Semidefinite Programming
and Nash Equilibria in Bimatrix Games

Amir Ali Ahmadi and Jeffrey Zhang *

Abstract

We explore the power of semidefinite programming (SDP) for finding additive \( \epsilon \)-approximate Nash equilibria in bimatrix games. We introduce an SDP relaxation for a quadratic programming formulation of the Nash equilibrium (NE) problem and provide a number of valid inequalities to improve the quality of the relaxation. If a rank-1 solution to this SDP is found, then an exact NE can be recovered. We show that for a (generalized) zero-sum game, our SDP is guaranteed to return a rank-1 solution. Furthermore, we prove that if a rank-2 solution to our SDP is found, then a \( \frac{5}{11} \)-NE can be recovered for any game, or a \( \frac{1}{3} \)-NE for a symmetric game. We propose two algorithms based on iterative linearization of smooth nonconvex objective functions which are designed to produce rank-1 solutions. Empirically, we show that these algorithms often recover solutions of rank at most two and \( \epsilon \) close to zero. We then show how our SDP approach can address two (NP-hard) problems of economic interest: finding the maximum welfare achievable under any NE, and testing whether there exists a NE where a particular set of strategies is not played. Finally, we show that by using the Lasserre/sum of squares hierarchy, we can get an arbitrarily close spectrahedral outer approximation to the convex hull of Nash equilibria, and that the SDP proposed in this paper dominates the first level of the sum of squares hierarchy.

1 Introduction

A bimatrix game is a game between two players (referred to in this paper as players A and B) defined by a pair of \( m \times n \) payoff matrices \( A \) and \( B \). Let \( \Delta_m \) and \( \Delta_n \) denote the \( m \)-dimensional and \( n \)-dimensional simplices

\[
\Delta_m = \{ x \in \mathbb{R}^m | x_i \geq 0, \forall i, \sum_{i=1}^{m} x_i = 1 \}, \quad \Delta_n = \{ y \in \mathbb{R}^n | y_i \geq 0, \forall i, \sum_{i=1}^{n} y_i = 1 \}.
\]

These form the strategy spaces of player A and player B respectively. For a strategy pair \( (x, y) \in \Delta_m \times \Delta_n \), the payoff received by player A (resp. player B) is \( x^T Ay \) (resp. \( x^T By \)). In particular, if the players pick vertices \( i \) and \( j \) of their respective simplices (also called pure strategies), their payoffs will be \( A_{i,j} \) and \( B_{i,j} \). One of the prevailing solution concepts for bimatrix games is the notion of Nash equilibrium. At such an equilibrium, the players are playing mutual best responses, i.e., a payoff maximizing strategy against the opposing player’s strategy. In our notation, a Nash equilibrium for the game \( (A, B) \) is a pair of strategies \( (x^*, y^*) \in \Delta_m \times \Delta_n \) such that

\[
x^{*T} Ay^* \geq x^T Ay^*, \forall x \in \Delta_m,
\]

*The authors are partially supported by the Young Investigator Award of the AFOSR, the CAREER Award of the NSF, the Google Faculty Award, and the Sloan Fellowship.
\[ x^*T By^* \geq x^*T B y, \forall y \in \triangle_n. \]

Nash [25] proved that for any bimatrix game, such pairs of strategies exist (in fact his result more generally applies to games with a finite number of players and a finite number of pure strategies). While existence of these equilibria is guaranteed, finding them is believed to be a computationally intractable problem. More precisely, a result of Daskalakis, Goldberg, and Papadimitriou [9] implies that computing Nash equilibria is PPAD-complete (see [9] for a definition) even when the number of players is 3. This result was later improved by Chen and Deng [6] who showed the same hardness result for bimatrix games.

These results motivate the notion of an approximate Nash equilibrium, a solution concept in which players receive payoffs “close” to their best response payoffs. More precisely, a pair of strategies \((x^*, y^*) \in \triangle_m \times \triangle_n\) is an \((\text{additive}) \epsilon\)-Nash equilibrium for the game \((A, B)\) if

\[ x^*T Ay^* \geq x^T Ay^* - \epsilon, \forall x \in \triangle_m, \]

and

\[ x^*T By^* \geq x^*T B y - \epsilon, \forall y \in \triangle_n. \]

Approximation of Nash equilibria has also shown to be computationally difficult. Cheng, Deng, and Teng proved in [7] that, unless PPAD \(\subseteq\) P, there cannot be a fully polynomial-time approximation scheme for computing Nash equilibria in bimatrix games. There have, however, been a series of constant factor approximation algorithms for this problem [11, 10, 17, 33], with the current best producing a .3393 approximation via an algorithm by Tsaknakis and Spirakis [33].

We remark that there are exponential-time algorithms for computing Nash equilibria, such as the Lemke-Howson algorithm [21, 30]. There are also certain subclasses of the problem which can be solved in polynomial time, the most notable example being the case of zero-sum games (i.e. when \(B = -A\)). This problem was shown to be solvable via linear programming by Dantzig [8], and later shown to be polynomially equivalent to linear programming by Adler [1]. Aside from computation of Nash equilibria, there are a number of related decision questions which are of economic interest but unfortunately NP-hard. Examples include deciding whether a player’s payoff exceeds a certain threshold in some Nash equilibrium, deciding whether a game has a unique Nash equilibrium, or testing whether there exists a Nash equilibrium where a particular set of strategies is not played [13].

Our focus in this paper is on understanding the power of semidefinite programming (SDP) for finding approximate Nash equilibria in bimatrix games or providing certificates for related decision problems. Semidefinite programming relaxations have been analyzed in the past for an array of intractable problems in computational mathematics (most notably in combinatorial optimization [14], [22] and systems theory [5]), but to our knowledge not for computation of Nash equilibria in general bimatrix games. SDPs have appeared however elsewhere in the literature on game theory for finding equilibria, e.g. by Stein for exchangeable equilibria in symmetric games [32], by Parrilo, Laraki, and Lasserre for Nash equilibria in zero-sum polynomial games [27, 18], or by Parrilo and Shah for zero-sum stochastic games [31].

\(^1\)In this paper we assume that all entries of \(A\) and \(B\) are between 0 and 1, and argue at the beginning of Section 2 why this is without loss of generality for the purpose of computing Nash equilibria.

\(^2\) The unfamiliar reader is referred to [34] for the theory of SDPs and a description of polynomial-time algorithms for them based on interior point methods.
1.1 Organization and Contributions of the Paper

In Section 2, we pose the problem of finding a Nash equilibrium in a bimatrix game as a nonconvex quadratically constrained quadratic program and pose an SDP relaxation for it. In Section 2.3, we exploit the structure of the Nash equilibrium problem to introduce a number of valid inequalities which strengthen our SDP relaxation. In Section 3, we show that our SDP is exact when the game is zero-sum (or generalized zero-sum; see Definition 3.3). In Section 4, we produce a number of bounds on the quality of the approximate Nash equilibria that our SDP produces. We show that if the SDP has a rank-\(k\) solution with nonzero eigenvalues \(\lambda_1, \ldots, \lambda_k\), then one can recover from it an \(\epsilon\)-Nash equilibrium with \(\epsilon \leq \frac{m+n}{2} \sum_{i=2}^{k} \lambda_i\) (Theorem 4.5). We then present an improved analysis in the rank-2 case which shows how one can recover a \(\frac{1}{11}\)-Nash equilibrium from the SDP solution (Theorem 4.12). We further prove that for symmetric games (i.e., when \(B = A^T\)), a \(\frac{1}{5}\)-Nash equilibrium can be recovered in the rank-2 case (Theorem 4.16). In Section 5, we design two algorithms based on iterative linearization of smooth nonconvex objective functions, which if minimized globally, would produce rank-1 solutions. We show empirically that these algorithms produce \(\epsilon\) very close to zero (on average in the order of \(10^{-3}\)). In Section 6, we show how our SDP formulation can be used to provide certificates for certain (NP-hard) questions of economic interest about Nash equilibria. These are the problems of testing whether the maximum welfare achievable under any Nash equilibrium exceeds some threshold, and whether a set of strategies is played in every Nash equilibrium. In Section 7, we show that by using the Lasserre/sum of squares hierarchy, we can get an arbitrarily close spectrahedral outer approximation to the convex hull of Nash equilibria (Theorem 7.5), and that the SDP analyzed in this paper dominates the first level of the Lasserre hierarchy (Proposition 7.1). Some directions for future research are discussed in Section 8.

2 The Formulation of our SDP Relaxation

In this section we present an SDP relaxation for the problem of finding Nash equilibria in bimatrix games. This is done after a straightforward reformulation of the problem as a nonconvex quadratically constrained quadratic program. Throughout the paper the following notation is used.

- \(A_i\) refers to the \(i\)-th row of a matrix \(A\).
- \(A_j\) refers to the \(j\)-th column of a matrix \(A\).
- \(e_i\) refers to the elementary vector \((0, \ldots, 0, 1, 0, \ldots, 0)^T\) with the 1 being in position \(i\).
- \(\Delta_k\) refers to the \(k\)-dimensional simplex.
- \(1_m\) refers to the \(m\)-dimensional vector of one’s.
- \(0_m\) refers to the \(m\)-dimensional vector of zero’s.
- \(J_{m,n}\) refers to the \(m \times n\) matrix of one’s.
- \(A \succeq 0\) denotes that the matrix \(A\) is positive semidefinite (psd), i.e., has nonnegative eigenvalues.
- \(A \geq 0\) denotes that the matrix \(A\) is nonnegative, i.e., has nonnegative entries.
- \(A \succeq B\) denotes that \(A - B \succeq 0\).
- \(S_{k \times k}\) denotes the set of symmetric \(k \times k\) matrices.
- \(\text{Tr}(A)\) denotes the trace of a matrix \(A\), i.e., the sum of its diagonal elements.
- \(A \otimes B\) denotes the Kronecker product of matrices \(A\) and \(B\).
- \(\text{vec}(M)\) denotes the vectorized version of a matrix \(M\), and \(\text{diag}(M)\) denotes the vector containing its diagonal entries.
We also assume that all entries of the payoff matrices $A$ and $B$ are between 0 and 1. This can be done without loss of generality because Nash equilibria are invariant under certain affine transformations in the payoffs. In particular, the games $(A, B)$ and $(cA + dJ_m \times n, eB + fJ_m \times n)$ have the same Nash equilibria for any scalars $c, d, e, f$, with $c$ and $e$ positive. This is because

$$x^TAy \geq x^TAy$$

$$\iff c(x^TAy^*) + d \geq c(x^TAy^*) + d$$

$$\iff c(x^TAy^*) + d(x^TJ_m \times ny^*) \geq c(x^TAy^*) + d(x^TJ_m \times ny^*)$$

$$\iff x^T(cA + dJ_m \times n)y^* \geq x^T(cA + dJ_m \times n)y$$

Identical reasoning applies for player B.

### 2.1 Nash Equilibria as Solutions to Quadratic Programs

Recall the definition of a Nash equilibrium from Section 1. The following offers a straightforward reformulation, which states that existence of a profitable deviation implies existence of a profitable pure strategy deviation.

**Proposition 2.1.** A strategy pair $(x^*, y^*) \in \triangle_m \times \triangle_n$ is a Nash equilibrium for the game $(A, B)$ if and only if

$$x^TAy^* \geq e_i^TAy^*, \forall i \in \{1, \ldots, m\},$$

$$x^TB y^* \geq x^TB e_i, \forall i \in \{1, \ldots, n\}.$$ 

**Proof.** The fact that we require $x^TAy^* \geq e_i^TAy^*, x^TB y^* \geq x^TB e_i$ follows from the definition of a Nash equilibrium as each $e_i$ is a simplex vector. For the other direction, observe that if $x^TAy^* \geq e_i^TAy^*, \forall i$, then we must have

$$x^TAy^* \geq \max_i e_i^TAy^* \geq x^TAy^*, \forall x \in \triangle_m,$$

where the second inequality follows from that for any $x \in \triangle_m$, the quantity $x^TAy$ is a convex combination of the scalars $e_i^TAy$. The same logic applies to $y$ and $B$ and this concludes the proof.

We now treat the Nash problem as the following quadratic programming (QP) feasibility problem:

$$\begin{align*}
\min_{x \in \mathbb{R}^m, y \in \mathbb{R}^n} & \quad 0 \\
\text{subject to} & \quad x^TAy \geq e_i^TAy, \forall i \in \{1, \ldots, m\}, \\
& \quad x^TB y \geq x^TB e_j, \forall j \in \{1, \ldots, n\}, \\
& \quad x_i \geq 0, \forall i \in \{1, \ldots, m\}, \\
& \quad y_j \geq 0, \forall j \in \{1, \ldots, n\}, \\
& \quad \sum_{i=1}^{m} x_i = 1,
& \quad \sum_{i=1}^{n} y_i = 1.
\end{align*}$$

(1)
An identical argument as in Proposition 2.1 shows that $x^* \in \triangle_m$ and $y^* \in \triangle_n$ form an $\epsilon$-Nash equilibrium for the game $(A, B)$ if and only if

\[ x^T Ay^* \geq e^T_i Ay - \epsilon, \forall i \in \{1, \ldots, m\}, \]
\[ x^T By^* \geq x^T Be_i - \epsilon, \forall i \in \{1, \ldots, n\}. \]

Observe that any pair of simplex vectors $(x, y)$ is an $\epsilon$-Nash equilibrium for the game $(A, B)$ for any $\epsilon$ that satisfies

\[ \epsilon \geq \max \{ \max_i e^T_i Ay - x^T Ay, \max_i x^T Be_i - x^T By \}. \]

We use the following notation throughout the paper:

\[ \epsilon_A(x, y) := \max_i e^T_i Ay - x^T Ay, \]
\[ \epsilon_B(x, y) := \max_i x^T Be_i - x^T By, \]
\[ \epsilon(x, y) := \max \{ \epsilon_A(x, y), \epsilon_B(x, y) \}, \]

and the function parameters are later omitted if they are clear from the context.

