = 0 )</td>
<td>-</td>
<td>( \lambda_k &lt; -\epsilon_H )</td>
</tr>
<tr>
<td>( |g_k| &gt; \epsilon_g )</td>
<td>( R_k &lt; -\epsilon_H ) ( R_k \in [-\epsilon_H, \epsilon_H] )</td>
<td>( \lambda_k &lt; -\epsilon_H ) ( \lambda_k \in [-\epsilon_H, \epsilon_H] ) ( \lambda_k &gt; \epsilon_H )</td>
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<tr>
<td>( |g_k| &gt; \epsilon_g )</td>
<td>( R_k &gt; \epsilon_H ) ( R_k &lt; -\epsilon_H ) ( R_k &lt; -\epsilon_H ) ( R_k \in [-\epsilon_H, \epsilon_H] )</td>
<td>( \lambda_k &lt; -\epsilon_H ) ( \lambda_k \in [-\epsilon_H, \epsilon_H] ) ( \lambda_k &gt; \epsilon_H )</td>
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Table 1: Steps and associated decrease lemmas for Algorithm 1.

Once a search direction has been selected, a backtracking line search is applied with an initial choice of 1. A sufficient condition related to the cube of the step norm must be satisfied; see (7). Such a condition has been instrumental in the complexity analysis of recently proposed Newton-type methods achieving the best known iteration complexity rates [2, 12].

At most one eigenvector computation and one linear system solve are needed per iteration of Algorithm 1, along with a gradient evaluation and the Hessian-vector multiplication required to calculate \( R_k \).
Algorithm 1 Second-Order Line Search Method

Init. Choose $x^0 \in \mathbb{R}^n$, $\theta \in (0,1)$, $\eta > 0$, $\epsilon_g \in (0,1)$, $\epsilon_H \in (0,1)$; for $k = 0, 1, 2, \ldots$ do

Step 1. (First-Order) Set $g_k = \nabla f(x_k)$;
if $\|g_k\| = 0$ then
  Go to Step 2;
end if
Compute $R_k = \frac{\nabla^2 f(x_k) g_k}{\|g_k\|^2}$;
if $R_k < -\epsilon_H$ then
  Set $d_k = \frac{R_k}{\|g_k\|^2} g_k$ and go to Step LS;
else if $R_k \in [-\epsilon_H, \epsilon_H]$ and $\|g_k\| > \epsilon_g$ then
  Set $d_k = -\frac{g_k}{\|g_k\|^2}$ and go to Step LS;
else
  Go to Step 2;
end if

Step 2. (Second-Order) Compute an eigenpair $(v_k, \lambda_k) \in \mathbb{R}^n \times \mathbb{R}$ where

$$\nabla^2 f(x_k) v_k = \lambda_k v_k, \quad v_k^T g_k \leq 0, \quad \|v_k\| = [-\lambda_k]_+;$$

if $\|g_k\| \leq \epsilon_g$ and $\lambda_k \geq -\epsilon_H$ then
  Terminate (or go to Local Phase);
else if $\lambda_k < -\epsilon_H$ then
  (Negative Curvature) Set $d_k = v_k$;
else if $\lambda_k > \epsilon_H$ then
  (Newton) Set $d_k = d^n_k$, where

$$\nabla^2 f(x_k) d^n_k = -g_k;$$
else
  (Regularized Newton) Set $d_k = d^r_k$, where

$$\left(\nabla^2 f(x_k) + 2\epsilon_H I\right) d^r_k = -g_k;$$
end if
Go to Step LS;

Step LS. (Line Search) Compute a step length $\alpha_k = \theta^j_k$, where $j_k$ is the smallest nonnegative integer such that

$$f(x_k + \alpha d_k) < f(x_k) - \frac{\eta}{6} \alpha^3 \|d_k\|^3$$

holds, and set $x_{k+1} = x_k + \alpha_k d_k$.
if $d_k = d^n_k$ or $d_k = d^r_k$ and $\|\nabla f(x_{k+1})\| \leq \epsilon_g$ then
  Terminate (or go to Local Phase);
end if
end for

The algorithm contains two tests for termination, with the option of switching to a “Local Phase” instead of terminating at a point that satisfies approximate
second-order conditions. The Local Phase aims for rapid local convergence to a point satisfying second-order necessary conditions for a local solution; it is detailed in Algorithm 2. Termination (or switch to the Local Phase) occurs at an iteration \(k\) at which an \((\epsilon_g, \epsilon_H)\)-approximate second-order critical point is reached, according to the following definition:

\[
\min \{\|g_k\|, \|g_{k+1}\|\} \leq \epsilon_g, \quad \text{and} \quad \lambda_{\text{min}}(\nabla^2 f(x_k)) \geq -\epsilon_H,
\]

where \(g_k = \nabla f(x_k)\), etc. As we see below, the quantity \(\min \{\|g_k\|, \|g_{k+1}\|\}\) arises naturally in the decrease formula we establish for the steps computed by Algorithm 1. In fact, for the methods we reviewed in introduction, one observes that the decreased formulas obtained for their steps either involve only \(\|g_k\|\) \([1, 4, 5, 15, 18]\), only \(\|g_{k+1}\|\) \([2, 12, 17]\), or the minimum of the two quantities \([8]\). The later case appears due to the presence of both gradient-type (see Lemma 2) and Newton-type steps (see Lemmas 3 and 4).

Algorithm 2 Local Phase

\[
\text{loop} \quad \text{Set } g_k = \nabla f(x_k);
\quad \text{if } \|g_k\| > \epsilon_g \text{ then }
\quad \quad \text{Return to Algorithm 1;}
\quad \text{end if}
\quad \text{Compute } \lambda_k \text{ and } v_k \text{ as in (4);}
\quad \text{if } \lambda_k < -\epsilon_H \text{ then }
\quad \quad \text{Return to Algorithm 1;}
\quad \text{else if } \lambda_k \in (-\epsilon_H, 0] \text{ then }
\quad \quad \text{Set } d_k = d_k^r \text{ from (6);}
\quad \text{else}
\quad \quad \text{Set } d_k = d_k^c \text{ from (5);}
\quad \text{end if}
\quad \text{Perform backtracking line search as in Step LS of Algorithm 1 to obtain } x_{k+1};
\quad k \leftarrow k + 1;
\text{end loop}
\]

The main convergence results of this section are complexity results on the number of iterations or function evaluations required to satisfy condition (8) for the first time. (Algorithm 2 makes provision for re-entering the main algorithm, if the approximate second-order conditions are violated at any point. This feature of re-entry is not covered by our complexity analysis.)

2.2. Iteration Complexity. We now establish a complexity bound for Algorithm 1, in the form of the maximum number of iterations that may occur before the Termination conditions are satisfied for the first time. To this end, we provide guarantees on the decrease that can be obtained for each of the possible choices of search direction. Being able to relate the function decrease to the optimality criterion or the tolerance of interest is a feature common to all methods with complexity guarantees.

In the rest of this paper, we make the following assumptions.

Assumption 1. The level set \(\mathcal{L}_f(x_0) = \{x|f(x) \leq f(x_0)\}\) is a compact set.
Assumption 2. The function $f$ is twice Lipschitz continuously differentiable on an open neighborhood of $L_f(x_0)$, and we denote by $L_g$ and $L_H$ the respective Lipschitz constants for $\nabla f$ and $\nabla^2 f$ on this set.

By the continuity of $f$ and its derivatives, Assumption 1 implies that there exist $f_{\text{low}} \in \mathbb{R}$, $U_g > 0$ and $U_H > 0$ such that for every $x \in \mathbb{R}^n$, one has

$$f(x) \geq f_{\text{low}}, \quad \|\nabla f(x)\| \leq U_g, \quad \|\nabla^2 f(x)\| \leq U_H. \tag{9}$$

We point out that the choice $U_H = L_g$ is a valid one for theoretical purposes. However, $U_H$ will serve as an explicit parameter of our inexact method in Section 3, so we distinguish between these two quantities in the rest of the paper.

An immediate consequence of these assumptions is that for any $x$ and $d$ such that Assumption 2 is satisfied at $x$ and $x + d$, we have

$$f(x + d) \leq f(x) + \nabla f(x)^T d + \frac{1}{2} d^T \nabla^2 f(x) d + \frac{L_H}{6} \|d\|^3. \tag{10}$$

The following four technical lemmas derive bounds on the decrease obtained from each type of step. The proofs are rather similar, and follow the usual template for backtracking line-search methods.

We begin with negative curvature directions, showing that our choices for initial scaling yield a decrease proportional to the cube of the (negative) curvature in that direction.

Lemma 1. Under Assumption 2, suppose that the search direction for the $k$-th iteration of Algorithm 1 is chosen either as $d_k = \frac{R_k}{\|g_k\|} g_k$ with $R_k < -c_H$ in Step 1 or $d_k = v_k$ in Step 2. Then the backtracking line search terminates with step length $\alpha_k$ after at most $j_\epsilon$ steps, with

$$j_\epsilon := \left[ \log_\theta \left( \frac{3}{L_H + \eta} \right) \right]_+,$$

and the decrease in the function value resulting from the chosen steplength satisfies

$$f(x_k) - f(x_k + \alpha_k d_k) \geq c_\epsilon \left( \frac{\|d_k^T \nabla^2 f(x_k) d_k\|}{\|d_k\|^2} \right)^3, \tag{12}$$

with

$$c_\epsilon := \frac{\eta}{6} \min \left\{ 1, \frac{27\theta^3}{(L_H + \eta)^3} \right\}.$$

Proof. For the direction $d_k = R_k g_k/\|g_k\|$, we have

$$d_k^T \nabla^2 f(x_k) d_k = \frac{R_k^2 \nabla^2 f(x_k) g_k}{\|g_k\|^2} = R_k^3 = -\|d_k\|^3.$$

For the other choice $d_k = v_k$, we have $d_k^T \nabla^2 f(x_k) d_k = \lambda_k^3 = -\|d_k\|^3$, so that in both cases we have

$$d_k^T \nabla^2 f(x_k) d_k = -\|d_k\|^3 \quad \text{and} \quad \frac{|d_k^T \nabla^2 f(x_k) d_k|}{\|d_k\|^2} = \|d_k\|.$$

Thus, if the unit value $\alpha_k = 1$ is accepted, the result trivially holds.
Suppose now that the unit step length is not accepted. Then the choice \( \alpha = \theta^j \)
does not satisfy the decrease condition (7) for some \( j \geq 0 \). Using (10) and the
definition of \( d_k \), we obtain

\[
-\frac{\eta}{6} \alpha^3 \|d_k\|^3 \leq f(x_k + \alpha d_k) - f(x_k) \leq \alpha g_k^\top d_k + \frac{\alpha^2}{2} d_k^\top \nabla^2 f(x_k) d_k + \frac{L_H}{6} \alpha^3 \|d_k\|^3 \\
\leq \frac{\alpha^2}{2} d_k^\top \nabla^2 f(x_k) d_k + \frac{L_H}{6} \alpha^3 \|d_k\|^3 \\
= -\frac{\alpha^2}{2} \|d_k\|^3 + \frac{L_H}{6} \alpha^3 \|d_k\|^3,
\]

where the last line follows from (13). Therefore, we have

\[
\alpha = \theta^j \geq \frac{3}{L_H + \eta},
\]

which does not hold for \( j > j_c \) by definition of \( j_c \), thus the line search must terminate.

