Closed Almost Knight's Tours on 2D and 3D Chessboards

Michael Firstein[1], Anja Fischer[2], and Philipp Hungerländer[1]

[1] Alpen-Adria-Universität Klagenfurt, Austria,
michaelfir@edu.aau.at, philipp.hungerlaender@aau.at
[2] Technische Universität Dortmund, Germany,
anja2.fischer@tu-dortmund.de

Abstract. Let a (generalized) chessboard in two or three dimensions be given. A closed knight's tour is defined as a Hamiltonian cycle over all cells of the chessboard, where all moves are knight's moves, i. e. have length $\sqrt{5}$. It is well-characterized for which chessboard sizes it is not possible to construct a closed knight's tour. On such chessboards a closed almost knight's tour is defined as a Hamiltonian cycle of minimal length, where only moves of length at least $\sqrt{5}$ are allowed. The problem of determining closed (almost) knight's tours on two-dimensional (2D) and three-dimensional (3D) chessboards is equivalent to the so called traveling salesman problem with forbidden neighborhoods on regular 2D and 3D grids, if the radius of the forbidden neighborhood is two and hence the minimal feasible distance of two consecutive cells in the tour equals the length of a knight's move. In this paper we present explicit construction schemes for closed almost knight's tours by extending ideas used for the construction of knight's tours.

Keywords: Knight's tour problem, generalized chessboards, traveling salesman problem, forbidden neighborhood, regular grids.

1 Introduction

The closed knight's tour problem is a well-studied problem [3,6,7]. Given a rectangular two-dimensional (2D) chessboard of size $m \times n$, does there exist a tour (a Hamiltonian cycle or closed knight's tour) over all cells of the chessboard such that each move is a knight's move? A knight's move is a step of length $\sqrt{5}$, where we move one cell in one direction and two cells in the other direction. There also exist extensions of the closed knight's tour problem to generalized chessboards in arbitrary dimension [3,4]. It is well-characterized for which 2D and 3D chessboards a closed knight's tour exists. For the 2D case Schwenk [7] proved the following result:

Theorem 1 ([7]). *An $m \times n$ chessboard with $m \leq n$ has a closed knight's tour if $(m, n) \notin \{(3, 4), (3, 6), (3, 8)\} \cup \{(m, n) \colon m \in \{1, 2, 4\}\}$ and if m, n are not both odd.*

DeMaio and Mathew [3] have extended Theorem 1 to 3D chessboards:

Theorem 2 ([3]). *An $m \times n \times \ell$ chessboard with $m, n, \ell \geq 2, m \leq n \leq \ell$, has a closed knight's tour if $(m, n, \ell) \neq (2, 3, 3), (m, n) \neq (2, 2)$ and at least one of m, n, ℓ is even.*

For chessboards that do not have a closed knight's tour we define a *closed almost knight's tour* as a Hamiltonian cycle of minimal length, where only moves of length at least $\sqrt{5}$ are allowed. The problem of determining closed (almost) knight's tours on 2D and 3D chessboards is equivalent to the so called traveling salesman problem with forbidden neighborhoods (TSPFN) with radius two on regular 2D and 3D grids. Given points in the Euclidean space and some radius $r \in \mathbb{R}_+$, the TSPFN asks for a shortest Hamiltonian cycle over these points, where connections between points with distance at most r are forbidden. The TSPFN was originally motivated by an application in laser beam melting, where a workpiece is built in several layers. By excluding the heating of positions that are too close during this process, one hopes to reduce the internal stresses of the workpiece. We refer to [5] and the references therein for details on this application and for TSPFN results for regular 2D grids and $r \in \{0, 1, \sqrt{2}\}$. In this paper we present construction schemes for closed almost knight's tour on 2D and 3D chessboards that are based on the ideas for constructing knight's tours suggested by Lin and Wei [6].