### 2.2 SDP Relaxation

The QP formulation in (1) lends itself to a natural SDP relaxation. We define a matrix

\[ M := \begin{bmatrix} X & P \\ Z & Y \end{bmatrix}, \]

and an augmented matrix

\[ M' := \begin{bmatrix} X & P & x \\ Z & Y & y \\ x & y & 1 \end{bmatrix}, \]

with $X \in S^{m \times m}, Z \in \mathbb{R}^{n \times m}, Y \in S^{n \times n}, x \in \mathbb{R}^m, y \in \mathbb{R}^n$ and $P = Z^T$.

The SDP relaxation can then be expressed as

\[
\begin{align*}
\min_{M' \in S^{m+n+1, m+n+1}} & 0 \\
\text{subject to} & \quad \text{Tr}(AZ) \geq e^T_i Ay, \forall i \in \{1, \ldots, m\}, \quad (2) \\
& \quad \text{Tr}(BZ) \geq x^T Be_j, \forall j \in \{1, \ldots, n\}, \quad (3) \\
& \quad \sum_{i=1}^m x_i = 1, \quad (4) \\
& \quad \sum_{i=1}^n y_i = 1, \quad (5) \\
& \quad M'_{ij} \geq 0 \quad \forall i, j \in \{1, \ldots, m+n+1\}, \quad (6) \\
& \quad M'_{m+n+1,m+n+1} = 1, \quad (7) \\
& \quad M' \succeq 0. \quad (8)
\end{align*}
\]

We refer to the constraints (2) and (3) as the relaxed Nash constraints and the constraints (4) and (5) as the unity constraints. This SDP is motivated by the following observation.
Proposition 2.2. Let $M'$ be any rank-1 feasible solution to $\text{SDP1}$. Then the vectors $x$ and $y$ from its last column constitute a Nash equilibrium for the game $(A, B)$.

Proof. We know that $x$ and $y$ are in the simplex from the constraints (4), (5), and (6). If the matrix $M'$ is rank-1, then it takes the form

$$
\begin{bmatrix}
xx^T & xy^T & x \\
yx^T & yy^T & y \\
x^T & y^T & 1
\end{bmatrix} =
\begin{bmatrix}
x \\
y \\
1
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
1
\end{bmatrix}^T.
$$

(9)

Then, from the relaxed Nash constraints we have that

$$
e_i^T Ay \leq \text{Tr}(AZ) = \text{Tr}(Ayx^T) = \text{Tr}(x^T Ay) = x^T Ay,
$$

$$
x^T Ae_i \leq \text{Tr}(BZ) = \text{Tr}(Byx^T) = \text{Tr}(x^T By) = x^T By.
$$

The claim now follows from Proposition 2.1. \qed

Remark 2.1. Because a Nash equilibrium always exists, there will always be a matrix of the form (9) which is feasible to $\text{SDP1}$. Thus we can disregard any concerns about $\text{SDP1}$ being feasible, even when we add valid inequalities to it in Section 2.3.

Remark 2.2. It is intuitive to note that the submatrix $P = Z^T$ of the matrix $M'$ corresponds to a probability distribution over the strategies, and that seeking a rank-1 solution to our SDP can be interpreted as making $P$ a product distribution.

The following theorem shows that $\text{SDP1}$ is a weak relaxation and stresses the necessity of additional valid constraints.

Theorem 2.3. Consider a bimatrix game with payoff matrices bounded in $[0, 1]$. Then for any two vectors $x \in \Delta_m$ and $y \in \Delta_n$, there exists a feasible solution $M'$ to $\text{SDP1}$ with $\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$ as its last column.

Proof. Consider any $x, y, \gamma > 0$, and the matrix

$$
\begin{bmatrix}
x \\
y \\
1
\end{bmatrix}
\begin{bmatrix}
x \\
y \\
1
\end{bmatrix}^T + \gamma J_{m+n,m+n} 0_{m+n}^T 0_{m+n}.
$$

This matrix is the sum of two nonnegative psd matrices and is hence nonnegative and psd. By assumption $x$ and $y$ are in the simplex, and so constraints (4) – (8) of $\text{SDP1}$ are satisfied. To check that constraints (2) and (3) hold, note that since $A$ and $B$ are nonnegative, as long as the matrices $A$ and $B$ are not the zero matrices, the quantities $\text{Tr}(AZ)$ and $\text{Tr}(BZ)$ will become arbitrarily large as $\gamma$ increases. Since $e_i^T Ay$ and $x^T Be_i$ are bounded by 1 by assumption, we will have that constraints (2) and (3) hold for $\gamma$ large enough. In the case where $A$ or $B$ is the zero matrix, the Nash constraints are trivially satisfied for the respective player. \qed

2.3 Valid Inequalities

In this subsection, we introduce a number of valid inequalities to improve upon the SDP relaxation in $\text{SDP1}$. These inequalities are justified by being valid if the matrix returned by the SDP is rank-1. The terminology we introduce here to refer to these constraints is used throughout the paper.
2.3.1 Distribution Constraints

**Proposition 2.4.** Any rank-1 solution \( M' \) to \( \text{SDP1} \) must satisfy the following:

\[
\begin{align*}
\sum_{i=1}^{m} \sum_{j=1}^{m} X_{i,j} &= 1, \\
\sum_{i=1}^{n} \sum_{j=1}^{m} Z_{i,j} &= 1, \\
\sum_{i=1}^{n} \sum_{j=1}^{n} Y_{i,j} &= 1.
\end{align*}
\]

(10)\( \ldots \)\( 12)

**Proof.** Recall from (9) that if \( M' \) is rank-1, it is of the form

\[
\begin{pmatrix}
x x^T & x y^T & x \\
y x^T & y y^T & y \\
x^T & y^T & 1
\end{pmatrix}
= \begin{pmatrix}
x \\
y \\
1
\end{pmatrix} \begin{pmatrix}
x \\
y \\
1
\end{pmatrix}^T.
\]

Then we must have

\[
\begin{align*}
\sum_{i=1}^{m} \sum_{j=1}^{m} X_{i,j} &= \sum_{i=1}^{m} \sum_{j=1}^{m} x_i x_j = \sum_{i=1}^{m} x_i \sum_{j=1}^{m} x_j = 1, \\
\sum_{i=1}^{n} \sum_{j=1}^{m} Z_{i,j} &= \sum_{i=1}^{n} \sum_{j=1}^{m} y_i x_j = \sum_{i=1}^{n} y_i \sum_{j=1}^{m} x_j = 1, \\
\sum_{i=1}^{n} \sum_{j=1}^{n} Y_{i,j} &= \sum_{i=1}^{n} \sum_{j=1}^{n} y_i y_j = \sum_{i=1}^{n} y_i \sum_{j=1}^{n} y_j = 1.
\end{align*}
\]

\( \blacksquare \)

The next proposition shows that we can drop constraint (11) from above.

**Proposition 2.5.** In \( \text{SDP1} \) the equality in (11) is implied by the equalities in (10) and (12) along with the unity and positive semidefiniteness constraints on \( M' \).

**Proof.** Denote \( s_X, s_Y, s_Z \) as the sums of all elements in \( X, Y \) and \( Z \) respectively (note \( s_X = s_Y = 1 \)). If \( s_z > 1 \), then

\[
\begin{pmatrix}
1_m \\
-1_n \\
0
\end{pmatrix}
T
\begin{pmatrix}
M' \\
1_m \\
0
\end{pmatrix}
= \begin{pmatrix}
x & s_X + s_Y - 2s_Z = 2 - 2s_z < 0.
\end{pmatrix}
\]

If \( s_z < 1 \), then

\[
\begin{pmatrix}
1_m \\
1_n \\
-2
\end{pmatrix}
T
\begin{pmatrix}
M' \\
1_m \\
1_n \\
-2
\end{pmatrix}
= \begin{pmatrix}
x + s_Y + 2s_Z + 4 - 4 \sum_{i=1}^{m} x - 4 \sum_{i=1}^{n} y = -2 + 2s_Z < 0.
\end{pmatrix}
\]

Since we require \( M' \) to be positive semidefinite, it must be that condition (11) holds. \( \blacksquare \)
2.3.2 Row Constraints

Proposition 2.6. Any rank-1 solution $M'$ to SDP1 must satisfy the following:

\[
\sum_{j=1}^{m} X_{i,j} = \sum_{j=1}^{n} P_{i,j} = x_i, \forall i \in \{1, \ldots, m\}, \tag{13}
\]

\[
\sum_{j=1}^{n} Y_{i,j} = \sum_{j=1}^{m} Z_{i,j} = y_i, \forall i \in \{1, \ldots, n\}. \tag{14}
\]

Proof. Recall the rank-1 form of $M'$. Then we must have

\[
\sum_{j=1}^{m} X_{i,j} = \sum_{j=1}^{m} x_i x_j = x_i, \forall i \in \{1, \ldots, m\},
\]

\[
\sum_{j=1}^{n} P_{i,j} = \sum_{j=1}^{n} y_j x_i = x_i, \forall i \in \{1, \ldots, m\},
\]

\[
\sum_{j=1}^{n} Y_{i,j} = \sum_{j=1}^{n} y_i y_j = y_i, \forall i \in \{1, \ldots, n\},
\]

\[
\sum_{j=1}^{m} Z_{i,j} = \sum_{j=1}^{m} y_i x_j = y_i, \forall i \in \{1, \ldots, n\}.
\]

Remark 2.3. One can easily show that the distribution constraints are also implied by the row constraints in conjunction with the unity constraints, but it will be useful for us later to refer to them separately.

2.3.3 Correlated Equilibrium Constraints

Proposition 2.7. Any rank-1 solution $M'$ to SDP1 must satisfy the following:

\[
\sum_{j=1}^{n} A_{i,j} P_{i,j} \geq \sum_{j=1}^{n} A_{k,j} P_{i,j}, \forall i, k \in \{1, \ldots, m\}, \tag{15}
\]

\[
\sum_{j=1}^{m} B_{j,i} P_{j,i} \geq \sum_{j=1}^{m} B_{j,k} P_{j,i}, \forall i, k \in \{1, \ldots, n\}. \tag{16}
\]

Proof. Note that the pair $(x, y)$ is a Nash equilibrium if and only if

\[
\forall i, x_i > 0 \Rightarrow e_i^T Ay = x^T Ay = \max_i e_i^T Ay,
\]

\[
\forall i, y_i > 0 \Rightarrow x^T Be_i = x^T By = \max_i x^T Be_i.
\]

This is because the Nash conditions require that $x^T Ay$, a convex combination of $e_i^T Ay$, be at least $e_i^T Ay$ for all $i$. Indeed, if $x_i > 0$ but $e_i^T Ay < x^T Ay$, the convex combination must be less than $\max_i x^T Ay$.
For each $i$ such that $x_i = 0$ or $y_i = 0$, inequalities (15) and (16) reduce to $0 \geq 0$, so we only need to consider strategies played with positive probability. Observe that if $\mathcal{M}'$ is rank-1, then

$$
\sum_{j=1}^{n} A_{i,j} P_{i,j} = x_i \sum_{j=1}^{n} A_{i,j} y_j = x_i e_i^T A y_i \geq x_i e_k^T A y_i = \sum_{j=1}^{n} A_{k,j} P_{i,j}, \forall i, k
$$

$$
\sum_{j=1}^{m} B_{j,i} P_{j,i} = y_i \sum_{j=1}^{m} B_{j,i} x_j = y_i x_i^T B e_i \geq y_i x_k^T B e_k = \sum_{j=1}^{m} B_{j,i} P_{j,k}, \forall i, k.
$$

There are two ways to interpret the inequalities in (15) and (16): the first is as a relaxation of the constraint $x_i(e_i^T A y - e_j^T A y) \geq 0, \forall i, j$, which must hold since any strategy played with positive probability must give the best response payoff. The other interpretation is to have the distribution over outcomes defined by $P$ be a correlated equilibrium [2]. This can be imposed by a set of linear constraints on the entries of $P$ as explained next.

Suppose the players have access to a public randomization device which prescribes a pure strategy to each of them (unknown to the other player). The distribution over the assignments can be given by a matrix $P$, where $P_{i,j}$ is the probability that strategy $i$ is assigned to player A and strategy $j$ is assigned to player B. This distribution is a correlated equilibrium if both players have no incentive to deviate from the strategy prescribed, that is, if the prescribed pure strategies $a$ and $b$ satisfy

$$
\sum_{j=1}^{n} A_{i,j} \text{Prob}(b = j|a = i) \geq \sum_{j=1}^{n} A_{k,j} \text{Prob}(b = j|a = i),
$$

$$
\sum_{i=1}^{m} B_{i,j} \text{Prob}(a = i|b = j) \geq \sum_{i=1}^{m} B_{i,k} \text{Prob}(a = i|b = j).
$$

If we interpret the $P$ submatrix in our SDP as the distribution over the assignments by the public device, then because of our row constraints, $\text{Prob}(b = j|a = i) = P_{i,j} x_i$ whenever $x_i \neq 0$ (otherwise the above inequalities are trivial). Similarly, $P(a = i|b = j) = P_{j,i} y_j$ for nonzero $y_j$. Observe now that the above two inequalities imply (15) and (16). Finally, note that every Nash equilibrium generates a correlated equilibrium, since if $P$ is a product distribution given by $xy^T$, then $\text{Prob}(b = j|a = i) = y_j$ and $P(a = i|b = j) = x_i$.

### 2.4 McCormick Inequalities

McCormick inequalities [23] are a well-known set of valid inequalities for box-constrained quadratic programs. However, they are not included explicitly in our SDP because they are redundant given the constraints we have, as shown in the following lemma.

**Lemma 2.8.** Let $z := \begin{bmatrix} x \\ y \end{bmatrix}$. The McCormick inequalities

$$
\mathcal{M}_{i,j} \leq z_i, \forall i, j \in \{1, \ldots, m+n\},
$$

$$
\mathcal{M}_{i,j} + 1 \geq z_i + z_j, \forall i, j \in \{1, \ldots, m+n\},
$$

are implied by the row constraints and nonnegativity of $\mathcal{M}'$. 


Proof. The first inequality is immediate as a consequence of (13) and (14), as all entries of $\mathcal{M}$ are nonnegative. To see why the second inequality holds, consider whichever submatrix $X, Y, P,$ or $Z$ that contains $\mathcal{M}_{i,j}$. Suppose that this submatrix is, e.g., $P$. Then, since $P$ is nonnegative,

$$0 \leq \sum_{k=1, k \neq i}^{m} \sum_{l=1, l \neq j}^{n} P_{k,l} \sum_{k=1, k \neq i}^{m} (x_k - P_{k,j}) (1 - x_i) - (y_j - P_{i,j}) = P_{i,j} + 1 - x_i - y_j.$$

The same argument holds for the other submatrices, and this concludes the proof.