Let \( j_k \) be the first integer such that the decrease condition (7) is satisfied. Because
the line search did not stop with step length \( \theta^{j_k-1} \), we must have

\[
\theta^{j_k-1} \geq \frac{3}{L_H + \eta} \Rightarrow \theta^{j_k} \geq \frac{3\theta}{L_H + \eta}.
\]

As a result, the decrease satisfied by the step \( \alpha_k d_k = \theta^{j_k} d_k \) is such that

\[
f(x_k) - f(x_k + \alpha_k d_k) \geq \frac{\eta}{6} \theta^{3j_k} \|d_k\|^3 \geq \frac{\eta}{6} \frac{27\theta^3}{(L_H + \eta)^3} \left[ \frac{d_k^\top \nabla^2 f(x_k) d_k}{\|d_k\|^2} \right]^3.
\]

This inequality, together with the analysis for the case of \( \alpha_k = 1 \), establishes the
desired result.

The second result concerns use of the step \( d_k = -g_k/\|g_k\|^{1/2} \) in the case in which
the curvature of the function along the gradient direction is small.

\textbf{Lemma 2.} Let Assumptions 1 and 2 hold. Then, if at the \( k \)-th iteration of Al-
gorithm 1, the search direction is \( d_k = -g_k/\|g_k\|^{1/2} \), the backtracking line search
terminates with step length \( \alpha_k \) in at most \( j_k \leq 1 + j_g \) steps, where

\[
j_g := \left[ \log_{\theta} \left( \min \left\{ \frac{5}{3}, \sqrt{\frac{1}{L_H + \eta}} \right\} \min \left\{ \epsilon_g^{1/2}, \epsilon_H^{-1} \right\} \right) + 1 \right],
\]

and the resulting step length \( \alpha_k \) is such that

\[
f(x_k) - f(x_k + \alpha_k d_k) \geq c_g \min \left\{ \epsilon_g^{-3}, \epsilon_H^{-1} \right\}
\]

where

\[
c_g := \frac{\eta}{6} \min \left\{ 1, \frac{\theta^3}{(L_H + \eta)^2}, \frac{125\theta^3}{27} \right\}.
\]

\textbf{Proof.} Recall that the choice \( d_k = -g_k/\|g_k\|^{1/2} \) is adopted only when \( \|g_k\| > \epsilon_g \)
and \( |R_k| \leq \epsilon_H \). If the unit step length \( \alpha_k = 1 \) is accepted, we have

\[
f(x_k) - f(x_k + d_k) \geq \frac{\eta}{6} \|d_k\|^3 = \frac{\eta}{6} \|g_k\|^{3/2} \geq \frac{\eta}{6} \epsilon_g^{3/2},
\]
satisfying (16). Otherwise, it means that there exists \( j \geq 0 \) for which the decrease condition (7) is not satisfied using the step size \( \theta^j \). For such \( j \), we have from (10) that

\[
-\frac{\eta}{6} \theta^j g_k \leq f(x_k - \theta^j g_k) - f(x_k)
\]

\[
\leq -\theta^j g_k \frac{\eta}{6} + \frac{\theta^j}{2} R_k \| g_k \| + \frac{L_H}{6} \theta^j \| g_k \|
\]

\[
\leq -\theta^j g_k \frac{\eta}{6} + \frac{\theta^j}{2} \epsilon_H \| g_k \| + \frac{L_H}{6} \theta^j \| g_k \|
\]

which leads to

\[
0 \leq \left[ -\frac{5}{6} \theta^j \| g_k \|^3 + \frac{\theta^j}{2} \epsilon_H \| g_k \| \right] + \left[ -\frac{1}{6} \theta^j \| g_k \|^3 + \frac{L_H + \eta}{6} \theta^j \| g_k \|^3 \right].
\]

Therefore, at least one of the two terms between brackets must be nonnegative. If

\[
\frac{\theta^j}{2} \epsilon_H \| g_k \| \geq 0,
\]

we have \( \theta^j \geq \frac{5}{3} \| g_k \|^3 \epsilon_H^{-1} \). On the other hand, if

\[
-\frac{1}{6} \theta^j \| g_k \|^3 + \frac{L_H + \eta}{6} \theta^j \| g_k \|^3 \geq 0,
\]

then \( \theta^j \geq \sqrt{\frac{1}{L_H + \eta}} \). Putting the two bounds together, we have that

\[
\theta^j \geq \min \left\{ \frac{5}{3} \| g_k \|^{1/2} \epsilon_H^{-1}, \sqrt{\frac{1}{L_H + \eta}} \right\}
\]

\[
\geq \min \left\{ \frac{5}{3} \sqrt{\frac{1}{L_H + \eta}} \right\} \min \left\{ \| g_k \|^{1/2} \epsilon_H^{-1}, 1 \right\}
\]

(18)

\[
\geq \min \left\{ \frac{5}{3} \sqrt{\frac{1}{L_H + \eta}} \right\} \min \left\{ \epsilon_g^{1/2} \epsilon_H^{-1}, 1 \right\}.
\]

Since \( j > j_g \) contradicts (18), the line search terminates for some \( j \leq 1 + j_g \). Let \( j_k \) be the smallest positive integer such that the decrease condition (7) holds. Since it did not hold for \( j_k - 1 \), we have that

\[
\theta^{j_k} \geq \min \left\{ \frac{5}{3} \sqrt{\frac{1}{L_H + \eta}} \right\} \min \left\{ \| g_k \|^{1/2} \epsilon_H^{-1}, 1 \right\}.
\]

The decrease obtained by the step length \( \alpha_k = \theta^{j_k} \) thus satisfies

\[
f(x_k) - f(x_k + \alpha_k d_k) \geq \frac{\eta}{6} \theta^{j_k} \| d_k \|^3
\]

\[
\geq \frac{\eta}{6} \left[ \theta \min \left\{ \frac{5}{3} \sqrt{\frac{1}{L_H + \eta}} \right\} \right]^3 \min \left\{ \| g_k \|^{3/2} \epsilon_H^{-3}, 1 \right\} \| g_k \|^{3/2}
\]

(19)

\[
\geq \frac{\eta}{6} \left[ \theta \min \left\{ \frac{5}{3} \sqrt{\frac{1}{L_H + \eta}} \right\} \right]^3 \min \left\{ \epsilon_g^{3/2} \epsilon_H^{-3}, \epsilon_g^{3/2} \right\}.
\]

Thus (16) is also satisfied in the case of \( \alpha_k < 1 \), completing the proof. \( \Box \)

Lemma 2 describes the reduction that can be achieved along the negative gradient direction when the curvature of the function in this direction is modest. When this
curvature is significantly positive (or when this curvature is slightly positive but the gradient is small), we compute the minimum Hessian eigenvalue (Step 2) and consider other options for the search direction.

Our next result concerns the decrease that can be guaranteed by the Newton step, when it is computed.

**Lemma 3.** Let Assumptions 1 and 2 hold. Suppose that the Newton direction $d_k = d_n^k$ is used at the $k$-th iteration of Algorithm 1. Then the backtracking line search terminates with step length $\alpha_k$ in at most $j_k \leq 1 + j_n$ steps, with

$$j_n := \left\lfloor \log_\theta \left( \sqrt{\frac{3}{L_H + \eta H_U g}} \right) \right\rfloor,$$

and we have

$$f(x_k) - f(x_k + \alpha_k d_k) \geq c_n \min \left\{ \| \nabla f(x_k + \alpha_k d_k) \|^{3/2}, \epsilon_H^3 \right\}.$$

where

$$c_n := \frac{\eta}{6} \min \left\{ \left[ \frac{2}{L_H} \right]^{3/2}, \frac{3 \eta}{L_H + \eta} \right\}.$$  

**Proof.** Note first that the Newton direction $d_k = d_n^k$ is computed only when $\nabla^2 f(x_k) \succ \epsilon_H I$, so we have

$$\|d_k\| \leq \| \nabla^2 f(x_k)^{-1} \| g_k \| \leq U_g / \epsilon_H.$$

Suppose first that the step length $\alpha_k = 1$ satisfies the decrease condition (7). Then from (5) and (10), we have

$$\| \nabla f(x_k + \alpha_k d_k) \| = \| \nabla f(x_k + d_k) - \nabla f(x_k) + \nabla f(x_k) \|$$

$$= \| \nabla f(x_k + d_k) - \nabla f(x_k) - \nabla^2 f(x_k) d_k \| \leq \frac{L_H}{2} \| d_k \|^2.$$

We thus have the following bound on the decrease obtained with the unitary Newton step:

$$f(x_k) - f(x_k + d_k) \geq \frac{\eta}{6} \left[ \frac{2}{L_H} \right]^{3/2} \| \nabla f(x_k + d_k) \|^{3/2}.$$

Suppose now that the unit step length does not allow for a sufficient decrease as measured by (7). Then this condition must fail for $\alpha_k = \theta^j$ for some $j \geq 0$. For this value, we have from (10) that

$$\frac{-\eta}{6} \theta^{3j} \| d_k \|^3 \leq f(x_k + \theta^j d_k) - f(x_k)$$

$$\leq \theta^j g_k^T d_k + \frac{\theta^{2j}}{2} d_k^T \nabla^2 f(x_k) d_k + \frac{L_H}{6} \theta^{3j} \| d_k \|^3$$

$$\leq \theta^j \left( \frac{\theta^j}{2} - 1 \right) d_k^T \nabla^2 f(x_k) d_k + \frac{L_H}{6} \theta^{3j} \| d_k \|^3$$

$$\leq -\frac{\theta^j}{2} d_k^T \nabla^2 f(x_k) d_k + \frac{L_H}{6} \theta^{3j} \| d_k \|^3$$

$$\leq -\frac{\theta^j}{2} \epsilon_H \| d_k \|^2 + \frac{L_H}{6} \theta^{3j} \| d_k \|^3,$$

