Semidefinite Programming and Nash Equilibria in Bimatrix Games

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Abstract
We explore the power of semidefinite programming (SDP) for finding additive $\epsilon$-approximate Nash equilibria in bimatrix games. We introduce an SDP relaxation for a quadratic programming formulation of the Nash equilibrium (NE) problem and provide a number of valid inequalities to improve the quality of the relaxation. If a rank-1 solution to this SDP is found, then an exact NE can be recovered. We show that for a strictly competitive game, our SDP is guaranteed to return a rank-1 solution. Furthermore, we prove that if a rank-2 solution to our SDP is found, then a $\frac{5}{11}$-NE can be recovered for any game, or a $\frac{1}{3}$-NE for a symmetric game. We propose two algorithms based on iterative linearization of smooth nonconvex objective functions that are designed so that their global minima coincide with rank-1 solutions. Empirically, we demonstrate that these algorithms often recover solutions of rank at most two and $\epsilon$ close to zero. We then show how our SDP approach can address two (NP-hard) problems of economic interest: finding the maximum welfare achievable under any NE, and testing whether there exists a NE where a particular set of strategies is not played. Finally, we show the connection between our SDP and the first level of the Lasserre/sum of squares hierarchy.

1 Introduction
A bimatrix game is a game between two players (referred to in this paper as players A and B) defined by a pair of $m \times n$ payoff matrices $A$ and $B$. Let $\Delta_m$ and $\Delta_n$ denote the $m$-dimensional and $n$-dimensional simplices

$$\Delta_m = \{ x \in \mathbb{R}^m | x_i \geq 0, \forall i, \sum_{i=1}^m x_i = 1 \}, \Delta_n = \{ y \in \mathbb{R}^n | y_i \geq 0, \forall i, \sum_{i=1}^n y_i = 1 \}.$$ 

These form the strategy spaces of player A and player B respectively. For a strategy pair $(x, y) \in \Delta_m \times \Delta_n$, the payoff received by player A (resp. player B) is $x^T A y$ (resp. $x^T B y$). In particular, if the players pick vertices $i$ and $j$ of their respective simplices (also called pure strategies), their payoffs will be $A_{i,j}$ and $B_{i,j}$. One of the prevailing solution concepts for bimatrix games is the notion of Nash equilibrium. At such an equilibrium, the players are playing mutual best responses, i.e., a payoff maximizing strategy against the opposing player’s strategy. In our notation, a Nash equilibrium for the game $(A, B)$ is a pair of strategies $(x^*, y^*) \in \Delta_m \times \Delta_n$ such that

$$x^* A y^* \geq x^T A y^*, \forall x \in \Delta_m,$$

and

$$x^* B y^* \geq x^T B y^*, \forall y \in \Delta_n.$$

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1 In this paper we assume that all entries of $A$ and $B$ are between 0 and 1, and argue at the beginning of Section 2 why this is without loss of generality for the purpose of computing Nash equilibria.

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Nash [29] proved that for any bimatrix game, such pairs of strategies exist (in fact his result more generally applies to games with a finite number of players and a finite number of pure strategies). While existence of these equilibria is guaranteed, finding them is believed to be a computationally intractable problem. More precisely, a result of Daskalakis, Goldberg, and Papadimitriou [12] implies that computing Nash equilibria is PPAD-complete (see [12] for a definition) even when the number of players is 3. This result was later improved by Chen and Deng [9] who showed the same hardness result for bimatrix games.

These results motivate the notion of an approximate Nash equilibrium, a solution concept in which players receive payoffs “close” to their best response payoffs. More precisely, a pair of strategies \((x^*, y^*) \in \Delta_m \times \Delta_n\) is an (additive) \(\epsilon\)-Nash equilibrium for the game \((A, B)\) if

\[
x^* A y^* \geq x^T A y^* - \epsilon, \forall x \in \Delta_m,
\]

and

\[
x^T B y^* \geq x^* T B y^* - \epsilon, \forall y \in \Delta_n.
\]

Note that when \(\epsilon = 0\), \((x^*, y^*)\) form an exact Nash equilibrium, and hence it is of interest to find \(\epsilon\)-Nash equilibria with \(\epsilon\) small. Unfortunately, approximation of Nash equilibria has also proved to be computationally difficult. Cheng, Deng, and Teng have shown in [10] that, unless \(\text{PPAD} \subseteq \text{P}\), there cannot be a fully polynomial-time approximation scheme for computing Nash equilibria in bimatrix games. There have, however, been a series of constant factor approximation algorithms for this problem [14, 13, 20, 37], with the current best producing a .3393 approximation via an algorithm by Tsaknakis and Spirakis [37].

We remark that there are exponential-time algorithms for computing Nash equilibria, such as the Lemke-Howson algorithm [25, 34]. There are also certain subclasses of the problem which can be solved in polynomial time, the most notable example being the case of zero-sum games (i.e. when \(B = -A\)). This problem was shown to be solvable via linear programming by Dantzig [11], and later shown to be polynomially equivalent to linear programming by Adler [2]. Aside from computation of Nash equilibria, there are a number of related decision questions which are of economic interest but unfortunately NP-hard. Examples include deciding whether a player’s payoff exceeds a certain threshold in some Nash equilibrium, deciding whether a game has a unique Nash equilibrium, or testing whether there exists a Nash equilibrium where a particular set of strategies is not played [16].

Our focus in this paper is on understanding the power of semidefinite programming (SDP) for finding approximate Nash equilibria in bimatrix games or providing certificates for related decision questions. The goal is not to develop a competitive solver, but rather to analyze the algorithmic power of SDP when applied to basic problems around computation of Nash equilibria. Semidefinite programming relaxations have been analyzed in the past for an array of intractable problems in computational mathematics (most notably in combinatorial optimization [17, 26] and systems theory [8]), but to our knowledge not for computation of Nash equilibria in general bimatrix games. SDPs have appeared however elsewhere in the literature on game theory for finding equilibria, e.g. by Stein for exchangeable equilibria in symmetric games [36], by Parrilo, Laraki, and Lasserre for Nash equilibria in zero-sum polynomial games [31, 21], or by Parrilo and Shah for zero-sum stochastic games [35].

1.1 Organization and Contributions of the Paper

In Section 2 we formulate the problem of finding a Nash equilibrium in a bimatrix game as a nonconvex quadratically constrained quadratic program and pose a natural SDP relaxation for it. In Section 3 we show that our SDP is exact when the game is strictly competitive; see Definition 3.3. [2] The unfamiliar reader is referred to [38] for the theory of SDPs and a description of polynomial-time algorithms for them based on interior point methods.
In Section 4, we establish a number of bounds on the quality of the approximate Nash equilibria that can be read off of feasible solutions to our SDP. We show that if the SDP has a rank-$k$ solution with nonzero eigenvalues $\lambda_1, \ldots, \lambda_k$, then one can recover from it an $\epsilon$-Nash equilibrium with $\epsilon \leq \frac{m+n}{2} \sum_{i=2}^{k} \lambda_i$ (Theorem 4.5). We then present an improved analysis in the rank-2 case which shows how one can recover a $\frac{\sqrt{2}}{4}$-Nash equilibrium from the SDP solution (Theorem 4.12). We further prove that for symmetric games (i.e., when $B = A^T$), a $\frac{1}{3}$-Nash equilibrium can be recovered in the rank-2 case (Theorem 4.16). We do not currently know of a polynomial-time algorithm to find rank-2 solutions to our SDP. If such an algorithm were found, it would, together with our analysis, improve the best known approximation bound for symmetric games. This motivates us to design, in Section 5, two continuous but nonconvex objective functions for our SDP whose global minima coincide with minimum-rank solutions. We prove an upper bound on $\epsilon$ based on the value of one of these objective functions (Theorem 5.6) and provide a heuristic for minimizing them both that is based on iterative linearization. We show empirically that these approaches produce $\epsilon$ very close to zero (on average in the order of $10^{-3}$). In Section 6, we show how our SDP formulation can be used to provide certificates for certain (NP-hard) questions of economic interest about Nash equilibria. These are the problems of testing whether the maximum welfare achievable under any Nash equilibrium exceeds some threshold, and whether a set of strategies is played in every Nash equilibrium. In Section 7, we show that the SDP analyzed in this paper dominates the first level of the Lasserre hierarchy (Proposition 7.1). Some directions for future research are discussed in Section 8.

2 The Formulation of our SDP Relaxation

In this section we present an SDP relaxation for the problem of finding Nash equilibria in bimatrix games. This is done after a straightforward reformulation of the problem as a nonconvex quadratically constrained quadratic program. Throughout the paper the following notation is used.

- $A_i$ refers to the $i$-th row of a matrix $A$.
- $A_{ij}$ refers to the $j$-th column of a matrix $A$.
- $e_i$ refers to the elementary vector $(0, \ldots, 0, 1, 0, \ldots, 0)^T$ with the 1 being in position $i$.
- $\Delta_k$ refers to the $k$-dimensional simplex.
- $1_m$ refers to the $m$-dimensional vector of one's.
- $0_m$ refers to the $m$-dimensional vector of zero's.
- $J_{m,n}$ refers to the $m \times n$ matrix of one's.
- $A \succeq 0$ denotes that the matrix $A$ is positive semidefinite (psd), i.e., has nonnegative eigenvalues.
- $A \succeq 0$ denotes that the matrix $A$ is nonnegative, i.e., has nonnegative entries.
- $A \succeq B$ denotes that $A - B \succeq 0$.
- $S_{k \times k}$ denotes the set of symmetric $k \times k$ matrices.
- $\text{Tr}(A)$ denotes the trace of a matrix $A$, i.e., the sum of its diagonal elements.
- $A \otimes B$ denotes the Kronecker product of matrices $A$ and $B$.
- $\text{vec}(M)$ denotes the vectorized version of a matrix $M$, and $\text{diag}(M)$ denotes the vector containing its diagonal entries.

We also assume that all entries of the payoff matrices $A$ and $B$ are between 0 and 1. This can be done without loss of generality because Nash equilibria are invariant under certain affine transformations in the payoffs. In particular, the games $(A, B)$ and $(cA + dJ_{m \times n}, eB + fJ_{m \times n})$
have the same Nash equilibria for any scalars $c, d, e,$ and $f$, with $c$ and $e$ positive. This is because

$$x^T Ay \geq x^T Ay$$

$$\Leftrightarrow c(x^T Ay^*) + d \geq c(x^T Ay^*) + d$$

$$\Leftrightarrow x^T (cA + dJ_{m \times n})y^* \geq x^T (cA + dJ_{m \times n})y$$

Identical reasoning applies for player B.

### 2.1 Nash Equilibria as Solutions to Quadratic Programs

Recall the definition of a Nash equilibrium from Section 1. An equivalent characterization is that a strategy pair $(x^*, y^*) \in \Delta_m \times \Delta_n$ is a Nash equilibrium for the game $(A, B)$ if and only if

$$x^T Ay^* \geq e_i^T Ay^*, \forall i \in \{1, \ldots, m\},$$

$$x^T By^* \geq x^T Be_i, \forall i \in \{1, \ldots, n\}. \quad (1)$$

The equivalence can be seen by noting that because the payoff from playing any mixed strategy is a convex combination of payoffs from playing pure strategies, there is always a pure strategy best response to the other player’s strategy.

We now treat the Nash problem as the following quadratic programming (QP) feasibility problem:

$$\min_{x \in \mathbb{R}^m, y \in \mathbb{R}^n} 0$$

subject to

$$x^T Ay \geq e_i^T Ay, \forall i \in \{1, \ldots, m\},$$

$$x^T By \geq x^T Be_j, \forall j \in \{1, \ldots, n\},$$

$$x_i \geq 0, \forall i \in \{1, \ldots, m\},$$

$$y_j \geq 0, \forall j \in \{1, \ldots, n\},$$

$$\sum_{i=1}^{m} x_i = 1,$$

$$\sum_{i=1}^{n} y_i = 1. \quad (2)$$

Similarly, a pair of strategies $x^* \in \Delta_m$ and $y^* \in \Delta_n$ form an $\epsilon$-Nash equilibrium for the game $(A, B)$ if and only if

$$x^* A y^* \geq e_i^T Ay^* - \epsilon, \forall i \in \{1, \ldots, m\},$$

$$x^* B y^* \geq x^* Be_i - \epsilon, \forall i \in \{1, \ldots, n\}.$$ Observe that any pair of simplex vectors $(x, y)$ is an $\epsilon$-Nash equilibrium for the game $(A, B)$ for any $\epsilon$ that satisfies

$$\epsilon \geq \max_i e_i^T Ay - x^T Ay, \max_i x^T Be_i - x^T By.$$

We use the following notation throughout the paper:

- $\epsilon_A(x, y) := \max_i e_i^T Ay - x^T Ay,$
- $\epsilon_B(x, y) := \max_i x^T Be_i - x^T By,$
- $\epsilon(x, y) := \max\{\epsilon_A(x, y), \epsilon_B(x, y)\},$

and the function parameters are later omitted if they are clear from the context.
2.2 SDP Relaxation

The QP formulation in (2) lends itself to a natural SDP relaxation. We define a matrix

\[ M := \begin{bmatrix} X & P \\ Z & Y \end{bmatrix}, \]

and an augmented matrix

\[ M' := \begin{bmatrix} X & P & x \\ Z & Y & y \\ x & y & 1 \end{bmatrix}, \]

with \( X \in S^{m \times m}, Z \in \mathbb{R}^{n \times m}, Y \in S^{n \times n}, x \in \mathbb{R}^m, y \in \mathbb{R}^n \) and \( P = Z^T \).