The paper is structured as follows. In Section 2 we consider 2D chessboards of size $m \times n$ with $m, n \geq 5$. For m and n odd, a closed almost knight's tour uses only knight's moves except for one move of length $\sqrt{8}$. In Section 3 we consider 3D chessboards of size $m \times n \times \ell, m, n \geq 5, \ell \geq 3$. For m, n, ℓ odd a closed almost knight's tour uses only knight's moves except for one move of length $\sqrt{6}$. In Section 4 we conclude the paper and give suggestions for future work.

2 Closed Almost Knight's Tours on 2D Chessboards

We consider an $m \times n$ chessboard, where each cell is denoted by a tuple $(i, j),\ i \in \{1, \ldots, m\},\ j \in \{1, \ldots, n\}$. First let us state a lower bound.

Lemma 1. *Given an $m \times n$ chessboard with $m, n \geq 5$ and m and n odd, $(mn - 1)\sqrt{5} + \sqrt{8}$ is a lower bound on the length of a closed almost knight's tour.*

This result follows directly from Th. 1, because there does not exist a closed knight's tour for the considered chessboard sizes and the shortest move longer than $\sqrt{5}$ has length $\sqrt{8}$. Next we show that there always exists a closed almost knight's tour with this length. For $m \times m$ chessboards the existence of such a closed almost knight's tour follows from a result in [1] on the existence of s-t-knights path on a quadratic chessboard between cells s and t.

Theorem 3. *Given an $m \times n$ chessboard with $m, n \geq 5$ and m and n odd, there always exists a closed almost knight's tour with length $(mn - 1)\sqrt{5} + \sqrt{8}$.*

1	22	11	16	3
12	17	2	21	10
23	8	25	4	15
18	13	6	9	20
7	24	19	14	5

1	12	27	22	3	10	29
26	23	2	11	28	21	4
13	16	35	24	7	30	9
34	25	18	15	32	5	20
17	14	33	6	19	8	31

1	42	21	26	3	40	35	6	11
22	27	2	41	20	5	10	39	34
43	18	45	4	25	36	31	12	7
28	23	16	19	30	9	14	33	38
17	44	29	24	15	32	37	8	13

33	12	29	42	3	14	27
30	43	32	13	28	5	2
11	34	41	4	1	26	15
44	31	10	25	40	19	6
35	24	49	46	9	16	39
48	45	22	37	18	7	20
23	36	47	8	21	38	17

31	62	27	40	1	16	23	52	45
28	39	30	37	26	51	44	15	22
61	32	63	2	41	24	17	46	53
6	29	38	25	36	43	50	21	14
33	60	7	42	3	18	11	54	47
8	5	58	35	10	49	56	13	20
59	34	9	4	57	12	19	48	55

53	64	79	58	1	8	17	32	41
80	59	54	63	78	31	40	9	18
65	52	81	2	57	16	7	42	33
60	55	4	77	62	39	30	19	10
51	66	61	56	3	6	15	34	43
68	27	72	5	76	29	38	11	20
73	50	67	28	71	14	23	44	35
26	69	48	75	24	37	46	21	12
49	74	25	70	47	22	13	36	45

Fig. 1. Closed almost knight's tours on $m \times n$ chessboards with $m, n \in \{5, 7, 9\}$.

Proof. We prove this result algorithmically by providing a construction scheme for building closed almost knight's tours. Closed almost knight's tours for $m \times n$ chessboards with $m, n \in \{5, 7, 9\}$ are depicted in Fig. 1. These basic tours for our construction scheme can, e. g., be derived by solving an integer linear program that is obtained by fixing certain variables to zero in the classic formulation of the traveling salesman problem by Dantzig et al. [2], see [5] for details. We want to emphasize that our closed almost knight's tours in Fig. 1 have some special structure, as they all contain the edges $\{(m-1, 1), (m, 3)\}$ and $\{(1, n-1), (3, n)\}$.