(24)
where we used $\nabla^2 f(x_k) \succeq \epsilon_H I$ for the final inequality. This relation holds in particular for $j = 0$, in which case it gives

$$-\frac{\eta}{6} \|d_k\|^3 \leq -\frac{\epsilon_H}{2} \|d_k\|^2 + \frac{L_H}{6} \|d_k\|^3$$

leading to the following lower bound on the norm of the Newton step:

$$\|d_k\| \geq \frac{3}{L_H + \eta} \epsilon_H.$$

More generally, for any integer $j$ such that the decrease condition is not satisfied, we have from (24) that

$$\theta_j \geq \sqrt{3} (L_H + \eta) \epsilon_H \frac{1}{2} \|d_k\|^{-1/2}.$$

For any $j > j_n$, the last inequality is violated since

$$\theta^* < \theta^{j_n} \leq \sqrt{3} \frac{\epsilon_H}{L_H + \eta} \sqrt{U_g} = \sqrt{3} \frac{\epsilon_H \frac{1}{2} \epsilon_H \frac{1}{2}}{L_H + \eta} \epsilon_H \frac{1}{2} \|d_k\|^{-1/2} \leq \sqrt{3} \frac{\epsilon_H}{L_H + \eta} \|d_k\|^{-1/2},$$

where we used (22) for the final inequality. This proves that the condition (7) will eventually be satisfied. Defining $j_k$ as the first integer for which it is satisfied, we know that $j_k - 1$ does not fulfill the decrease requirement, so it follows from (26) that

$$\theta_{j_k} \geq \theta \sqrt{3} \frac{\epsilon_H}{L_H + \eta} \|d_k\|^{-1/2}.$$

By substituting this lower bound into the sufficient decrease condition, and then using (25), we obtain

$$f(x_k) - f(x_k + \alpha_k d_k) = f(x_k) - f(x_k + \theta_{j_k} d_k)$$

$$\geq \frac{\eta}{6} \theta_{j_k}^3 \|d_k\|^3$$

$$\geq \frac{\eta}{6} \theta_{j_k}^3 \left[ \frac{3}{L_H + \eta} \epsilon_H \frac{1}{2} \|d_k\|-\frac{3}{2} \|d_k\| \right]^3$$

$$\geq \frac{\eta}{6} \theta_{j_k}^3 \left[ \frac{3}{L_H + \eta} \epsilon_H \frac{1}{2} \|d_k\|^{-3/2} \|d_k\|^3 \right],$$

where the final inequality is from (25). We obtain the required result by combining this equality with the bound (23) for the case of $\alpha_k = 1$. \[ \square \]

Our last intermediate result addresses the case of a regularized Newton step.

**Lemma 4.** Let Assumptions 1 and 2 hold. Suppose that $d_k = d^*_k$ at the $k$-th iteration of Algorithm 1. Then the backtracking line search terminates with step length $\alpha_k$ in at most $j_k \leq 1 + j_r$ steps, with

$$j_r := \left\lfloor \log_{\theta} \left( \frac{6}{L_H + \eta} \epsilon_H \frac{1}{2} \sqrt{U_g} \right) \right\rfloor_+,$$

and we have

$$f(x_k) - f(x_k + \alpha_k d_k) \geq c_r \min \left\{ \varphi \left( \|\nabla f(x_k + \alpha_k d_k)\|, \epsilon_H \right), \epsilon_H^3 \right\},$$
where
\[ c_r := \frac{\eta}{6} \min \left\{ 1, \left[ \frac{6\theta}{L_H + \eta} \right]^3 \right\} \]
and the function \( \varphi : \mathbb{R}^2_+ \to \mathbb{R} \) is defined by
\[ \varphi(t, u) := \frac{-2u + \sqrt{4u^2 + 2L_H t}}{L_H}, \] for all \( t > 0 \) and \( u > 0 \).

Proof. Note first that the regularized Newton step is taken only when \( \nabla^2 f(x_k) \succeq -\epsilon_H I \). Thus the minimum eigenvalue of the coefficient matrix in (6) is \( \lambda_k + 2\epsilon_H \geq \epsilon_H \), and we have
\[ \|d_k\| \leq \frac{\|g_k\|}{\lambda_k + 2\epsilon_H} \leq \frac{\|g_k\|}{\epsilon_H} \leq \frac{U_g}{\epsilon_H}. \]

Suppose first that the unit step is accepted. Then the gradient norm at the new point satisfies
\[
\|\nabla f(x_k + d_k)\| = \|\nabla f(x_k + d_k) - \nabla f(x_k) + \nabla f(x_k)\|
\leq \|\nabla f(x_k + d_k) - \nabla f(x_k) - \nabla^2 f(x_k) d_k - 2\epsilon_H d_k\|
\leq \frac{L_H}{2} \|d_k\|^2 + 2\epsilon_H \|d_k\|,
\]
and therefore
\[ \frac{L_H}{2} \|d_k\|^2 + 2\epsilon_H \|d_k\| - \|\nabla f(x_k + d_k)\| \geq 0. \]

By treating the left-hand side as a quadratic in \( \|d_k\| \), and using the definition (29) of \( \varphi \), we obtain from this bound that
\[ \|d_k\| \geq -\frac{2\epsilon_H + \sqrt{4\epsilon_H^2 + 2L_H \|\nabla f(x_k + d_k)\|}}{L_H} = \varphi(\|\nabla f(x_k + d_k)\|, \epsilon_H). \]
Therefore, if the unit step is accepted, we have
\[ f(x_k) - f(x_k + d_k) \geq \frac{\eta}{6} \|d_k\|^3 \geq \frac{\eta}{6} \varphi(\|\nabla f(x_k + d_k)\|, \epsilon_H)^3. \]

If the unit step does not yield a sufficient decrease, there must be a value \( j \geq 0 \) such that (7) is not satisfied for \( \alpha = \theta^j \). For such \( j \), and using again (10), we have
\[
-\frac{\eta}{6} \theta^{3j} \|d_k\|^3 \leq f(x_k + \theta^j d_k) - f(x_k)
\leq \theta^j g_k^\top d_k + \frac{\theta^{2j}}{2} d_k^\top \nabla^2 f(x_k) d_k + \frac{L_H}{6} \theta^{3j} \|d_k\|^3
= \theta^j \left( 1 - \frac{\theta^j}{2} \right) g_k^\top d_k - \epsilon_H \theta^{2j} \|d_k\|^2 + \frac{L_H}{6} \theta^{3j} \|d_k\|^3.
\leq -\epsilon_H \theta^{2j} \|d_k\|^2 + \frac{L_H}{6} \theta^{3j} \|d_k\|^3.
\]
Thus, for any \( j \geq 0 \) for which sufficient decrease is not obtained, one has

(33) \[ \theta^j \geq \frac{6}{L_H + \eta} \epsilon_H \|d_k\|^{-1}. \]

Meanwhile, we have from the definition of \( j_r \) that

\[ \theta^j < \theta^j_r \leq \frac{6}{L_H + \eta} \epsilon_H \|d_k\|^{-1}, \]

using the upper bound (30). By comparing this bound with (33), we deduce that \( j \leq j_r \), that is, the backtracking line-search procedure terminates in at most \( j_r \) steps.

Denoting by \( j_k \) the first positive integer for which sufficient decrease is obtained, we have \( j_k \geq 1 \), and thus (at least) the preceding step corresponding to \( j_k - 1 \) satisfies (33).

Therefore, we have

\[ \theta^{j_k} \geq \frac{6 \theta}{L_H + \eta} \epsilon_H \|d_k\|^{-1} \]

and

\[ f(x_k) - f(x_k + \theta^{j_k} d_k) \geq \frac{\eta}{6} \theta^{3j_k} \|d_k\|^3 \geq \frac{\eta}{6} \left[ \frac{6 \theta}{L_H + \eta} \right]^3 \epsilon_H^3. \]

By combining this bound with the bound (32) obtained for the unit-step case, we obtain the result.

By combining the estimates of function decrease proved in the lemmas above, bound the number of iterations needed by Algorithm 1 to satisfy the approximate second-order optimality conditions (8).

**Theorem 5.** Let Assumptions 1 and 2 hold. Then Algorithm 1 reaches an iterate that satisfies (8) in at most

(34) \[ C \max \left\{ \epsilon_g^{-3}, \epsilon_H^{-3/2}, \epsilon^{-3}, \varphi(\epsilon_g, \epsilon_H)^{-3} \right\} \]

iterations,

where

(35) \[ C := c^{-1}(f(x_0) - f_{low}), \quad c := \min \{ c_g, c_v, c_n, c_r \}. \]

**Proof.** Suppose \( l \) is at iteration at which the conditions for termination are not satisfied. We consider in turn the various types of steps that could have been taken at iteration \( l \), and obtain a lower bound on the amount of decrease obtained from each. Table 1 is helpful in working through the various cases. We consider two main cases, and several subcases.

**Case 1:** \( \lambda_l < -\epsilon_H \).

From Table 1, we see that in this case, the search direction is either a scaling of \(-g_k\), or the most-negative-curvature direction \(v_k\). When \( R_l < -\epsilon_H \), we have \( d_l = \frac{R_l}{\|g_l\|} g_l \), and Lemma 1 indicates the following bound on function decrease:

\[ f(x_l) - f(x_{l+1}) \geq c \epsilon_g^3. \]

When \( R_l \in [-\epsilon_H, \epsilon_H] \) and \( \|g_l\| > \epsilon_g \), we have \( d_l = -g_l/\|g_l\|^{1/2} \). Thus, using Lemma 2, we have

\[ f(x_l) - f(x_{l+1}) \geq c_g \min \left\{ \epsilon_g^3 \epsilon_H^{-3}, \epsilon_H^{3/2} \right\}. \]
For the remaining cases of \( \|g_l\| \leq \epsilon_g \) and \( R_l \in [-\epsilon_H, \epsilon_H] \) and \( \|g_l\| > \epsilon_g \) and \( R_l > \epsilon_H \), the search direction is necessarily \( v_l \). We have from Lemma 1 that
\[
f(x_l) - f(x_{l+1}) \geq c_c \left( \frac{d_l^T \nabla^2 f(x_l) d_l}{\|d_l\|^2} \right)^3 = c_c |\lambda_l|^3 \geq c_c \epsilon_H^3.
\]

**Case 2:** \( \lambda_l \geq -\epsilon_H, \|g_l\| > \epsilon_g, \text{ and } \|g_{l+1}\| > \epsilon_g \).

In this case, we have three possible choices for the search direction. The first one is \( d_l = -g_l/\|g_l\|^{1/2} \), in which case we have from Lemma 2 that
\[
f(x_l) - f(x_{l+1}) \geq c_g \min \left\{ \epsilon_g^3 \epsilon_H^{-3}, \epsilon_g^{3/2} \right\}.
\]
The second possible choice is the Newton direction \( d_l = d_l^h \). Using Lemma 3, we obtain
\[
f(x_l) - f(x_{l+1}) \geq c_n \min \left\{ \epsilon_g^{3/2}, \epsilon_H^3 \right\}.
\]
The third choice is the regularized Newton direction \( d_l = d_l^r \), for which Lemma 4 yields
\[
f(x_l) - f(x_{l+1}) \geq c_r \min \left\{ \varphi(\|g_{l+1}\|, \epsilon_H)^3, \epsilon_H^3 \right\} \geq c_r \min \left\{ \varphi(\epsilon_g, \epsilon_H)^3, \epsilon_H^3 \right\},
\]
since the function \( \varphi \) defined in (29) is increasing in its first argument.