2.5 The Effect of Valid Inequalities on an Example Game

Consider the following randomly-generated $5 \times 5$ bimatrix game:

$$A = \begin{bmatrix} 0.42 & 0.46 & 0.03 & 0.77 & 0.33 \\ 0.54 & 0.03 & 0.71 & 0.06 & 0.56 \\ 0.53 & 0.43 & 0.17 & 0.85 & 0.30 \\ 0.19 & 0.56 & 0.59 & 0.38 & 0.37 \\ 0.08 & 0.64 & 0.61 & 0.40 & 0.35 \end{bmatrix}, \quad B = \begin{bmatrix} 0.63 & 0.19 & 0.09 & 0.22 & 0.33 \\ 0.66 & 0.92 & 0.26 & 0.97 & 0.43 \\ 0.99 & 0.29 & 0.43 & 0.43 & 0.72 \\ 0.94 & 0.55 & 0.58 & 0.78 & 0.92 \\ 0.35 & 0.92 & 0.90 & 0.53 & 0.89 \end{bmatrix}.$$

Figure 1 demonstrates the shrinkage in the feasible set of our SDP, projected onto the first two pure strategies of player A, as valid inequalities are added. Subfigure (a) depicts the Nash equilibria and the feasible set of SDP1 (recall from 2.3 that the feasible region without valid inequalities is the projection of the entire simplex). The row and distribution constraints are added for subfigure (b), and the correlated equilibrium constraints are further added for subfigure (c). Subfigure (d) depicts the true projection of the convex hull of Nash equilibria.

![Figure 1](image-url)

Figure 1: Reduction in the size of our spectrahedral outer approximation to the convex hull of Nash equilibria through the addition of valid inequalities.
2.6 Strengthened SDP Relaxation

We now write out our new SDP with all constraints in one place. Recall the representation of the matrix

\[ \mathcal{M} := \begin{bmatrix} X & P & x \\ Z & Y & y \\ x^T & y^T & 1 \end{bmatrix}, \]

with \( P = Z^T \). The improved SDP is now:

\[
\begin{align*}
\min_{\mathcal{M} \in S^{(m+n+1) \times (m+n+1)}} & \quad 0 \\
\text{subject to} & \quad \mathcal{M} \succeq 0, \\
& \quad \mathcal{M}_{ij} \geq 0, i, j \in \{1, \ldots, m+n+1\}, \\
& \quad \mathcal{M}_{m+n+1,m+n+1} = 1, \\
& \quad \sum_{i=1}^{m} x_i = 1, \\
& \quad \sum_{i=1}^{n} y_i = 1, \\
& \quad \text{Tr}(AZ) - e_i^T Ay \geq 0, \forall i \in \{1, \ldots, m\}, \\
& \quad \text{Tr}(BZ) - x^T Be_i \geq 0, \forall i \in \{1, \ldots, n\}, \\
& \quad \sum_{j=1}^{m} X_{i,j} = \sum_{j=1}^{n} Z_{j,i} = x_i, \forall i \in \{1, \ldots, m\}, \\
& \quad \sum_{j=1}^{n} Y_{i,j} = \sum_{j=1}^{m} Z_{i,j} = y_i, \forall i \in \{1, \ldots, n\}, \\
& \quad \sum_{j=1}^{m} A_{i,j} P_{i,j} \geq \sum_{j=1}^{n} A_{k,j} P_{i,j}, \forall i, k \in \{1, \ldots, m\}, \\
& \quad \sum_{j=1}^{m} B_{j,i} P_{j,i} \geq \sum_{j=1}^{n} B_{j,k} P_{j,i}, \forall i, k \in \{1, \ldots, n\}.
\end{align*}
\]

Constraints (22) and (23) are the relaxed Nash constraints, constraints (24) and (25) are the row constraints, and constraints (26) and (27) are the correlated equilibrium constraints.

3 Exactness for Zero-Sum Games

In this section, we show that the semidefinite programs we have proposed are guaranteed to recover a Nash equilibrium for zero-sum games. This class of games is solvable in polynomial time with linear programming. Nonetheless, it is reassuring to know that our SDPs recover this important special case.

**Definition 3.1.** A zero-sum game is a game in which the payoff matrices satisfy \( A = -B \).
Remark 3.1. This definition captures a game in which two players are in direct competition, however, it clearly cannot be applied in the context of games where all payoffs are assumed to be nonnegative. We later define (see Definition 3.3) a generalized zero-sum game, which preserves this property of direct competition but is more general and allows for nonnegative payoffs.

Theorem 3.2. For a zero-sum game, the vectors \( x \) and \( y \) from the last column of any feasible solution \( \mathcal{M}' \) to SDP1 constitute a Nash equilibrium.

Proof. Recall that the relaxed Nash constraints (2) and (3) read
\[
\text{Tr}(AZ) \geq e_i^T Ay, \forall i \in \{1, \ldots, m\},
\]
\[
\text{Tr}(BZ) \geq x^T Be_j, \forall j \in \{1, \ldots, n\}.
\]
Since \( B = -A \), the latter statement is equivalent to
\[
\text{Tr}(AZ) \leq x^T Ae_j, \forall j \in \{1, \ldots, n\}.
\]
In conjunction these imply
\[
e_i^T Ay \leq \text{Tr}(AZ) \leq x^T Ae_j, \forall i \in \{1, \ldots, m\}, \forall j \in \{1, \ldots, n\}.
\]
(28)

We claim that any pair \( x \in \Delta_m \) and \( y \in \Delta_n \) which satisfies the above condition is a Nash equilibrium. To see that \( x^T Ay \geq e_i^T Ay, \forall i \in \{1, \ldots, m\} \), observe that \( x^T Ay \) is a convex combination of \( x^T Ae_j \), which are at most \( e_i^T Ay \) by (28). To see that \( x^T By \geq x^T Be_j \Leftrightarrow x^T Ay \leq x^T Ae_j, \forall j \in \{1, \ldots, n\} \), observe that \( x^T Ay \) is a convex combination of \( e_i^T Ay \), which are at most \( x^T Ae_j \) by (28).

Definition 3.3. We say that a game \((A, B)\) is generalized zero-sum if there exist scalars \( c, d, e, \) and \( f \), with \( c > 0, e > 0 \), such that \( cA + dJ_{m \times n} = -eB + fJ_{m \times n} \).

One can easily show that there exist generalized zero-sum games for which not all feasible solutions to SDP1 have Nash equilibria as their last columns (see Theorem 2.3). However, we show that this is the case for SDP2. To prove this, we need the following lemma, which shows that feasibility of a matrix \( \mathcal{M}' \) in SDP2 is invariant under certain transformations of \( A \) and \( B \).

Lemma 3.4. Let \( c, d, e, \) and \( f \) be any set of scalars with \( c > 0 \) and \( e > 0 \). If a matrix \( \mathcal{M}' \) is feasible to SDP2 with input payoff matrices \( A \) and \( B \), then it is also feasible to SDP2 with input matrices \( cA + dJ_{m \times n} \) and \( eB + fJ_{m \times n} \).

Proof. It suffices to check that constraints (22), (23), (26), and (27) of SDP2 still hold, as only the relaxed Nash and correlated equilibrium constraints use the matrices \( A \) and \( B \). We only show that constraints (22) and (26) still hold because the arguments for constraints (23) and (27) are identical.

First recall that due to constraints (21) and (25) of SDP2 \( \text{Tr}(J_{m \times n} Z) = 1 \). To check that the relaxed Nash constraints hold, observe that for scalars \( c > 0 \) and \( d \), and for all \( i \in \{1, \ldots, m\} \),
\[
\text{Tr}(AZ) - e_i^T Ay \geq 0,
\]
\[
\Leftrightarrow c\text{Tr}(AZ) + d - c(e_i^T Ay) - d \geq 0,
\]
\[
\Leftrightarrow c\text{Tr}(AZ) + d(\text{Tr}(J_{m \times n} Z)) - c(e_i^T Ay) - d(\text{Tr}(J_{m \times n} Z)) \geq 0,
\]
\[
\Leftrightarrow \text{Tr}((cA + dJ_{m \times n}) Z) - e_i^T (cA + dJ_{m \times n}) y \geq 0.
\]
Now recall from constraint (24) of SDP2 (keeping in mind that \( P = Z^T \)) that \( \sum_{j=1}^{n} (J_{m \times n})_{i,j} P_{i,j} = x_i \). To check that the correlated equilibrium constraints hold, observe that for scalars \( c > 0 \), \( d \), and for all \( i, k \in \{1, \ldots, m\} \),

\[
\sum_{j=1}^{n} A_{i,j} P_{i,j} \geq \sum_{j=1}^{n} A_{k,j} P_{i,j},
\]

\[
\iff c \sum_{j=1}^{n} A_{i,j} P_{i,j} + dx_i \geq c \sum_{j=1}^{n} A_{k,j} P_{i,j} + dx_i
\]

\[
\iff c \sum_{j=1}^{n} A_{i,j} P_{i,j} + d \sum_{j=1}^{n} (J_{m \times n})_{i,j} P_{i,j} \geq c \sum_{j=1}^{n} A_{k,j} P_{i,j} + d \sum_{j=1}^{n} (J_{m \times n})_{k,j} P_{i,j}
\]

\[
\iff \sum_{j=1}^{n} (cA_{i,j} + dJ_{m \times n})_{k,j} P_{i,j} \geq \sum_{j=1}^{n} (cA_{i,j} + dJ_{m \times n})_{k,j} P_{i,j}.
\]

\[
\frac{1}{n} \sum_{j=1}^{n} (cA_{i,j} + dJ_{m \times n})_{k,j} P_{i,j} \geq \frac{1}{n} \sum_{j=1}^{n} (cA_{i,j} + dJ_{m \times n})_{k,j} P_{i,j}.
\]

**Theorem 3.5.** For a generalized zero-sum game, the vectors \( x \) and \( y \) from the last column of any feasible solution \( \mathcal{M}' \) to SDP2 constitute a Nash equilibrium.

**Proof.** Let \( A \) and \( B \) be the payoff matrices of the given generalized zero-sum game and let \( \mathcal{M}' \) be a feasible solution to SDP2. Since the game is generalized zero-sum, we know that \( cA + dJ_{m \times n} = -eB + fJ_{m \times n} \) for some scalars \( c > 0 \), \( e > 0 \), \( d \), \( f \). Consider a new game with input matrices \( \tilde{A} = cA + dJ_{m \times n} \) and \( \tilde{B} = eB - fJ_{m \times n} \). By Lemma 3.4, \( \mathcal{M}' \) is still feasible to SDP2 with input matrices \( \tilde{A} \) and \( \tilde{B} \). Furthermore, since the constraints of SDP1 are a subset of the constraints of SDP2, \( \mathcal{M}' \) is also feasible to SDP1. Now notice that since \( A = -\tilde{B} \), Theorem 3.2 implies that the vectors \( x \) and \( y \) in the last column form a Nash equilibrium to the game \( (\tilde{A}, \tilde{B}) \). Finally recall from Section 2 that Nash equilibria are invariant to scaling and shifting of the payoff matrices, and hence \( (x, y) \) is a Nash equilibrium to the game \( (A, B) \).

**Remark 3.2.** We later show in Proposition 5.1 that using the trace of \( \mathcal{M} \) as the objective function to SDP2 will guarantee that SDP2 returns a rank-1 solution.

**4 Bounds on \( \epsilon \) for General Games**

In this section, we provide upper bounds on the \( \epsilon \) returned by SDP2 for an arbitrary bimatrix game. Since the problem of computing a Nash equilibrium in such a game is PPAD-complete, it is unlikely that one can find rank-1 solutions to this SDP in polynomial time. In Section 5, we design objective functions (such as variations of the nuclear norm) that empirically do very well in finding low-rank solutions to SDP2. Nevertheless, it is of interest to know if the solution returned by SDP2 is not rank-1, whether one can recover an \( \epsilon \)-Nash equilibrium from it and have a guarantee on \( \epsilon \).

Recall our notation for the matrices

\[
\mathcal{M} := \begin{bmatrix} X & P \\ Z & Y \end{bmatrix},
\]

and
\[
\mathcal{M} := \begin{bmatrix} X & P & x \\ Z & Y & y \\ x^T & y^T & 1 \end{bmatrix}.
\]

Throughout this section, any matrices \(X, Z, P = Z^T\) and \(Y\) or vectors \(x\) and \(y\) are assumed to be taken from a feasible solution to SDP2. The ultimate results of this section are Theorems 4.5, 4.12, and 4.16. To work towards them, we need a number of preliminary lemmas which we present in Section 4.1.

4.1 Lemmas Towards Bounds on \(\epsilon\)

We first observe the following connection between the approximate payoffs \(\text{Tr}(AZ)\) and \(\text{Tr}(BZ)\), and \(\epsilon(x, y)\), as defined in Section 2.1.

**Lemma 4.1.** Consider a feasible solution \(\mathcal{M}'\) to SDP2 and the vectors \(x\) and \(y\) and the matrix \(Z\) from that solution. Then

\[
\epsilon(x, y) \leq \max\{\text{Tr}(AZ) - x^T Ay, \text{Tr}(BZ) - x^T By\}.
\]

**Proof.** Note that since \(\text{Tr}(AZ) \geq e_i^T Ay\) and \(\text{Tr}(BZ) \geq x^T Be_i\) from constraints (22) and (23) of SDP2, we have \(\epsilon_A \leq \text{Tr}(AZ) - x^T Ay\) and \(\epsilon_B \leq \text{Tr}(BZ) - x^T By\).

We thus are interested in the difference of the two matrices \(P = Z^T\) and \(xy^T\). These two matrices can be interpreted as two different probability distributions over the strategy outcomes. The matrix \(P\) is the probability distribution from the SDP which generates the approximate payoffs \(\text{Tr}(AZ)\) and \(\text{Tr}(BZ)\), while \(xy^T\) is the product distribution that would have resulted if the matrix had been rank-1. We will see that the difference of these distributions is key in studying the \(\epsilon\) which results from the SDP. Hence, we first take steps to represent this difference.

