Gradient Descent using Duality Structures

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Abstract
Gradient descent is commonly used to solve optimization problems arising in machine learning, such as training neural networks. Although it seems to be effective for many different neural network training problems, it is unclear if the effectiveness of gradient descent can be explained using existing performance guarantees for the algorithm. We argue that existing analyses of gradient descent rely on assumptions that are too strong to be applicable in the case of multi-layer neural networks. To address this, we propose an algorithm, duality structure gradient descent (DSGD), that is amenable to a non-asymptotic performance analysis, under mild assumptions on the training set and network architecture. The algorithm can be viewed as a form of layer-wise coordinate descent, where at each iteration the algorithm chooses one layer of the network to update. The decision of what layer to update is done in a greedy fashion, based on a rigorous lower bound of the function decrease for each possible choice of layer. In the analysis, we bound the time required to reach approximate stationary points, in both the deterministic and stochastic settings. The convergence is measured in terms of a Finsler geometry that is derived from the network architecture and designed to confirm a Lipschitz-like property on the gradient of the training objective function. Numerical experiments in both the full batch and mini-batch settings suggest that the algorithm is a promising step towards methods for training neural networks that are both rigorous and efficient.

1 Introduction
Gradient descent and its variants are often used to train machine learning models, and these algorithms have led to impressive results in many different applications. These include the problems of training neural networks for tasks such as representation learning [30, 23], image classification [15, 20], scene labeling [13], and multimodal signal processing [32], just to name a few. In each case, these systems employ some form gradient based optimization, and the algorithm settings must be carefully tuned to guarantee success. For example, choosing small step-sizes leads to slow optimization, and step-sizes that are too large result in unstable algorithm behavior. Therefore it would be useful to have a theory that provides a rule for the step-sizes and other settings that guarantees the success of optimization.

Existing approaches to the analysis of gradient descent for nonconvex functions, some of which are reviewed below, utilize bounds on one or more derivatives of the objective function in order to obtain non-asymptotic guarantees. For example, if the objective has a Lipschitz continuous gradient, with Lipschitz constant $L$, then gradient descent with a step-size of $1/L$ is guaranteed to find approximate stationary points [26, Section 1.2.3]. However, it is doubtful that this approach can be applied to multi-layer neural networks, since there are very simple neural network training problems where the objective function does not admit global bounds on its second (or third) derivatives. In Section 2 we present an example of a training problem of this type.

Our approach to this problem has three main components. The starting point is a “Layer wise-Lipschitz” property that is satisfied by the neural network optimization objective. Motivated
by this we design our algorithm to choose one layer of the network to update at each iteration, using a lower bound for the amount of function decrease that each choice of layer would yield. The second component is an analytical framework based on Finsler structures that is used to discuss the convergence of our algorithm, and that we believe may also be of general interest. Thirdly, the geometric point of view is not just a tool for analysis but offers flexibility, as a variety of algorithms with convergence guarantees can be described this way: by defining the search directions within each layer according to a possibly non-Euclidean norm one can generate a variety of different update rules, and these can all be accommodated in our Finsler structures framework. We now consider these three components in more detail.

**Layer-wise Lipschitz property** A neural network with no hidden layers presents a relatively straightforward optimization problem (under mild assumptions on the loss function). Typically, the resulting objective function has a Lipschitz gradient. But when multiple layers are connected in the typical feed-forward fashion, the result is a hierarchical system that as a whole does not appear to satisfy the property of having a Lipschitz gradient (See Proposition 2.1 below.) However, if we focus our attention to only one layer of the multi-layer network, then the task is somewhat simplified. Specifically, consider a neural network with the weight matrices ordered from input to output as \( w_1, w_2, \ldots, w_L \). Then under mild assumptions, the magnitude of the second derivative (Hessian matrix) of the objective function restricted to the weights in layer \( i \) can be bounded by a polynomial in the norms of the successive matrices \( w_{i+1}, \ldots, w_L \). This is formalized below in Proposition 4.3. This fact can be used to infer a lower bound on the function decrease that will happen when taking a gradient descent step on layer \( i \). By computing this bound for each possible layer \( i \in \{1, \ldots, L\} \) we can choose which update to perform using the greedy heuristic of picking the layer that maximizes the lower bound. The pseudocode for the procedure is presented in Algorithm 4.1 below.