In our construction scheme, which is similar to the approach for constructing knight's tours suggested by Lin and Wei [6], we extend our basic tours from Fig. 1 from the right and from below in order to derive a closed almost knight's tour for the whole chessboard. This is done iteratively, increasing the size in horizontal or vertical direction by six in each step. For these extensions we use open knight's tours on the 5×6, the 6×6, the 7×6 and the 9×6 chessboard, see Fig. 2, and also [6], for a visualization. We combine the open knight's tours and our basic tours as outlined in the construction depicted in Fig. 3, where we delete the dotted edges and connect the single tours and paths by knight's moves. It is easy to check that this is always possible for the considered open knight's tours and our basic tours, partially mirrored.

In summary starting with one of the basic tours from Fig. 1 we can construct a closed almost knight's tour for all $m \times n$ chessboards with $m, n \geq 5$ and m and n odd by iteratively adding open knight's tours from Fig. 2. □

Corollary 4 *Optimal TSPFN tours with $r = 2$ on regular $m \times n$ grids with $m, n \geq 5$ have length $mn\sqrt{5}$ for m or n even and length $(mn-1)\sqrt{5} + \sqrt{8}$ for m and n odd.*

1	14	29	20	3	12
30	21	2	13	28	19
15	8	17	24	11	4
22	25	6	9	18	27
7	16	23	26	5	10

1	18	9	26	3	20
36	25	2	19	10	27
17	8	35	28	21	4
24	31	22	13	34	11
7	16	29	32	5	14
30	23	6	15	12	33

1	14	39	26	3	16
42	25	2	15	36	27
13	38	29	40	17	4
30	41	24	37	28	35
9	12	31	20	5	18
32	23	10	7	34	21
11	8	33	22	19	6

1	26	21	16	3	28
54	15	2	27	20	17
25	22	53	18	29	4
14	41	24	51	34	19
23	52	35	42	5	30
40	13	50	31	46	33
49	10	47	36	43	6
12	39	8	45	32	37
9	48	11	38	7	44

Fig. 2. Open knight's tours on the 5×6, the 6×6, the 7×6 and the 9×6 chessboard that are needed (sometimes mirrored at the diagonal) in the proof of Th. 3. They start at $(1, 1)$ and end in $(2, 1)$.

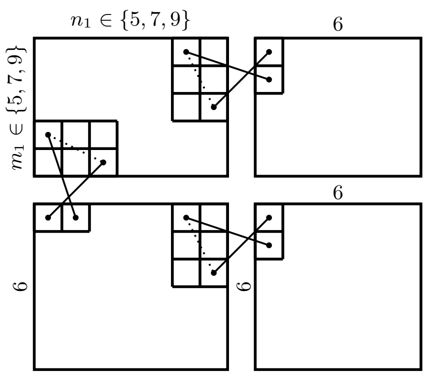

Fig. 3. Combining an $m_1 \times n_1$ basic tour with $m_1 \times 6$, $6 \times n_1$ and 6×6 open knight's tours to obtain a closed almost knight's tour on an $m \times n$ chessboard with $m, n \geq 5$ and m and n odd. For large values of m, n several open knight's tour have to be included. To improve visibility we separated the single building blocks by some space.

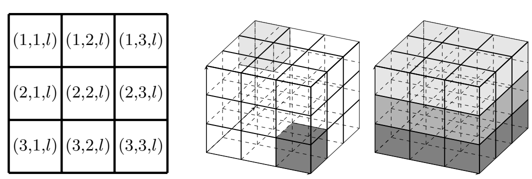

Fig. 4. Visualizations of the $3 \times 3 \times 3$ chessboard. The left picture shows the numbering of layers $l \in \{1, 2, 3\}$. In the middle picture, the dark gray cell is $(3, 3, 1)$ and the light gray one is $(1, 1, 3)$. In the right picture, each layer has one color, where layers with smaller numbers are darker.