By putting all these bounds together, we obtain the following lower bound on the decrease in \( f \) on iteration \( l \):
\[
f(x_l) - f(x_{l+1}) \geq c_0 \min \left\{ \epsilon_g^3 \epsilon_H^{-3}, \epsilon_g^{3/2}, \epsilon_H^3, \varphi(\epsilon_g, \epsilon_H)^3 \right\},
\]
where \( c_0 \) is defined in (35). Consequently, summing across all iterations up to \( k \) yields
\[
f(x_0) - f_{low} \geq \sum_{l=0}^{k-1} f(x_l) - f(x_{l+1}) \geq k c_0 \min \left\{ \epsilon_g^3 \epsilon_H^{-3}, \epsilon_g^{3/2}, \epsilon_H^3, \varphi(\epsilon_g, \epsilon_H)^3 \right\},
\]
which implies that \( k \) is bounded above by (34). Therefore, there must exist a finite index \( k_0 \) such that (8) is satisfied. For this index, the bound (8) applies, hence the result. \( \square \)

We obtain two corollaries from Theorem 5.

**Corollary 6.** Suppose the assumptions of Theorem 5 hold, and define \( \epsilon_g = \epsilon \) and \( \epsilon_H = \sqrt{\epsilon} \), for some \( \epsilon \in (0, 1) \). Then Algorithm 1 reaches an iterate that satisfies (8) in at most
\[
C_1 \epsilon^{-3/2} \text{ iterations,}
\]

where
\[
C_1 := c_1^{-1}(f(x_0) - f_{low}), \quad c_1 := \min \left\{ \epsilon_g, c_v, c_n, c_r, \epsilon_g \frac{-2 + \sqrt{4 + 2L_H}}{L_H^3} \right\}.
\]

**Proof.** For the first three quantities in the iteration bound (34) in Theorem 5, we have for our choices of \( \epsilon_g \) and \( \epsilon_H \) that
\[
\epsilon_g^{-3/2} \epsilon_H^{-3} = \epsilon_g^{-3/2} = \epsilon_H^{-3} = \epsilon^{-3/2}.
\]

The result therefore follows from an appropriate bound for the term \( \varphi(\epsilon_g, \epsilon_H)^{-3} \) in Theorem 5. We have for our particular choices of \( \epsilon_g \) and \( \epsilon_H \) and the definition (29) that

\[
\varphi(\epsilon_g, \epsilon_H) = -\frac{2\epsilon_H + \sqrt{4\epsilon_H^2 + 2L_H \epsilon_g}}{L_H} = -\frac{2 + \sqrt{4 + 2L_H \epsilon}}{L_H} \epsilon^{1/2}.
\]

Thus, using (28), we can incorporate the coefficient \( \left( (-2 + \sqrt{4 + 2L_H})/L_H \right)^3 \) into \( c \) in (35) to obtain \( c_1 \), and the proof is complete. 

**Corollary 7.** Suppose the assumptions of Theorem 5 hold, and let \( \epsilon_g = \epsilon_H = \epsilon \) for some \( \epsilon \in (0, 1) \). Then Algorithm 1 reaches an iterate that satisfies (8) in at most

\[
C_1 \epsilon^{-3} \text{ iterations},
\]

where \( C_1 \) is defined as in Corollary 6.

**Proof.** Choosing \( \epsilon_g = \epsilon_H = \epsilon \) yields

\[
\varphi(\epsilon_g, \epsilon_H) = -\frac{2\epsilon + \sqrt{4\epsilon^2 + 2L_H \epsilon}}{L_H} = -\frac{2 + \sqrt{4 + 2L_H \epsilon}}{L_H} \epsilon \geq -\frac{2 + \sqrt{4 + 2L_H \epsilon}}{L_H} \epsilon
\]

since \( \epsilon \in (0, 1) \). For the bound (34), we therefore have

\[
\max \left\{ \epsilon^{-3}, \epsilon^{-3/2}, \epsilon^{-3}, \varphi(\epsilon_g, \epsilon_H)^{-3} \right\} \leq \max \left\{ \epsilon^{-3}, \left(-\frac{2 + \sqrt{4 + 2L_H \epsilon}}{L_H} \right)^{-3} \epsilon^{-3} \right\}.
\]

The bound follows by incorporating the coefficient \( \left( (-2 + \sqrt{4 + 2L_H})/L_H \right)^3 \) into the quantity \( c \) in (35) to obtain \( c_1 \), as in Corollary 6. 

The bounds obtained in Corollaries 6 and 7 thus match the optimal ones known for second-order globally convergent methods in terms of iteration count.

**2.3. Evaluation/Inner Iteration Complexity.** We now discuss the function evaluation complexity of Algorithm 1, which counts the number of function calls required by the algorithm before its termination conditions are satisfied. We need to refine the iteration complexity analysis of Section 2.2 to take into account the function evaluations associated with the backtracking line-search process.

**Theorem 8.** Suppose that Assumptions 1 and 2 hold. The number of function evaluations required by Algorithm 1 prior to reaching a point that satisfies (8) is at most

\[
(1 + \mathcal{K} + \log_\theta \left( \min \{ \epsilon_H^{-1/2}, \epsilon_g^{-1/2}, \epsilon_H^{-1} \} \right) ) C \max \left\{ \epsilon_g^{-3}, \epsilon_H^{-3/2}, \varphi(\epsilon_g, \epsilon_H)^{-3} \right\},
\]

where

\[
\mathcal{K} := \left\lfloor \log_\theta \left( \frac{5}{3} \left( L_H + \eta \right)^{1/2} \right) \right\rfloor + \frac{3}{6 \left( L_H + \eta \right)^{1/2}} \sqrt{3 \left( L_H + \eta \right)^{1/2}}
\]

and \( C \) is defined as in Theorem 5.

**Proof.** A bound on the number of evaluations is obtained from Theorem 5. We obtain

\[
(1 + \max \{ j_g, j_e, j_n, j_r \}) C \max \left\{ \epsilon_g^{-3}, \epsilon_H^{-3/2}, \varphi(\epsilon_g, \epsilon_H)^{-3} \right\}.
\]
By using the definitions of $j_g, j_e, j_n,$ and $j_r$ from Lemmas 1, 2, 3, and 4, and using $\epsilon_g, \epsilon_H \in (0, 1)$, we obtain the result. □

With our specific choices of $\epsilon_g$ and $\epsilon_H$ made in Corollaries 6 and 7, we observe that the evaluation complexity bounds are in $O\left(\log\left(\frac{1}{\epsilon}\right)\epsilon^{-3/2}\right)$ and $O\left(\log\left(\frac{1}{\epsilon}\right)\epsilon^{-3}\right)$, respectively.

2.4. Local Convergence. In the previous sections, we have derived global complexity guarantees for Algorithm 1. We now aim to show rapid local convergence for the variant of the algorithm that invokes the Local Phase, Algorithm 2, rather than terminating as soon as the conditions (8) are satisfied. We note that local convergence results like the one we prove here have in the past gone hand-in-hand with global convergence results in smooth nonconvex optimization (see for example [19]). More recently, several works in the optimization literature establish rapid local convergence alongside global complexity guarantees [2, 7, 12].

For this section, we will make the following additional assumption.

Assumption 3. The sequence of iterates generated by Algorithm 1 in conjunction with Algorithm 2 converges to a local minimizer, that is, a point $x^*$ at which $\nabla f(x^*) = 0$ and $\nabla^2 f(x^*) \succ 0$.

Under this assumption, the following result is immediate.

Lemma 9. Under Assumptions 1, 2 and 3, there exists $k_0 \in \mathbb{N}$ such that for every $k \geq k_0$, we have for $\mu := \frac{1}{2} \min\left(1, \lambda_{\min}\left(\nabla^2 f(x^*)\right)\right) > 0$ that

(40) $\mu I \preceq \nabla^2 f(x_k) \preceq U_H I,$

and

(41) $\|g_k\| < \min\left\{\frac{3\mu}{4L_H + \eta}, \epsilon_g\right\}.$

Note that the conditions on $k_0$ in Lemma 9 are such that the combined strategy of Algorithm 1-Algorithm 2 will have entered the Local Phase (Algorithm 2) before iteration $k_0$, and will stay in this phase at all subsequent iterations.

We now establish a local quadratic convergence result.