**Lemma 4.2.** Consider any feasible matrix \(\mathcal{M}'\) to SDP2, and its submatrix \(\mathcal{M}\). Let the matrix \(\mathcal{M}\) be given by an eigendecomposition

\[
\mathcal{M} = \sum_{i=1}^{k} \lambda_i v_i v_i^T =: \sum_{i=1}^{k} \lambda_i \begin{bmatrix} a_i \\ b_i \end{bmatrix} \begin{bmatrix} a_i \\ b_i \end{bmatrix}^T,
\]

(29)

so that the eigenvectors \(v_i \in \mathbb{R}^{m+n}\) are partitioned into \(a_i \in \mathbb{R}^m\) and \(b_i \in \mathbb{R}^n\). Then for all \(i, \sum_{j=1}^{m}(a_i)_j = \sum_{j=1}^{n}(b_i)_j\).

**Proof.** We know from the distribution constraints from Section 2.3.1 that

\[
\sum_{i=1}^{k} \lambda_i 1_m^T a_i a_i^T 1_m \overset{10}{=} 1,
\]

(30)

\[
\sum_{i=1}^{k} \lambda_i 1_n^T b_i b_i^T 1_n \overset{12}{=} 1.
\]

(33)
Then by subtracting terms we have

\[(30) - (31) = \sum_{i=1}^{k} \lambda_i a_i^T (a_i^T 1_m - b_i^T 1_n) = 0,\]  
\[(32) - (33) = \sum_{i=1}^{k} \lambda_i b_i^T (a_i^T 1_m - b_i^T 1_n) = 0.\]  

By subtracting again these imply

\[(34) - (35) = \sum_{i=1}^{k} \lambda_i (a_i^T a_i - b_i^T b_i)^2 = 0.\]  

As all \(\lambda_i\) are nonnegative due to positive semidefiniteness of \(M\), the only way for this equality to hold is to have \(1_m^T a_i = 1_n^T b_i\), \(\forall i\). This is equivalent to the statement of the claim. \(\square\)

From Lemma 4.2, we can let \(s_i := \sum_{j=1}^{m} (a_i)_j = \sum_{j=1}^{n} (b_i)_j\), and furthermore we assume without loss of generality that each \(s_i\) is nonnegative. Note that from the row constraint \((13)\) we have

\[x_i = \sum_{j=1}^{m} X_{ij} = \sum_{i=1}^{k} \sum_{j=1}^{m} \lambda_i (a_i)_i (a_i)_j = \sum_{j=1}^{k} \lambda_j s_j (a_i)_i.\]  

Hence,

\[x = \sum_{i=1}^{k} \lambda_i s_i a_i.\]  

Similarly,

\[y = \sum_{i=1}^{k} \lambda_i s_i b_i.\]  

Finally note from the distribution constraint \((10)\) that this implies

\[\sum_{i=1}^{k} \lambda_i s_i^2 = 1.\]  

**Lemma 4.3.** Let

\[\mathcal{M} = \sum_{i=1}^{k} \lambda_i \begin{bmatrix} a_i \\ b_i \end{bmatrix} \begin{bmatrix} a_i \\ b_i \end{bmatrix}^T,\]

be a feasible solution to \(\text{SDP2}\) such that the eigenvectors of \(\mathcal{M}\) are partitioned into \(a_i\) and \(b_i\) with \(\sum_{i=1}^{m} a_i = \sum_{i=1}^{n} b_i = s_i, \forall i\). Then

\[P - xy^T = \sum_{i=1}^{k} \sum_{j>i}^{k} \lambda_i \lambda_j (s_j a_i - s_i a_j)(s_j b_i - s_i b_j)^T.\]
Proof. Using equations (38) and (39) we can write
\[
P - xy^T = \sum_{i=1}^{k} \lambda_i a_i b_i^T - (\sum_{i=1}^{k} \lambda_i s_i a_i)(\sum_{j=1}^{k} \lambda_j s_j b_j)^T
\]
\[
= \sum_{i=1}^{k} \lambda_i (b_i - s_i \sum_{j=1}^{k} \lambda_j s_j b_j)^T
\]
\[
\geq \sum_{i=1}^{k} \lambda_i a_i (s_j b_j - s_i b_j)^T
\]
\[
= \sum_{i=1}^{k} \sum_{j=1}^{k} \lambda_i \lambda_j (s_j a_i - s_i a_j)(s_j b_i - s_i b_j)^T.
\]

We can relate \(\epsilon\) and \(P - xy^T\) with the following lemma.

**Lemma 4.4.** Consider any feasible solution \(M'\) to \(\text{SDP}_2\) and the matrix \(P - xy^T\). Then
\[
\epsilon \leq \|P - xy^T\|_1^2,
\]
where \(\|\cdot\|_1\) here denotes the entrywise L-1 norm, i.e., the sum of the absolute values of the entries of the matrix.

Proof. Let \(D := P - xy^T\). From Lemma 4.1,
\[
\epsilon_A \leq \text{Tr}(A(Z) - x^T Ay) = \text{Tr}(A(Z - yx^T)).
\]
If we then hold \(D\) fixed and restrict that \(A\) has entries bounded in \([0,1]\), the quantity \(\text{Tr}(AD^T)\) is maximized when
\[
A_{i,j} = \begin{cases} 
1 & D_{i,j} \geq 0 \\
0 & D_{i,j} < 0 
\end{cases}.
\]
The resulting quantity \(\text{Tr}(AD^T)\) will then be the sum of all nonnegative elements of \(D\). Since the sum of all elements in \(D\) is zero, this quantity will be equal to \(\frac{1}{2}\|D\|_1\).
The proof for \(\epsilon_B\) is identical, and the result follows from that \(\epsilon\) is the maximum of \(\epsilon_A\) and \(\epsilon_B\).

4.2 Bounds on \(\epsilon\)

We provide a number of bounds on \(\epsilon(x, y)\), where \(x\) and \(y\) are taken from the last column of a feasible solution to \(\text{SDP}_2\). The first is a theorem stating that solutions which are “close” to rank-1 provide small \(\epsilon\).

**Theorem 4.5.** Let \(M'\) be a feasible solution to \(\text{SDP}_2\). Suppose \(M\) is rank-\(k\) and its eigenvalues are \(\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k \geq 0\). Then, the \(x\) and \(y\) from the last column of \(M'\) constitute an \(\epsilon\)-NE to the game \((A, B)\) with \(\epsilon \leq \frac{m+n}{2} \sum_{i=2}^{k} \lambda_i\).
Finally note that a consequence of the nonnegativity of $s$, we know that
\[ s_i \leq \min\{\|a_i\|_1, \|b_i\|_1\} \leq \frac{\sqrt{m+n}}{2}. \]  
This then gives us
\[ s_i^2 \leq \|a_i\|_1\|b_i\|_1 \leq \frac{m+n}{4}. \]

Now we let $D := P - xy^T$ and upper bound $\frac{1}{2}\|D\|_1$ using Lemma 4.3.
\[
\frac{1}{2}\|D\|_1 = \frac{1}{2} \left\| \sum_{i=1}^{k} \sum_{j \succ i} \lambda_i \lambda_j (s_j a_i - s_i a_j)(s_j b_i - s_i b_j)^T \right\|_1
\]
\[
\leq \frac{1}{2} \sum_{i=1}^{k} \sum_{j \succ i} \|\lambda_i \lambda_j (s_j a_i - s_i a_j)(s_j b_i - s_i b_j)^T\|_1
\]
\[
\leq \frac{1}{2} \sum_{i=1}^{k} \sum_{j \succ i} \lambda_i \lambda_j \|s_j a_i - s_i a_j\|_1 \|s_j b_i - s_i b_j\|_1
\]
\[
\leq \frac{1}{2} \sum_{i=1}^{k} \sum_{j \succ i} \lambda_i \lambda_j (s_j \|a_i\|_1 + s_i \|a_j\|_1)(s_j \|b_i\|_1 + s_i \|b_j\|_1)
\]
\[
\leq \frac{1}{2} \sum_{j=2}^{k} \lambda_1 s_1^2 \lambda_j (s_j + \|a_j\|_1)(s_j + \|b_j\|_1)
\]
\[
+ \frac{1}{2} \sum_{i=2}^{k} \sum_{j \succ i} \lambda_i \lambda_j (s_j^2 + \frac{m+n}{4}) + s_i s_j \|a_i\|_1 \|b_j\|_1 + s_i s_j \|a_j\|_1 \|b_i\|_1)
\]
\[
\leq \frac{m+n}{2} \lambda_1 s_1^2 \sum_{i=2}^{k} \lambda_i
\]
\[
+ \frac{1}{2} \sum_{i=2}^{k} \sum_{j \succ i} \lambda_i \lambda_j \left( \frac{m+n}{4} s_i^2 + s_j^2 \right) + \lambda_1 \lambda_j s_i s_j (m+n)
\]
\[
\leq \frac{m+n}{2} \lambda_1 s_1^2 \sum_{i=2}^{k} \lambda_i + \frac{m+n}{2} \sum_{i=2}^{k} \sum_{j \succ i} \lambda_i \lambda_j \left( \frac{s_i^2 + s_j^2}{4} + \frac{s_i^2 + s_j^2}{2} \right).
\]
4.3 Bounds on $\epsilon$ in the Rank-2 Case

We now give a series of bounds on $\epsilon$ which hold for feasible solutions to SDP2 that are rank-2 (note that due to the row inequalities, $M$ will have the same rank as $M'$). This is motivated by our ability to show stronger (constant) bounds in this case, and the fact that we often recover rank-2 (or rank-1) solutions with our algorithms in Section 5. As it will be important in our study of this particular case, we first recall the definition of a completely positive factorization/rank of a matrix.

**Definition 4.6.** A matrix $M$ is **completely positive (CP)** if it admits a decomposition $M = UU^T$ for some nonnegative matrix $U$.

**Definition 4.7.** The **CP-rank** of an $n \times n$ CP matrix $M$ is the smallest $k$ for which there exists a nonnegative $n \times k$ matrix $U$ such that $M = UU^T$.

**Theorem 4.8** (see e.g. [16] or Theorem 2.1 in [4]). A rank-2, nonnegative, and positive semidefinite matrix is CP and has CP-rank 2.

It is also known (see e.g., Section 4 in [16]) that the CP factorization of a rank-2 CP matrix can be found to arbitrary accuracy in polynomial time. With these preliminaries in mind, we present lemmas which are similar to Lemmas 4.2 and 4.3, but provide a decomposition which is specific to the rank-2 case.

**Lemma 4.9.** Suppose that a feasible solution $M'$ to SDP2 is rank-2. Then there exists a decomposition $M = \sigma_1 \begin{bmatrix} a \\ b \end{bmatrix} \begin{bmatrix} a^T \\ b \end{bmatrix}^T + \sigma_2 \begin{bmatrix} c \\ d \end{bmatrix} \begin{bmatrix} c^T \\ d \end{bmatrix}^T$, where $\sigma_1$ and $\sigma_2$ are nonnegative, $\sigma_1 + \sigma_2 = 1$, $a, c \in \Delta_m$, and $b, d \in \Delta_n$.
Proof. Since the matrix $M$ is rank-2, nonnegative, and positive semidefinite, from Theorem 4.8 we can decompose the matrix $M$ into $v_1v_1^T + v_2v_2^T$ where $v_1$ and $v_2$ are nonnegative. We can then partition $v_1$ and $v_2$ into vectors $\begin{bmatrix} a \\ b \end{bmatrix}$ and $\begin{bmatrix} c \\ d \end{bmatrix}$ as we did in Lemma 4.2. Furthermore, we note that because the distribution constraints (10)-(12) still hold, by repeating the proof of Lemma 4.2 we find that $\sum_{i=1}^{m} a_i' = \sum_{i=1}^{n} b_i'$, and $\sum_{i=1}^{m} c_i' = \sum_{i=1}^{n} d_i'$. By letting $\sigma_1 = (\sum_{i=1}^{m} a_i')^2, \sigma_2 = (\sum_{i=1}^{m} c_i')^2, a = \frac{a'}{\sqrt{\sigma_1}}, b = \frac{b'}{\sqrt{\sigma_1}}, c = \frac{c'}{\sqrt{\sigma_2}}, d = \frac{d'}{\sqrt{\sigma_2}},$ we have found constants $\sigma_1$ and $\sigma_2$, and vectors $a, b, c$ and $d$ which satisfy

$$M = \sigma_1 \begin{bmatrix} a \\ b \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}^T + \sigma_2 \begin{bmatrix} c \\ d \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix}^T,$$

and $a, b, c$, and $d$ are simplex vectors. Note that we can assume $\sigma_1$ and $\sigma_2$ are both positive, as otherwise we are in the rank-1 case. Finally, to show that $\sigma_1 + \sigma_2 = 1$, recall that the sum of all elements in $M$ is 4, as are the sum of the elements of $\begin{bmatrix} a \\ b \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}^T$ and $\begin{bmatrix} c \\ d \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix}^T$. 

\[\square\]

**Lemma 4.10.** Suppose a feasible solution $M'$ to $SDP2$ is rank-2, and that

$$M = \sigma_1 \begin{bmatrix} a \\ b \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}^T + \sigma_2 \begin{bmatrix} c \\ d \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix}^T,$$

where $\sigma_1$ and $\sigma_2$ are nonnegative, $\sigma_1 + \sigma_2 = 1$, $a, c \in \Delta_m$, and $b, d \in \Delta_n$. Then,

$$P - xy^T = \sigma_1 \sigma_2 (a - c)(b - d)^T.$$

**Proof.** Recall that in our notation,

$$P = \sigma_1 ab^T + \sigma_2 cd^T,$$

$$x = \sigma_1 a + \sigma_2 c,$$

$$y = \sigma_1 b + \sigma_2 d.$$

Then,

$$P - xy^T = \sigma_1 ab^T + \sigma_2 cd^T - (\sigma_1 a + \sigma_2 c)(\sigma_1 b + \sigma_2 d)^T$$

$$\begin{align*}
= \sigma_1 ab^T + \sigma_2 cd^T - \sigma_1^2 ab^T - \sigma_1 \sigma_2 ab^T - \sigma_1 \sigma_2 cd^T - \sigma_1^2 cd^T \\
= \sigma_1 (1 - \sigma_1) ab^T - \sigma_1 \sigma_2 ab^T - \sigma_1 \sigma_2 cd^T + \sigma_2 (1 - \sigma_2) cd^T \\
= \sigma_1 \sigma_2 ab^T - \sigma_1 \sigma_2 ab^T - \sigma_1 \sigma_2 cd^T + \sigma_2 \sigma_1 cd^T \\
= \sigma_1 \sigma_2 (a - c)(b - d)^T.
\end{align*}$$

\[\square\]

With this decomposition we can present a series of constant additive factor approximations for the rank-2 case. To do so we apply Lemma 4.4 to the decomposition in Lemma 4.10. All the following proofs will use the notation with $\sigma, a, b, c, d$ as in Lemma 4.10.