**Duality Structure Gradient Descent** A widely used success criteria for non-convex optimization is that the algorithm yields a point where the Euclidean norm of the derivative of the objective is small. It appears difficult to establish this sort of guarantee in the situation described above, where one layer at a time is updated according to a greedy criteria. However, the analysis becomes simpler if we are willing to adjust the geometric framework used to define convergence. In the geometry we introduce, the norm at each point in the parameter space is determined by the weights of the neural network, and the convergence guarantee we seek is that our algorithm generates a point with a small derivative relative to these local norms. This notion of success is used in the theory of non-convex optimization on manifolds [7, 34].

The geometry used in our analysis is a special case of a Finsler structure and is designed in response to the structure of the neural network, taking into account our bound on the Lipschitz constants, and our greedy ‘maximum update’ criterion. The Finsler structure encodes our algorithm in the sense that one step of the algorithm corresponds to taking a step in the steepest descent direction as defined by the Finsler structure. The steepest descent directions with respect to this geometry are computed by solving a secondary optimization problem at each iteration, in order to identify which layer maximizes the lower bound on the function decrease. Formally, the solutions to this sub-problem are represented with a duality structure, hence the title of this paper.

**Intralayer update rules** A third component of our approach, which turns out to be key to obtaining an algorithm that is not only theoretically convergent but also effective in practice, is to consider the geometry within each layer of the weight matrices. Typically in first order gradient descent, the update direction is the vector of partial derivatives of the objective function. This can be motivated using Taylor’s theorem: if it is known that the spectral norm of the Hessian matrix of a given function \( f \) is bounded by a constant \( L \) then Taylor’s theorem provides a quadratic upper bound for the objective of the form \( f(w - \Delta) \leq f(w) - \frac{\partial f}{\partial w}(w) \cdot \Delta + \frac{L}{2} \| \Delta \|^2 \), and setting \( \Delta = \frac{1}{L} \frac{\partial f}{\partial w}(w) \) results in...
in a function decrease of magnitude at least $\frac{1}{2} \| \frac{\partial f}{\partial w} (w) \|^2_2$. Using a different norm when applying Taylor’s theorem results in a different quadratic upper bound. A general theorem about gradient descent for arbitrary norms is stated in Lemma 4.4. The basic idea is that if $\| \cdot \|$ is an arbitrary norm and $L$ is a global bound on the norm of the bilinear maps $\frac{\partial^2 f}{\partial w^2}(w)$, as measured with respect to $\| \cdot \|$, then $f$ satisfies a quadratic bound of the form $f(w - \Delta) \leq f(w) - \frac{\partial f}{\partial w}(w) \cdot \Delta + \frac{L}{2} \| \Delta \|^2$. Using the notion of a duality map $\rho$ for the norm $\| \cdot \|$ (see Equation (9) for a formal definition), the update $\Delta = \frac{1}{L} \rho(\frac{\partial f}{\partial w}(w))$ leads to a decrease of magnitude at least $\frac{1}{2} \frac{\| \partial f}{\partial w} (w) \|^2_2$. For example, when the argument $w$ has a matrix structure and $\| \cdot \|$ is the spectral norm, then the update direction is a spectrally-normalized version of the Euclidean gradient, in which the matrix of partial derivatives has its non-zero singular values set to 1 (See Proposition 4.5 for duality maps corresponding to the matrix 2 and $\infty$ norms.) The choice of norm for the weights can be encoded in the Finsler structure, and each norm leads to a different provably convergent variant of the algorithm. In our experiments we considered update rules based on the spectral norm and $\infty$-norm.