Theorem 10. Suppose that Assumptions 1, 2, and 3 are satisfied, and let $\mu$ and $k_0$ be as defined in Lemma 9. Then for every $k \geq k_0$, the method always takes the Newton direction with a unit step length, and we have

(42) $\|g_{k+1}\| \leq \frac{L_H}{2\mu^2} \|g_k\|^2 \leq \frac{3}{8} \|g_k\|.$

Proof. Let $k \geq k_0$. As noted above, we are in the Local Phase (Algorithm 2) at iteration $k$. By Lemma 9, the Hessian at $\nabla^2 f(x_k)$ is positive definite, with smallest eigenvalue bounded below by $\mu > 0$. Thus Algorithm 2 computes the Newton direction $d_k = d^*_k$, and we have

$\|d_k\| \leq \|g_k\|/\mu, \quad g_k^T d_k \leq -\mu \|g_k\|^2.$

We thus have

$f(x_k + d_k) - f(x_k) \leq g_k^T d_k + \frac{1}{2} d_k^T \nabla^2 f(x_k) d_k + \frac{L_H}{6} \|d_k\|^3$

$= \frac{1}{2} g_k^T d_k + \frac{L_H}{6} \|d_k\|^3 \leq -\mu \|g_k\|^2 + \frac{L_H}{6} \|d_k\|^3.$
Thus if the sufficient decrease condition \( f(x_k + d_k) - f(x_k) \geq -\frac{\eta}{6} \|d_k\|^3 \) is not satisfied for the unit step, we must have

\[
\frac{L_H + \eta \|d_k\|^3}{6} \geq \frac{\mu}{2} \|g_k\|^2,
\]

which by the bound \( \|d_k\| \leq \|g_k\|/\mu \) can be true only if

\[
\frac{L_H + \eta \|g_k\|^3}{\mu^3} \geq \frac{\mu}{2} \|g_k\|^2 \quad \Leftrightarrow \quad \|g_k\| \geq \frac{3\mu^4}{L_H + \eta},
\]

which contradicts (41). Thus the unit Newton step is taken, and we have

\[
\|g_{k+1}\| = \|\nabla f(x_k + d_k)\| = \|\nabla f(x_k + d_k) - \nabla f(x_k) - \nabla^2 f(x_k) d_k\|
\leq \frac{L_H}{2} \|d_k\|^2
\leq \frac{L_H}{2\mu^2} \|g_k\|^2
\leq \frac{L_H}{2\mu^2} \frac{3\mu^4}{L_H + \eta} \|g_k\|
\leq \frac{3}{2} \mu^2 \|g_k\| \leq \frac{3}{8} \|g_k\|,
\]

completing the proof.

3. A Variant with Inexact Directions. In Section 2, we have assumed that certain linear-algebra operations in Algorithm 1 — the linear system solves of (5) and (6) and the eigenvalue / eigenvector computation of (4) — are performed exactly. In a large-scale setting, the cost of these operations can be prohibitive, so iterative techniques that perform these operations \textit{inexactly} are of interest. In this section, we describe inexact methods for these key operations, and examine their consequences for the complexity analysis.

3.1. Inexact Eigenvector Calculation: Randomized Lanczos Method.

The problem of finding the minimum eigenvalue of the matrix in (4) and its associated eigenvector can be reformulated as one of finding the maximum eigenvalue and eigenvector of a positive semidefinite matrix. The Lanczos algorithm with a random starting vector is an appealing option for the latter problem, yielding an \( \epsilon \)-approximate eigenvector in \( O(\log(n/\delta)\epsilon^{-1/2}) \) iterations, with probability at least \( 1 - \delta \) [16]. This fact has been used in several methods that achieve fast convergence rates [1, 4, 5]. In order to apply this method to a matrix that is not positive definite, one must make use of a bound on the Hessian norm. For sake of completeness, we spell out the procedure in the following lemma.

**Lemma 11.** Let \( H \) be a symmetric matrix satisfying \( \|H\| \leq M \) for some \( M > 0 \). Suppose that the Lanczos procedure is applied to find the largest eigenvalue of \( MI - H \) starting at a random vector uniformly distributed over the unit sphere. Then, for any \( \varepsilon > 0 \) and \( \delta \in (0, 1) \), there is a probability at least \( 1 - \delta \) that the procedure outputs a vector \( v \) such that

\[
v^\top Hv \leq \lambda_{\text{min}}(H) + \varepsilon
\]

in at most

\[
\min \left\{ n, \frac{\ln(n/\delta^2)}{2\sqrt{2}} \sqrt{\frac{M}{\varepsilon}} \right\} \text{ iterations.}
\]
After at most \( n \) iterations, the procedure obtains a vector \( v \) such that \( v^\top H v = \lambda_{\min}(H) \), with probability 1.

Proof. By definition, the matrix \( H' = MI - H \) is a symmetric positive semidefinite matrix with its spectrum lying in \([0, 2M]\). Applying the Lanczos procedure to this matrix from a starting point drawn randomly from the unit sphere yields a unitary vector \( v \) such that
\[
(45) \quad v^\top H' v \geq \left(1 - \frac{\varepsilon}{2M}\right) \lambda_{\max}(H') \geq \left(1 - \frac{\varepsilon}{2M}\right) (M - \lambda_{\min}(H))
\]
in no more than \( \min\left\{ n, \frac{\ln(n/\delta^2)}{4\sqrt{\varepsilon}/(2M)} \right\} \) iterations with probability at least \( 1 - \delta \). (This result is from [16, Theorem 4.2] extended by a continuity argument from the positive definite case to the positive semidefinite case; see [16, Remark 7.5].) Moreover, using (45), we have
\[
v^\top H v = -v^\top H' v + M \\
\leq -\left(1 - \frac{\varepsilon}{2M}\right) (M - \lambda_{\min}(H)) + M \\
= -M + \lambda_{\min}(H) + \frac{\varepsilon}{2M} \lambda_{\min}(H) + M \\
= \lambda_{\min}(H) + \frac{\varepsilon}{2M} \lambda_{\min}(H) \\
\leq \lambda_{\min}(H) + \frac{\varepsilon}{2M} M \\
= \lambda_{\min}(H) + \varepsilon,
\]
as required. \( \square \)

Lemma 11 admits the following variant, for the case in which we fix the number of Lanczos iterations.

LEMMA 12. Let \( H \) be a matrix such that \( \|H\| \leq M \). Suppose that \( q \) iterations of the Lanczos procedure are applied to find the largest eigenvalue of \( MI - H \) starting at a random vector uniformly distributed over the unit sphere. Then for any \( \varepsilon > 0 \), the procedure outputs a vector \( v \) such that \( v^\top H v \leq \lambda_{\min}(H) + \varepsilon \) with probability at least
\[
(46) \quad 1 - \delta = 1 - \sqrt{n} \exp \left[-\sqrt{2q} \sqrt{\frac{\varepsilon}{M}} \right].
\]

We point out that the choice \( \delta = 0 \) (or, equivalently, \( q = n \)) is possible, that is, after \( n \) iterations, the Lanczos procedure started with a random vector uniformly generated over the unit sphere returns an approximate eigenvector with probability one [16, Theorem 4.2 (a)].

3.2. Inexact Newton and Regularized Newton Directions: Conjugate Gradient Method. Here we describe the use of the conjugate gradient (CG) algorithm to solve the symmetric positive definite linear systems (5) or (6) — the Newton and regularized Newton equations, respectively. The conjugate gradient method is the most popular iterative method for positive definite linear systems, due to its rich convergence theory and strong practical performance. It has also been popular in the context of nonconvex smooth minimization; see [20]. It requires only matrix-vector operations involving the coefficient matrix (often these can be found or approximated without explicit knowledge of the matrix) together with some vector operations. It does not require knowledge or estimation of the extreme eigenvalues of the matrix.
We apply CG to a system $Hd = -g$ where there are positive quantities $m$ and $M$ such that $mI \preceq H \preceq MI$, so that the condition number $\kappa$ of $H$ is bounded above by $M/m$. Standard convergence indicates that CG outputs a vector $d$ such that $\|Hy + g\| \leq \zeta \|g\|$ (for $\zeta \in (0,1)$) in
\[ O\left( \min\left\{ n, \kappa^{1/2} \log(\kappa/\zeta) \right\} \right) \] iterations, with $\kappa$ being the condition number of $H$ [3]. We use a different stopping criterion, namely
\[ \|Hd + g\| \leq \frac{1}{2} \zeta \min\{\|g\|, m\|d\|\} \] for some $\zeta \in (0,1)$. This criterion is stronger than the one typically used in truncated Newton-Krylov methods, in that we require the residual norm to be bounded by a multiple of the norm of the approximate direction, as well as being bounded by a specified fraction of the initial residual norm. The extra criterion resembles the so-called $s$-condition arising in cubic regularization techniques, where the approximate minimizer $s_k$ of the cubic model $m_k$ is required to satisfy
\[ \|\nabla m_k(s_k)\| \leq O\left( \|s_k\|^2 \right). \] This property provides a lower bound on $\|s_k\|$, that is instrumental in obtaining the optimal complexity order of $O(\epsilon^{-3/2})$ for first-order convergence [8]. Our condition replaces $\|s_k\|^2$ by $m\|d_k\|$, but serves a similar purpose.

The next lemma establishes a bound on the number of CG iterations needed to reach the desired accuracy.

**Lemma 13.** Let $Hd = -g$ be a linear system with $H$ symmetric and $mI \preceq H \preceq MI$, where $m \in (0,1)$, $M > 0$, and $\|g\| > 0$. Then the conjugate gradient algorithm computes a vector $d$ such that $\|Hd + g\| \leq \frac{1}{2} \zeta m \|d\|$ for some $\zeta \in (0,1)$ in at most
\[ \min\left\{ n, \frac{1}{2} \sqrt{\kappa} \ln \left( \frac{4 \kappa^5/2}{\zeta} \right) \right\}, \] iterations, where $\kappa = M/m$.

**Proof.** Let $d^{(q)}$ be the iterated obtained at the $q$-th iteration of the conjugate gradient method applied to $Hd = -g$, with $d^{(0)} = 0$. The classical bound on the behavior of the conjugate gradient residual [19, Section 5.1] yields
\[ \left\|d^{(q)} + H^{-1}g\right\| \leq 2 \left( \frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1} \right)^q \|H^{-1}g\|, \] where $\|x\|_H = \sqrt{x^THx}$. From this definition and the bounds on the spectrum of $H$, we have
\[ \left\|d^{(q)} + H^{-1}g\right\|^2 \geq m \left\|d^{(q)} + H^{-1}g\right\|^2 \geq m \frac{\|Hd^{(q)} + g\|^2}{\|H\|^2} \geq \frac{m}{M^2} \left\|Hd^{(q)} + g\right\|^2 = \frac{1}{M\kappa} \|Hd^{(q)} + g\|^2, \]
as well as
\[ \|H^{-1}g\|_H \leq M\|H^{-1}g\|^2 \leq M\|H^{-1}\|^2\|g\|^2 \leq \frac{M}{m^2}\|g\|^2 = \frac{\kappa}{m}\|g\|^2. \]

By substituting these bounds into (50), we obtain the following relation:

\[ (51) \quad \|Hd^{(q)} + g\| \leq 2\kappa^{3/2}\left(\frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1}\right)^q \|g\|. \]

Thus, as long as our stopping criterion is not satisfied, we have

\[ (52) \quad \frac{1}{2}\zeta \min\{\|g\|, m\|d^{(q)}\|\} \leq 2\kappa^{3/2}\left(\frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1}\right)^q \|g\|. \]