**Theorem 4.11.** If a feasible solution $M'$ to $SDP2$ is rank-2, then the $x$ and $y$ from the last column of $M'$ constitute a $\frac{1}{2}$-Nash equilibrium.

19
Proof. We can use Lemma [4.10] to represent

\[ D := P - xy^T = \sigma_1 \sigma_2 (a - c)(b - d)^T, \]

where \( \sigma_1, \sigma_2 \geq 0, \sigma_1 + \sigma_2 = 1, a, c \in \Delta_m, \) and \( b, d \in \Delta_n. \) Then we can use Lemma [4.14] to get that

\[ \epsilon \leq \frac{\|D\|_1}{2} = \frac{1}{2} \|\sigma_1 \sigma_2 (a - c)(b - d)^T\|_1 \leq \frac{1}{2} \sigma_1 \sigma_2 \|a - c\|_1 \|b - d\|_1. \]

Since we have \( \|a\|_1 = \|b\|_1 = \|c\|_1 = \|d\|_1 = 1, \) we get the following bound for \( \epsilon: \)

\[ \frac{1}{2} \sigma_1 \sigma_2 \|a - c\|_1 \|b - d\|_1 \leq \frac{1}{2} \sigma_1 \sigma_2 \cdot 2 \cdot 2 = 2 \sigma_1 \sigma_2. \]

Since \( \sigma_1 \) and \( \sigma_2 \) sum to one and are nonnegative, we know that \( \sigma_1 \sigma_2 \leq \frac{1}{4}, \) and hence we have \( \epsilon \leq \frac{1}{2}. \)

Theorem 4.12. If a feasible solution \( \mathcal{M}' \) to \( \text{SDP}_2 \) is rank-2, then either the \( x \) and \( y \) from its last column constitute a \( \frac{5}{11} \)-NE, or a \( \frac{5}{11} \)-NE can be recovered from \( \mathcal{M}' \) in polynomial time.

Proof. We consider 3 cases, depending on whether \( \epsilon_A(x, y) \) and \( \epsilon_B(x, y) \) are greater than or less than .4. If \( \epsilon_A \leq .4, \epsilon_B \leq .4, \) then \( (x, y) \) is already a .4-Nash equilibrium. Now consider the case when \( \epsilon_A \geq .4, \epsilon_B \geq .4. \) Since \( \epsilon_A \leq \text{Tr}(A(P - xy^T)^T) \) and \( \epsilon_B \leq \text{Tr}(B(P - xy^T)^T), \) we have

\[ \sigma_1 \sigma_2 (a - c)^T A(b - d) \geq .4, \sigma_1 \sigma_2 (a - c)^T B(b - d) \geq .4. \]

Since \( A, a, b, c, \) and \( d \) are all nonnegative and \( \sigma_1 \sigma_2 \leq \frac{1}{4}, \)

\[ a^T Ab + c^T Ad \geq (a - c)^T A(b - d) \geq 1.6, \]

and the same inequalities hold for for player B. In particular, since \( A \) and \( B \) have entries bounded in \([0,1]\) and \( a, b, c, \) and \( d \) are simplex vectors, all the quantities \( a^T Ab, c^T Ad, a^T Bb, \) and \( c^T Bd \) are at most 1, and consequently at least .6. Hence \( (a, b) \) and \( (c, d) \) are both .4-Nash equilibria.

Now suppose that \((x, y)\) is a .4-NE for one player (without loss of generality player A) but not for the other (without loss of generality player B). Then \( \epsilon_A \leq .4, \) and \( \epsilon_B \geq .4. \) Let \( y^* \) be a best response for player B to \( x, \) and let \( p = \frac{1}{1 + \epsilon_B - \epsilon_A}. \) Consider the strategy profile \((\tilde{x}, \tilde{y}) := (x, py + (1 - p)y^*). \) This can be interpreted as the outcome \((x, y)\) occurring with probability \( p, \) and the outcome \((x, y^*)\) happening with probability \( 1 - p. \) In the first case, player A will have \( \epsilon_A(x, y) = \epsilon_A \) and player B will have \( \epsilon_B(x, y) = \epsilon_B. \) In the second outcome, player A will have \( \epsilon_A(x, y^*) \) at most 1, while player B will have \( \epsilon_B(x, y^*) = 0. \) Then under this strategy profile, both players have the same upper bound for \( \epsilon, \) which equals \( \epsilon_Bp = \frac{\epsilon_B}{1 + \epsilon_B - \epsilon_A}. \) To find the worst case for this value, let \( \epsilon_B = .5 \) (note from Theorem [4.11] that \( \epsilon_B \leq \frac{1}{2} ) \) and \( \epsilon_A = .4, \) and this will return \( \epsilon = \frac{5}{11}. \)

We now show a stronger result in the case of symmetric games.

Definition 4.13. A symmetric game is a game in which the payoff matrices \( A \) and \( B \) satisfy \( B = A^T. \)

Definition 4.14. A Nash equilibrium strategy \((x, y)\) is said to be symmetric if \( x = y. \)

Theorem 4.15 (see Theorem 2 in [25]). Every symmetric bimatrix game has a symmetric Nash equilibrium.
For the proof of Theorem 4.16 below we modify SDP2 so that we are seeking a symmetric solution.

**Theorem 4.16.** Suppose the constraints \( x = y \) and \( X = P = Y \) are added to SDP2. Then if a feasible solution \( \mathcal{M}' \) to this new SDP is rank-2, either the \( x \) and \( y \) from its last column constitute a symmetric \( \frac{1}{3} \)-NE, or a symmetric \( \frac{1}{3} \)-NE can be recovered from \( \mathcal{M}' \) in polynomial time.

**Proof.** If \( (x, y) \) is already a symmetric \( \frac{1}{3} \)-NE, then the claim is established. Now suppose that \( (x, y) \) does not constitute a \( \frac{1}{3} \)-Nash equilibrium. Observe that since \( x = y \), we must have \( a = b \) and \( c = d \). Then following the same reasoning as in the proof of Theorem 4.12 we have

\[
\sigma_1 \sigma_2 (a - c)^T A (a - c) \geq \frac{1}{3}.
\]

Since \( A, a, \) and \( c \) are all nonnegative, and \( \sigma_1 \sigma_2 \leq \frac{1}{4} \), we get

\[
a^T A a + c^T A c \geq (a - c)^T A (a - c) \geq \frac{4}{3}.
\]

In particular, at least one of \( a^T A a \) and \( c^T A c \) is at least \( \frac{2}{3} \). Since the maximum possible payoff is 1, at least one of \( (a, a) \) and \( (c, c) \) is a (symmetric) \( \frac{1}{3} \)-Nash equilibrium.

---

**5 Algorithms for Lowering the Rank**

In this section, we present heuristics which aim to find low-rank solutions to SDP2 and present some empirical results. Recall that our SDP2 in Section 2.6 did not have an objective function. Hence, we can encourage low-rank solutions by choosing certain objective functions, for example the nuclear norm (i.e. trace) of the matrix \( \mathcal{M} \), which is a general heuristic for rank minimization [29, 12]. This simple objective function is already guaranteed to produce a rank-1 solution in the case of generalized zero-sum games (see Proposition 5.1 below). For general games, however, one can design better objective functions in an iterative fashion (see Section 5.1).

**Proposition 5.1.** For a generalized zero-sum game, any optimal solution to SDP2 with \( \text{Tr}(\mathcal{M}) \) as the objective function must be rank-1.

**Proof.** Let

\[
\mathcal{M} := \begin{bmatrix} X & P \\ Z & Y \end{bmatrix}, \quad \mathcal{M}' := \begin{bmatrix} X & P \\ \begin{bmatrix} x \\ y \end{bmatrix} & \begin{bmatrix} x \\ y \end{bmatrix}^T \end{bmatrix},
\]

with \( P = Z^T \), be a feasible solution to SDP2. In the case of generalized zero-sum games, from Theorem 3.5 we know that that \( (x, y) \) is a Nash equilibrium. Then because the matrix \( \mathcal{M}' \) must be psd, by applying the Schur complement, we have that \( \mathcal{M} \succeq \begin{bmatrix} x \\ y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}^T \), and therefore \( \mathcal{M} = \begin{bmatrix} xx^T & xy^T \\ yx^T & yy^T \end{bmatrix} + \mathcal{P} \) for some psd matrix \( \mathcal{P} \) and some Nash equilibrium \( (x, y) \). Given this expression, the objective value is then \( x^T x + y^T y + \text{Tr}(\mathcal{P}) \). As \( (x, y) \) is a Nash equilibrium, the choice of \( \mathcal{P} = 0 \) results in a feasible solution. Since the zero matrix has the minimum possible trace among all psd matrices, the solution will be the rank-1 matrix

\[
\begin{bmatrix} x \\ y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}^T.
\]

---
5.1 Linearization Algorithms

The algorithms we present in this section are based on iterative linearization of certain nonconvex objective functions. Motivated by the next proposition, we design two continuous (nonconvex) objective functions that, if minimized exactly, would guarantee rank-1 solutions. We will then linearize these functions iteratively.

**Proposition 5.2.** Let the matrices $X$ and $Y$ and vectors $x$ and $y$ be taken from a feasible solution $M'$ to $\text{SDP}_2$. Then the matrix $M'$ is rank-1 if and only if $X_{i,i} = x_i^2$ and $Y_{i,i} = y_i^2$ for all $i$.

**Proof.** Necessity of the condition is trivial; we argue sufficiency. Denote the vector $z = \begin{bmatrix} x \\ y \end{bmatrix}$. First recall from the row constraints of Section 2.3.2 that $M$ will have the same rank as $M'$, as the last column is a linear combination of the columns of $X$ and $Y$. Since $M$ is psd, we have that $M_{i,j} \leq \sqrt{M_{i,i}M_{j,j}}$, which implies $M_{i,j} \leq z_iz_j$ by the assumption of the proposition. Further recall that a consequence of the unity constraints (20) and (21) is that $\sum_{i=1}^{m+n} \sum_{j=1}^{m+n} z_iz_j = 4$, and that we require from the distribution constraints from Section 2.3.1 that $\sum_{i=1}^{m+n} \sum_{j=1}^{m+n} M_{i,j} = 4$.

Now we can see that in order to have the equality $4 = \sum_{i=1}^{m+n} \sum_{j=1}^{m+n} M_{i,j} \leq \sum_{i=1}^{m+n} \sum_{j=1}^{m+n} z_iz_j = 4$, we must have $M_{i,j} = z_iz_j$ for each $i$ and $j$. Consequently $M$ is rank-1.

We focus now on two nonconvex objectives that as a consequence of the above proposition would return rank-1 solutions:

**Proposition 5.3.** All optimal solutions to $\text{SDP}_2$ with the objective function $\sum_{i=1}^{m+n} \sqrt{M_{i,i}}$ or $\text{Tr}(M) - x^Tx - y^Ty$ are rank-1.

**Proof.** We show that each of these objectives has a specific lower bound which is achieved if and only if the matrix is rank-1.

Observe that since $M \succeq \begin{bmatrix} x \\ y \end{bmatrix}^T \begin{bmatrix} x \\ y \end{bmatrix}$, we have $\sqrt{X_{i,i}} \geq x_i$ and $\sqrt{Y_{i,i}} \geq y_i$, and hence

$$\sum_{i=1}^{m+n} \sqrt{M_{i,i}} \geq \sum_{i=1}^{m} x_i + \sum_{i=1}^{n} y_i = 2.$$  

Further note that

$$\text{Tr}(M) - \begin{bmatrix} x \\ y \end{bmatrix}^T \begin{bmatrix} x \\ y \end{bmatrix} \geq \begin{bmatrix} x \\ y \end{bmatrix}^T \begin{bmatrix} x \\ y \end{bmatrix} - \begin{bmatrix} x \\ y \end{bmatrix}^T \begin{bmatrix} x \\ y \end{bmatrix} = 0.$$

We can see that the lower bounds are achieved if and only if $X_{i,i} = x_i^2$ and $Y_{i,i} = y_i^2$ for all $i$, which by Proposition 5.2 happens if and only if $M$ is rank-1.

We refer to our two objective functions in Proposition 5.3 as the “square root objective” and the “diagonal gap objective” respectively. While these are both nonconvex, we will attempt to iteratively minimize them by linearizing them through a first order Taylor expansion. For example, at iteration $k$ of the algorithm,

$$\sum_{i=1}^{m+n} \sqrt{M_{i,i}^{(k)}} \approx \sum_{i=1}^{m+n} \sqrt{M_{i,i}^{(k-1)}} + \frac{1}{2 \sqrt{M_{i,i}^{(k-1)}}} (M_{i,i}^{(k)} - M_{i,i}^{(k-1)}).$$
Note that for the purposes of minimization, this reduces to minimizing \( \sum_{i=1}^{m+n} \frac{1}{\sqrt{M_{i,i}^{(k)}}} \).