The SDP relaxation can then be expressed as

\[
\begin{align*}
\min_{M' \in S^{m+n+1,m+n+1}} & \quad 0 \\
\text{subject to} & \quad \text{Tr}(AZ) \geq e_i^T Ay, \forall i \in \{1, \ldots, m\}, \quad (3) \\
& \quad \text{Tr}(BZ) \geq x^T Be_j, \forall j \in \{1, \ldots, n\}, \quad (4) \\
& \quad \sum_{i=1}^m x_i = 1, \quad (5) \\
& \quad \sum_{i=1}^n y_i = 1, \quad (6) \\
& \quad M'_{ij} \geq 0 \quad \forall i, j \in \{1, \ldots, m + n + 1\}, \quad (7) \\
& \quad M'_{m+n+1,m+n+1} = 1, \quad (8) \\
& \quad M' \succeq 0. \quad (9)
\end{align*}
\]

We refer to the constraints (3) and (4) as the relaxed Nash constraints and the constraints (5) and (6) as the unity constraints. This SDP is motivated by the following observation.

**Proposition 2.1.** Let \( M' \) be any rank-1 feasible solution to SDP1. Then the vectors \( x \) and \( y \) from its last column constitute a Nash equilibrium for the game \((A, B)\).

**Proof.** We know that \( x \) and \( y \) are in the simplex from the constraints (5), (6), and (7). If the matrix \( M' \) is rank-1, then it takes the form

\[
\begin{bmatrix}
xx^T & xy^T & x \\
x^Ty^T & y^T & y \\
x^T & y & 1
\end{bmatrix} =
\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}^T .
\]

Then, from the relaxed Nash constraints we have that

\[
\begin{align*}
e_i^T Ay & \leq \text{Tr}(AZ) = \text{Tr}(Ayx^T) = \text{Tr}(x^T Ay) = x^T Ay, \\
x^T Ae_i & \leq \text{Tr}(BZ) = \text{Tr}(Byx^T) = \text{Tr}(x^T By) = x^T By.
\end{align*}
\]

The claim now follows from the characterization given in (1). \( \square \)

**Remark 2.1.** Because a Nash equilibrium always exists, there will always be a matrix of the form (10) which is feasible to SDP1. Thus we can disregard any concerns about SDP1 being feasible, even when we add valid inequalities to it in Section 2.3.
Remark 2.2. It is intuitive to note that the submatrix $P = Z^T$ of the matrix $\mathcal{M}'$ corresponds to a probability distribution over the strategies, and that seeking a rank-1 solution to our SDP can be interpreted as making $P$ a product distribution.

The following theorem shows that [SDP1] is a weak relaxation and stresses the necessity of additional valid constraints.

**Theorem 2.2.** Consider a bimatrix game with payoff matrices bounded in $[0, 1]$. Then for any two vectors $x \in \Delta_m$ and $y \in \Delta_n$, there exists a feasible solution $\mathcal{M}'$ to [SDP1] with $\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$ as its last column.

**Proof.** Consider any $x, y, \gamma > 0$, and the matrix

$$\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}^T + \begin{bmatrix} \gamma J_{m+n,m+n} & 0_{m+n} \\ 0_{m+n} & 0 \end{bmatrix}.$$

This matrix is the sum of two nonnegative psd matrices and is hence nonnegative and psd. By assumption $x$ and $y$ are in the simplex, and so constraints $(5) - (9)$ of [SDP1] are satisfied. To check that constraints $(3)$ and $(4)$ hold, note that since $A$ and $B$ are nonnegative, as long as the matrices $A$ and $B$ are not the zero matrices, the quantities $\text{Tr}(AZ)$ and $\text{Tr}(BZ)$ will become arbitrarily large as $\gamma$ increases. Since $e_i^T Ay$ and $x^T Be_i$ are bounded by 1 by assumption, we will have that constraints $(3)$ and $(4)$ hold for $\gamma$ large enough. In the case where $A$ or $B$ is the zero matrix, the Nash constraints are trivially satisfied for the respective player.

### 2.3 Valid Inequalities

In this subsection, we introduce a number of valid inequalities to improve upon the SDP relaxation in [SDP1]. These inequalities are justified by being valid if the matrix returned by the SDP is rank-1. The terminology we introduce here to refer to these constraints is used throughout the paper. Constraints $(11)$ and $(12)$ will be referred to as the row inequalities, and $(13)$ and $(14)$ will be referred to as the correlated equilibrium inequalities.

**Proposition 2.3.** Any rank-1 solution $\mathcal{M}'$ to [SDP1] must satisfy the following:

1. \[ \sum_{j=1}^m X_{i,j} = \sum_{j=1}^n P_{i,j} = x_i, \forall i \in \{1, \ldots, m\}, \]  
2. \[ \sum_{j=1}^n Y_{i,j} = \sum_{j=1}^m Z_{i,j} = y_i, \forall i \in \{1, \ldots, n\}. \]  
3. \[ \sum_{j=1}^n A_{i,j} P_{i,j} \geq \sum_{j=1}^n A_{k,j} P_{i,j}, \forall i, k \in \{1, \ldots, m\}, \]  
4. \[ \sum_{j=1}^m B_{j,i} P_{j,i} \geq \sum_{j=1}^m B_{j,k} P_{j,i}, \forall i, k \in \{1, \ldots, n\}. \]

**Proof.** Recall from $(10)$ that if $\mathcal{M}'$ is rank-1, it is of the form

$$\begin{bmatrix} xx^T & xy^T & x \\ yx^T & yy^T & y \\ x^T & y^T & 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}^T.$$
To show (11), observe that
\[ \sum_{j=1}^{m} X_{i,j} = \sum_{j=1}^{m} x_{i} x_{j} = x_{i}, \forall i \in \{1, \ldots, m\}. \]

An identical argument works for the remaining matrices \( P, Z, \) and \( Y \). To show (13) and (14), observe that a pair \((x, y)\) is a Nash equilibrium if and only if
\[ \forall i, x_{i} > 0 \Rightarrow e_{i}^{T} A y = x^{T} A y = \max_{i} e_{i}^{T} A y, \]
\[ \forall i, y_{i} > 0 \Rightarrow x^{T} B e_{i} = x^{T} B y = \max_{i} x^{T} B e_{i}. \]

This is because the Nash conditions require that \( x^{T} A y \), a convex combination of \( e_{i}^{T} A y \), be at least \( e_{i}^{T} A y \) for all \( i \). Indeed, if \( x_{i} > 0 \) but \( e_{i}^{T} A y < x^{T} A y \), the convex combination must be less than \( \max_{i} x^{T} A y \).

For each \( i \) such that \( x_{i} = 0 \) or \( y_{i} = 0 \), inequalities (13) and (14) reduce to \( 0 \geq 0 \), so we only need to consider strategies played with positive probability. Observe that if \( M' \) is rank-1, then
\[ \sum_{j=1}^{n} A_{i,j} P_{i,j} = x_{i} \sum_{j=1}^{n} A_{i,j} y_{j} = x_{i} e_{i}^{T} A y \geq x_{i} e_{k}^{T} A y = \sum_{j=1}^{n} A_{k,j} P_{i,j}, \forall i, k \]
\[ \sum_{j=1}^{m} B_{j,i} P_{j,i} = y_{i} \sum_{j=1}^{m} B_{j,i} x_{j} = y_{i} x^{T} B e_{i} \geq y_{i} x^{T} B e_{k} = \sum_{j=1}^{m} B_{j,i} P_{j,k}, \forall i, k. \]

Remark 2.3. There are two ways to interpret the inequalities in (13) and (14): the first is as a relaxation of the constraint \( x_{i}(e_{i}^{T} A y - e_{j}^{T} A y) \geq 0, \forall i, j \), which must hold since any strategy played with positive probability must give the best response payoff. The other interpretation is to have the distribution over outcomes defined by \( P \) be a correlated equilibrium [4]. This can be imposed by a set of linear constraints on the entries of \( P \) as explained next.

Suppose the players have access to a public randomization device which prescribes a pure strategy to each of them (unknown to the other player). The distribution over the assignments can be given by a matrix \( P \), where \( P_{i,j} \) is the probability that strategy \( i \) is assigned to player A and strategy \( j \) is assigned to player B. This distribution is a correlated equilibrium if both players have no incentive to deviate from the strategy prescribed, that is, if the prescribed pure strategies \( a \) and \( b \) satisfy
\[ \sum_{j=1}^{n} A_{i,j} \text{Prob}(b = j|a = i) \geq \sum_{j=1}^{n} A_{k,j} \text{Prob}(b = j|a = i), \]
\[ \sum_{i=1}^{m} B_{i,j} \text{Prob}(a = i|b = j) \geq \sum_{i=1}^{m} B_{i,k} \text{Prob}(a = i|b = j). \]

If we interpret the \( P \) submatrix in our SDP as the distribution over the assignments by the public device, then because of our row constraints, \( \text{Prob}(b = j|a = i) = \frac{P_{i,j}}{x_{i}} \) whenever \( x_{i} \neq 0 \) (otherwise the above inequalities are trivial). Similarly, \( P(a = i|b = j) = \frac{P_{i,j}}{y_{j}} \) for nonzero \( y_{j} \). Observe now that the above two inequalities imply (13) and (14). Finally, note that every Nash equilibrium generates a correlated equilibrium, since if \( P \) is a product distribution given by \( xy^{T} \), then \( \text{Prob}(b = j|a = i) = y_{j} \) and \( P(a = i|b = j) = x_{i} \).
2.3.1 Implied Inequalities

In addition to those explicitly mentioned in the previous section, there are other natural valid inequalities which are omitted because they are implied by the ones we have already proposed. We give two examples of such inequalities in the next proposition. We refer to the constraints in (15) below as the distribution constraints. The constraints in (16) are the familiar McCormick inequalities [27] for box-constrained quadratic programming.

**Proposition 2.4.** Let $z := \begin{bmatrix} x \\ y \end{bmatrix}$. Any rank-1 solution $M'$ to SDP1 must satisfy the following:

\[
\sum_{i=1}^{m} \sum_{j=1}^{m} X_{i,j} = \sum_{i=1}^{n} \sum_{j=1}^{m} Z_{i,j} = \sum_{i=1}^{n} \sum_{j=1}^{n} Y_{i,j} = 1.
\]  

(15)

\[
M_{i,j} \leq z_{i}, \forall i, j \in \{1, \ldots, m + n\},
\]

\[
M_{i,j} + 1 \geq z_{i} + z_{j}, \forall i, j \in \{1, \ldots, m + n\}.
\]  

(16)

**Proof.** The distribution constraints follow immediately from the row constraints (11) and (12), along with the unity constraints (5) and (6).

The first McCormick inequality is immediate as a consequence of (11) and (12), as all entries of $M$ are nonnegative. To see why the second inequality holds, consider whichever submatrix $X, Y, P$, or $Z$ that contains $M_{i,j}$. Suppose that this submatrix is, e.g., $P$. Then, since $P$ is nonnegative,

\[
0 \leq \sum_{k=1, k \neq i}^{m} \sum_{l=1, l \neq j}^{n} P_{k,l} \sum_{k=1, k \neq i}^{m} (x_{k} - P_{k,j}) \sum_{k=1, k \neq i}^{m} (1 - x_{i}) - (y_{j} - P_{i,j}) = P_{i,j} + 1 - x_{i} - y_{j}.
\]

The same argument holds for the other submatrices, and this concludes the proof. \( \square \)

2.3.2 The Effect of Valid Inequalities on an Example Game

Consider the following randomly-generated 5 × 5 bimatrix game:

\[
A = \begin{bmatrix}
0.42 & 0.46 & 0.03 & 0.77 & 0.33 \\
0.54 & 0.03 & 0.71 & 0.06 & 0.56 \\
0.53 & 0.43 & 0.17 & 0.85 & 0.30 \\
0.19 & 0.56 & 0.59 & 0.38 & 0.37 \\
0.08 & 0.64 & 0.61 & 0.40 & 0.35
\end{bmatrix},
\]

\[
B = \begin{bmatrix}
0.63 & 0.19 & 0.09 & 0.22 & 0.33 \\
0.66 & 0.92 & 0.26 & 0.97 & 0.43 \\
0.99 & 0.29 & 0.43 & 0.43 & 0.72 \\
0.94 & 0.55 & 0.58 & 0.78 & 0.92 \\
0.35 & 0.92 & 0.90 & 0.53 & 0.89
\end{bmatrix}.
\]

Figure 1 demonstrates the shrinkage in the feasible set of our SDP, projected onto the first two pure strategies of player A, as valid inequalities are added. Subfigure (a) depicts the Nash equilibria and the feasible set of SDPI (recall from 2.2 that the feasible region without valid inequalities is the projection of the entire simplex). The row and distribution constraints are added for subfigure (b), and the correlated equilibrium constraints are further added for subfigure (c). Subfigure (d) depicts the true projection of the convex hull of Nash equilibria.