Furthermore, defining \( r^{(q)} = Hd^{(q)} + g \), we have

\[ \|d^{(q)}\| = \|H^{-1}(-g + r^{(q)})\| \geq \frac{\|g - r^{(q)}\|}{M} \geq \frac{\sqrt{\|g\|^2 - 2g^TR^{(q)} + \|r^{(q)}\|^2}}{M} \geq \|g\|, \]

for all \( q \geq 1 \), where we used the fact that using the facts that \( r^{(0)} = g \) and that in CG, the residuals are orthogonal: \( (r^{(i)})^TR^{(j)} = 0 \) for \( i \neq j \). Using this bound within (52), we obtain

\[ \frac{\zeta}{2} \min\{1, m/M\}\|g\| \leq 2\kappa^{3/2}\left(\frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1}\right)^q \|g\| \Leftrightarrow \frac{\zeta}{4\kappa^{5/2}} \leq \left(\frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1}\right)^q. \]

By taking logarithms on both sides, we arrive at

\[ q \leq \frac{\ln(\zeta/(4\kappa^{5/2}))}{\ln\left(\frac{\sqrt{\kappa} - 1}{\sqrt{\kappa} + 1}\right)} = \frac{\ln(4\kappa^{5/2}/\zeta)}{\ln\left(1 + \frac{2}{\sqrt{\kappa} - 1}\right)} \leq \frac{1}{2}\sqrt{\kappa}\ln\left(\frac{4\kappa^{5/2}}{\zeta}\right), \]

where the bound \( \ln(1 + \frac{1}{t}) \geq \frac{1}{t+1/2} \) was used to obtain the last inequality. \( \square \)

### 3.3. Complexity Analysis Based on Inexact Computations.

We present a variant of our main algorithm, specified as Algorithm 3, in which computation of approximate eigenvectors and linear system solves are performed inexactly by the means described above. We provide a complexity analysis of this modified scheme. Algorithm 3 requires two additional parameters: an upper bound on the Hessian norms of the iterates and a probability threshold \( \delta \). As we expect only to recover inexact global complexity guarantees, the method does not exploit a local phase.

Note that when Algorithm 3 terminates, condition (8) must hold. This follows from the fact that termination occurs when \( \min(\|g_k\|, \|g_{k+1}\|) \leq \epsilon_g \) and \( \lambda_k^1 \geq -\frac{1}{2}\epsilon_H \). Since \( \lambda_k^1 \) is within \( \frac{1}{2}\epsilon_H \) of \( \lambda_{\min}(\nabla^2 f(x_k)) \), we must have \( \lambda_{\min}(\nabla^2 f(x_k)) \geq -\epsilon_H \), thus satisfying (8).
Algorithm 3 Inexact Second-Order Line Search Method

Init. Choose $x^0 \in \mathbb{R}^n$, $\theta \in (0,1)$, $\zeta, \delta \in [0,1)$, $\eta > 0$, $\epsilon_g \in (0,1)$, $\epsilon_H \in (0,1)$, $U_H > 0$;

for $k = 0, 1, 2, \ldots$ do

Step 1. (First-Order) Set $g_k = \nabla f(x_k)$;
if $\|g_k\| = 0$ then
  Go to Step 2;
end if

Compute $R_k = \frac{g_k^\top \nabla^2 f(x_k) g_k}{\|g_k\|^2}$;
if $R_k < -\epsilon_H$ then
  Set $d_k = \frac{R_k}{\|g_k\|^2} g_k$ and go to Step LS;
else if $R_k \in [-\epsilon_H, \epsilon_H]$ and $\|g_k\| > \epsilon_g$ then
  Set $d_k = \frac{g_k}{\|g_k\|^2}$ and go to Step LS;
else
  Go to Step 2;
end if

Step 2. (Inexact Second-Order) Compute an inexact eigenvector $v_i^k$ such that (with probability $1 - \delta$)

\begin{equation}
[v_i^k]_j^\top g_k \leq 0, \quad \|v_i^k\| = 1, \quad [v_i^k]_j^\top \nabla^2 f(x_k)v_i^k \leq \lambda_{\min} \left(\nabla^2 f(x_k)\right) + \frac{\epsilon_H}{2};
\end{equation}

Let $\lambda_i^k = [v_i^k]_j^\top \nabla^2 f(x_k)v_i^k$;
if $\|g_k\| \leq \epsilon_g$ and $\lambda_i^k \geq -\frac{1}{2} \epsilon_H$ then
  Terminate;
else if $\lambda_i^k < -\frac{1}{2} \epsilon_H$ then
  (Negative Curvature) Set $d_k = v_i^k$;
else if $\lambda_i^k > \frac{3}{2} \epsilon_H$ then
  (Inexact Newton) Use conjugate gradient to calculate $d_k = d_i^n_k$, where

\begin{equation}
\|\nabla^2 f(x_k)d_i^n_k + g_k\| \leq \frac{\zeta}{2} \min \left\{ \|g_k\|, \epsilon_H \|d_i^n_k\| \right\};
\end{equation}

else
  (Inexact regularized Newton) Use conjugate gradient to calculate $d_k = d_r^r_k$, where

\begin{equation}
\|\left(\nabla^2 f(x_k) + 2 \epsilon_H I\right)d_r^r_k + g_k\| \leq \frac{\zeta}{2} \min \left\{ \|g_k\|, \epsilon_H \|d_r^r_k\| \right\};
\end{equation}

end if

Go to Step LS;

Step LS. (Line Search) Compute a step length $\alpha_k = \theta^j_k$, where $j_k$ is the smallest nonnegative integer such that

\begin{equation}
f(x_k + \alpha_k d_k) < f(x_k) - \frac{\eta}{6} \alpha^3 \|d_k\|^3
\end{equation}

holds, and set $x_{k+1} = x_k + \alpha_k d_k$;
if $d_k = d_i^n_k$ or $d_k = d_r^r_k$ and $\|\nabla f(x_{k+1})\| \leq \epsilon_g$ then
  Terminate;
end if

end for
Table 2 shows a summary of the possible choices for the search direction. It possesses the same number of cases than in Table 1, but the context is now determined from the eigenvalue estimate $\lambda_k$. Table 2 mentions two additional lemmas, that respectively replace Lemmas 3 and 4 in order to take inexactness into account. We state and prove these results next.

**Lemma 14.** Let Assumptions 1 and 2 hold. Suppose that an inexact Newton direction $d_k = d_k^{\text{in}}$ is computed at the $k$-th iteration of Algorithm 3. Then with probability at least $1 - \delta$, the backtracking line search terminates with step length $\alpha_k$ in at most $j_k \leq 1 + j_{\text{in}}$ steps, with

$$j_{\text{in}} := \left\lceil \frac{1}{2} \log \left( \frac{3}{L_H + \eta} \frac{(1 - \zeta)\epsilon_H^2}{1 + \zeta^2/4} \right) \right\rceil,$$

and we have

$$f(x_k) - f(x_k + \alpha_k d_k) \geq c_{\text{in}} \min \left\{ \varphi \left( \|\nabla f(x_k + \alpha_k d_k)\|, \frac{\zeta \epsilon_H^3}{2} \right), \epsilon_H^3 \right\},$$

where $\varphi$ is defined in (29) and

$$c_{\text{in}} := \frac{\eta}{6} \min \left\{ 1, \left( \frac{3\theta(1 - \zeta)}{L_H + \eta} \right)^3 \right\}.$$

*Proof.* We observe first that when the Newton step is computed in Algorithm 3, we have from (43) that

$$\frac{3\epsilon_H}{2} < \lambda_k \leq \lambda_{\text{min}} \left(\nabla^2 f(x_k)\right) + \frac{\epsilon_H}{2} \Rightarrow \lambda_{\text{min}} \left(\nabla^2 f(x_k)\right) \geq \epsilon_H,$$

with probability $1 - \delta$. On the one hand, suppose that the step length $\alpha_k = 1$ satisfies the decrease condition (56). Then, defining $r_k := \nabla^2 f(x_k) d_k + g_k$ and using the inexactness criterion for the inexact Newton step $d_k$, we find that the gradient at the next point $x_k + d_k$ satisfies

$$\|\nabla f(x_k + d_k)\| = \|\nabla f(x_k + d_k) - \nabla f(x_k) + \nabla f(x_k)\|
\leq \frac{L_H}{2} \|d_k\|^2 + \|r_k\| \leq \frac{L_H}{2} \|d_k\|^2 + \frac{\epsilon_H}{2} \|d_k\|.$$
As in the proof of Lemma 4, we obtain a lower bound on \(\|d_k\|\):
\[
\|d_k\| \geq \varphi \left( \|\nabla f(x_k + d_k)\|, \frac{\zeta}{2} \epsilon_H \right).
\]
Therefore, taking the inexact Newton step with a unit step length guarantees
\[
f(x_k) - f(x_k + d_k) \geq \frac{\eta}{6} \|d_k\|^3 \geq \frac{\eta}{6} \varphi \left( \|\nabla f(x_k + d_k)\|, \frac{\zeta}{2} \epsilon_H \right)^3,
\]
so the inequality (58) is satisfied in the case of a unit step length.

To complete the proof, consider the case in which the unit step length does not lead to sufficient decrease. In that case, for any value \(j > 0\) such that (56) is not satisfied, we have
\[
-\frac{\eta}{6} \theta^3 \|d_k\|^3 \leq f(x_k + \theta d_k) - f(x_k)
\]
\[
\leq \theta^3 g_k^\top d_k + \frac{\theta^2 d_k^\top \nabla^2 f(x_k) d_k + \frac{L_H}{6} \theta^3 \|d_k\|^3}
\]
\[
\leq \theta^3 (-\nabla^2 f(x_k) d_k + r_k)^\top d_k + \frac{\theta^2}{2} d_k^\top \nabla^2 f(x_k) d_k + \frac{L_H}{6} \theta^3 \|d_k\|^3
\]
\[
= -\theta^3 \left( 1 - \theta^3 \right) d_k^\top \nabla^2 f(x_k) d_k + \theta^3 d_k^\top r_k + \frac{L_H}{6} \theta^3 \|d_k\|^3
\]
\[
\leq -\frac{\theta^3}{2} \epsilon_H \|d_k\|^2 + \theta^3 \|d_k\| \|r_k\| + \frac{L_H}{6} \theta^3 \|d_k\|^3
\]
\[
\leq -\frac{\theta^3}{2} (1 - \zeta) \epsilon_H \|d_k\|^2 + \frac{L_H}{6} \theta^3 \|d_k\|^3.
\]
Thus, for any \(j > 0\) for which sufficient decrease is not obtained, we have
\[
\theta^{2j} \geq \frac{3}{L_H + \eta} (1 - \zeta) \epsilon_H \|d_k\|^{-1}.
\]
In particular, since (60) holds for \(j = 0\), we have
\[
\|d_k\| \geq \frac{3}{L_H + \eta} (1 - \zeta) \epsilon_H.
\]
By definition of \(d_k\), we also have the following upper bound on its norm:
\[
\|d_k\| = \|\nabla^2 f(x_k)^{-1} (g_k + r_k)\| \leq \|\nabla^2 f(x_k)^{-1}\| \|g_k + r_k\| \leq \frac{1}{\epsilon_H} \sqrt{\|g_k\|^2 + \|r_k\|^2}
\]
\[
\leq \frac{\sqrt{1 + \zeta^2 / 4}}{\epsilon_H} \|g_k\|
\]
\[
\leq \frac{\sqrt{1 + \zeta^2 / 4}}{\epsilon_H} U_g,
\]
using again the fact that \(g_k\) and \(r_k\) are orthogonal (by the properties of the CG algorithm), as well as the criterion (54) and the bound (9).