Despite the possible complexity of the Finsler structure, the analysis is straightforward and mimics the standard proof of convergence for Euclidean gradient descent. In the resulting convergence theory, we study how quickly the norm of the gradient tends to zero. In this analysis we measure the magnitude of the gradient with respect to the local norms $\| \cdot \|_{w(t)}$, as in other works non-convex optimization in manifolds. Roughly speaking, the quantity that is proved to tend to zero is $\| \frac{\partial f}{\partial w} (w(t)) \| / p(\| w(t) \|)$, where $\| w(t) \|$ is the norm of the network parameters and $p$ is an increasing function that depends on the architecture of the neural network. See Proposition 4.3 and the discussion following it for more details. This is in contrast to the usual Euclidean non-asymptotic performance analysis, which tracks the gradient measured with respect to a fixed norm, that is, $\| \frac{\partial f}{\partial w} (w(t)) \|$. 

1.1 Outline

After reviewing some related work, in Section 2 we present an example of a neural network training problem where the objective function does not have bounded second or third order derivatives. In Section 3 we introduce the abstract duality structure gradient descent (DSGD) algorithms and the convergence analyses. The main results in this section are Theorem 3.5 concerning the number of iterations needed to reach an approximate stationary point in deterministic gradient descent, and Corollary 3.12 which considers the expected number of iterations needed to reach approximate stationary points in stochastic duality structure gradient descent (SDSGD). We formally apply the DSGD framework to a neural network with multiple hidden layers in Section 4. The main results in this section are convergence analyses for the neural network training procedure presented in Algorithm 4.1, both in the deterministic case (Proposition 3.5) and a corresponding analysis for the mini-batch variant of the algorithm (Proposition 4.11.) Numerical experiments on the MNIST, CIFAR, and SVHN benchmark data sets are presented in Section 5. We finish with a discussion in Section 6. Several proofs are deferred to an appendix.

1.2 Related work

There are a number of performance analyses of gradient descent for non-convex functions which utilize the assumption that one or more of the functions derivatives are bounded. Perhaps the simplest among them is as follows.

**Proposition 1.1.** Let $f : \mathbb{R}^n \rightarrow \mathbb{R}$ be a continuously differentiable function whose values are bounded from below by $f^* \in \mathbb{R}$ and suppose that the gradient of $f$ is $L$-Lipschitz continuous. Starting from a point $w(1) \in \mathbb{R}^n$, define $w(2), w(3), \ldots$ as

$$w(t + 1) = w(t) - \epsilon \frac{\partial f}{\partial w}(w(t)).$$
If the step-size is set to $\epsilon = 1/L$, then $\|\frac{\partial f}{\partial w}(w(t))\| \to 0$ and

$$\min_{1 \leq t \leq T} \left\| \frac{\partial f}{\partial w}(w(t)) \right\|^2 \leq \frac{2L(f(w(0)) - f^*)}{T}.$$ 

The proof of this result can be derived from, for instance, the results in [26, Section 1.2.3]. Note the inverse relationship between the Lipschitz constant $L$ and the step-size $1/L$, which is characteristic of results that rely on a Lipschitz property of the gradient for non-asymptotic analysis.

Many practical algorithms use approximations to the gradient, and can be modeled as stochastic versions of gradient descent (SGD). The Randomized Stochastic Gradient (RSG) algorithm [14] and is an analytically tractable variant of SGD. In RSG, a stochastic gradient update is run for $T$ steps, and then a random iterate is returned. In [14] it was proved that the expected squared-norm of the gradient tends to zero as $T$ increases, with a convergence rate of $\frac{1}{T}$. Their assumptions include a Lipschitz gradient and uniformly bounded variance of gradient estimates.