2.4 Strengthened SDP Relaxation

We now write out our new SDP with all constraints in one place. Recall the representation of the matrix

\[
M' := \begin{bmatrix} X & P & x \\ Z & Y & y \\ x^T & y^T & 1 \end{bmatrix},
\]
Figure 1: Reduction in the size of our spectrahedral outer approximation to the convex hull of Nash equilibria through the addition of valid inequalities.

with $P = Z^T$. The improved SDP is now:

$$\min_{\mathcal{M}' \in S^{(m+n+1) \times (m+n+1)}} 0$$

subject to

$$\mathcal{M}' \succeq 0,$$  \hspace{1cm}  \text{(17)}

$$\mathcal{M}'_{ij} \geq 0, \ i, j \in \{1, \ldots, m + n + 1\},$$  \hspace{1cm}  \text{(18)}

$$\mathcal{M}'_{m+n+1, m+n+1} = 1,$$  \hspace{1cm}  \text{(19)}

$$\sum_{i=1}^{m} x_i = \sum_{i=1}^{n} y_i = 1,$$  \hspace{1cm}  \text{(20)}

$$\text{Tr}(AZ) - e_i^T A y \geq 0, \ \forall i \in \{1, \ldots, m\},$$  \hspace{1cm}  \text{(21)}

$$\text{Tr}(BZ) - x^T B e_i \geq 0, \ \forall i \in \{1, \ldots, n\},$$  \hspace{1cm}  \text{(22)}

$$\sum_{j=1}^{m} X_{i,j} = \sum_{j=1}^{n} Z_{j,i} = x_i, \ \forall i \in \{1, \ldots, m\},$$  \hspace{1cm}  \text{(23)}

$$\sum_{j=1}^{n} Y_{i,j} = \sum_{j=1}^{m} Z_{i,j} = y_i, \ \forall i \in \{1, \ldots, n\},$$  \hspace{1cm}  \text{(24)}

$$\sum_{j=1}^{m} A_{i,j} P_{i,j} \geq \sum_{j=1}^{n} A_{k,j} P_{i,j}, \ \forall i, k \in \{1, \ldots, m\},$$  \hspace{1cm}  \text{(25)}

$$\sum_{j=1}^{m} B_{j,i} P_{j,i} \geq \sum_{j=1}^{n} B_{j,k} P_{j,i}, \ \forall i, k \in \{1, \ldots, n\}.$$  \hspace{1cm}  \text{(26)}

Constraints (21) and (22) are the relaxed Nash constraints, constraints (23) and (24) are the row constraints, and constraints (25) and (26) are the correlated equilibrium constraints.
3 Exactness for Strictly Competitive Games

In this section, we show that SDP1 recovers a Nash equilibrium for any zero-sum game, and that SDP2 recovers a Nash equilibrium for any strictly competitive game (see Definition 3.3 below). Both these notions represent games where the two players are in direct competition, but strictly competitive games are more general and for example, allow both players to have nonnegative payoff matrices. These classes of games are solvable in polynomial time via linear programming. Nonetheless, it is reassuring to know that our SDPs recover these important special cases.

Definition 3.1. A zero-sum game is a game in which the payoff matrices satisfy \( A = -B \).

Theorem 3.2. For a zero-sum game, the vectors \( x \) and \( y \) from the last column of any feasible solution \( M' \) to SDP1 constitute a Nash equilibrium.

Proof. Recall that the relaxed Nash constraints (3) and (4) read

\[
\text{Tr}(AZ) \geq e^T_i Ay, \forall i \in \{1, \ldots, m\},
\]

\[
\text{Tr}(BZ) \geq x^T Be_j, \forall j \in \{1, \ldots, n\}.
\]

Since \( B = -A \), the latter statement is equivalent to

\[
\text{Tr}(AZ) \leq x^T A e_j, \forall j \in \{1, \ldots, n\}.
\]

In conjunction these imply

\[
e^T_i Ay \leq \text{Tr}(AZ) \leq x^T A e_j, \forall i \in \{1, \ldots, m\}, j \in \{1, \ldots, n\}. \tag{27}
\]

We claim that any pair \( x \in \triangle_m \) and \( y \in \triangle_n \) which satisfies the above condition is a Nash equilibrium. To see that \( x^T Ay \geq e^T_i Ay, \forall i \in \{1, \ldots, m\} \), observe that \( x^T Ay \) is a convex combination of \( x^T A e_j \), which are at least \( e^T_i Ay \) by (27). To see that \( x^T By \geq x^T Be_j \Leftrightarrow x^T Ay \leq x^T A e_j, \forall j \in \{1, \ldots, n\} \), observe that \( x^T Ay \) is a convex combination of \( e^T_i Ay \), which are at most \( x^T A e_j \) by (27).

Definition 3.3. A game \((A, B)\) is strictly competitive if for all \( x, x' \in \triangle_m, y, y' \in \triangle_n \), \( x^T Ay - x'^T Ay' \) and \( x^T By - x'^T By' \) have the same sign.

The interpretation of this definition is that if one player benefits from changing from one outcome to another, the other player must suffer. Adler, Daskalakis, and Papadimitriou show in [3] that the following much simpler characterization is equivalent.

Theorem 3.4 (Theorem 1 of [3]). A game is strictly competitive if and only if there exist scalars \( c, d, e, \) and \( f \), with \( c > 0, e > 0 \), such that \( cA + dJ_{m \times n} = -eB + fJ_{m \times n} \).

One can easily show that there exist strictly competitive games for which not all feasible solutions to SDP1 have Nash equilibria as their last columns (see Theorem 2.2). However, we show that this is the case for SDP2.

Theorem 3.5. For a strictly competitive game, the vectors \( x \) and \( y \) from the last column of any feasible solution \( M' \) to SDP2 constitute a Nash equilibrium.

To prove Theorem 3.5 we need the following lemma, which shows that feasibility of a matrix \( M' \) in SDP2 is invariant under certain transformations of \( A \) and \( B \).

Lemma 3.6. Let \( c, d, e, \) and \( f \) be any set of scalars with \( c > 0 \) and \( e > 0 \). If a matrix \( M' \) is feasible to SDP2 with input payoff matrices \( A \) and \( B \), then it is also feasible to SDP2 with input matrices \( cA + dJ_{m \times n} \) and \( eB + fJ_{m \times n} \).
Proof. It suffices to check that constraints (21), (22), (25), and (26) of SDP2 still hold, as only the relaxed Nash and correlated equilibrium constraints use the matrices $A$ and $B$. We only show that constraints (21) and (25) still hold because the arguments for constraints (22) and (26) are identical.

First recall that due to constraints (20) and (24) of SDP2, $\text{Tr}(J_{m \times n} Z) = 1$. To check that the relaxed Nash constraints hold, observe that for scalars $c > 0$ and $d$, and for all $i \in \{1, \ldots, m\}$,

$$\text{Tr}(AZ) - e_i^T Ay \geq 0$$

$$\iff c \text{Tr}(AZ) + d - c(e_i^T Ay) - d \geq 0$$

$$\iff c \text{Tr}(AZ) + d(\text{Tr}(J_{m \times n} Z)) - c(e_i^T Ay) - d(\text{Tr}(J_{m \times n} Z)) \geq 0$$

$$\iff \text{Tr}((cA + dJ_{m \times n}) Z) - e_i^T (cA + dJ_{m \times n}) y \geq 0.$$  

Now recall from constraint (23) of SDP2 (keeping in mind that $P = Z^T$) that $\sum_{j=1}^{n} (J_{m \times n})_{i,j} P_{i,j} = x_i$. To check that the correlated equilibrium constraints hold, observe that for scalars $c > 0, d$, and for all $i, k \in \{1, \ldots, m\}$,

$$\sum_{j=1}^{n} A_{i,j} P_{i,j} \geq \sum_{j=1}^{n} A_{k,j} P_{i,j}$$

$$\iff c \sum_{j=1}^{n} A_{i,j} P_{i,j} + dx_i \geq c \sum_{j=1}^{n} A_{k,j} P_{i,j} + dx_i$$

$$\iff c \sum_{j=1}^{n} A_{i,j} P_{i,j} + d \sum_{j=1}^{n} (J_{m \times n})_{i,j} P_{i,j} \geq c \sum_{j=1}^{n} A_{k,j} P_{i,j} + d \sum_{j=1}^{n} (J_{m \times n})_{k,j} P_{i,j}$$

$$\iff \sum_{j=1}^{n} (cA_{i,j} + dJ_{m \times n})_{k,j} P_{i,j} \geq \sum_{j=1}^{n} (cA_{i,j} + dJ_{m \times n})_{k,j} P_{i,j}.$$

\(\square\)

Proof (of Theorem 3.3). Let $A$ and $B$ be the payoff matrices of the given strictly competitive game and let $\mathcal{M}'$ be a feasible solution to SDP2. Since the game is strictly competitive, we know that $cA + dJ_{m \times n} = -eB + fJ_{m \times n}$ for some scalars $c > 0, e > 0, d, f$. Consider a new game with input matrices $\tilde{A} = cA + dJ_{m \times n}$ and $\tilde{B} = eB - fJ_{m \times n}$. By Lemma 3.6, $\mathcal{M}'$ is still feasible to SDP2 with input matrices $\tilde{A}$ and $\tilde{B}$. Furthermore, since the constraints of SDP1 are a subset of the constraints of SDP2, $\mathcal{M}'$ is also feasible to SDP1. Now notice that since $A = -\tilde{B}$, Theorem 3.2 implies that the vectors $x$ and $y$ in the last column form a Nash equilibrium to the game $(\tilde{A}, \tilde{B})$. Finally recall from Section 2 that Nash equilibria are invariant to scaling and shifting of the payoff matrices, and hence $(x, y)$ is a Nash equilibrium to the game $(A, B)$.  

\(\square\)

Remark 3.1. We later show in Proposition 5.1 that using the trace of $\mathcal{M}$ as the objective function to SDP2 will guarantee that SDP2 returns a rank-1 solution.

4 Bounds on $\epsilon$ for General Games

In this section, we provide upper bounds on the $\epsilon$ returned by SDP2 for an arbitrary bimatrix game. Since the problem of computing a Nash equilibrium in such a game is PPAD-complete, it is unlikely that one can find rank-1 solutions to this SDP in polynomial time. In Section 5, we design objective functions (such as variations of the nuclear norm) that empirically do very well in finding
low-rank solutions to SDP2. Nevertheless, it is of interest to know if the solution returned by SDP2 is not rank-1, whether one can recover an $\epsilon$-Nash equilibrium from it and have a guarantee on $\epsilon$.

Recall our notation for the matrices

$$M := \begin{bmatrix} X & P \\ Z & Y \end{bmatrix},$$

and

$$M' := \begin{bmatrix} X & P & x \\ Z & Y & y \\ x^T & y^T & 1 \end{bmatrix}.$$ 

Throughout this section, any matrices $X, Z, P = Z^T$ and $Y$ or vectors $x$ and $y$ are assumed to be taken from a feasible solution to SDP2. The ultimate results of this section are Theorems 4.5, 4.12, and 4.16. To work towards them, we need a number of preliminary lemmas which we present in Section 4.1.

### 4.1 Lemmas Towards Bounds on $\epsilon$

We first observe the following connection between the approximate payoffs $\text{Tr}(AZ)$ and $\text{Tr}(BZ)$, and $\epsilon(x, y)$, as defined in Section 2.1.

**Lemma 4.1.** Consider a feasible solution $M'$ to SDP2 and the vectors $x$ and $y$ and the matrix $Z$ from that solution. Then

$$\epsilon(x, y) \leq \max \{\text{Tr}(AZ) - x^T Ay, \text{Tr}(BZ) - x^T By\}.$$ 

**Proof.** Note that since $\text{Tr}(AZ) \geq e_i^T Ay$ and $\text{Tr}(BZ) \geq x^T Be_i$ from constraints (21) and (22) of SDP2, we have $\epsilon_A \leq \text{Tr}(AZ) - x^T Ay$ and $\epsilon_B \leq \text{Tr}(BZ) - x^T By$. 

We thus are interested in the difference of the two matrices $P = Z^T$ and $xy^T$. These two matrices can be interpreted as two different probability distributions over the strategy outcomes. The matrix $P$ is the probability distribution from the SDP which generates the approximate payoffs $\text{Tr}(AZ)$ and $\text{Tr}(BZ)$, while $xy^T$ is the product distribution that would have resulted if the matrix had been rank-1. We will see that the difference of these distributions is key in studying the $\epsilon$ which results from the SDP. Hence, we first take steps to represent this difference.

**Lemma 4.2.** Consider any feasible matrix $M'$ to SDP2 and its submatrix $M$. Let the matrix $M$ be given by an eigendecomposition

$$M = \sum_{i=1}^{k} \lambda_i v_i v_i^T =: \sum_{i=1}^{k} \lambda_i \begin{bmatrix} a_i \\ b_i \end{bmatrix} \begin{bmatrix} a_i \\ b_i \end{bmatrix}^T,$$

(28)

so that the eigenvectors $v_i \in \mathbb{R}^{m+n}$ are partitioned into $a_i \in \mathbb{R}^m$ and $b_i \in \mathbb{R}^n$. Then for all $i$, $\sum_{j=1}^{m} (a_i)_j = \sum_{j=1}^{n} (b_i)_j$. 

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Proof. We know from the distribution constraints from Section 2.3.1 that

\begin{align}
\sum_{i=1}^{k} \lambda_i 1^T_m a_i a_i^T 1_m & = 1, \tag{15} \\
\sum_{i=1}^{k} \lambda_i 1^T_m a_i b_i^T 1_n & = 1, \tag{16} \\
\sum_{i=1}^{k} \lambda_i 1^T_n b_i a_i^T 1_m & = 1, \tag{17} \\
\sum_{i=1}^{k} \lambda_i 1^T_n b_i b_i^T 1_n & = 1. \tag{18}
\end{align}

Then by subtracting terms we have

\begin{align}
(29) - (30) & = \sum_{i=1}^{k} \lambda_i 1^T_m a_i (a_i^T 1_m - b_i^T 1_n) = 0, \tag{33} \\
(31) - (32) & = \sum_{i=1}^{k} \lambda_i 1^T_n b_i (a_i^T 1_m - b_i^T 1_n) = 0. \tag{34}
\end{align}

By subtracting again these imply

\begin{equation}
(33) - (34) = \sum_{i=1}^{k} \lambda_i (1^T_m a_i - 1^T_n b_i)^2 = 0. \tag{35}
\end{equation}

As all \( \lambda_i \) are nonnegative due to positive semidefiniteness of \( M \), the only way for this equality to hold is to have \( 1^T_m a_i = 1^T_n b_i, \forall i. \) This is equivalent to the statement of the claim.