In similar fashion, for the second objective function, at iteration \( k \) we can make the approximation

\[
\text{Tr}(\mathcal{M}) - \begin{bmatrix} x \end{bmatrix}^{(k)} \begin{bmatrix} x \\ y \end{bmatrix}^{(k)} \simeq \text{Tr}(\mathcal{M}) - \begin{bmatrix} x \end{bmatrix}^{(k-1)} \begin{bmatrix} x \\ y \end{bmatrix}^{(k-1)T} \begin{bmatrix} x \\ y \end{bmatrix}^{(k)} - 2 \begin{bmatrix} x \end{bmatrix}^{(k-1)T} \begin{bmatrix} x \end{bmatrix}^{(k)} - \begin{bmatrix} x \end{bmatrix}^{(k-1)}. \]

Once again, for the purposes of minimization this reduces to minimizing \( \text{Tr}(\mathcal{M}) - 2 \begin{bmatrix} x \end{bmatrix}^{(k-1)T} \begin{bmatrix} x \end{bmatrix}^{(k)} \).

This approach then leads to the following two algorithms.\footnote{An algorithm similar to Algorithm 2 is used in \cite{15}.}

**Algorithm 1** Square Root Minimization Algorithm

1: Let \( x^{(0)} = 1_m, y^{(0)} = 1_n, k = 1 \).
2: while !convergence do
3: Solve SDP2 with \( \sum_{i=1}^{m} \frac{1}{\sqrt{x_{i,i}^{(k-1)}}} X_{i,i} + \sum_{i=1}^{n} \frac{1}{\sqrt{y_{i,i}^{(k-1)}}} Y_{i,i} \) as the objective, and denote the optimal solution by \( \mathcal{M}^* \).
4: Let \( x^{(k)} = \text{diag}(X^*), y^{(k)} = \text{diag}(Y^*) \).
5: Let \( k = k + 1 \).
6: end while

**Algorithm 2** Diagonal Gap Minimization Algorithm

1: Let \( x^{(0)} = 0_m, y^{(0)} = 0_n, k = 1 \).
2: while !convergence do
3: Solve SDP2 with \( \text{Tr}(X) + \text{Tr}(Y) - 2 \begin{bmatrix} x \end{bmatrix}^{(k-1)T} \begin{bmatrix} x \\ y \end{bmatrix}^{(k)} \) as the objective, and denote the optimal solution by \( \mathcal{M}^* \).
4: Let \( x^{(k)} = x^*, y^{(k)} = y^* \).
5: Let \( k = k + 1 \).
6: end while

**Remark 5.1.** Note that the first iteration of both algorithms uses the nuclear norm (i.e. trace) of \( \mathcal{M} \) as the objective.

The square root algorithm has the following property.

**Theorem 5.4.** Let \( \mathcal{M}^{(1)}, \mathcal{M}^{(2)}, \ldots \) be the sequence of optimal matrices obtained from the square root algorithm. Then the sequence

\[
\left\{ \sum_{i=1}^{m+n} \sqrt{M_{i,i}^{(k)}} \right\} \quad (46)
\]

is nonincreasing and is lower bounded by two. If it reaches two at some iteration \( t \), then the matrix \( \mathcal{M}^{(t)} \) is rank-1.

**Proof.** Observe that for any \( k > 1 \),

\[
\sum_{i=1}^{m+n} \sqrt{M_{i,i}^{(k)}} \leq \frac{1}{2} \sum_{i=1}^{m+n} \left( \frac{M_{i,i}^{(k)}}{\sqrt{M_{i,i}^{(k-1)}}} \right) + \sqrt{M_{i,i}^{(k-1)}} \leq \frac{1}{2} \sum_{i=1}^{m+n} \left( \frac{M_{i,i}^{(k-1)}}{\sqrt{M_{i,i}^{(k-1)}}} \right) + \sqrt{M_{i,i}^{(k-1)}} = \sum_{i=1}^{m+n} \sqrt{M_{i,i}^{(k-1)}}.
\]
where the first inequality follows from the arithmetic-mean-geometric-mean inequality, and the second follows from that $M_{i,i}^{(k)}$ is chosen to minimize $\sum_{i=1}^{m+n} \frac{M_{i,i}^{(k)}}{\sqrt{M_{i,i}^{(k-1)}}}$ and hence achieves a no larger value than the feasible solution $M^{(k-1)}$. This shows that the sequence is nonincreasing.

The proof of Proposition 5.3 already shows that the sequence is lower bounded by two, and Proposition 5.3 itself shows that reaching two is sufficient to have the matrix be rank-1.

The diagonal gap algorithm has the following property.

**Theorem 5.5.** Let $M'(1), M'(2), \ldots$ be the sequence of optimal matrices obtained from the diagonal gap algorithm. Then the sequence

$$\{\text{Tr}(M^{(k)}) - \begin{bmatrix} x(k) \\ y(k) \end{bmatrix}^T \begin{bmatrix} x(k) \\ y(k) \end{bmatrix} \}$$

(47)

is nonincreasing and is lower bounded by zero. If it reaches zero at some iteration $t$, then the matrix $M'(t)$ is rank-1.

**Proof.** Observe that

$$\text{Tr}(M^{(k)}) - \begin{bmatrix} x(k) \\ y(k) \end{bmatrix}^T \begin{bmatrix} x(k) \\ y(k) \end{bmatrix} \leq \text{Tr}(M^{(k)}) - \begin{bmatrix} x(k) \\ y(k) \end{bmatrix}^T \begin{bmatrix} x(k) \\ y(k) \end{bmatrix} + \begin{bmatrix} x(k) \\ y(k) \end{bmatrix}^T \begin{bmatrix} x(k) \\ y(k) \end{bmatrix} - \begin{bmatrix} x(k) \\ y(k) \end{bmatrix}^T \begin{bmatrix} x(k) \\ y(k) \end{bmatrix}$$

$$= \text{Tr}(M^{(k)}) - 2 \begin{bmatrix} x(k) \\ y(k) \end{bmatrix}^T \begin{bmatrix} x(k) \\ y(k) \end{bmatrix}$$

$$\leq \text{Tr}(M^{(k-1)}) - 2 \begin{bmatrix} x(k-1) \\ y(k-1) \end{bmatrix}^T \begin{bmatrix} x(k-1) \\ y(k-1) \end{bmatrix}$$

where the second inequality follows from that $M'(k)$ is chosen to minimize $\text{Tr}(M^{(k-1)}) - 2 \begin{bmatrix} x(k-1) \\ y(k-1) \end{bmatrix}^T \begin{bmatrix} x(k-1) \\ y(k-1) \end{bmatrix}$ and hence achieves a no larger value than the feasible solution $M'(k-1)$. This shows that the sequence is nonincreasing.

The proof of Proposition 5.3 already shows that the sequence is lower bounded by zero, and Proposition 5.3 itself shows that reaching zero is sufficient to have the matrix be rank-1.

**5.2 Numerical Experiments**

We tested Algorithms 1 and 2 on games coming from 100 randomly generated payoff matrices with entries bounded in $[0,1]$ of varying sizes. Below is a table of statistics for $20 \times 20$ matrices; the data for the rest of the sizes can be found in Appendix A.4

We can see that our algorithms return approximate Nash equilibria with fairly low $\epsilon$ (recall the definition from Section 2.1). We ran 20 iterations of each algorithm on each game.

---

4The code that produced these results is publicly available at [aaa.princeton.edu/software](aaa.princeton.edu/software). The function nash.m computes an approximate Nash equilibrium using one of our two algorithms as specified by the user.
Table 1: Statistics on $\epsilon$ for $20 \times 20$ games after 20 iterations.

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Max</th>
<th>Mean</th>
<th>Median</th>
<th>StDev</th>
</tr>
</thead>
<tbody>
<tr>
<td>Square Root</td>
<td>0.0198</td>
<td>0.0046</td>
<td>0.0039</td>
<td>0.0034</td>
</tr>
<tr>
<td>Diagonal Gap</td>
<td>0.0159</td>
<td>0.0032</td>
<td>0.0024</td>
<td>0.0032</td>
</tr>
</tbody>
</table>

The histograms below show the effect of increasing the number of iterations on lowering $\epsilon$ on $20 \times 20$ games. For both algorithms, there was a clear improvement of the $\epsilon$ by increasing the number of iterations.

Figure 2: Distribution of $\epsilon$ over numbers of iterations for the square root algorithm.

Figure 3: Distribution of $\epsilon$ over numbers of iterations for the diagonal gap algorithm.
6 Bounding Payoffs and Strategy Exclusion

In addition to finding \( \epsilon \)-additive Nash equilibria, our SDP approach can be used to answer certain questions of economic interest about Nash equilibria without actually computing them. For instance, economists often would like to know the maximum welfare (sum of the two players’ payoffs) achievable under any Nash equilibrium, or whether there exists a Nash equilibrium in which a given subset of strategies (corresponding, e.g., to undesirable behavior) is not played. Both these questions are NP-hard for bimatrix games [13]. In this section, we show how our SDP can be applied to these problems and given some numerical experiments.

6.1 Bounding Payoffs

When designing policies that are subject to game theoretic behavior by agents, economists would often like to find one with a good socially optimal outcome, which usually corresponds to an equilibrium giving the maximum welfare. Hence, given a game, it is of interest to know the highest achievable welfare under any Nash equilibrium.

To address this problem, we begin as we did in Section 2.1 by posing the question of maximizing the welfare under any Nash equilibrium as a quadratic program. Since the feasible set of this program is the set of Nash equilibria, the constraints are the same as those in the formulation in (1), though the objective function is now the welfare:

\[
\begin{align*}
\max_{x,y} & \quad x^T Ay + x^T By \\
\text{subject to} & \quad x^T Ay \geq e_i^T Ay, \forall i \in \{1, \ldots, m\}, \\
& \quad x^T By \geq x^T Be_j, \forall j \in \{1, \ldots, n\}, \\
& \quad x_i \geq 0, \forall i \in \{1, \ldots, m\}, \\
& \quad y_j \geq 0, \forall j \in \{1, \ldots, n\}, \\
& \quad \sum_{i=1}^{m} x_i = 1, \\
& \quad \sum_{i=1}^{n} y_i = 1.
\end{align*}
\]  

(48)

The SDP relaxation of this quadratic program will then be given by

\[
\max_{M' \in S_{m+n+1,m+n+1}} \quad \text{Tr}(AZ) + \text{Tr}(BZ) \\
\text{subject to} \quad (17) - (27).
\]  

(SDP3)  

One can easily see that the optimal value of this SDP is an upper bound on the welfare achievable under any Nash equilibrium. To test the quality of this upper bound, we tested this SDP on a random sample of one hundred 5 \times 5 and 10 \times 10 games. The results are in Figures 4 and 5, which show that the bound returned by SDP3 was exact in a large number of the experiments.

\footnote{The matrices were randomly generated with uniform and independent entries in [0,1]. The computation of the upper bounds on the maximum payoffs was done with the function \texttt{nashbound.m}, which computes an SDP lower bound on the problem of minimizing a quadratic function over the set of Nash equilibria of a bimatrix game. This code is publicly available at \url{aaa.princeton.edu/software}. The exact computation of the maximum payoffs was done with the \texttt{lrsnash} software [3], which computes extreme Nash equilibria. For a definition of extreme Nash equilibria and for understanding why it is sufficient for us to compare against extreme Nash equilibria (both in Section 6.1 and in Section 6.2), see Appendix 17.}
Figure 4: The quality of the upper bound on the maximum welfare obtained by SDP3 on 100 $5 \times 5$ games.

Figure 5: The quality of the upper bound on the maximum welfare obtained by SDP3 on 100 $10 \times 10$ games.
6.2 Strategy Exclusion

The strategy exclusion problem asks, given a subset of strategies \( S = (S_x, S_y) \), with \( S_x \subseteq \{1, \ldots, m\} \) and \( S_y \subseteq \{1, \ldots, n\} \), is there a Nash equilibrium in which no strategy in \( S \) is played with positive probability. We will call a set \( S \) “persistent” if the answer to this question is negative, i.e. at least one strategy in \( S \) is played with positive probability in every Nash equilibrium. One application of the strategy exclusion problem is to understand whether certain strategies can be discouraged in the design of a game, such as reckless behavior in a game of chicken or defecting in a game of prisoner’s dilemma. In these particular examples these strategy sets are persistent and cannot be discouraged.

A quadratic program which can address the strategy exclusion problem is as follows:

\[
\begin{align*}
\min_{x,y} & \quad \sum_{i \in S_x} x_i + \sum_{i \in S_y} y_i \\
\text{subject to} & \quad x^T A y \geq \epsilon_i^T A y, \forall i \in \{1, \ldots, m\}, \\
& \quad x^T B y \geq x^T B e_j, \forall j \in \{1, \ldots, n\}, \\
& \quad x_i \geq 0, \forall i \in \{1, \ldots, m\}, \\
& \quad y_j \geq 0, \forall j \in \{1, \ldots, n\}, \\
& \quad \sum_{i=1}^m x_i = 1, \\
& \quad \sum_{i=1}^n y_i = 1.
\end{align*}
\]  

(50)

Observe that by design, \( S \) is persistent if and only if this quadratic program has a positive optimal value. The SDP relaxation of this problem is given by

\[
\begin{align*}
\min_{M'} & \quad \sum_{i \in S} x_i + \sum_{i \in S} y_i \\
\text{subject to} & \quad (17) - (27). \\
& \quad (SDP4)
\end{align*}
\]  

(51)

Our approach for the strategy exclusion problem would be to declare that a strategy set is persistent if and only if (SDP4) has positive optimal value.

Note that since the optimal value of (SDP4) is a lower bound for that of (50), (SDP4) carries over the property that if a set \( S \) is not persistent, then the SDP for sure returns zero. Thus, when using (SDP4) on a set which is not persistent, our algorithm will always be correct. However, this is not necessarily the case for a persistent set. While we can be certain that a set is persistent if (SDP4) returns a positive optimal value (again, because the optimal value of (SDP4) is a lower bound for that of (50)), there is still the possibility that for a persistent set (SDP4) will have optimal value zero.