A variety of other, more specialized algorithms have also been analyzed under the Lipschitz-gradient assumption. The Stochastic Variance Reduced Gradient (SVRG) algorithm combine features of deterministic and stochastic gradient descent, alternating between full gradient calculations and SGD iterations [17]. In some remarkable recent works [3] [29], it was shown that SVRG for non-convex functions requires fewer gradient evaluations on average compared to RSG. The step-sizes follow a $1/L$ rule, and the variance assumptions are weaker compared to RSG. For machine learning on a large scale, distributed and decentralized algorithms become of interest. Decentralized SGD was analyzed in [24], leading to a $1/L$-type result for this setting.

One approach to extend the results on gradient descent is to augment or replace the assumption on the second derivative with an analogous assumption on third order derivatives. In an analysis of cubic regularization methods, [8] proved a bound on the asymptotic rate of convergence for nonconvex functions that have a Lipschitz-continuous Hessian. In a non-asymptotic analysis of a trust region algorithm in [10], convergence was shown to points that approximately satisfy a second order optimality condition, assuming a Lipschitz gradient and Lipschitz hessian.

A natural question is whether these results can be generalized to exploit the Lipschitz properties of derivatives of arbitrary order. A recent work considered this scenario, where it is assumed that the derivative of order $p$ is Lipschitz continuous, for arbitrary $p \geq 1$ [6]. In their algorithm, they construct a $p + 1$ degree polynomial majorizing the objective at each iteration, and the next parameter value is obtained by approximately minimizing this polynomial. The algorithm in a sense generalizes the classical $1/L$ iterations of Proposition 1.1 and as well as cubic regularization methods. A remarkable feature of the analysis is that the convergence rate improves as $p$ increases. However, the practicality of the method is unclear, as it requires the optimization of potentially high degree multivariate polynomials.

The above results all require a bound on a derivative to obtain a convergence guarantee. One approach to generalize this is with the concept of relative smoothness, defined in [25] (and closely related to condition $LC$ in [5]). Roughly speaking, a function $f$ is defined to be relatively smooth relative to a reference function $h$ if the the Hessian of $f$ is upper bounded by the hessian of $h$ (see Proposition 1.1 in [25].) In the optimization procedure, one solves sub-problems that involve the function $h$ instead of $f$, and if $h$ is significantly simpler than $f$ the procedure can be practical. A non-asymptotic convergence guarantee is established under an additional relative-convexity condition. Our work in this paper is related in spirit to relative smoothness, as we are also concerned with how to overcome the use of restrictive Lipschitz-gradient assumptions. However, this work has two notable differences. The first is that we don’t assume the function is convex in any sense. The second is the geometric interpretation of our approach in terms of Finsler structures, which proves a link with optimization on manifolds.

The notion of convergence we employ is formulated using ideas from optimization on manifolds. Many optimization problems involve search spaces with a natural structure as a smooth manifold,
and specialized algorithms that exploit this structure have been developed. Notable instances include optimizing over spaces of structured matrices [2], and parameterized probability distributions, as in information geometry [3]. In the context of neural networks, natural gradient approaches to optimization have been explored [21, 4], and recently [28] considered some practical variants of this, while also extending it to networks with multiple hidden layers. Our work is related to the manifold perspective in that we borrow ideas from that domain to formulate our notion of convergence. The main idea that we borrow is that of a Finsler structure, which is a family of norms that parameterized by points in the search space, subject to a continuity condition. When we discuss convergence, it is measured with respect to a Finsler structure, rather than a fixed norm. This notion of convergence is similar to what is used to formulate convergence bounds in several algorithms for non-convex optimization on Riemannian manifolds [34, 4]. The setting in this work is slightly different. Firstly, we are concerned with unconstrained optimization over Euclidean space, which has a trivial manifold structure. Secondly, while a Riemannian metric specifies an inner-product norm that various continuously across the domain, the Finsler structure approach used in the present work does not require the norms at each location to be inner product norms. This is important for the analysis, since there are key features of our algorithm that can not be be encoded using a Riemannian metric. Primarily, this is the layer-wise update rule, as explained in Remark 4.9 below.