From Lemma 4.2, we can let \( s_i := \sum_{j=1}^{m}(a_i)_j = \sum_{j=1}^{n}(b_i)_j, \) and furthermore we assume without loss of generality that each \( s_i \) is nonnegative. Note that from the row constraint (11) we have

\begin{equation}
x_i = \sum_{j=1}^{m} X_{ij} = \sum_{j=1}^{m} \sum_{i=1}^{k} \lambda_i (a_i)_j (a_i)_j = \sum_{j=1}^{k} \lambda_j s_j (a_i)_j. \tag{36}
\end{equation}

Hence,

\begin{equation}
x = \sum_{i=1}^{k} \lambda_i s_i a_i. \tag{37}
\end{equation}

Similarly,

\begin{equation}
y = \sum_{i=1}^{k} \lambda_i s_i b_i. \tag{38}
\end{equation}

Finally note from the distribution constraint (15) that this implies

\begin{equation}
\sum_{i=1}^{k} \lambda_i s_i^2 = 1. \tag{39}
\end{equation}
Lemma 4.3. Let
\[ M = \sum_{i=1}^{k} \lambda_i \begin{bmatrix} a_i \\ b_i \end{bmatrix} \begin{bmatrix} a_i^T \\ b_i^T \end{bmatrix}, \]
be a feasible solution to SDP2, such that the eigenvectors of \( M \) are partitioned into \( a_i \) and \( b_i \) with \( \sum_{j=1}^{m} (a_i)_j = \sum_{j=1}^{n} (b_i)_j = s_i, \forall i \). Then
\[ P - xy^T = \sum_{i=1}^{k} \sum_{j>i} \lambda_i \lambda_j (s_j a_i - s_i a_j)(s_j b_i - s_i b_j)^T. \]

Proof. Using equations (37) and (38) we can write
\[
P - xy^T = \sum_{i=1}^{k} \lambda_i a_i b_i^T - (\sum_{i=1}^{k} \lambda_i s_i a_i)(\sum_{j=1}^{k} \lambda_j s_j b_j)^T
\]
\[
= \sum_{i=1}^{k} \lambda_i a_i (b_i - s_i \sum_{j=1}^{k} \lambda_j s_j b_j)^T
\]
\[
= \sum_{i=1}^{k} \sum_{j=1}^{k} \lambda_i \lambda_j a_j s_j (b_i - s_i b_j)^T
\]
\[
= \sum_{i=1}^{k} \sum_{j>i} \lambda_i \lambda_j (s_j a_i - s_i a_j)(s_j b_i - s_i b_j)^T,
\]
where the last line follows from observing that terms where \( i \) and \( j \) are switched can be combined.

We can relate \( \epsilon \) and \( P - xy^T \) with the following lemma.

Lemma 4.4. Consider any feasible solution \( M' \) to SDP2 and the matrix \( P - xy^T \). Then
\[ \epsilon \leq \frac{\|P - xy^T\|_1}{2}, \]
where \( \| \cdot \|_1 \) here denotes the entrywise L-1 norm, i.e., the sum of the absolute values of the entries of the matrix.

Proof. Let \( D := P - xy^T \). From Lemma 4.1,
\[ \epsilon_A \leq \text{Tr}(AZ) - x^T Ay = \text{Tr}(A(Z - yx^T)). \]
If we then hold \( D \) fixed and restrict that \( A \) has entries bounded in \([0,1]\), the quantity \( \text{Tr}(AD^T) \) is maximized when
\[ A_{i,j} = \begin{cases} 
1 & D_{i,j} \geq 0 \\
0 & D_{i,j} < 0.
\end{cases} \]
The resulting quantity \( \text{Tr}(AD^T) \) will then be the sum of all nonnegative elements of \( D \). Since the sum of all elements in \( D \) is zero, this quantity will be equal to \( \frac{1}{2} \|D\|_1 \).
The proof for \( \epsilon_B \) is identical, and the result follows from that \( \epsilon \) is the maximum of \( \epsilon_A \) and \( \epsilon_B \).
4.2 Bounds on $\epsilon$

We provide a number of bounds on $\epsilon(x, y)$, where $x$ and $y$ are taken from the last column of a feasible solution to SDP2. The first is a theorem stating that solutions which are “close” to rank-1 provide small $\epsilon$.

**Theorem 4.5.** Let $M'$ be a feasible solution to SDP2. Suppose $M$ is rank-$k$ and its eigenvalues are $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_k \geq 0$. Then, the $x$ and $y$ from the last column of $M'$ constitute an $\epsilon$-NE to the game $(A, B)$ with $\epsilon \leq \frac{m+n}{2} \sum_{i=2}^{k} \lambda_i$.

**Proof.** By the Perron Frobenius theorem (see e.g. [28, Chapter 8.3]), the eigenvector corresponding to $\lambda_1$ can be assumed to be nonnegative, and hence

$$s_1 = \|a_1\|_1 = \|b_1\|_1. \quad (40)$$

We further note that for all $i$, since $\begin{bmatrix} a_i \\ b_i \end{bmatrix}$ is a vector of length $m + n$ with 2-norm equal to 1, we must have

$$\left\| \begin{bmatrix} a_i \\ b_i \end{bmatrix} \right\|_1 \leq \sqrt{m+n}. \quad (41)$$

Since $s_i$ is the sum of the elements of $a_i$ and $b_i$, we know that

$$s_i \leq \min\{\|a_i\|_1, \|b_i\|_1\} \leq \frac{\sqrt{m+n}}{2}. \quad (42)$$

This then gives us

$$s_i^2 \leq \|a_i\|_1 \|b_i\|_1 \leq \frac{m+n}{4}, \quad (43)$$

with the first inequality following from (42) and the second from (41). Finally note that a consequence of the nonnegativity of $\|\cdot\|_1$ and (41) is that for all $i, j$,

$$\|a_i\|_1 \|b_j\|_1 + \|b_i\|_1 \|a_j\|_1 \leq (\|a_i\|_1 + \|b_i\|_1)(\|a_j\|_1 + \|b_j\|_1) = \left\| \begin{bmatrix} a_i \\ b_i \end{bmatrix} \right\|_1 \left\| \begin{bmatrix} a_j \\ b_j \end{bmatrix} \right\|_1 \leq m+n. \quad (44)$$

Now we let $D := P - xy^T$ and upper bound $\frac{1}{2} \|D\|_1$ using Lemma 4.3.

$$\frac{1}{2} \|D\|_1 = \frac{1}{2} \| \sum_{i=1}^{k} \sum_{j>i}^{k} \lambda_i \lambda_j (s_j a_i - s_i a_j)(s_j b_i - s_i b_j)^T \|_1$$

$$\leq \frac{1}{2} \sum_{i=1}^{k} \sum_{j>i}^{k} \|\lambda_i \lambda_j (s_j a_i - s_i a_j)(s_j b_i - s_i b_j)^T\|_1$$

$$\leq \frac{1}{2} \sum_{i=1}^{k} \sum_{j>i}^{k} \lambda_i \lambda_j \|s_j a_i - s_i a_j\|_1 \|s_j b_i - s_i b_j\|_1$$

$$\leq \frac{1}{2} \sum_{i=1}^{k} \sum_{j>i}^{k} \lambda_i \lambda_j (s_i \|a_j\|_1 + s_j \|a_i\|_1)(s_i \|b_j\|_1 + s_j \|b_i\|_1)$$

$$\leq \frac{1}{2} \sum_{j=2}^{k} \lambda_1 s_1^2 \lambda_j (s_j + \|a_j\|_1)(s_j + \|b_j\|_1) \quad (40, 43)$$
\[
\begin{align*}
&+ \frac{1}{2} \sum_{i=2}^{k} \sum_{j>i} \lambda_i \lambda_j \left( \frac{s^2_i + s^2_j}{4} \right) + s_i s_j \|a_i\|_1 \|b_j\|_1 + s_i s_j \|a_j\|_1 \|b_i\|_1 \\
&\leq \frac{m + n}{2} \lambda_1 s_1^2 \sum_{i=2}^{k} \lambda_i
\end{align*}
\]

\[
\begin{align*}
&+ \frac{1}{2} \sum_{i=2}^{k} \sum_{j>i} \lambda_i \lambda_j \left( \frac{s^2_i + s^2_j}{4} \right) + \lambda_i \lambda_j s_i s_j (m + n) \\
&\leq \frac{m + n}{2} \lambda_1 s_1^2 \sum_{i=2}^{k} \lambda_i + \frac{3}{8} \sum_{i=2}^{k} \sum_{j>i} \lambda_i \lambda_j (s_i^2 + s_j^2)
\end{align*}
\]

\[
\begin{align*}
&= \frac{m + n}{2} \lambda_1 s_1^2 \sum_{i=2}^{k} \lambda_i + \frac{3}{8} \sum_{i=2}^{k} \sum_{j>i} \lambda_i \lambda_j (s_i^2 + s_j^2)
\end{align*}
\]

\[
\begin{align*}
&= \frac{m + n}{2} \lambda_1 s_1^2 \sum_{i=2}^{k} \lambda_i + \frac{3}{8} \left( \sum_{i=2}^{k} \lambda_i s_i^2 \sum_{j>i} \lambda_j + \sum_{i=2}^{k} \lambda_i \sum_{j>i} \lambda_j s_j^2 \right)
\end{align*}
\]

\[
\begin{align*}
&= \frac{m + n}{2} \lambda_1 s_1^2 \sum_{i=2}^{k} \lambda_i + \frac{3}{8} \left( \sum_{j=2}^{k} \lambda_j \sum_{i=2}^{k} \lambda_i s_i^2 + \sum_{i=2}^{k} \lambda_i \sum_{j>i} \lambda_j s_j^2 \right)
\end{align*}
\]

\[
\begin{align*}
&\leq \frac{m + n}{2} \lambda_1 s_1^2 \sum_{i=2}^{k} \lambda_i + \frac{3}{8} \left( \sum_{j=2}^{k} \lambda_j s_j^2 \sum_{i=2}^{k} \lambda_i \right)
\end{align*}
\]

\[
\begin{align*}
&= \frac{m + n}{2} \lambda_1 s_1^2 \sum_{i=2}^{k} \lambda_i + \frac{3}{8} \left( \sum_{j=2}^{k} \lambda_j s_j^2 \right) \sum_{i=2}^{k} \lambda_i
\end{align*}
\]

\[
\begin{align*}
&\leq \frac{m + n}{2} \lambda_1 s_1^2 \sum_{i=2}^{k} \lambda_i + \frac{3}{8} \left( \sum_{j=2}^{k} \lambda_j s_j^2 \right) \sum_{i=2}^{k} \lambda_i
\end{align*}
\]

\[
\begin{align*}
&\leq \frac{m + n}{2} \sum_{i=2}^{k} \lambda_i.
\end{align*}
\]

4.3 Bounds on $\epsilon$ in the Rank-2 Case

We now give a series of bounds on $\epsilon$ which hold for feasible solutions to the SDP formulation that are rank-2 (note that due to the row inequalities, $M$ will have the same rank as $M'$). This is motivated by our ability to show stronger (constant) bounds in this case, and the fact that we often recover rank-2 (or rank-1) solutions with our algorithms in Section 5. As it will be important in our study of this particular case, we first recall the definition of a completely positive factorization/rank of a matrix.

**Definition 4.6.** A matrix $M$ is completely positive (CP) if it admits a decomposition $M = U U^T$ for some nonnegative matrix $U$.

**Definition 4.7.** The CP-rank of an $n \times n$ CP matrix $M$ is the smallest $k$ for which there exists a nonnegative $n \times k$ matrix $U$ such that $M = U U^T$.

\(\text{AMGM}\) is used to denote the arithmetic-mean-geometric-mean inequality.
Theorem 4.8 (see e.g. [19] or Theorem 2.1 in [6]). A rank-2, nonnegative, and positive semidefinite matrix is CP and has CP-rank 2.

It is also known (see e.g., Section 4 in [19]) that the CP factorization of a rank-2 CP matrix can be found to arbitrary accuracy in polynomial time. With these preliminaries in mind, we present lemmas which are similar to Lemmas 4.2 and 4.3, but provide a decomposition which is specific to the rank-2 case.

Lemma 4.9. Suppose that a feasible solution $\mathcal{M}'$ to $\text{SDP}_2$ is rank-2. Then there exists a decomposition $\mathcal{M} = \begin{bmatrix} a & b \\ b & a \end{bmatrix}^T + \begin{bmatrix} c & d \\ d & c \end{bmatrix}^T$, where $\sigma_1$ and $\sigma_2$ are nonnegative, $\sigma_1 + \sigma_2 = 1$, $a, c \in \Delta_m$, and $b, d \in \Delta_n$.