Meanwhile, for any \(j > j_0\), we have
\[
\theta^{2j} \leq \frac{3}{L_H + \eta} (1 - \zeta) \frac{\epsilon_H}{U_g \sqrt{1 + \zeta^2 / 4}} \leq \frac{3}{L_H + \eta} (1 - \zeta) \epsilon_H \frac{\epsilon_H}{U_g \sqrt{1 + \zeta^2 / 4}}
\]
\[
\leq \frac{3}{L_H + \eta} (1 - \zeta) \epsilon_H \|d_k\|^{-1}.\]
As a result, (60) is violated for \( j > j_{in} \), which means that the line search must terminate with a step length satisfying the decrease condition. Denoting by \( j_k \) the first positive integer for which sufficient decrease is obtained, one has \( j_k \geq 1 \), and the preceding step corresponding to \( j_k - 1 \) satisfies (60). Therefore, we have

\[
\theta^{j_k} \geq \sqrt{\frac{3\theta}{L_H + \eta}} (1 - \zeta) \epsilon_H^{1/2} \|d_k\|^{-1/2},
\]

and from the sufficient decrease condition, we have

\[
f(x_k) - f(x_k + \theta^{j_k} d_k) \geq \frac{\eta}{6} \theta^{j_k} \|d_k\|^3 \geq \frac{\eta}{6} \left[ \frac{3\theta}{L_H + \eta} (1 - \zeta) \right]^{3/2} \epsilon_H^{3/2} \|d_k\|^{3/2} \geq \frac{\eta}{6} \left[ \frac{3\theta}{L_H + \eta} (1 - \zeta) \right]^3 \epsilon_H^3,
\]

where the second inequality follows from (62) and the third inequality follows from (61) (using the fact that \( \theta \in (0, 1) \)). Hence, the claim (58) is satisfied in the case of non-unit step length \( \alpha_k \) too, and the proof is complete. \( \Box \)

**Lemma 15.** Let Assumptions 1 and 2 hold. Suppose that an inexact regularized Newton direction \( d_k = d_k^\ast \) is computed at the \( k \)-th iteration of Algorithm 3. Then with probability at least \( 1 - \delta \), the backtracking line search terminates with step length \( \alpha_k \) in at most \( j_{ir} = 1 + j_{ir} \) steps, where

\[
j_{ir} := \left[ \log_\theta \left( \frac{3}{L_H + \eta} \frac{3 - \zeta}{Ug \sqrt{1 + \zeta^2/4}} \right) \right]_+,
\]

and we have

\[
f(x_k) - f(x_k + \alpha_k d_k) \geq c_{ir} \min \left\{ \varphi \left( \|\nabla f(x_k + \alpha_k d_k)\|, \frac{4 + \zeta}{2} \epsilon_H \right)^3, \epsilon_H^3 \right\}.
\]

where \( \varphi \) is defined as in Lemma 4 and

\[
c_{ir} := \frac{\eta}{6} \min \left\{ 1, \left[ \frac{3\theta (3 - \zeta)}{L_H + \eta} \right]^3 \right\}.
\]

**Proof.** The inexact regularized Newton step is computed only when \( -\frac{1}{2} \epsilon_H \leq \lambda_k^1 \leq \frac{3}{2} \epsilon_H \), so from (43) with \( \epsilon = \frac{1}{2} \epsilon_H \), we have

\[
\lambda_{\min}(\nabla^2 f(x_k)) + 2\epsilon_H \geq \lambda_k^1 - \frac{1}{2} \epsilon_H + 2\epsilon_H \geq \epsilon_H,
\]

with probability at least \( 1 - \delta \). Suppose first that the step length \( \alpha_k = 1 \) satisfies the decrease condition (56). Then, defining \( r_k = (\nabla^2 f(x_k) + 2\epsilon_H I) d_k + g_k \), we have that

\[
\|\nabla f(x_k + d_k)\| = \|\nabla f(x_k + d_k) - \nabla f(x_k) + \nabla f(x_k)\| = \|\nabla f(x_k + d_k) - \nabla f(x_k) - \nabla^2 f(x_k) d_k - 2\epsilon_H d_k + r_k\| \leq \frac{L_H}{2} \|d_k\|^2 + 2\epsilon_H \|d_k\| + \|r_k\| \leq \frac{L_H}{2} \|d_k\|^2 + 4 + \frac{\zeta}{2} \epsilon_H \|d_k\|.
\]
As in the proof of Lemma 4, this leads to the following lower bound on \( \|d_k\| \):

\[
(66) \quad \|d_k\| \geq \varphi \left( \|\nabla f(x_k + d_k)\|, \frac{4 + \zeta}{2} \epsilon_H \right).
\]

Therefore, taking the unit regularized Newton step guarantees

\[
f(x_k) - f(x_k + d_k) \geq \frac{\eta}{6} \|d_k\|^3 \geq \frac{\eta}{6} \varphi \left( \|\nabla f(x_k + d_k)\|, \frac{4 + \zeta}{2} \epsilon_H \right)^3,
\]

so the result of the theorem holds in the case in which the unit step satisfies the sufficient decrease condition.

To complete the proof, we consider the case in which \( \alpha_k < 1 \). In that case, for any value \( j \geq 0 \) such that (56) is not satisfied, we have from the definition of \( r_k \), the bound on \( \|r_k\| \) in the definition of \( d_k^* \), and (65) that

\[
-\frac{\eta}{6} \theta^{3j} \|d_k\|^3 \leq f(x_k + \theta^j d_k) - f(x_k)
\]

\[
\leq \theta^j g_k^\top d_k + \frac{\theta^{2j}}{2} d_k^\top \nabla^2 f(x_k) d_k + \frac{L_H}{6} \theta^{3j} \|d_k\|^3
\]

\[
= -\theta^j \left[ \nabla^2 f(x_k) d_k + 2 \epsilon_H I - r_k \right]^\top d_k + \frac{\theta^{2j}}{2} d_k^\top \nabla^2 f(x_k) d_k + \frac{L_H}{6} \theta^{3j} \|d_k\|^3
\]

\[
= -\theta^j \left[ 1 - \theta^j \right] d_k^\top \left[ \nabla^2 f(x_k) + 2 \epsilon_H I \right] d_k - \theta^j \epsilon_H \|d_k\|^2 + \theta^j r_k^\top d_k
\]

\[
\leq \frac{L_H}{6} \theta^{3j} \|d_k\|^3
\]

\[
\leq -\frac{\theta^j}{2} \epsilon_H \|d_k\|^2 - \theta^j \epsilon_H \|d_k\|^2 + \theta^j \|r_k\| \|d_k\| + \frac{L_H}{6} \theta^{3j} \|d_k\|^3
\]

\[
\leq \left( \frac{1}{2} \theta^j - \theta^j \right) \epsilon_H \|d_k\|^2 + \theta^j \frac{\zeta}{2} \epsilon_H \|d_k\|^2 + \frac{L_H}{6} \theta^{3j} \|d_k\|^3
\]

\[
= \left( \frac{1}{2} \theta^j (1 - \zeta) - \theta^{2j} \right) \epsilon_H \|d_k\|^2 + \frac{L_H}{6} \theta^{3j} \|d_k\|^3
\]

\[
\leq -\frac{\theta^j}{2} (3 - \zeta) \epsilon_H \|d_k\|^2 + \frac{L_H}{6} \theta^{3j} \|d_k\|^3.
\]

Thus, for any \( j \geq 0 \) for which sufficient decrease is not obtained, one has

\[
(67) \quad \theta^j \geq \frac{3}{L_H + \eta} (3 - \zeta) \epsilon_H \|d_k\|^{-1}.
\]

Note that the right-hand side is bounded below from

\[
\|d_k\| = \left\| \left[ \nabla^2 f(x_k) + 2 \epsilon_H I \right]^{-1} (-g_k + r_k) \right\| \leq \left\| \left[ \nabla^2 f(x_k) + 2 \epsilon_H I \right]^{-1} \right\| \| -g_k + r_k \|
\]

\[
\leq \frac{1}{\epsilon_H} \sqrt{\|g_k\|^2 + \|r_k\|^2}
\]

\[
\leq \frac{\sqrt{1 + \zeta^2/4}}{\epsilon_H} \|g_k\|
\]

\[
\leq \frac{\sqrt{1 + \zeta^2/4}}{\epsilon_H} U_g,
\]

(68)
where we used again the orthogonality of \(g_k\) and \(r_k\) (from the properties of conjugate gradient) as well as the condition (55). For any \(j > j_{ir}\), we have

\[
\theta^j < \theta^{j_{ir}} \leq 3(3 - \zeta) \frac{\epsilon_H^2}{L_H + \eta U_g \sqrt{1 + \zeta^2/4}} \leq \frac{3(3 - \zeta)}{L_H + \eta} \epsilon_H \|d_k\|^{-1},
\]

where the last inequality follows from (68). Therefore, (67) is violated for \(j > j_{ir}\), which means that the line search must terminate with a step length satisfying the decrease condition in a most \(j_{ir}\) backtracking steps. Denoting by \(j_k\) the first positive integer for which sufficient decrease is obtained, one has \(j_k \geq 1\), and the preceding step corresponding to \(j_k - 1\) satisfies (67). Therefore, we have

\[
\theta^{j_k} \geq \frac{3 \theta}{L_H + \eta} (3 - \zeta) \epsilon_H \|d_k\|^{-1},
\]

so that

\[
f(x_k) - f(x_k + \theta^{j_k} d_k) \geq \frac{\eta}{6} \theta^{j_k} \|d_k\|^3 \geq \frac{\eta}{6} \left[ \frac{3 \theta}{L_H + \eta} (3 - \zeta) \right]^3 \epsilon_H.
\]

Thus, condition (64) also holds in the case of \(\alpha_k < 1\), and the proof is complete.