Finally, we note that several heuristics for step-size selection in the specific case of gradient descent for neural networks have been proposed, including [31, 12, 18], but the theoretical analyses in these works is limited to convex functions.

Notation  
$f$: an objective function to minimize. $f^*$: a lower bound on values of the objective function. $w$: the parameter we are optimizing over. $n$: dimensionality of parameter space. $t$: iteration number in an optimization algorithm. $\mathcal{L}(\mathbb{R}^d, \mathbb{R})$: the set of linear maps from $\mathbb{R}^d$ to $\mathbb{R}$. $\ell$: a generic element in the space of linear maps $\mathcal{L}(\mathbb{R}^d, \mathbb{R})$. $\epsilon$: step-size in an optimization algorithm. $g(t)$: an approximate derivative of the objective function. $\delta$: error of an approximate derivative. $L$: Lipschitz-type constant. $K$: number of layers in a neural network. $n_k$: number of nodes in layer $k$ of a neural network. $x^k$: state of layer $k$ of neural network. $y$: input to a neural network. $z$: output target for a neural network. $m$: number of examples in a training set. $f_i$: loss function for training example $i$. $\|\cdot\|$: a norm. $\|\cdot\|_w$: a norm that depends on a parameter $w$. $\rho$: a duality map. $\rho_w$: duality map that depends on a parameter $w$. $h$: the function computed by one layer in a neural network. $q$: a choice of norm in $(2, \infty]$. $\text{tr}$: trace of a matrix. $b$: batch size. $B(t)$: random variable representing the batch at time $t$. $T$: final iterate of an optimization algorithm. $r, v, s$: auxiliary polynomials used to define bounds on neural network derivatives. $w_{1:k}$: if $w$ is a vector $w = (w_1, \ldots, w_n)$ with $n$ components and $k \leq n$, then $w_{1:k} = (w_1, \ldots, w_k)$. $J$: loss function used for neural network training. $A_1 \oplus A_2$: given two linear maps $A_1: X \rightarrow U$ and $A_2: Z \rightarrow U$, the direct sum $A_1 \oplus A_2$ is the linear map from $X \times Z$ to $U \times U$ that maps a vector $(z_1, z_2)$ to $(A_1 z_1, A_2 z_2)$. $\|A\|$; if $A$ is a linear map $A: X \rightarrow Y$ between normed spaces $X$ and $Y$ then $\|A\| = \sup_{\|x\|_X = 1} \|Ax\|_Y$. $C(x, y)$: the result of applying the bilinear map $C$ to the argument consisting of two vectors $x, y$. $\|C\|$; if $C$ is a bilinear map $C: X \times Y \rightarrow Z$ then $\|C\| = \sup_{\|x\|_X = \|y\|_Y = 1} \|C(u_1, u_2)\|_Z$. $\text{sgn}$: if $x$ is a non-negative number then $\text{sgn}(x) = 1$, and $\text{sgn}(x) = -1$ otherwise.

If $\ell: \mathbb{R}^n \rightarrow \mathbb{R}$ is a linear functional, then we represent the value of $\ell$ at the point $u \in \mathbb{R}^n$ by $\ell(u)$. For the sake of clarity, when a linear functional or any other linear map depends on a parameter (as is the case for the derivatives of functions) we use a slightly different notation, which is the same as that used in [11]. The derivative of a function $f: \mathbb{R}^n \rightarrow \mathbb{R}^m$ at point $x_0 \in \mathbb{R}^n$ is a linear map from $\mathbb{R}^n$ to $\mathbb{R}^m$, denoted by $\frac{\partial f}{\partial x}(x_0)$. The result of applying this linear map to a vector $u \in \mathbb{R}^n$ is a vector in $\mathbb{R}^m$ denoted $\frac{\partial f}{\partial x}(x_0) \cdot u$. The second derivative of a function $f$ at $x_0$ is a bilinear map from $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}^m$, denoted by $\frac{\partial^2 f}{\partial x^2}(x_0)$, and we use the notation $\frac{\partial^2 f}{\partial x^2}(x_0) \cdot (u, v)$ to represent the $\mathbb{R}^m$-valued result of applying this bilinear map to the pair of vectors $(u, v)$. 