Proof. Since the matrix $\mathcal{M}$ is rank-2, nonnegative, and positive semidefinite, from Theorem 4.8 we can decompose the matrix $\mathcal{M}$ into $v_1v_1^T + v_2v_2^T$ where $v_1$ and $v_2$ are nonnegative. We can then partition $v_1$ and $v_2$ into vectors $\begin{bmatrix} a' \\ b' \end{bmatrix}$ and $\begin{bmatrix} c' \\ d' \end{bmatrix}$ as we did in Lemma 4.2. Furthermore, we note that because the distribution constraints (15) still hold, by repeating the proof of Lemma 4.2, we find that $\sum_{i=1}^m a_i' = \sum_{i=1}^n b_i'$, and $\sum_{i=1}^m c_i' = \sum_{i=1}^n d_i'$. By letting

$$\sigma_1 = \left( \sum_{i=1}^m a_i' \right)^2, \sigma_2 = \left( \sum_{i=1}^m c_i' \right)^2, a = \frac{a'}{\sqrt{\sigma_1}}, b = \frac{b'}{\sqrt{\sigma_1}}, c = \frac{c'}{\sqrt{\sigma_2}}, d = \frac{d'}{\sqrt{\sigma_2}},$$

we have found constants $\sigma_1$ and $\sigma_2$, and vectors $a, b, c$ and $d$ which satisfy

$$\mathcal{M} = \sigma_1 \begin{bmatrix} a & b \\ b & a \end{bmatrix} + \sigma_2 \begin{bmatrix} c & d \\ d & c \end{bmatrix},$$

and $a, b, c$, and $d$ are simplex vectors. Note that we can assume $\sigma_1$ and $\sigma_2$ are both positive, as otherwise we are in the rank-1 case. Finally, to show that $\sigma_1 + \sigma_2 = 1$, recall that the sum of all elements in $\mathcal{M}$ is 4, as are the sum of the elements of $\begin{bmatrix} a & a \\ b & b \end{bmatrix}$ and $\begin{bmatrix} c & c \\ d & d \end{bmatrix}$. \hfill \qed

Lemma 4.10. Suppose a feasible solution $\mathcal{M}'$ to $\text{SDP}_2$ is rank-2, and that

$$\mathcal{M} = \sigma_1 \begin{bmatrix} a & b \\ b & a \end{bmatrix} + \sigma_2 \begin{bmatrix} c & d \\ d & c \end{bmatrix},$$

where $\sigma_1$ and $\sigma_2$ are nonnegative, $\sigma_1 + \sigma_2 = 1$, $a, c \in \Delta_m$, and $b, d \in \Delta_n$. Then,

$$P - xy^T = \sigma_1 \sigma_2 (a - c)(b - d)^T.$$

Proof. Recall that in our notation,

$$P = \sigma_1 a b^T + \sigma_2 c d^T,$$

$$x = \sigma_1 a + \sigma_2 c,$$

$$y = \sigma_1 b + \sigma_2 d.$$

Then,

$$P - xy^T = \sigma_1 a b^T + \sigma_2 c d^T - (\sigma_1 a + \sigma_2 c)(\sigma_1 b + \sigma_2 d)^T$$

$$= \sigma_1 a b^T + \sigma_2 c d^T - \sigma_1^2 ab - \sigma_1 \sigma_2 ad - \sigma_1 \sigma_2 cb - \sigma_2^2 cd$$

$$= \sigma_1 (1 - \sigma_1) ab - \sigma_1 \sigma_2 ad - \sigma_1 \sigma_2 cb + \sigma_2 (1 - \sigma_2) cd$$

$$= \sigma_1 \sigma_2 ab + \sigma_1 \sigma_2 ad + \sigma_1 \sigma_2 cb + \sigma_2 \sigma_1 cd$$

$$= \sigma_1 \sigma_2 (a - c)(b - d)^T.$$

\hfill \qed
With this decomposition we can present a series of constant additive factor approximations for the rank-2 case. To do so we apply Lemma 4.4 to the decomposition in Lemma 4.10. All the following proofs will use the notation with \( \sigma, a, b, c, \) and \( d \) as in Lemma 4.10.

**Theorem 4.11.** If a feasible solution \( \mathcal{M}' \) to \( \text{SDP}^2 \) is rank-2, then the \( x \) and \( y \) from the last column of \( \mathcal{M}' \) constitute a \( \frac{1}{2} \)-Nash equilibrium.

**Proof.** We can use Lemma 4.10 to represent

\[
D := P - xy^T = \sigma_1 \sigma_2 (a - c)(b - d)^T,
\]

where \( \sigma_1, \sigma_2 \geq 0, \sigma_1 + \sigma_2 = 1, a, c \in \Delta_m, \) and \( b, d \in \Delta_n. \) Then we can use Lemma 4.4 to get that

\[
\epsilon \leq \frac{\|D\|_1}{2} = \frac{1}{2} \|\sigma_1 \sigma_2 (a - c)(b - d)^T\|_1 \leq \frac{1}{2} \sigma_1 \sigma_2 \|a - c\|_1 \|b - d\|_1.
\]

Since we have \( \|a\|_1 = \|b\|_1 = \|c\|_1 = \|d\|_1 = 1, \) we get the following bound for \( \epsilon: \)

\[
\frac{1}{2} \sigma_1 \sigma_2 \|a - c\|_1 \|b - d\|_1 \leq \frac{1}{2} \sigma_1 \sigma_2 \cdot 2 \cdot 2 = 2 \sigma_1 \sigma_2.
\]

Since \( \sigma_1 \) and \( \sigma_2 \) sum to one and are nonnegative, we know that \( \sigma_1 \sigma_2 \leq \frac{1}{4}, \) and hence we have \( \epsilon \leq \frac{1}{2}. \) \qed

**Theorem 4.12.** If a feasible solution \( \mathcal{M}' \) to \( \text{SDP}^2 \) is rank-2, then either the \( x \) and \( y \) from its last column constitute a \( \frac{5}{11} \)-NE, or a \( \frac{5}{11} \)-NE can be recovered from \( \mathcal{M}' \) in polynomial time.

**Proof.** We consider 3 cases, depending on whether \( \epsilon_A(x, y) \) and \( \epsilon_B(x, y) \) are greater than or less than \( .4. \) If \( \epsilon_A \leq .4, \epsilon_B \leq .4, \) then \( (x, y) \) is already a .4-Nash equilibrium. Now consider the case when \( \epsilon_A \geq .4, \epsilon_B \geq .4. \) Since \( \epsilon_A \leq \text{Tr}(A(P - xy^T)^T) \) and \( \epsilon_B \leq \text{Tr}(B(P - xy^T)^T), \) we have

\[
\sigma_1 \sigma_2 (a - c)^T A (b - d) \geq .4, \sigma_1 \sigma_2 (a - c)^T B (b - d) \geq .4.
\]

Since \( A, a, b, c, \) and \( d \) are all nonnegative and \( \sigma_1 \sigma_2 \leq \frac{1}{4}, \)

\[
a^T Ab + c^T Ad \geq (a - c)^T A (b - d) \geq 1.6,
\]

and the same inequalities hold for for player B. In particular, since \( A \) and \( B \) have entries bounded in \([0,1]\) and \( a, b, c, \) and \( d \) are simplex vectors, all the quantities \( a^T Ab, c^T Ad, a^T Bb, \) and \( c^T Bd \) are at most 1, and consequently at least .6. Hence \( (a, b) \) and \( (c, d) \) are both .4-Nash equilibria.

Now suppose that \( (x, y) \) is a .4-NE for one player (without loss of generality player A) but not for the other (without loss of generality player B). Then \( \epsilon_A \leq .4, \) and \( \epsilon_B \geq .4. \) Let \( y^* \) be a best response for player B to \( x, \) and let \( p = \frac{1}{1 + \epsilon_B - \epsilon_A}. \) Consider the strategy profile \( (\hat{x}, \hat{y}) := (x, py + (1 - p)y^*). \) This can be interpreted as the outcome \( (x, y) \) occurring with probability \( p, \) and the outcome \( (x, y^*) \) happening with probability \( 1 - p. \) In the first case, player A will have \( \epsilon_A(x, y) = \epsilon_A \) and player B will have \( \epsilon_B(x, y) = \epsilon_B. \) In the second outcome, player A will have \( \epsilon_A(x, y^*) = \) at most 1, while player B will have \( \epsilon_B(x, y^*) = 0. \) Then under this strategy profile, both players have the same upper bound for \( \epsilon, \) which equals \( \epsilon_B p = \frac{\epsilon_B}{1 + \epsilon_B - \epsilon_A}. \) To find the worst case for this value, let \( \epsilon_B = .5 \) (note from Theorem 4.11 that \( \epsilon_B \leq \frac{1}{2} \)) and \( \epsilon_A = .4, \) and this will return \( \epsilon = \frac{5}{11}. \) \qed

We now show a stronger result in the case of symmetric games.

**Definition 4.13.** A symmetric game is a game in which the payoff matrices \( A \) and \( B \) satisfy \( B = A^T. \)
**Definition 4.14.** A Nash equilibrium strategy \((x, y)\) is said to be symmetric if \(x = y\).

**Theorem 4.15** (see Theorem 2 in [29]). Every symmetric bimatrix game has a symmetric Nash equilibrium.

For the proof of Theorem 4.16 below we modify \(\text{SDP2}\) so that we are seeking a symmetric solution.

**Theorem 4.16.** Suppose the constraints \(x = y\) and \(X = P = Y\) are added to \(\text{SDP2}\). Then if a feasible solution \(M'\) to this new SDP is rank-2, either the \(x\) and \(y\) from its last column constitute a symmetric \(\frac{1}{3}\)-NE, or a symmetric \(\frac{1}{3}\)-NE can be recovered from \(M'\) in polynomial time.

**Proof.** If \((x, y)\) is already a symmetric \(\frac{1}{3}\)-NE, then the claim is established. Now suppose that \((x, y)\) does not constitute a \(\frac{1}{3}\)-Nash equilibrium. Observe that since \(x = y\), we must have \(a = b\) and \(c = d\). Then following the same reasoning as in the proof of Theorem 4.12, we have

\[
\sigma_1\sigma_2(a-c)^T A(a-c) \geq \frac{1}{3}.
\]

Since \(A, a,\) and \(c\) are all nonnegative, and \(\sigma_1\sigma_2 \leq \frac{1}{4}\), we get

\[
a^T Aa + c^T Ac \geq (a-c)^T A(a-c) \geq \frac{4}{3}.
\]

In particular, at least one of \(a^T Aa\) and \(c^T Ac\) is at least \(\frac{2}{3}\). Since the maximum possible payoff is 1, at least one of \((a, a)\) and \((c, c)\) is a (symmetric) \(\frac{1}{3}\)-Nash equilibrium. \(\square\)

## 5 Algorithms for Lowering the Rank

In this section, we present heuristics which aim to find low-rank solutions to \(\text{SDP2}\) and present some empirical results. Recall that our \(\text{SDP2}\) in Section 2.4 did not have an objective function. Hence, we can encourage low-rank solutions by choosing certain objective functions, for example the nuclear norm (i.e. trace) of the matrix \(M\), which is a general heuristic for rank minimization [33, 15]. This simple objective function is already guaranteed to produce a rank-1 solution in the case of strictly competitive games (see Proposition 5.1 below). For general games, however, one can design better objective functions in an iterative fashion (see Section 5.1).

**Proposition 5.1.** For a strictly competitive game, any optimal solution to \(\text{SDP2}\) with \(\text{Tr}(M)\) as the objective function must be rank-1.

**Proof.** Let

\[
M := \begin{bmatrix} X & P \\ Z & Y \end{bmatrix}, M' := \begin{bmatrix} X & P & x \\ Z & Y & y \\ x^T & y^T & 1 \end{bmatrix},
\]

with \(P = Z^T\), be a feasible solution to \(\text{SDP2}\). In the case of strictly competitive games, from Theorem 3.5 we know that that \((x, y)\) is a Nash equilibrium. Then because the matrix \(M'\) must be psd, by applying the Schur complement (see, e.g. [7, Sect. A.5.5]), we have that \(M \succeq \begin{bmatrix} x \ & x^T \\ y & y^T \end{bmatrix}\), and therefore \(M = \begin{bmatrix} xx^T & xy^T \\ yx^T & yy^T \end{bmatrix} + P\) for some psd matrix \(P\) and some Nash equilibrium \((x, y)\). Given this expression, the objective value is then \(x^T x + y^T y + \text{Tr}(P)\). As \((x, y)\) is a Nash equilibrium, the choice of \(P = 0\) results in a feasible solution. Since the zero matrix has the minimum possible trace among all psd matrices, the solution will be the rank-1 matrix \(\begin{bmatrix} x \\ x^T \\ y \\ y^T \end{bmatrix}\). \(\square\)
5.1 Linearization Algorithms

The algorithms we present in this section are based on iterative linearization of certain nonconvex objective functions. Motivated by the next proposition, we design two continuous (nonconvex) objective functions that, if minimized exactly, would guarantee rank-1 solutions. We will then linearize these functions iteratively.

**Proposition 5.2.** Let the matrices \( X \) and \( Y \) and vectors \( x \) and \( y \) be taken from a feasible solution \( \mathcal{M}' \) to \( \text{SDP2} \). Then the matrix \( \mathcal{M}' \) is rank-1 if and only if \( X_{i,i} = x_i^2 \) and \( Y_{i,i} = y_i^2 \) for all \( i \).

**Proof.** Necessity of the condition is trivial; we argue sufficiency. Denote the vector \( z = \begin{bmatrix} x & y \end{bmatrix} \). First recall from the row constraints of Section 2.3 that \( \mathcal{M} \) will have the same rank as \( \mathcal{M}' \), as the last column is a linear combination of the columns of \( X \) and \( Y \). Since \( \mathcal{M} \) is psd, we have that \( M_{i,j} \leq \sqrt{M_{i,i} M_{j,j}} \), which implies \( M_{i,j} \leq z_i z_j \) by the assumption of the proposition. Further recall that a consequence of the unity constraint (20) is that \( \sum_{i=1}^{m+n} \sum_{j=1}^{m+n} z_i z_j = 4 \), and that we require from the distribution constraints from Section 2.3.1 that \( \sum_{i=1}^{m+n} \sum_{j=1}^{m+n} M_{i,j} = 4 \). Now we can see that in order to have the equality

\[
4 = \sum_{i=1}^{m+n} \sum_{j=1}^{m+n} M_{i,j} \leq \sum_{i=1}^{m+n} \sum_{j=1}^{m+n} z_i z_j = 4,
\]

we must have \( M_{i,j} = z_i z_j \) for each \( i \) and \( j \). Consequently \( M \) is rank-1. \( \square \)

We focus now on two nonconvex objectives that as a consequence of the above proposition would return rank-1 solutions:

**Proposition 5.3.** All optimal solutions to \( \text{SDP2} \) with the objective function \( \sum_{i=1}^{m+n} \sqrt{M_{i,i}} \) or \( \text{Tr}(\mathcal{M}) - x^T x - y^T y \) are rank-1.

**Proof.** We show that each of these objectives has a specific lower bound which is achieved if and only if the matrix is rank-1.