3 Closed Almost Knight's Tours on 3D Chessboards

In this section we consider $m \times n \times \ell$ chessboards, where ℓ is the number of layers. For an illustration of the $3 \times 3 \times 3$ chessboard we refer to Figure 4.

Lemma 2. *Given an $m \times n \times \ell$ chessboard with $m, n, \ell \geq 3$ and m, n, ℓ odd, $(mn\ell - 1)\sqrt{5} + \sqrt{6}$ is a lower bound on the length of a closed almost knight's tour.*

This result follows directly from Th. 2, because there does not exist a closed knight's tour for the considered chessboard sizes and the shortest move longer than $\sqrt{5}$ has length $\sqrt{6}$. Next we show that there always exists a closed almost knight's tour with this length. The proof idea is, similar to [4], to use slightly adapted 2D tours in each layer and to delete some specific edges so that cells from different layers can be connected by some new edges.

Theorem 5. *Given an $m \times n \times \ell$ chessboard with $m, n \geq 5, \ell \geq 3$ and m, n, ℓ odd, there always exists a closed almost knight's tour with length $(mn\ell - 1)\sqrt{5} + \sqrt{6}$.*

Proof. We prove this result algorithmically by providing a construction scheme for building closed almost knight's tours. First, we use specific knight's tours for every layer and then we connect different layers via new edges.

We build an $m \times n$ closed almost knight's tour using the construction scheme described in the proof of Th. 3 and then modify the upper left area that belongs to our basic tours with $m, n \in \{5, 7, 9\}$, see Fig. 1, as follows. We delete the edge of length $\sqrt{8}$ connected to Cell 1 and one further edge and adapt the cell sequence of the new tours (subtours for the 3D chessboard) as follows:

- $(1, \ldots, 21, 22, 25, 24, 23, 1)$ for the 5×5 chessboard,
- $(1, \ldots, 12, 35, 34, \ldots, 13, 1)$ for the 5×7 chessboard,
- $(1, \ldots, 42, 45, 44, 43, 1)$ for the 5×9 chessboard,
- $(1, \ldots, 40, 49, 48, \ldots, 41, 1)$ for the 7×7 chessboard,
- $(1, \ldots, 40, 63, 62, \ldots, 41, 1)$ for the 7×9 chessboard, and
- $(1, \ldots, 78, 81, 80, 79, 1)$ for the 9×9 chessboard,

where the numbering refers to Fig. 1. After the first modification, each such 2D tour has length $(mn - 1)\sqrt{5} + 2$ instead of $(mn - 1)\sqrt{5} + \sqrt{8}$.

Now we use these modified tours for all layers except for Layer ℓ and connect Layers k and $k + 1$, k odd, by replacing the edges $\{(i, j, k), (i + 2, j, k)\}$ and $\{(i, j, k+1), (i+2, j, k+1)\}$ of length 2 in Layers k and $k+1$ that lie directly above each other by the edges $\{(i, j, k), (i + 2, j, k + 1)\}$ and $\{(i, j, k + 1), (i + 2, j, k)\}$ of length $\sqrt{5}$ that switch the layer.

Next we connect Layers k and $k + 1$, k even, by using the same exchange as in [4]. We delete the two edges of a so called *bi-site* (see [4]), where two knight's moves of our subtours cross each other. A bi-site can be found in the lower left corner of each basic tour in Fig. 1, e. g. $\{23, 24\}$ and $\{7, 8\}$ for the 5×5 chessboard or $\{35, 36\}$ and $\{23, 24\}$ for the 7×7 chessboard. In general let the bi-sites be $\{(i, j, k), (i + 2, j + 1, k)\}$ and $\{(i, j + 1, k), (i + 2, j, k)\}$ for $k = 1, \ldots, \ell - 1$, then we replace them by the edges $\{(i, j, k), (i+2, j, k+1)\}$ and $\{(i + 2, j + 1, k), (i, j + 1, k + 1)\}$.