To test the performance of (SDP4), we generated 100 random games of size 5 \( \times \) 5 and 10 \( \times \) 10 and computed all their extreme Nash equilibria.\(^6\) We then, for every strategy set \( S \) of cardinality one and two, checked whether that set of strategies was persistent, first by checking among the extreme

\(^6\)The exact computation of the Nash equilibria was done again with the lrsnash software \(^3\), which computes extreme Nash equilibria. To understand why this suffices for our purposes see Appendix B.
Nash equilibria, then through \textbf{SDP4}. The results are presented in Tables 2 and 3. As motivated by the discussion above, we separately show the performance on all instances and the performances on persistent input instances. As can be seen, \textbf{SDP4} was quite effective for the strategy exclusion problem. In particular, for $10 \times 10$ games, we have a perfect identification rate.

|S| | 1 | 2 |
|---|---|---|
|Number of Total Sets| 1000 | 4500 |
|Number Correct| 996 | 4465 |
|Percent Correct| 99.6 % | 99.2 % |

|S| | 1 | 2 |
|---|---|---|
|Number of Persistent Sets| 22 | 1478 |
|Number Correct| 18 | 1443 |
|Percent Correct| 81.8% | 97.6% |

Table 2: Performance of \textbf{SDP4} on $5 \times 5$ games

|S| | 1 | 2 |
|---|---|---|
|Number of Total Sets| 2000 | 19000 |
|Number Correct| 2000 | 19000 |
|Percent Correct| 100 % | 100 % |

|S| | 1 | 2 |
|---|---|---|
|Number of Persistent Sets| 11 | 841 |
|Number Correct| 11 | 841 |
|Percent Correct| 100% | 100% |

Table 3: Performance of \textbf{SDP4} on $10 \times 10$ games

7 Higher Order Semidefinite Relaxations and the Sum of Squares Hierarchy

The SDPs we have presented thus far have all had semidefinite constraints of size $(m + n + 1) \times (m + n + 1)$. In this section, we demonstrate how increasing the size of the SDP can provide better approximations to the convex hull of Nash equilibria.

7.1 Sum of Squares/Lasserre Hierarchy

The sum of squares/Lasserre hierarchy\footnote{The unfamiliar reader is referred to \cite{19, 26, 20} for an introduction to the sum of squares/Lasserre hierarchy and the related theory of moment relaxations.} gives a recipe for constructing a sequence of SDPs whose optimal values converge to the optimal value of a given polynomial optimization problem. In this section, we will discuss the implications of this hierarchy for the Nash equilibrium problem. We start with a very brief review of the Lasserre hierarchy.

Recall that a polynomial optimization problem (pop) is a problem of minimizing a polynomial over a basic semialgebraic set, i.e., a problem of the form

\[
\min_{x \in \mathbb{R}^n} f(x) \\
\text{subject to } g_i(x) \geq 0, \forall i \in \{1, \ldots, m\},
\]

where $f, g_i$ are polynomial functions. We say that a polynomial $p$ is a sum of squares (sos) if there exist polynomials $q_1, \ldots, q_r$ such that $p = \sum_{i=1}^r q_i^2$. A basic semialgebraic set $\{x \in \mathbb{R}^n| g_i(x) \geq 0\}$ is said to be Archimedean if there exist a scalar $R > 0$ and sos polynomials $\sigma_0, \ldots, \sigma_m$ such that

\[
R - \sum_{i=1}^n x_i^2 = \sigma_0(x) + \sum_{i=1}^m \sigma_i(x)g_i(x).
\]
Note that any Archimedean set is compact.

In this section, when we refer to the $k$-th level of the Lasserre hierarchy, we mean the optimization problem

$$
\gamma^k_{sos} := \max_{\gamma, \sigma_i} \gamma
$$

subject to

$$f(x) - \gamma = \sigma_0(x) + \sum_{i=1}^m \sigma_i(x)g_i(x),$$

$\sigma_i$ is sos, $\forall i \in \{0, \ldots, m\}$,

$\sigma_0, g_i \sigma_i$ have degree at most $2k$, $\forall i \in \{1, \ldots, m\}$.  

There are two primary properties of the Lasserre hierarchy which are of interest. The first is that when the degree of the polynomials $f$ and $g_i$ are fixed, and when $k$ is fixed, the $k$-th level of this hierarchy is an SDP of size polynomial in $n$. The second is that, if the set $\{x \in \mathbb{R}^n | g_i(x) \geq 0\}$ is Archimedean, then $\lim_{k \to \infty} \gamma^k_{sos} = p^*$, where $p^*$ is the optimal value of the pop in (52). This is a consequence of Putinar’s positivstellensatz [28], [19].

### 7.2 The Lasserre Hierarchy and SDP1

We show in this section that the first level of the Lasserre hierarchy is dominated by our SDP1.

**Proposition 7.1.** Consider the problem of minimizing any quadratic objective function over the set of Nash equilibria of a bimatrix game. Then, SDP1 (and hence SDP2) gives a lower bound on this problem which is no worse than that produced by the first level of the Lasserre hierarchy.

**Proof.** To prove this proposition we show that the first level of the Lasserre hierarchy is dual to a weakened version of SDP1.

**Explicit parametrization of first level of the Lasserre hierarchy.** Consider the formulation of the Lasserre hierarchy in (53) with $k = 1$. Suppose we are minimizing a quadratic function

$$f(x, y) = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}^T C \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

over the set of Nash equilibria as described by the linear and quadratic constraints in (1). If we apply the first level of the Lasserre hierarchy to this particular pop, we get...
max \quad \gamma

\text{subject to } \begin{bmatrix} x^T & y^T & 1 \end{bmatrix} C \begin{bmatrix} x^T \\ y^T \\ 1 \end{bmatrix} - \gamma = \begin{bmatrix} x^T \\ y^T \\ 1 \end{bmatrix} Q \begin{bmatrix} x^T \\ y^T \\ 1 \end{bmatrix} + \sum_{i=1}^m \alpha_i (x^T Ay - e_i^T Ay) + \sum_{i=1}^n \beta_i (x^T By - x^T Be_i) + \sum_{i=1}^m \chi_i x_i + \sum_{i=1}^n \psi_i y_i + \eta_1 (\sum_{i=1}^m x_i - 1) + \eta_2 (\sum_{i=1}^n y_i - 1),

Q \succeq 0, 
\alpha, \chi, \beta, \psi \geq 0,

\text{where } Q \in S^{m+n+1 \times m+n+1}, \alpha, \chi \in \mathbb{R}^m, \beta, \psi \in \mathbb{R}^n, \eta \in \mathbb{R}^2.

By matching coefficients of the two quadratic functions on the left and right hand sides of (54), this SDP can be written as

max \quad \gamma 

\text{subject to } H \succeq 0,
\alpha, \beta, \chi, \psi \geq 0,

where

\begin{align*}
H := \frac{1}{2} & \begin{bmatrix} 0 & 0 & 0 \\ -\sum_{i=1}^m \alpha_i A + (-\sum_{i=1}^n \beta_i) B & -\sum_{i=1}^n \beta_i B^T - \chi^T - \eta_1 1_m^T & \sum_{i=1}^m \alpha_i A_i \end{bmatrix} + \begin{bmatrix} 0 & B & -B_i \\ B^T & 0 & 0 \\ -B^T_i & 0 & 0 \end{bmatrix} + C.
\end{align*}

\text{Dual of a weakened version of SDP1.} \quad \text{With this formulation in mind, let us consider a weakened version of SDP1 with only the relaxed Nash constraints, unity constraints, and nonnegativity constraints on } x \text{ and } y \text{ in the last column (i.e., the nonegativity constraint is not applied to the entire matrix). Let the objective be } \text{Tr}(CM'). \text{ To write this new SDP in standard form, let }

\begin{align*}
A_i := \frac{1}{2} \begin{bmatrix} 0 & A & 0 \\ A^T & 0 & -A_i^T \\ 0 & -A_i & 0 \end{bmatrix}, & \quad B_i := \frac{1}{2} \begin{bmatrix} 0 & B & -B_i \\ B^T & 0 & 0 \\ -B^T_i & 0 & 0 \end{bmatrix},
S_1 := \frac{1}{2} \begin{bmatrix} 0 & 0 & 1_m \\ 0 & 0 & 0 \\ 1_m^T & 0 & -2 \end{bmatrix}, & \quad S_2 := \frac{1}{2} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1_n \\ 0 & 1_n^T & -2 \end{bmatrix}.
\end{align*}

Let } \mathcal{N}_i \text{ be the matrix with all zeros except a } \frac{1}{2} \text{ at entry } (i, m+n+1) \text{ and } (m+n+1, i) \text{ (or a 1 if } i = m+n+1). \text{ Then this SDP can be written as}
\[
\begin{align*}
\min_{\mathcal{M}'} \quad & \quad \text{Tr}(CM') \quad \text{(SDP0)} \\
\text{subject to} \quad & \quad \mathcal{M}' \succeq 0, \quad (57) \\
& \quad \text{Tr}(N_i \mathcal{M}') \geq 0, \forall i \in \{1, \ldots, m + n\}, \quad (58) \\
& \quad \text{Tr}(A_i \mathcal{M}') \geq 0, \forall i \in \{1, \ldots, m\}, \quad (59) \\
& \quad \text{Tr}(B_i \mathcal{M}') \geq 0, \forall i \in \{1, \ldots, n\}, \quad (60) \\
& \quad \text{Tr}(S_1 \mathcal{M}') = 0, \quad (61) \\
& \quad \text{Tr}(S_2 \mathcal{M}') = 0, \quad (62) \\
& \quad \text{Tr}(N_{m+n+1}) = 1. \quad (63)
\end{align*}
\]

We now create dual variables for each constraint; we choose \(\alpha_i\) and \(\beta_i\) for the relaxed Nash constraints (59) and (60), \(\eta_1\) and \(\eta_2\) for the unity constraints (61) and (62), \(\chi\) for the nonnegativity of \(x\) (58), \(\psi\) for the nonnegativity of \(y\) (58), and \(\gamma\) for the final constraint on the corner (63). These variables are chosen to coincide with those used in the parametrization of the first level of the Lasserre hierarchy, as can be seen more clearly below.

We then write the dual of the above SDP as

\[
\begin{align*}
\max_{\alpha, \beta, \lambda, \gamma} \quad & \quad \gamma \\
\text{subject to} \quad & \quad \sum_{i=1}^{m} \alpha_i A_i + \sum_{i=1}^{n} \beta_i B_i + \sum_{i=1}^{2} \eta_i S_i + \sum_{i=1}^{m} N_{i+n} \chi_i + \sum_{i=1}^{n} N_i \psi_i + \gamma \leq C, \\
& \quad \alpha, \beta, \chi, \psi \geq 0.
\end{align*}
\]

which can be rewritten as

\[
\begin{align*}
\max_{\alpha, \beta, \chi, \psi, \gamma} \quad & \quad \gamma \\
\text{subject to} \quad & \quad G \succeq 0, \\
& \quad \alpha, \beta, \chi, \psi \geq 0,
\end{align*}
\]

where

\[
G := \frac{1}{2} \begin{bmatrix}
0 & (-\sum_{i=1}^{m} \alpha_i) A + (-\sum_{i=1}^{n} \beta_i) B & (-\sum_{i=1}^{m} \alpha_i) A + (-\sum_{i=1}^{m} \beta_i) B & \sum_{i=1}^{m} \beta_i B_i - \chi - \eta_1 1_m \\
\sum_{i=1}^{m} \beta_i B_i^T - \chi^T - \eta_1 1_m & \sum_{i=1}^{n} \alpha_i A_i - \psi^T - \eta_2 1_n & \sum_{i=1}^{m} \alpha_i A_i - \psi^T - \eta_2 1_n & 2\eta_1 + 2\eta_2 - 2\gamma \end{bmatrix} + C.
\]

We can now see that the matrix \(G\) coincides with the matrix \(H\) in the SDP (55). Then we have

\[
(54)^{opt} = (55)^{opt} = (64)^{opt} \leq SDP_0^{opt} \leq SDP_1^{opt},
\]

where the first inequality follows from weak duality, and the second follows from that the constraints of \(SDP_0\) are a subset of the constraints of \(SDP_1\).

Remark 7.1. One can see, either by inspection or as an implication of the proof of Theorem 2.3, that in the case where the objective function corresponds to maximizing player A’s and/or B’s
SDPs (55) and (64) are infeasible. This means that for such problems the first level of the Lasserre hierarchy gives a trivial upper bound of $+\infty$ on the maximum payoff. On the other hand, the additional valid inequalities in SDP2 guarantee that the resulting bound is always finite.

Remark 7.2. The Lasserre hierarchy can be viewed in each step as a pair of primal-dual SDPs: the sum of squares formulation which we have just presented, and a moment formulation which is dual to the sos formulation [19]. All our SDPs in this paper can be viewed more directly as an improvement upon the moment formulation. The valid inequalities in Section 2.3 were improvements upon the first level of the Lasserre hierarchy. One can make similar improvements in higher order relaxations. For example, since any strategy played with positive probability must give the same payoff, one can add a relaxed version of the cubic constraints

$$x_i x_j (e_i^T A y - e_j^T A y) = 0, \forall i, j \in \{1, \ldots, m\},$$

to improve the second level of the Lasserre hierarchy.

7.3 The Lasserre Hierarchy and Convergence to Convex Hull of Nash Equilibria

Our goal for this section is to show that for any bimatrix game, there exists a sequence of SDPs which provide an arbitrarily tight outer approximation to the convex hull of Nash equilibria. This is obtained by a direct application of Lassere’s hierarchy to (1), but we fill in the details. Before we apply the Lasserre hierarchy, we need to first demonstrate the Archimedean property.

Lemma 7.2. The simplex constraints given by

$$\{ x \in \mathbb{R}^m \mid \sum_{i=1}^{m} x_i = 1, x_i \geq 0, i = 1, \ldots, m \}$$

are Archimedean.

Proof. A basic semialgebraic set $\{ x \in \mathbb{R}^m \mid g_i(x) \geq 0, i = 1, \ldots, m, h_j(x) = 0, j = 1, \ldots, r \}$ is Archimedean if there exist a scalar $R > 0$ and sos polynomials $\sigma_0, \ldots, \sigma_m$ such that

$$R - \sum_{i=1}^{n} x_i^2 = \sigma_0(x) + \sum_{i=1}^{m} \sigma_i(x)g_i(x) + \sum_{i=1}^{r} p_i(x)h_i(x).$$

One can verify the following identity,

$$1 - \sum_{i=1}^{m} x_i^2 = \sum_{i=1}^{m} x_i (\sum_{j=1,j\neq i}^{m} 2x_j^2) + (1 - \sum_{i=1}^{m} x_i)(1 + \sum_{i=1}^{m} x_i + 2\sum_{i=1}^{m} \sum_{j=1,j\neq i}^{m} x_i x_j),$$

which proves that the simplex constraints are Archimedean since each polynomial $\sum_{j=1,j\neq i}^{m} 2x_j^2$ is sos.