Observe that since \( \mathcal{M} \succeq \begin{bmatrix} x & y \\ y & y \end{bmatrix}^T \), we have \( \sqrt{X_{i,i}} \geq x_i \) and \( \sqrt{Y_{i,i}} \geq y_i \), and hence

\[
\sum_{i=1}^{m+n} \sqrt{M_{i,i}} \geq \sum_{i=1}^{m} x_i + \sum_{i=1}^{n} y_i = 2.
\]

Further note that

\[
\text{Tr}(\mathcal{M}) - \begin{bmatrix} x & y \\ y & y \end{bmatrix}^T \begin{bmatrix} x & y \\ y & y \end{bmatrix} = \begin{bmatrix} x & y \\ y & y \end{bmatrix}^T \begin{bmatrix} x & x \\ y & y \end{bmatrix} - 2 = 0.
\]

We can see that the lower bounds are achieved if and only if \( X_{i,i} = x_i^2 \) and \( Y_{i,i} = y_i^2 \) for all \( i \), which by Proposition 5.2 happens if and only if \( \mathcal{M} \) is rank-1. \( \square \)

We refer to our two objective functions in Proposition 5.3 as the “square root objective” and the “diagonal gap objective” respectively. While these are both nonconvex, we will attempt to iteratively minimize them by linearizing them through a first order Taylor expansion. For example, at iteration \( k \) of the algorithm,

\[
\sum_{i=1}^{m+n} \sqrt{M_{i,i}^{(k)}} \approx \sum_{i=1}^{m+n} \sqrt{M_{i,i}^{(k-1)}} + \frac{1}{2\sqrt{M_{i,i}^{(k-1)}}} (M_{i,i}^{(k)} - M_{i,i}^{(k-1)}).
\]
Note that for the purposes of minimization, this reduces to minimizing \( \sum_{i=1}^{m+n} \frac{1}{\sqrt{M_{i,i}^{(k)}}} \).

In similar fashion, for the second objective function, at iteration \( k \) we can make the approximation
\[
\text{Tr}(\mathcal{M}) - \begin{bmatrix} x \\ y \end{bmatrix}^{(k)} \begin{bmatrix} x \\ y \end{bmatrix} \simeq \text{Tr}(\mathcal{M}) - \begin{bmatrix} x \\ y \end{bmatrix}^{(k-1)} \begin{bmatrix} x \\ y \end{bmatrix}^{(k-1)T} - 2 \begin{bmatrix} x \\ y \end{bmatrix}^{(k)} \left( \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} x \\ y \end{bmatrix}^{(k-1)} \right).
\]

Once again, for the purposes of minimization this reduces to minimizing \( \text{Tr}(\mathcal{M}) - 2 \begin{bmatrix} x \\ y \end{bmatrix}^{(k-1)T} \begin{bmatrix} x \\ y \end{bmatrix}^{(k)} \).

This approach then leads to the following two algorithms.

**Algorithm 1 Square Root Minimization Algorithm**

1. Let \( x^{(0)} = 1_m, y^{(0)} = 1_n, k = 1 \).
2. **while** !convergence **do**
   3. Solve SDP2 with \( \sum_{i=1}^{m} \frac{1}{\sqrt{M_{i,i}^{(k-1)}}} X_{i,i} + \sum_{i=1}^{n} \frac{1}{\sqrt{M_{i,i}^{(k-1)}}} Y_{i,i} \) as the objective, and denote the optimal solution by \( \mathcal{M}^* \).
   4. Let \( x^{(k)} = \text{diag}(X^*), y^{(k)} = \text{diag}(Y^*) \).
   5. Let \( k = k + 1 \).
3. **end while**

**Algorithm 2 Diagonal Gap Minimization Algorithm**

1. Let \( x^{(0)} = 0_m, y^{(0)} = 0_n, k = 1 \).
2. **while** !convergence **do**
   3. Solve SDP2 with \( \text{Tr}(X) + \text{Tr}(Y) - 2 \begin{bmatrix} x \\ y \end{bmatrix}^{(k-1)T} \begin{bmatrix} x \\ y \end{bmatrix}^{(k)} \) as the objective, and denote the optimal solution by \( \mathcal{M}^* \).
   4. Let \( x^{(k)} = x^*, y^{(k)} = y^* \).
   5. Let \( k = k + 1 \).
3. **end while**

**Remark 5.1.** Note that the first iteration of both algorithms uses the nuclear norm (i.e. trace) of \( \mathcal{M} \) as the objective.

The square root algorithm has the following property.

**Theorem 5.4.** Let \( \mathcal{M}^{(1)}, \mathcal{M}^{(2)}, \ldots \) be the sequence of optimal matrices obtained from the square root algorithm. Then the sequence
\[
\left\{ \sum_{i=1}^{m+n} \sqrt{M_{i,i}^{(k)}} \right\}
\]
is nonincreasing and is lower bounded by two. If it reaches two at some iteration \( t \), then the matrix \( \mathcal{M}^{(t)} \) is rank-1.

**Proof.** Observe that for any \( k > 1 \),
\[
\sum_{i=1}^{m+n} \sqrt{M_{i,i}^{(k)}} \leq \frac{1}{2} \sum_{i=1}^{m+n} \left( \frac{\mathcal{M}_{i,i}^{(k)}}{\sqrt{M_{i,i}^{(k-1)}}} + \sqrt{M_{i,i}^{(k-1)}} \right) \leq \frac{1}{2} \sum_{i=1}^{m+n} \left( \sqrt{\mathcal{M}_{i,i}^{(k-1)}} + M_{i,i}^{(k-1)} \right) = \sum_{i=1}^{m+n} \sqrt{M_{i,i}^{(k-1)}}.
\]

\footnote{An algorithm similar to Algorithm 2 is used in [18].}
where the first inequality follows from the arithmetic-mean-geometric-mean inequality, and the second follows from that $M_{i,i}^{(k)}$ is chosen to minimize $\sum_{i=1}^{m+n} \frac{M_{i,i}^{(k)}}{\sqrt{M_{i,i}^{(k-1)}}}$ and hence achieves a no larger value than the feasible solution $M^{(k-1)}$. This shows that the sequence is nonincreasing.

The proof of Proposition 5.3 already shows that the sequence is lower bounded by two, and Proposition 5.3 itself shows that reaching two is sufficient to have the matrix be rank-1. \qed

The diagonal gap algorithm has the following property.

**Theorem 5.5.** Let $M^{(1)}, M^{(2)}, \ldots$ be the sequence of optimal matrices obtained from the diagonal gap algorithm. Then the sequence

$$\{\text{Tr}(M^{(k)}) - \left[\begin{array}{c} x \\ y \end{array}\right]^{(k)T} \left[\begin{array}{c} x \\ y \end{array}\right]^{(k)}\}$$

is nonincreasing and is lower bounded by zero. If it reaches zero at some iteration $t$, then the matrix $M^{(t)}$ is rank-1.

**Proof.** Observe that

$$\text{Tr}(M^{(k)}) - \left[\begin{array}{c} x \\ y \end{array}\right]^{(k)T} \left[\begin{array}{c} x \\ y \end{array}\right]^{(k)} \leq \text{Tr}(M^{(k)}) - \left[\begin{array}{c} x \\ y \end{array}\right]^{(k)T} \left[\begin{array}{c} x \\ y \end{array}\right]^{(k)} + \left(\left[\begin{array}{c} x \\ y \end{array}\right]^{(k)} - \left[\begin{array}{c} x \\ y \end{array}\right]^{(k-1)}\right)^{T} \left(\left[\begin{array}{c} x \\ y \end{array}\right]^{(k)} - \left[\begin{array}{c} x \\ y \end{array}\right]^{(k-1)}\right)$$

$$= \text{Tr}(M^{(k)}) - 2 \left[\begin{array}{c} x \\ y \end{array}\right]^{(k)T} \left[\begin{array}{c} x \\ y \end{array}\right]^{(k-1)} + \left[\begin{array}{c} x \\ y \end{array}\right]^{(k-1)T} \left[\begin{array}{c} x \\ y \end{array}\right]^{(k-1)}$$

$$\leq \text{Tr}(M^{(k-1)}) - 2 \left[\begin{array}{c} x \\ y \end{array}\right]^{(k-1)T} \left[\begin{array}{c} x \\ y \end{array}\right]^{(k-1)} + \left[\begin{array}{c} x \\ y \end{array}\right]^{(k-1)T} \left[\begin{array}{c} x \\ y \end{array}\right]^{(k-1)}$$

$$= \text{Tr}(M^{(k-1)}) - \left[\begin{array}{c} x \\ y \end{array}\right]^{(k-1)T} \left[\begin{array}{c} x \\ y \end{array}\right]^{(k-1)}$$

where the second inequality follows from that $M^{(k)}$ is chosen to minimize $\text{Tr}(M^{(k-1)}) - 2 \left[\begin{array}{c} x \\ y \end{array}\right]^{(k-1)T} \left[\begin{array}{c} x \\ y \end{array}\right]^{(k-1)}$ and hence achieves a no larger value than the feasible solution $M^{(k-1)}$. This shows that the sequence is nonincreasing.

The proof of Proposition 5.3 already shows that the sequence is lower bounded by zero, and Proposition 5.3 itself shows that reaching zero is sufficient to have the matrix be rank-1. \qed

Our last theorem further quantifies how making the objective of the diagonal gap algorithm small makes $\epsilon$ small. The proof is similar to the proof of Theorem 4.5.

**Theorem 5.6.** Let $M'$ be a feasible solution to $SDP_2$. Then, the $x$ and $y$ from the last column of $M'$ constitute an $\epsilon$-NE to the game $(A, B)$ with $\epsilon \leq \frac{3(m+n)}{8} (\text{Tr}(M') - x^T x - y^T y)$.

**Proof.** Let $k$ be the rank of $M'$ with the eigenvalues of $M'$ given by $\lambda_1, \ldots, \lambda_k$ and the eigenvectors $v_1, \ldots, v_k$ partitioned as in Lemma 4.2 so that $v_i = \left[\begin{array}{c} a_i \\ b_i \end{array}\right]$ with $\sum_{j=1}^{m} (a_i)_j = \sum_{j=1}^{n} (b_i)_j = s_i, \forall i$.

Then we have $\text{Tr}(M') = \sum_{i=1}^{k} \lambda_i$, and

$$x^T x + y^T y = \sum_{i=1}^{k} \lambda_i s_i v_i^T (\sum_{i=1}^{k} \lambda_i s_i v_i) = \sum_{i=1}^{k} \lambda_i^2 s_i^2.$$

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We now get the following chain of inequalities (the first one follows from Lemma 4.4 and the proof of Theorem 4.5):

\[ \epsilon \leq \frac{1}{2} \sum_{i=1}^{k} \sum_{j>i}^{k} \lambda_i \lambda_j (s_j \|a_i\|_1 + s_i \|a_j\|_1)(s_j \|b_i\|_1 + s_i \|b_j\|_1) \]

\[ \leq \frac{1}{2} \sum_{i=1}^{k} \sum_{j>i}^{k} \lambda_i \lambda_j \left( s_j^2 \frac{m+n}{4} + s_i^2 \frac{m+n}{4} + s_i s_j \|a_i\|_1 \|b_j\|_1 + s_i s_j \|a_j\|_1 \|b_i\|_1 \right) \]

\[ \leq \frac{1}{2} \sum_{i=1}^{k} \sum_{j>i}^{k} \lambda_i \lambda_j \left( m+n \right) \left( s_i^2 + s_j^2 \right) + \lambda_i \lambda_j s_i s_j (m+n) \]

\[ \leq \frac{3(m+n)}{8} \sum_{i=1}^{k} \sum_{j>i}^{k} \lambda_i \lambda_j \left( s_i^2 + s_j^2 \right) \]

\[ = \frac{3(m+n)}{8} \left( \sum_{j=1}^{k} \lambda_j \left( \sum_{i=1}^{k} \lambda_i s_i^2 \right) + \sum_{i=1}^{k} \lambda_i \left( \sum_{j>i}^{k} \lambda_j s_j^2 \right) \right) \]

\[ = \frac{3(m+n)}{8} \left( \sum_{i=1}^{k} \lambda_i \left( \sum_{1<i<j}^{k} \lambda_j s_j^2 \right) \right) \]

\[ = \frac{3(m+n)}{8} \left( \sum_{i=1}^{k} \lambda_i (1 - \lambda_i s_i^2) \right) \]

\[ = \frac{3(m+n)}{8} \left( \sum_{i=1}^{k} \lambda_i - \sum_{i=1}^{k} \lambda_i^2 s_i^2 \right) = \frac{3(m+n)}{8} \left( \text{Tr}(M') - x^T x - y^T y \right). \]

\[ \square \]

5.2 Numerical Experiments

We tested Algorithms 1 and 2 on games coming from 100 randomly generated payoff matrices with entries bounded in \([0,1]\) of varying sizes. Below is a table of statistics for \(20 \times 20\) matrices; the data for the rest of the sizes can be found in Appendix \[A\]. We can see that our algorithms return approximate Nash equilibria with fairly low \(\epsilon\) (recall the definition from Section 2.1). We ran 20 iterations of each algorithm on each game. Using the SDP solver of MOSEK \[1\], each iteration takes on average under 4 seconds to solve on a standard personal machine with a 3.4 GHz processor and 16 GB of memory.

\[5\] The code that produced these results is publicly available at [aaa.princeton.edu/software]. The function nash.m computes an approximate Nash equilibrium using one of our two algorithms as specified by the user.
Table 1: Statistics on $\epsilon$ for $20 \times 20$ games after 20 iterations.

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Max</th>
<th>Mean</th>
<th>Median</th>
<th>StDev</th>
</tr>
</thead>
<tbody>
<tr>
<td>Square Root</td>
<td>0.0198</td>
<td>0.0046</td>
<td>0.0039</td>
<td>0.0034</td>
</tr>
<tr>
<td>Diagonal Gap</td>
<td>0.0159</td>
<td>0.0032</td>
<td>0.0024</td>
<td>0.0032</td>
</tr>
</tbody>
</table>

The histograms below show the effect of increasing the number of iterations on lowering $\epsilon$ on $20 \times 20$ games. For both algorithms, there was a clear improvement of the $\epsilon$ by increasing the number of iterations.