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2 Non-applicability of existing theory to neural networks

In this section we consider whether the “$1/L$”-type results like Proposition 1.1 could be applied in the case of neural networks. Consider the neural network depicted in Figure 1, which maps a real-valued input to a hidden layer with one node and produces a real-valued output. Suppose that the sigmoid activation function $\sigma(u) = 1/(1 + e^{-u})$ is used, so that the function computed by the network is

$$f(w_1, b_1, w_2, b_2; x) = \sigma(w_2 \sigma(w_1 x + b_1) + b_2).$$

Consider training the network to map the input $x = 1$ to the output $0$, using a squared-error loss function. This leads to the optimization objective $E : \mathbb{R}^4 \to \mathbb{R}$ defined by

$$E(w_1, b_1, w_2, b_2) = |f(w_1, b_1, w_2, b_2; 1)|^2. \quad (1)$$

In Proposition 2.1 we establish that $E$ does not have bounded second or third derivatives. This means that the results from $1/L$-theory cannot be used to guarantee the convergence of gradient descent in these cases.

**Proposition 2.1.** The function $E$ defined in Equation (1) has unbounded second and third order derivatives: $\sup_{z \in \mathbb{R}^4} \| \frac{\partial^2 E}{\partial z^2} (z) \| = \infty$ and $\sup_{z \in \mathbb{R}^4} \| \frac{\partial^3 E}{\partial z^3} (z) \| = \infty$.

**Proof.** We will use $z$ to denote a particular choice of parameters $z = (w_1, b_1, w_2, b_2)$. Focusing on the weights $w_1$ and $w_2$, we will to define a family of parameters $z_\epsilon$ such that

$$\lim_{\epsilon \to \infty} \frac{\partial^2 E}{\partial w_1 \partial w_2} (z_\epsilon) = +\infty \quad (2)$$

and

$$\lim_{\epsilon \to \infty} \frac{\partial^3 E}{\partial w_1 \partial w_2^2} (z_\epsilon) = +\infty. \quad (3)$$

Beginning with the second derivative term (2), the chain-rule gives

$$\frac{\partial^2 E}{\partial w_1 \partial w_2} (z) = 2f(z) \frac{\partial^2 f}{\partial w_1 \partial w_2} (z) + 2 \frac{\partial f}{\partial w_1} (z) \frac{\partial f}{\partial w_2} (z).$$
The derivatives of $f$ appearing above in this equation are as follows:

$$\frac{\partial f}{\partial w_1}(z) = \sigma'(w_2\sigma(w_1 + b_1) + b_2)w_2\sigma'(w_1 + b_1), \quad (4a)$$

$$\frac{\partial f}{\partial w_2}(z) = \sigma'(w_2\sigma(w_1 + b_1) + b_2)\sigma(w_1 + b_1), \quad (4b)$$

$$\frac{\partial^2 f}{\partial w_1 \partial w_2}(z) = \sigma''(w_2\sigma(w_1 + b_1) + b_2)\sigma(w_1 + b_1)w_2\sigma'(w_1 + b_1) + \sigma'(w_2\sigma(w_1 + b_1) + b_2)\sigma'(w_1 + b_1), \quad (4c)$$

Set $y$ to be the number

$$y = \arg \max_{u \in \mathbb{R}} \sigma''(u).$$

Observe that if $b_2 = y - w_2\sigma(w_1 + b_1)$ then these equations hold:

$$f(z) = \sigma(w_2\sigma(w_1 + b_1) + b_2) = \sigma(y), \quad (5a)$$

$$\sigma''(w_2\sigma(w_1 + b_1) + b_2) = \|\sigma''\|_{\infty}, \quad (5b)$$

$$\sigma'''(w_2\sigma(w_1 + b_1) + b_2) = 0. \quad (5c)$$

In each of these equations, the right-hand side is a constant independent of $w_1, b_1$, and $w_2$.