**Theorem 16.** Let Assumptions 1 and 2 hold. Then, Algorithm 3 returns a point \(x_k\) satisfying (2) in at most

\[
\hat{K} := \hat{c} \max \left\{ \epsilon_g^{-3} \epsilon_H, \epsilon_g^{-3/2}, \epsilon_H^{-3}, \varphi \left( \frac{\epsilon_g, \frac{\zeta}{2} \epsilon_H}{\epsilon_g, \frac{4 + \zeta}{2} \epsilon_H} \right)^{-3}, \varphi \left( \epsilon_g, \frac{4 + \zeta}{2} \epsilon_H \right)^{-3} \right\}
\]

iterations, where

\[
\hat{c} := \frac{f(x_0) - f_{low}}{\tilde{c}}, \quad \tilde{c} := \min \left\{ c_g, \frac{c_e}{8}, c_{in}, c_{ir} \right\},
\]

with probability at least \(1 - \hat{K} \delta\). The constants \(c_g\), \(c_e\), \(c_{in}\), and \(c_{ir}\) are defined in Lemmas 2, 1, 14, and 15, respectively.

**Proof.** For any iteration \(l\) such that \(x_l\) does not satisfy (8), we must have that either \(\min(\|g_l\|, \|g_{l+1}\|) > \epsilon_g\) or \(\lambda_{\min}(\nabla^2 f(x_l)) < -\epsilon_H\), where the latter implies that \(\lambda_l^i < -\frac{1}{2} \epsilon_H\). Thus, similarly to the proof of Theorem 5, we can consider the following two cases.

**Case 1:** \(\lambda_l^i < -\frac{1}{2} \epsilon_H\).

From Table 2, we see that the same three choices for \(d_l\) as in the exact version are possible. If \(d_l = \frac{R_l}{\|g_l\|} g_l\), we have exactly as in Lemma 1 that

\[
f(x_l) - f(x_{l+1}) \geq c_e \epsilon_H^3.
\]

When \(d_l = -g_l/\|g_l\|^{1/2}\), we have from Lemma 2 that

\[
f(x_l) - f(x_{l+1}) \geq c_g \min \left\{ \epsilon_g^{-3} \epsilon_H, \epsilon_g^{3/2} \right\}.
\]

The remaining case corresponds to the choice \(d_l = v_l^i\). Since

\[
\frac{d_l^T \nabla^2 f(x_l) d_l}{\|d_l\|^2} = \lambda_l^i \leq -\frac{\epsilon_H}{2},
\]

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we have from Lemma 1 that
\[ f(x_l) - f(x_{l+1}) \geq \frac{c_{\varepsilon}}{8} \varepsilon_H^2. \]

**Case 2:** \( \lambda_l^1 \geq -\frac{1}{2} \varepsilon_H, \|g_l\| > \varepsilon_g, \) and \( \|g_{l+1}\| > \varepsilon_g. \)

In this situation, we have three possible choices of search direction \( d_l \). If \( d_l = -g_l/\|g_l\|^{1/2} \), we have from Lemma 2 that
\[ f(x_l) - f(x_{l+1}) \geq c_g \min \{ \varepsilon_g^{-3}, \varepsilon_g^{-3/2} \}. \]

If the inexact Newton direction is taken, we obtain by Lemma 14 and the monotonic nature of \( \varphi \) with respect to its first argument that
\[ f(x_k) - f(x_{k+1}) \geq c_{in} \min \left\{ \varphi \left( \frac{\varepsilon_g}{2}, \frac{\varepsilon_H}{2} \right)^3, \varepsilon_H^3 \right\} \]

Finally, if the search direction is the inexact regularized Newton direction, that is, \( d_l = d_l^r \), we have from Lemma 15 that
\[ f(x_k) - f(x_{k+1}) \geq c_{ir} \min \left\{ \varphi \left( \frac{\varepsilon_g}{4}, \frac{\varepsilon_H}{2} \right)^3, \varepsilon_H^3 \right\} \]

By putting all these bounds together, as in the proof of Theorem 5, we obtain that the number of iterations before reaching a point satisfying (8) is bounded above by \( \hat{K} \) defined in the statement of the theorem.

Recalling that for each of these iterations, there is a probability \( \delta \) that the randomized Lanczos iteration in (53) will fail, we bound the probability of failure during the course of the algorithm by \( \hat{K} \delta \).

Note that if \( \delta \) is chosen large enough such that \( 1 - \hat{K} \delta < 0 \), Theorem 16 is not informative. The same remark holds for the corollary below, that makes use of the results from Sections 3.1 and 3.2 to obtain a bound on the total number of Hessian-vector multiplications and gradient evaluations needed by the procedure (assuming that these operations cost roughly the same).

**Corollary 17.** Suppose the assumptions of Theorem 16 hold, and let \( \delta \in (0,1) \) be given. Then the total number of gradient evaluations and Hessian-vector multiplications requires by Algorithm 3 to reach an iterate satisfying (8) is bounded by
\[
\left[ 2 + \min \left\{ n, (U_H + 2)^{1/2} \varepsilon_H^{-1/2} \ln \left( \frac{4(U_H + 2)^{1/2} \varepsilon_H^{-1/2}}{\varepsilon} \right) \right\} + \right.
\]
\[
\left. \min \left\{ n, (U_H + 2)^{1/2} \varepsilon_H^{-1/2} \ln \left( \frac{n/\delta^2}{2} \right) \right\} \right] \times \hat{K},
\]

with probability \( 1 - \hat{K} \delta \).

**Proof.** The proof follows directly from Lemmas 11 and 13, setting \( M = U_H + 2 \) and \( \varepsilon = \varepsilon_H/2 \), noting that for both Newton and regularized Newton steps, the condition number of the respective coefficient matrices can be bounded by \( (U_H + 2)/\varepsilon_H \).

As in Section 2.2, we can particularize this result to a specific choice of tolerances.
Corollary 18. Suppose that the assumptions of Theorem 16 hold, and let \( \delta \in (0, 1) \) be given. Define \( \epsilon_g = \epsilon \) and \( \epsilon_H = \sqrt{\epsilon} \), for some \( \epsilon \in (0, 1) \). Then the number of gradient evaluations and Hessian-vector products needed to Algorithm 3 to satisfy (8) is bounded by

\[
\begin{align*}
2 + \min \left\{ n, (U_H + 2)^{1/2} \epsilon^{-1/4} \ln \left( \frac{4(U_H+2)^{5/2} \epsilon^{-5/4}}{\zeta} \right) \right\} + \\
\min \left\{ n, (U_H + 2)^{1/2} \epsilon^{-1/4} \frac{\ln(n/\delta^2)}{2} \right\} \times \hat{C}_1 \epsilon^{-3/2},
\end{align*}
\]

where \( \hat{C}_1 = (f(x_0) - f_{\text{low}})/\hat{c}_1 \) and

\[
\hat{c}_1 = \min \left\{ c_g, c_{\text{ie}}, c_{\text{ir}}, \frac{\zeta + \sqrt{\zeta^2 + 2L_H}}{L_H}, \frac{-c_{\text{in}} - (4 + \zeta) + \sqrt{(4 + \zeta)^2 + 2L_H}}{L_H} \right\},
\]

with probability at least \( 1 - \hat{C}_1 \epsilon^{-3/2} \delta \).

This result is meaningful when \( \delta \ll \epsilon^{3/2} \). In terms of the complexity bound, such a choice is not prohibitively small, because \( \delta \) enters into the bound (71) only inside a log term.

We can obtain a bound for the case of \( \delta = 0 \) (that is, complete certainty), at the cost of taking \( n \) Lanczos iterations whenever the smallest eigenvalue is needed (see Lemma 11). In this case, the bound (71) either becomes \( O \left( (n + \ln(\epsilon^{-1})) \epsilon^{-7/4} \right) \) or \( O(n \epsilon^{-3/2}) \), depending on which term dominates in the quantity corresponding to conjugate gradient iterations.

For very large \( n \) and \( \delta > 0 \), we can consider that the term involving \( \epsilon \) is smaller than \( n \) in both minimum expressions. In that case, the bound is

\[
O \left( \ln \left( \frac{1}{\min\{\epsilon, \delta/\sqrt{n}\}} \right) \epsilon^{-7/4} \right).
\]

This rate matches the recent findings of [1] and [5].

As a final note, we observe that one could also include the number of line-search iterations into our complexity bound. However, this cost is essentially logarithmic in \( 1/\epsilon \), therefore it is dominated by the cost of the linear algebra techniques.

4. Discussion. Among the many algorithmic frameworks that have been proposed for smooth nonconvex optimization with second-order complexity guarantees, it can be difficult to determine the algorithmic features that affect the complexity analysis, and how the guarantees provided by different algorithms relate to each other. We have presented a second-order complexity analysis of a framework that is based exclusively on line searches along certain directions. It does not require solution of cubic-regularized or trust-region subproblems, or minimization of convexified functions — operations that are needed by other approaches. Our search directions are of several types — gradient, negative-curvature, Newton, and regularized Newton — and we present a variant of our method that allows inexact direction computation using iterative methods. We believe that ours is the first approach of line-search type to achieve known optimal complexity, among methods that identify points that satisfy approximate second-order necessary conditions.

In designing the framework of Algorithms 1 and 3, we have made some choices to give preference to one direction choice over another, and we have also incorporated several types of steps. Given the recent literature in this area, our proposed scheme
is actually one particular instance of a broader class of methods with similar complexity guarantees but possibly diverse practical performance. An implementation of our approach would raise several delicate issues, for example, issues associated with failure of the randomized Lanczos procedure for obtaining an estimate of the smallest eigenvalue. An incorrect estimate here could lead to the conjugate gradient method subsequently being applied to an indefinite matrix; a robust implementation would need to detect and recover from such an occurrence. Additionally, the choice of suitable values for the bound on the Hessian norm is likely to be of critical importance. Addressing these concerns in the aim of developing a practical algorithm with good complexity guarantees is the subject of ongoing research.

REFERENCES