Figure 2: Distribution of $\epsilon$ over numbers of iterations for the square root algorithm (left) and the diagonal gap algorithm (right).

6 Bounding Payoffs and Strategy Exclusion

In addition to finding $\epsilon$-additive Nash equilibria, our SDP approach can be used to answer certain questions of economic interest about Nash equilibria without actually computing them. For instance, economists often would like to know the maximum welfare (sum of the two players’ payoffs) achievable under any Nash equilibrium, or whether there exists a Nash equilibrium in which a given subset of strategies (corresponding, e.g., to undesirable behavior) is not played. Both these questions are NP-hard for bimatrix games [16]. In this section, we show how our SDP can be applied to these problems and given some numerical experiments.

6.1 Bounding Payoffs

When designing policies that are subject to game theoretic behavior by agents, economists would often like to find one with a good socially optimal outcome, which usually corresponds to an equilibrium giving the maximum welfare. Hence, given a game, it is of interest to know the highest achievable welfare under any Nash equilibrium.

To address this problem, we begin as we did in Section 2.1 by posing the question of maximizing the welfare under any Nash equilibrium as a quadratic program. Since the feasible set of this program is the set of Nash equilibria, the constraints are the same as those in the formulation in [2], though the objective function is now the welfare:
\[
\begin{align*}
\max_{x,y} & \quad x^T Ay + x^T By \\
\text{subject to} & \quad x^T Ay \geq e_i^T Ay, \forall i \in \{1, \ldots, m\}, \\
& \quad x^T By \geq x^T Be_j, \forall j \in \{1, \ldots, n\}, \\
& \quad x_i \geq 0, \forall i \in \{1, \ldots, m\}, \\
& \quad y_j \geq 0, \forall j \in \{1, \ldots, n\}, \\
& \quad \sum_{i=1}^{m} x_i = 1, \\
& \quad \sum_{i=1}^{n} y_i = 1.
\end{align*}
\]

(47)

The SDP relaxation of this quadratic program will then be given by

\[
\begin{align*}
\max_{M' \in \mathbb{S}^{m+n+1}} & \quad \text{Tr}(AZ) + \text{Tr}(BZ) \\
\text{subject to} & \quad (17) - (26).
\end{align*}
\]

(SDP3) (48)

One can easily see that the optimal value of this SDP is an upper bound on the welfare achievable under any Nash equilibrium. To test the quality of this upper bound, we tested this SDP on a random sample of one hundred 5 \times 5 and 10 \times 10 games\footnote{The matrices were randomly generated with uniform and independent entries in [0,1]. The computation of the upper bounds on the maximum payoffs was done with the function \texttt{nashbound.m}, which computes an SDP lower bound on the problem of minimizing a quadratic function over the set of Nash equilibria of a bimatrix game. This code is publicly available at \texttt{aaa.princeton.edu/software}. The exact computation of the maximum payoffs was done with the \texttt{lrsnash} software\textsuperscript{b}, which computes extreme Nash equilibria. For a definition of extreme Nash equilibria and for understanding why it is sufficient for us to compare against extreme Nash equilibria (both in Section 6.1 and in Section 6.2), see Appendix B.}. The results are in Figures 3, which show that the bound returned by SDP3 was exact in a large number of the experiments.

Figure 3: The quality of the upper bound on the maximum welfare obtained by SDP3 on 100 5 \times 5 games (left) and 100 10 \times 10 games (right).
6.2 Strategy Exclusion

The strategy exclusion problem asks, given a subset of strategies $\mathcal{S} = (\mathcal{S}_x, \mathcal{S}_y)$, with $\mathcal{S}_x \subseteq \{1, \ldots, m\}$ and $\mathcal{S}_y \subseteq \{1, \ldots, n\}$, is there a Nash equilibrium in which no strategy in $\mathcal{S}$ is played with positive probability. We will call a set $\mathcal{S}$ “persistent” if the answer to this question is negative, i.e. at least one strategy in $\mathcal{S}$ is played with positive probability in every Nash equilibrium. One application of the strategy exclusion problem is to understand whether certain strategies can be discouraged in the design of a game, such as reckless behavior in a game of chicken or defecting in a game of prisoner’s dilemma. In these particular examples these strategy sets are persistent and cannot be discouraged.

A quadratic program which can address the strategy exclusion problem is as follows:

$$\min_{x \in \Delta_m, y \in \Delta_n} \sum_{i \in \mathcal{S}_x} x_i + \sum_{i \in \mathcal{S}_y} y_i$$
subject to
$$x^T Ay \geq e_i^T Ay, \forall i \in \{1, \ldots, m\},$$
$$x^T B y \geq x^T B e_j, \forall j \in \{1, \ldots, n\}.$$  \hspace{1cm} (49)

Observe that by design, $\mathcal{S}$ is persistent if and only if this quadratic program has a positive optimal value. The SDP relaxation of this problem is given by

$$\min_{M' \in \mathbb{S}^{m+n+1}} \sum_{i \in \mathcal{S}} x_i + \sum_{i \in \mathcal{S}} y_i$$
subject to
$$\mathcal{M}' \in \mathbb{S}^{m+n+1}, \mathcal{M}+1,$$
$$\{17\} - \{26\}.\hspace{1cm} (50)$$

Our approach for the strategy exclusion problem would be to declare that a strategy set is persistent if and only if $\text{SDP4}$ has positive optimal value.

Note that since the optimal value of $\text{SDP4}$ is a lower bound for that of $\text{49}$, $\text{SDP4}$ carries over the property that if a set $\mathcal{S}$ is not persistent, then the SDP for sure returns zero. Thus, when using $\text{SDP4}$ on a set which is not persistent, our algorithm will always be correct. However, this is not necessarily the case for a persistent set. While we can be certain that a set is persistent if $\text{SDP4}$ returns a positive optimal value (again, because the optimal value of $\text{SDP4}$ is a lower bound for that of $\text{49}$), there is still the possibility that for a persistent set $\text{SDP4}$ will have optimal value zero.

To test the performance of $\text{SDP4}$ we generated 100 random games of size $5 \times 5$ and $10 \times 10$ and computed all their extreme Nash equilibria.\footnote{The exact computation of the Nash equilibria was done again with the lrsnash software \cite{5}, which computes extreme Nash equilibria. To understand why this suffices for our purposes see Appendix \ref{app}.} We then, for every strategy set $\mathcal{S}$ of cardinality one and two, checked whether that set of strategies was persistent, first by checking among the extreme Nash equilibria, then through $\text{SDP4}$. The results are presented in Tables \ref{table2} and \ref{table3}. As motivated by the discussion above, we separately show the performance on all instances and the performances on persistent input instances. As can be seen, $\text{SDP4}$ was quite effective for the strategy exclusion problem. In particular, for $10 \times 10$ games, we have a perfect identification rate.

7 Connection to the Sum of Squares/Lasserre Hierarchy

In this section, we clarify the connection of the SDPs we have proposed in this paper to those arising in the sum of squares/Lasserre hierarchy. We start by briefly reviewing this hierarchy.
Table 2: Performance of SDP4 on 5 × 5 games

<table>
<thead>
<tr>
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</tr>
</thead>
<tbody>
<tr>
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<tr>
<td>Number Correct</td>
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<td>4465</td>
<td></td>
</tr>
<tr>
<td>Percent Correct</td>
<td>99.6 %</td>
<td>99.2 %</td>
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</tbody>
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<table>
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<tr>
<th></th>
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</tr>
</thead>
<tbody>
<tr>
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<td></td>
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<tr>
<td>Number Correct</td>
<td>18</td>
<td>1443</td>
<td></td>
</tr>
<tr>
<td>Percent Correct</td>
<td>81.8%</td>
<td>97.6%</td>
<td></td>
</tr>
</tbody>
</table>

Table 3: Performance of SDP4 on 10 × 10 games

<table>
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<td></td>
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<tr>
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<td></td>
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<tr>
<td>Percent Correct</td>
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<td>100 %</td>
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<tr>
<td>Percent Correct</td>
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<td>100%</td>
<td></td>
</tr>
</tbody>
</table>

7.1 **Sum of Squares/Lasserre Hierarchy**

The sum of squares/Lasserre hierarchy\(^8\) gives a recipe for constructing a sequence of SDPs whose optimal values converge to the optimal value of a given polynomial optimization problem. Recall that a polynomial optimization problem (pop) is a problem of minimizing a polynomial over a basic semialgebraic set, i.e., a problem of the form

\[
\min_{x \in \mathbb{R}^n} f(x) \\
\text{subject to } g_i(x) \geq 0, \forall i \in \{1, \ldots, m\}, \tag{51}
\]

where \(f, g_i\) are polynomial functions. In this section, when we refer to the \(k\)-th level of the Lasserre hierarchy, we mean the optimization problem

\[
\gamma^k_{sos} := \max_{\gamma, \sigma_i} \gamma \\
\text{subject to } f(x) - \gamma = \sigma_0(x) + \sum_{i=1}^m \sigma_i(x)g_i(x), \tag{52}
\]

\[
\sigma_i \text{ is sos, } \forall i \in \{0, \ldots, m\}, \\
\sigma_0, g_i \sigma_i \text{ have degree at most } 2k, \forall i \in \{1, \ldots, m\}.
\]

Here, the notation “sos” stands for sum of squares. We say that a polynomial \(p\) is a sum of squares if there exist polynomials \(q_1, \ldots, q_r\) such that \(p = \sum_{i=1}^r q_i^2\). There are two primary properties of the Lasserre hierarchy which are of interest. The first is that any fixed level of this hierarchy gives an SDP of size polynomial in \(n\). The second is that, if the set \(\{x \in \mathbb{R}^n|g_i(x) \geq 0\}\) is Archimedean (see, e.g. \([24]\) for definition), then \(\lim_{k \to \infty} \gamma^k_{sos} = p^*,\) where \(p^*\) is the optimal value of the pop in (51).

The latter statement is a consequence of Putinar’s positivstellensatz \([32], [22]\).

7.2 **The Lasserre Hierarchy and SDP1**

One can show, e.g. via the arguments in \([23]\), that the feasible sets of the SDPs dual to the SDPs underlying the hierarchy we summarized above produce an arbitrarily tight outer approximation to the convex hull of the set of Nash equilibria of any game. The downside of this approach, however, is that the higher levels of the hierarchy can get expensive very quickly. This is why the approach we took in this paper was instead to improve the first level of the hierarchy. The next proposition formalizes this connection.

\(^8\)The unfamiliar reader is referred to \([22, 30, 24]\) for an introduction to this hierarchy and the related theory of moment relaxations.
Proposition 7.1. Consider the problem of minimizing any quadratic objective function over the set of Nash equilibria of a bimatrix game. Then, \[ \text{SDP1} \] (and hence \[ \text{SDP2} \]) gives a lower bound on this problem which is no worse than that produced by the first level of the Lasserre hierarchy.

Proof. To prove this proposition we show that the first level of the Lasserre hierarchy is dual to a weakened version of \[ \text{SDP1} \].

Explicit parametrization of first level of the Lasserre hierarchy. Consider the formulation of the Lasserre hierarchy in (52) with \( k = 1 \). Suppose we are minimizing a quadratic function

\[
\begin{bmatrix}
  x \\
  y \\
  1 
\end{bmatrix}^T \begin{bmatrix}
  C \\
  y \\
  1 
\end{bmatrix}
\]

over the set of Nash equilibria as described by the linear and quadratic constraints in (2). If we apply the first level of the Lasserre hierarchy to this particular pop, we get

\[
\max_{\gamma, \alpha, \chi, \beta, \psi, \eta} \gamma
\]

subject to

\[
\begin{bmatrix}
  x \\
  y \\
  1 
\end{bmatrix}^T \begin{bmatrix}
  C \\
  y \\
  1 
\end{bmatrix} - \gamma = \begin{bmatrix}
  x \\
  y \\
  1 
\end{bmatrix}^T \begin{bmatrix}
  Q \\
  y \\
  1 
\end{bmatrix} + \sum_{i=1}^m \alpha_i (x^T A y - e_i^T A y)
\]

\[
+ \sum_{i=1}^n \beta_i (x^T B y - x^T B e_i)
\]

\[
+ \sum_{i=1}^m \chi_i x_i + \sum_{i=1}^n \psi_i y_i
\]

\[
+ \eta_1 (m \sum_{i=1}^m x_i - 1) + \eta_2 (n \sum_{i=1}^n y_i - 1),
\]

\[
Q \succeq 0,
\]

\[
\alpha, \chi, \beta, \psi \geq 0,
\]

where \( Q \in \mathbb{S}^{m+n+1 \times m+n+1} \), \( \alpha, \chi \in \mathbb{R}^m \), \( \beta, \psi \in \mathbb{R}^n \), \( \eta \in \mathbb{R}^2 \).