It remains to connect Layer ℓ to the other layers. To do so we use a closed almost knight's tour in Layer ℓ and then delete the edge of length $\sqrt{8}$ and one appropriate edge in Layer $\ell - 1$ (depending on the position of the edge of length 2) such that the exchange visualized in Fig. 5 is possible. The squares highlighted

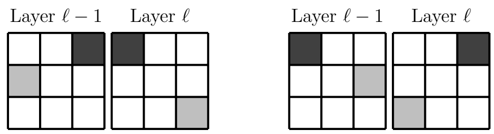

Fig. 5. Moves connecting Layers $\ell - 1$ and ℓ in closed almost knight's tours.

1	22	11	16	3
12	17	2	21	10
25	8	23	4	15
18	13	6	9	20
7	24	19	14	5

26	47	36	41	28
37	42	27	46	35
125	33	48	29	40
43	38	31	34	45
32	49	44	39	30

118	64	53	58	120
54	59	119	63	52
67	50	65	121	57
60	55	123	51	62
124	66	61	56	122

68	114	78	108	70
104	109	69	113	77
117	75	115	71	107
110	105	73	76	112
74	116	111	106	72

79	100	89	94	81
90	95	80	99	88
101	86	103	82	93
96	91	84	87	98
85	102	97	92	83

Fig. 6. Visualization of a closed almost knight's tour on a $5 \times 5 \times 5$ chessboard.

light gray are connected by a move of length $\sqrt{6}$ and the squares highlighted dark gray are connected by a move of length $\sqrt{5}$. It is easy to check that the respective edges exist in each of these tours and that we do not create subtours. □

For illustration purposes we depict a closed almost knight's tour for the $5 \times 5 \times 5$ chessboard in Fig. 6.

Corollary 6 *Optimal TSPFN tours with* $r = 2$ *on regular* $m \times n \times \ell$ *grids with* $m, n \geq 5$ *and* $\ell \geq 3$ *have length* $mn\ell\sqrt{5}$ *for* m *or* n *or* ℓ *even and length* $(mn\ell - 1)\sqrt{5} + \sqrt{6}$ *for* m, n, ℓ *odd.*

4 Conclusion and Future Work

In this paper we introduced the concept of closed almost knight's tours and proposed construction schemes for such tours on 2D and 3D chessboards, where no classic closed knight's tour exists. We restricted our analysis to the case with $m, n \geq 5$ in 2D and $m, n \geq 5$ and $\ell \geq 3$ in 3D. It remains to determine closed almost knight's tours for other chessboard sizes, where no classic closed knight's tour exists. Furthermore, one can think of extending the results to arbitrary chessboard dimensions. For the closely related traveling salesman problem with forbidden neighborhoods it would be interesting to study also other values for r.

References

1. A. Conrad, T. Hindrichs, M. Hussein and I. Wegener. Solution of the knight's Hamiltonian path problem on chessboards. *Discrete Applied Mathematics*, 50(2): 125–134, 1994.
2. G. Dantzig, R. Fulkerson, and S. Johnson. Solution of a large-scale traveling-salesman problem. *Operations Research*, 2:393–410, 1954.
3. J. DeMaio and B. Mathew. Which chessboards have a closed knight's tour within the rectangular prism?. *The Electronic Journal of Combinatorics* 18, 2011.

4. J. Erde, B. Golénia and S. Golénia. The closed knight tour problem in higher dimensions. *The Electronic Journal of Combinatorics*, 16(4), 2012.
5. A. Fischer and P. Hungerländer. The traveling salesman problem on grids with forbidden neighborhoods. *Journal of Combinatorial Optimization*, 34:(3):891–915, 2017.
6. S. Lin and C. Wei. Optimal algorithms for constructing knight's tours on arbitrary $n \times m$ chessboards. *Discrete Applied Mathematics*, 146:219–232, 2005.
7. A. J. Schwenk. Which rectangular chessboards have a knight's tour?. *Mathematics Magazine*, 64(5):325–332, 1991.