Corollary 7.3. The constraints of (1) are Archimedean.

This would be the case, for example, in the maximum social welfare problem of Section 6.1, where the matrix of the quadratic form in the objective function is given by

$$C = \begin{bmatrix} 0 & -A - B & 0 \\ -A - B & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}. $$
Proof. This is an immediate consequence of the following identity:

\[
2 - \sum_{i=1}^{m} x_i^2 - \sum_{i=1}^{n} y_i^2 = 0 + \sum_{i=1}^{m} x_i \left( \sum_{j=1, j \neq i}^{m} 2x_j^2 \right) + \left(1 - \sum_{i=1}^{m} x_i\right) \left(1 + \sum_{i=1}^{m} x_i + 2 \sum_{i=1}^{m} \sum_{j=1, j \neq i}^{m} x_i x_j\right) + \sum_{i=1}^{n} y_i \left( \sum_{j=1, j \neq i}^{n} 2y_j^2 \right) + \left(1 - \sum_{i=1}^{n} y_i\right) \left(1 + \sum_{i=1}^{n} y_i + 2 \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{n} y_i y_j\right) + \sum_{i=1}^{m} (x^T Ay - e_i^T Ay) \cdot 0 + \sum_{i=1}^{n} (x^T By - x^T Ae_i) \cdot 0.
\] (65)

\[\square\]

**Definition 7.4.** For a set \( S \subseteq \mathbb{R}^n \), we define its \( \epsilon \)-neighborhood

\[
N_\epsilon(S) := \{ x \in \mathbb{R}^n \mid \exists s \in S \text{ such that } \|x - s\|_2 \leq \epsilon \}
\]

to be the set of points that are within distance \( \epsilon \) of \( S \).

In the sequel, we denote the boundary of a set \( S \) by \( \partial(S) \).

**Proposition 7.5.** For a bimatrix game, denote the convex hull of the set of its Nash equilibria by \( \text{convNE} := \text{convNE}(A, B) \). Denote by \( P_k \) the projection to the \( x \) and \( y \) variables of the feasible set of the SDP generated by the dual of the \( k \)-th level Lasserre hierarchy applied to (1). Then, for any \( \epsilon > 0 \), there exists \( K \) such that \( P_k \subset N_\epsilon(\text{convNE}) \) for all \( k \geq K \).

**Proof.** Consider \( \epsilon > 0 \). If, for some \( k \), no point on the boundary of \( N_\epsilon(\text{convNE}) \) is in \( P_k \), then \( P_k \subset N_\epsilon(\text{convNE}) \). This is because \( P_k \) is convex and contains \( \text{convNE} \).

Now suppose that for some \( \epsilon \) this is not the case, i.e., for any \( k \), there exists some point \( p_k \in \partial(N_\epsilon(\text{convNE})) \cap P_k \). Consider the sequence of points \( \{p_k\} \). Since \( \partial(N_\epsilon(\text{convNE})) \) is a closed and bounded set, the sequence \( \{p_k\} \) has a convergent subsequence \( \{p_{k_n}\} \) which converges to a point \( p^* \in \partial(N_\epsilon(\text{convNE})) \). We claim that \( p^* \in P_k \), \( \forall k \).

Now suppose for some \( K \), we had \( p^* \notin P_K \). Then because the complement of \( P_K \) is open, for \( \delta \) small enough, \( N_\delta(p^*) \cap P_K = \emptyset \). Thus because \( P_k \) is a monotonically shrinking sequence, \( N_\delta(p^*) \cap P_k = \emptyset \) for all \( k > K \). However, this contradicts the construction of \( p^* \) as the limit of \( \{p_{k_n}\} \), since for any \( \delta \) there is \( M \) large enough such that \( \{p_{k_m}\} \in N_\delta(p^*) \) for any \( m > M \). Thus, it must be that \( p^* \in P_k \) for all \( k > K \).

Now note that such a point \( p^* \in \partial(N_\epsilon(\text{convNE})) \) has nonzero distance to \( \text{convNE} \), and therefore there exists a hyperplane \( a^T \begin{bmatrix} x \\ y \end{bmatrix} = b \) which separates \( p^* \) and the convex hull of Nash equilibria.

Without loss of generality suppose \( a^T p^* < b \) and \( a^T c > b \) for any \( c \) in \( \text{convNE} \). Now consider the dual of Lasserre hierarchy applied to (1), but with the objective being the minimization of \( a^T \begin{bmatrix} x \\ y \end{bmatrix} \).

By assumption, the maximum value achieved by any Nash equilibrium is strictly larger than \( b \), but the optimal value of the dual of the Lasserre hierarchy will converge to some number at most \( b \) (since \( p^* \in P_k \), \( \forall k \)), and so will the Lasserre hierarchy by weak duality. Since the Lasserre hierarchy must converge to the correct value under the Archimedean assumption, we have a contradiction.

Hence such an \( \epsilon \) cannot exist. \( \square \)
8 Future Work

Our work leaves many avenues of further research. Are there other interesting subclasses of games (besides generalized zero-sum games) for which our SDP is guaranteed to recover an exact Nash equilibrium? Can the guarantees on $\epsilon$ in Section 4 be improved in the rank-2 case (or the general case), for example by using the correlated equilibrium constraints (which we did not use)? Is there a polynomial time algorithm that is guaranteed to find a rank-2 solution to SDP2? Such an algorithm, together with our analysis, would improve the best known approximation bound for symmetric games (see Theorem 4.16). Can SDPs in a higher level of the Lasserre hierarchy be used to achieve better $\epsilon$ guarantees? What are systematic ways of adding valid inequalities to these higher-order SDPs by exploiting the structure of the Nash equilibrium problem? Finally, our algorithms were specifically designed for two-player one-shot games. This leaves open the design and analysis of semidefinite relaxations for repeated games or games with more than two players.

References


A Statistics on $\epsilon$ from Algorithms in Section 5

Below are statistics for the $\epsilon$ recovered in 100 random games of varying sizes using the algorithms of Section 5.

Table 4: Statistics on $\epsilon$ for $5 \times 5$ games after 20 iterations.

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Max</th>
<th>Mean</th>
<th>Median</th>
<th>StDev</th>
</tr>
</thead>
<tbody>
<tr>
<td>Square Root</td>
<td>0.0702</td>
<td>0.0040</td>
<td>0.0004</td>
<td>0.0099</td>
</tr>
<tr>
<td>Diagonal Gap</td>
<td>0.0448</td>
<td>0.0027</td>
<td>0.0000</td>
<td>0.0061</td>
</tr>
</tbody>
</table>

Table 5: Statistics on $\epsilon$ for $10 \times 5$ games after 20 iterations.

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Max</th>
<th>Mean</th>
<th>Median</th>
<th>StDev</th>
</tr>
</thead>
<tbody>
<tr>
<td>Square Root</td>
<td>0.0327</td>
<td>0.0044</td>
<td>0.0021</td>
<td>0.0064</td>
</tr>
<tr>
<td>Diagonal Gap</td>
<td>0.0267</td>
<td>0.0033</td>
<td>0.0006</td>
<td>0.0053</td>
</tr>
</tbody>
</table>

Table 6: Statistics on $\epsilon$ for $10 \times 10$ games after 20 iterations.

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Max</th>
<th>Mean</th>
<th>Median</th>
<th>StDev</th>
</tr>
</thead>
<tbody>
<tr>
<td>Square Root</td>
<td>0.0373</td>
<td>0.0058</td>
<td>0.0039</td>
<td>0.0065</td>
</tr>
<tr>
<td>Diagonal Gap</td>
<td>0.0266</td>
<td>0.0043</td>
<td>0.0026</td>
<td>0.0051</td>
</tr>
</tbody>
</table>

Table 7: Statistics on $\epsilon$ for $15 \times 10$ games after 20 iterations.

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Max</th>
<th>Mean</th>
<th>Median</th>
<th>StDev</th>
</tr>
</thead>
<tbody>
<tr>
<td>Square Root</td>
<td>0.0206</td>
<td>0.0050</td>
<td>0.0034</td>
<td>0.0045</td>
</tr>
<tr>
<td>Diagonal Gap</td>
<td>0.0212</td>
<td>0.0038</td>
<td>0.0025</td>
<td>0.0039</td>
</tr>
</tbody>
</table>

Table 8: Statistics on $\epsilon$ for $15 \times 15$ games after 20 iterations.

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Max</th>
<th>Mean</th>
<th>Median</th>
<th>StDev</th>
</tr>
</thead>
<tbody>
<tr>
<td>Square Root</td>
<td>0.0169</td>
<td>0.0051</td>
<td>0.0042</td>
<td>0.0039</td>
</tr>
<tr>
<td>Diagonal Gap</td>
<td>0.0159</td>
<td>0.0038</td>
<td>0.0029</td>
<td>0.0034</td>
</tr>
</tbody>
</table>
Table 9: Statistics on $\epsilon$ for $20 \times 15$ games after 20 iterations.

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Max</th>
<th>Mean</th>
<th>Median</th>
<th>StDev</th>
</tr>
</thead>
<tbody>
<tr>
<td>Square Root</td>
<td>0.0152</td>
<td>0.0046</td>
<td>0.0035</td>
<td>0.0036</td>
</tr>
<tr>
<td>Diagonal Gap</td>
<td>0.0119</td>
<td>0.0032</td>
<td>0.0022</td>
<td>0.0027</td>
</tr>
</tbody>
</table>

Table 10: Statistics on $\epsilon$ for $20 \times 20$ games after 20 iterations.

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>Max</th>
<th>Mean</th>
<th>Median</th>
<th>StDev</th>
</tr>
</thead>
<tbody>
<tr>
<td>Square Root</td>
<td>0.0198</td>
<td>0.0046</td>
<td>0.0039</td>
<td>0.0034</td>
</tr>
<tr>
<td>Diagonal Gap</td>
<td>0.0159</td>
<td>0.0032</td>
<td>0.0024</td>
<td>0.0032</td>
</tr>
</tbody>
</table>

B Lemmas for Extreme Nash Equilibria

The results reported in Section 6 were found using the lrsnash software which computes extreme Nash equilibria (see definition below). In particular the true maximum welfare and the persistent strategy sets were found in relation to extreme Nash equilibria only. We show in this appendix why this is sufficient for the claims we made about all Nash equilibria.

Definition B.1. An extreme Nash equilibrium is a Nash equilibrium which cannot be expressed as a convex combination of other Nash equilibria.

Lemma B.2. All Nash equilibria are convex combinations of extreme Nash equilibria.

Proof. It suffices to show that any extreme point of the convex hull of the set of Nash equilibria must be an extreme Nash equilibrium, as any point in a convex set can be written as a convex combination of its extreme points. Suppose for the purpose of contradiction that this was not the case, i.e. there is a point $x$ which is an extreme point of the convex hull of Nash equilibria but is not an extreme Nash equilibrium. Then either it is not a Nash equilibrium, or it is a Nash equilibrium which is not extreme. In both cases, $x$ can be written as a convex combination of other Nash equilibria, and so cannot be an extreme point for the convex hull. For the former case it is because its membership in the convex hull must be due to an expression of it as a convex combination of Nash equilibria, and in the latter it is due to the definition of extreme Nash equilibria.

The next lemma shows that checking extreme Nash equilibria are sufficient for the maximum welfare problem.

Lemma B.3. For any bimatrix game, there exists an extreme Nash equilibrium giving the maximum welfare among all Nash equilibria.

Proof. Consider any Nash equilibrium $(\tilde{x}, \tilde{y})$, and let it be written as $[\tilde{x} \ y] = \sum_{i=1}^{r} \lambda_i [x^i \ y^i]$ for some set of extreme Nash equilibria $[x^1 \ y^1], \ldots, [x^r \ y^r]$ and $\lambda \in \triangle_r$. Observe that for any $i, j$,

$$x^iT A y^j \leq x^iT A y^i, x^iT B y^j \leq x^iT B y^i,$$

from the definition of a Nash equilibrium. Now note that
\[
\tilde{x}^T (A + B) \tilde{y} = \left( \sum_{i=1}^{r} \lambda_i x_i^i \right)^T (A + B) \left( \sum_{i=1}^{r} \lambda_i y_i^i \right)
\]

\[
= \sum_{i=1}^{r} \sum_{j=1}^{r} \lambda_i \lambda_j x_i^i y_j^j
\]

\[
= \sum_{i=1}^{r} \sum_{j=1}^{r} \lambda_i \lambda_j x_i^i A y_j^j + \sum_{i=1}^{r} \sum_{j=1}^{r} \lambda_i \lambda_j x_i^i B y_j^j
\]

\[
\leq \sum_{i=1}^{r} \sum_{j=1}^{r} \lambda_i \lambda_j x_i^i A y_j^j + \sum_{i=1}^{r} \sum_{j=1}^{r} \lambda_i \lambda_j x_i^i B y_j^j
\]

\[
= \sum_{i=1}^{r} \lambda_i x_i^i A y_i^i + \sum_{i=1}^{r} \lambda_i x_i^i B y_i^i
\]

\[
= \sum_{i=1}^{r} \lambda_i x_i^i (A + B) y_i^i.
\]

In particular, since each \((x_i^i, y_i^i)\) is an extreme Nash equilibrium, this tells us for any Nash equilibrium \((\tilde{x}, \tilde{y})\) there must be an extreme Nash equilibrium which has at least as much welfare.

Similarly for the results for persistent sets in Section 6.2, there is no loss in restricting attention to extreme Nash equilibria.

**Lemma B.4.** For a given strategy set \(S\), if every extreme Nash equilibrium plays at least one strategy in \(S\) with positive probability, then every Nash equilibrium plays at least one strategy in \(S\) with positive probability.

**Proof.** Let \(S\) be a persistent set of strategies. Since all Nash equilibria are composed of nonnegative entries, and every extreme Nash equilibrium has positive probability on some entry in \(S\), any convex combination of extreme Nash equilibria must have positive probability on some entry in \(S\). 

39