By matching coefficients of the two quadratic functions on the left and right hand sides of (53), this SDP can be written as

\[
\max_{\gamma, \alpha, \beta, \chi, \psi, \eta} \gamma
\]

subject to

\[
\mathcal{H} \succeq 0,
\]

\[
\alpha, \beta, \chi, \psi \geq 0,
\]

where

\[
\mathcal{H} := \frac{1}{2} \begin{bmatrix}
  0 & - \sum_{i=1}^m \alpha_i A + (- \sum_{i=1}^m \beta_i) B & - \sum_{i=1}^m \beta_i B_i^T - \chi^T - \eta_1 1_m & \zeta \left( - \sum_{i=1}^m \beta_i B_i^T - \chi^T - \eta_1 1_m \right)
  \\
  \sum_{i=1}^m \alpha_i A_i - \psi^T - \eta_2 1_n & 0 & \sum_{i=1}^m \alpha_i A_i^T - \psi - \eta_2 1_n & \zeta \left( \sum_{i=1}^m \alpha_i A_i^T - \psi - \eta_2 1_n \right)
  \\
  \end{bmatrix} + C.
\]

Dual of a weakened version of SDP1. With this formulation in mind, let us consider a weakened version of \[ \text{SDP1} \] with only the relaxed Nash constraints, unity constraints, and nonnegativity constraints on \( x \) and \( y \) in the last column (i.e., the nonegativity constraint is not applied to the entire matrix). Let the objective be \( \text{Tr}(CM') \). To write this new SDP in standard form, let

\[
28
\]
\[ A_i := \frac{1}{2} \begin{bmatrix} 0 & A_i & 0 \\ A_i^T & 0 & -A_i^T \\ 0 & -A_i & 0 \end{bmatrix}, \quad B_i := \frac{1}{2} \begin{bmatrix} 0 & B_i & -B_i \\ B_i^T & 0 & 0 \\ -B_i^T & 0 & 0 \end{bmatrix}, \]
\[ S_1 := \frac{1}{2} \begin{bmatrix} 0 & 0 & 1_m \\ 0 & 0 & 0 \\ 1_m^T & 0 & -2 \end{bmatrix}, \quad S_2 := \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1_n \\ 1_n^T & 0 & -2 \end{bmatrix}. \]

Let \( N_i \) be the matrix with all zeros except a \( \frac{1}{2} \) at entry \((i, m + n + 1)\) and \((m + n + 1, i)\) (or a 1 if \( i = m + n + 1 \)).
Then this SDP can be written as

\[
\begin{align*}
\min \quad & \text{Tr}(CM') \\
\text{subject to} \quad & \mathcal{M}' \succeq 0, \\
& \text{Tr}(N_i\mathcal{M}') \geq 0, \forall i \in \{1, \ldots, m + n\}, \\
& \text{Tr}(A_i\mathcal{M}') \geq 0, \forall i \in \{1, \ldots, m\}, \\
& \text{Tr}(B_i\mathcal{M}') \geq 0, \forall i \in \{1, \ldots, n\}, \\
& \text{Tr}(S_1\mathcal{M}') = 0, \\
& \text{Tr}(S_2\mathcal{M}') = 0, \\
& \text{Tr}(N_{m+n+1}) = 1.
\end{align*}
\]

We now create dual variables for each constraint; we choose \( \alpha_i \) and \( \beta_i \) for the relaxed Nash constraints (58) and (59), \( \eta_1 \) and \( \eta_2 \) for the unity constraints (60) and (61), \( \chi \) for the nonnegativity of \( x \) (57), \( \psi \) for the nonnegativity of \( y \) (57), and \( \gamma \) for the final constraint on the corner (62). These variables are chosen to coincide with those used in the parametrization of the first level of the Lasserre hierarchy, as can be seen more clearly below.

We then write the dual of the above SDP as

\[
\max_{\alpha, \beta, \chi, \psi, \gamma} \quad \gamma
\]

\[
\text{subject to} \quad G \succeq 0, \\
\alpha, \beta, \chi, \psi \geq 0, \quad \begin{align*}
\sum_{i=1}^{m} \alpha_i A_i + \sum_{i=1}^{n} \beta_i B_i + \sum_{i=1}^{2} \eta_i S_i + \sum_{i=1}^{m} N_i + \sum_{i=1}^{n} \chi_i + \sum_{i=1}^{m} N_i + \sum_{i=1}^{n} \chi_i + \gamma N_{m+n+1} & \preceq C, \\
\end{align*}
\]

which can be rewritten as

\[
\max_{\alpha, \beta, \chi, \psi, \gamma} \quad \gamma
\]

\[
\text{subject to} \quad G \succeq 0, \\
\alpha, \beta, \chi, \psi \geq 0, \quad \begin{align*}
\sum_{i=1}^{m} \alpha_i A_i + \sum_{i=1}^{n} \beta_i B_i + \sum_{i=1}^{2} \eta_i S_i + \sum_{i=1}^{m} N_i + \sum_{i=1}^{n} \chi_i + \gamma N_{m+n+1} & \preceq C, \\
\end{align*}
\]

where

\[
G := \frac{1}{2} \left[ \begin{array}{ccc}
0 & (-\sum_{i=1}^{m} \alpha_i) A + (-\sum_{i=1}^{n} \beta_i) B & (-\sum_{i=1}^{m} \alpha_i) A + (-\sum_{i=1}^{m} \beta_i) B \\
(-\sum_{i=1}^{m} \beta_i) B^T + \chi T - \eta_1 1_m^T & 0 & \sum_{i=1}^{m} \beta_i B_i + \chi - \eta_1 1_m \\
\sum_{i=1}^{m} \beta_i B_i^T - \chi T - \eta_1 1_m & \sum_{i=1}^{m} \alpha_i A_i + \psi T - \eta_2 1_n & 2\eta_1 + 2\eta_2 - 2\gamma \end{array} \right] + C.
\]

We can now see that the matrix \( G \) coincides with the matrix \( H \) in the SDP (54). Then we have

\[
\begin{align*}
\text{(53) } \text{opt} &= \text{(54) } \text{opt} = \text{(63) } \text{opt} & \leq & \text{SDP0 opt} \leq \text{SDP1 opt},
\end{align*}
\]

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where the first inequality follows from weak duality, and the second follows from that the constraints of \( \text{SDP0} \) are a subset of the constraints of \( \text{SDP1} \).

**Remark 7.1.** The Lasserre hierarchy can be viewed in each step as a pair of primal-dual SDPs: the sum of squares formulation which we have just presented, and a moment formulation which is dual to the sos formulation \([22]\). All our SDPs in this paper can be viewed more directly as an improvement upon the moment formulation.

**Remark 7.2.** One can see, either by inspection or as an implication of the proof of Theorem 2.2, that in the case where the objective function corresponds to maximizing player A’s and/or B’s payoffs\(^9\) SDPs (54) and (63) are infeasible. This means that for such problems the first level of the Lasserre hierarchy gives an upper bound of \(+\infty\) on the maximum payoff. On the other hand, the additional valid inequalities in \( \text{SDP2} \) guarantee that the resulting bound is always finite.

## 8 Future Work

Our work leaves many avenues of further research. Are there other interesting subclasses of games (besides strictly competitive games) for which our SDP is guaranteed to recover an exact Nash equilibrium? Can the guarantees on \( \epsilon \) in Section 4 be improved in the rank-2 case (or the general case), by improving our analysis or for example by using the correlated equilibrium constraints (which we did not use)? Is there a polynomial time algorithm that is guaranteed to find a rank-2 solution to \( \text{SDP2} \)? Such an algorithm, together with our analysis, would improve the best known approximation bound for symmetric games (see Theorem 4.16). Can SDPs in a higher level of the Lasserre hierarchy be used to achieve better \( \epsilon \) guarantees? What are systematic ways of adding valid inequalities to these higher-order SDPs by exploiting the structure of the Nash equilibrium problem? For example, since any strategy played with positive probability must give the same payoff, one can add a relaxed version of the cubic constraints

\[
x_ix_j(e_i^T Ay - e_j^T Ay) = 0, \forall i, j \in \{1, \ldots, m\}
\]

to the SDP underlying the second level of the Lasserre hierarchy. What are other valid inequalities for the second level? Finally, our algorithms were specifically designed for two-player one-shot games. This leaves open the design and analysis of semidefinite relaxations for repeated games or games with more than two players.

## References


\(^9\)This would be the case, for example, in the maximum social welfare problem of Section 6.1 where the matrix of the quadratic form in the objective function is given by

\[
C = \begin{bmatrix}
0 & -A - B & 0 \\
-A - B & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]


A Statistics on $\epsilon$ from Algorithms in Section 5

Below are statistics for the $\epsilon$ recovered in 100 random games of varying sizes using the algorithms of Section 5.

Table 4: Statistics on $\epsilon$ for 5 x 5 games after 20 iterations.

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Max</th>
<th>Mean</th>
<th>Median</th>
<th>StDev</th>
</tr>
</thead>
<tbody>
<tr>
<td>Square Root</td>
<td>0.0702</td>
<td>0.0040</td>
<td>0.0004</td>
<td>0.0099</td>
</tr>
<tr>
<td>Diagonal Gap</td>
<td>0.0448</td>
<td>0.0027</td>
<td>0</td>
<td>0.0061</td>
</tr>
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</table>
B Lemmas for Extreme Nash Equilibria

The results reported in Section 6 were found using the lrsnash [5] software which computes extreme Nash equilibria (see definition below). In particular the true maximum welfare and the persistent strategy sets were found in relation to extreme Nash equilibria only. We show in this appendix why this is sufficient for the claims we made about all Nash equilibria.

**Definition B.1.** An extreme Nash equilibrium is a Nash equilibrium which cannot be expressed as a convex combination of other Nash equilibria.

**Lemma B.2.** All Nash equilibria are convex combinations of extreme Nash equilibria.

<table>
<thead>
<tr>
<th>Table 5: Statistics on $\epsilon$ for $10 \times 5$ games after 20 iterations.</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Algorithm</strong></td>
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<td>Square Root</td>
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<td>Diagonal Gap</td>
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<table>
<thead>
<tr>
<th>Table 6: Statistics on $\epsilon$ for $10 \times 10$ games after 20 iterations.</th>
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<tr>
<td><strong>Algorithm</strong></td>
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<tr>
<td>Square Root</td>
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<td>Diagonal Gap</td>
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<table>
<thead>
<tr>
<th>Table 7: Statistics on $\epsilon$ for $15 \times 10$ games after 20 iterations.</th>
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</thead>
<tbody>
<tr>
<td><strong>Algorithm</strong></td>
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<td>Square Root</td>
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<td>Diagonal Gap</td>
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<thead>
<tr>
<th>Table 8: Statistics on $\epsilon$ for $15 \times 15$ games after 20 iterations.</th>
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<td><strong>Algorithm</strong></td>
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<td>Square Root</td>
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<td>Diagonal Gap</td>
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</table>

<table>
<thead>
<tr>
<th>Table 9: Statistics on $\epsilon$ for $20 \times 15$ games after 20 iterations.</th>
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<tr>
<td><strong>Algorithm</strong></td>
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<tr>
<td>Square Root</td>
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<td>Diagonal Gap</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table 10: Statistics on $\epsilon$ for $20 \times 20$ games after 20 iterations.</th>
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</thead>
<tbody>
<tr>
<td><strong>Algorithm</strong></td>
</tr>
<tr>
<td>Square Root</td>
</tr>
<tr>
<td>Diagonal Gap</td>
</tr>
</tbody>
</table>

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Proof. It suffices to show that any extreme point of the convex hull of the set of Nash equilibria must be an extreme Nash equilibrium, as any point in a convex set can be written as a convex combination of its extreme points. Suppose for the purpose of contradiction that this was not the case, i.e. there is a point \( x \) which is an extreme point of the convex hull of Nash equilibria but is not an extreme Nash equilibrium. Then either it is not a Nash equilibrium, or it is a Nash equilibrium which is not extreme. In both cases, \( x \) can be written as a convex combination of other Nash equilibria, and so cannot be an extreme point for the convex hull. For the former case it is because its membership in the convex hull must be due to an expression of it as a convex combination of Nash equilibria, and in the latter it is due to the definition of extreme Nash equilibria.

The next lemma shows that checking extreme Nash equilibria are sufficient for the maximum welfare problem.

Lemma B.3. For any bimatrix game, there exists an extreme Nash equilibrium giving the maximum welfare among all Nash equilibria.

Proof. Consider any Nash equilibrium \((\tilde{x}, \tilde{y})\), and let it be written as \[
\begin{bmatrix}
\tilde{x} \\
\tilde{y}
\end{bmatrix}
= \sum_{i=1}^{r} \lambda_i \begin{bmatrix}
x^i \\
y^i
\end{bmatrix}
\] for some set of extreme Nash equilibria \(\begin{bmatrix}
x^1 \\
y^1
\end{bmatrix}, \ldots, \begin{bmatrix}
x^r \\
y^r
\end{bmatrix}\) and \(\lambda \in \Delta_r\). Observe that for any \(i,j\),

\[
x^iT(A + B)y^j \leq x^iT Ay^j, x^iT B y^j \leq x^iT B y^i,
\] (64)

from the definition of a Nash equilibrium. Now note that

\[
\tilde{x}^T (A + B)\tilde{y} = \left( \sum_{i=1}^{r} \lambda_i x^i \right)^T (A + B) \left( \sum_{i=1}^{r} \lambda_i y^i \right)
= \sum_{i=1}^{r} \sum_{j=1}^{r} \lambda_i \lambda_j x^iT (A + B) y^j
= \sum_{i=1}^{r} \sum_{j=1}^{r} \lambda_i \lambda_j x^iT Ay^j + \sum_{i=1}^{r} \sum_{j=1}^{r} \lambda_i \lambda_j x^iT By^j
\leq \sum_{i=1}^{r} \sum_{j=1}^{r} \lambda_i \lambda_j x^iT Ay^j + \sum_{i=1}^{r} \sum_{j=1}^{r} \lambda_i \lambda_j x^iT By^i
= \sum_{i=1}^{r} \lambda_i x^iT Ay^i + \sum_{i=1}^{r} \lambda_i x^iT By^i
= \sum_{i=1}^{r} \lambda_i x^iT (A + B) y^i.
\]

In particular, since each \((x^i, y^i)\) is an extreme Nash equilibrium, this tells us for any Nash equilibrium \((\tilde{x}, \tilde{y})\) there must be an extreme Nash equilibrium which has at least as much welfare.

Similarly for the results for persistent sets in Section B.2, there is no loss in restricting attention to extreme Nash equilibria.

Lemma B.4. For a given strategy set \(S\), if every extreme Nash equilibrium plays at least one strategy in \(S\) with positive probability, then every Nash equilibrium plays at least one strategy in \(S\) with positive probability.
Proof. Let $S$ be a persistent set of strategies. Since all Nash equilibria are composed of nonnegative entries, and every extreme Nash equilibrium has positive probability on some entry in $S$, any convex combination of extreme Nash equilibria must have positive probability on some entry in $S$.

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