Starting from an an arbitrary choice of $w_1, b_1$, consider the following definition of $z_\epsilon$:

$$z_\epsilon = (w_1, b_1, \epsilon, y - \epsilon \sigma(w_1 + b_1)) \quad (6)$$

Using Equations (4a)-(4c), (5a), and (5b) we conclude that $f(z_\epsilon)$ and $\frac{\partial f}{\partial w_1}(z_\epsilon)$ are positive constants for all $\epsilon$, while $\frac{\partial f}{\partial w_2}(z_\epsilon)$ and $\frac{\partial^2 f}{\partial w_1 \partial w_2}(z_\epsilon)$ are increasing affine functions of $\epsilon$. This means Equation (2) holds.

Continuing to the third derivative term (3), we have

$$\frac{\partial^3 E}{\partial w_1 \partial w_2^2}(z) = 4 \frac{\partial f}{\partial w_2}(z) \frac{\partial^2 f}{\partial w_1 \partial w_2}(z) + 2f(z) \frac{\partial^3 f}{\partial w_1 \partial w_2^2}(z) + 2 \frac{\partial f}{\partial w_1}(z) \frac{\partial^2 f}{\partial w_2^2}(z). \quad (7)$$

In addition to (4a)-(4c), the derivatives of $f$ that appear in the above are $\frac{\partial^3 f}{\partial w_1 \partial w_2}$ and $\frac{\partial^3 f}{\partial w_2^2}$, which are

$$\frac{\partial^3 f}{\partial w_1 \partial w_2}(z) = \sigma'''(w_2\sigma(w_1 + b_1) + b_2)\sigma(w_1 + b_1)\sigma'(w_1 + b_1)w_2\sigma'(w_1 + b_1) + \sigma''(w_2\sigma(w_1 + b_1) + b_2)\sigma'(w_1 + b_1)$$

$$+ \sigma'(w_2\sigma(w_1 + b_1) + b_2)\sigma'(w_1 + b_1) + \sigma''(w_2\sigma(w_1 + b_1) + b_2)\sigma'(w_1 + b_1)w_2\sigma'(w_1 + b_1), \quad (7)$$

$$\frac{\partial^2 f}{\partial w_2^2}(z) = \sigma''(w_2\sigma(w_1 + b_1) + b_2)\sigma(w_1 + b_1)\sigma'(w_1 + b_1).$$

We consider the same points $z_\epsilon$ defined in (6). Again using Equations (5a)-(5c) we can see that (7) is a sum of three terms, each of which is product of a constant term and increasing affine function of $\epsilon$. This establishes Equation (3).

In this proposition we showed that the second (and third) derivatives of the function $E$ are unbounded by constructing a trajectory in the weight space along which the norm of the second (third) derivative grows without bound. Note that this does not mean that simple constant step-size methods will fail in neural network training problems arising in practice, but it does suggest that for these problems a more sophisticated theory would be needed to analyze their convergence. To overcome this problem, in this paper we introduce the DSGD algorithm, with the benefit that it allows us to prove convergence for a more general class of functions, including the one defined in Equation (1), with abstract convergence proofs that mimic those used in $1/L$-type results like Proposition 1.1.
3 Finsler Gradient Descent

The analysis framework involves two geometric structures: A Finsler structure and an associated Finsler duality structure. For our purposes, a Finsler structure $\|\cdot\|_w$ is simply a family of norms parameterized by points $w$ in some Euclidean space, subject to a continuity condition. The notation $\|u\|_w$ refers to the norm of the vector $u$ at the parameter $w$. The continuity condition roughly means that is that the norms $\|\cdot\|_{w_1}$ and $\|\cdot\|_{w_2}$ should be similar if $w_1$ and $w_2$ are close.

**Definition 3.1.** Let $\|\cdot\|_w$ be an assignment of a norm on $\mathbb{R}^n$ to each point of $\mathbb{R}^n$. We say that $\|\cdot\|_w$ is a Finsler structure if the map $(w, u) \mapsto \|u\|_w$ is continuous on $\mathbb{R}^n \times \mathbb{R}^n$.

The Finsler structure induces a norm on the dual space $\mathcal{L}(\mathbb{R}^n, \mathbb{R})$ at each $w \in \mathbb{R}^n$; if $\ell \in \mathcal{L}(\mathbb{R}^n, \mathbb{R})$ then

$$\|\ell\|_w = \sup_{\|u\|_w=1} \ell(u).$$

It is the case that for any Finsler structure the dual norm map $(w, \ell) \mapsto \|\ell\|_w$ is continuous on $\mathbb{R}^n \times \mathcal{L}(\mathbb{R}^n, \mathbb{R})$. This follows from [11, Proposition 27.7]. For completeness, a proof of this fact is included in the appendix (see Lemma A.1).

A vector achieving the supremum in Equation (8) always exists, since the dual norm is defined as the supremum of a continuous function over a compact set. We represent (scaled versions of) solutions to this optimization problem using a duality map. Formally, a duality map at $w$ is a function from $\mathcal{L}(\mathbb{R}^n, \mathbb{R})$ to $\mathbb{R}^n$, denoted $\rho_w$, with the following properties: For all $\ell \in \mathcal{L}(\mathbb{R}^n, \mathbb{R})$,

$$\|\rho_w(\ell)\|_w = \|\ell\|_w,$$

$$\rho_w(\ell) = \|\ell\|_w^2.$$  \hfill (9a)

If the underlying norm $\|\cdot\|_w$ is an inner product norm, then it can be shown that there is a unique choice for the duality map. In detail, let $Q_w$ be the positive definite matrix such that $\|u\|_w = \sqrt{u^TQ_wu}$ for all vectors $u$. Then the duality mapping is

$$\rho(\ell) = Q_w^{-1}\ell.$$  \hfill (10)

However, in general there might be more than one choice for the duality map. For example, consider the norm $\|\cdot\|_{\infty}$ on $\mathbb{R}^2$, and let $\ell$ be the linear functional $\ell((x_1, x_2)) = x_1$. Then both of the vectors $u = (1, 1)$ and $u' = (1, -1)$ satisfy properties (9a) and (9b).

To any Finsler structure we can associate a duality structure:

**Definition 3.2.** A duality structure is an assignment of a duality map to each $w \in \mathbb{R}^n$. The notation $\rho_w(\ell)$ refers to the value of the duality map at $w$ applied to the functional $\ell$. That is, a duality structure is a function $\rho : \mathbb{R}^n \times \mathcal{L}(\mathbb{R}^n, \mathbb{R}) \to \mathbb{R}^n$ such that for all $w \in \mathbb{R}^n$, the function $\rho_w \in \mathcal{L}(\mathbb{R}^n, \mathbb{R})$ satisfies the two properties (9a) and (9b).

The simplest Finsler structure is the one which assigns the Euclidean norm to each point of the space. In this situation, the dual norm is also the Euclidean norm and the duality map at each point is simply the identity function. Before continuing, let us consider a less trivial example.

**Example 3.3.** Let $h : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be any continuous functions. Consider the following family of norms on $\mathbb{R}^2$:

$$\|\delta x, \delta y\|_{(x,y)} = \sqrt{1 + |h(y)||\delta x| + \sqrt{1 + |g(x)||\delta y|}}.$$

As the function $(x, y, \delta x, \delta y) \mapsto \|\delta x, \delta y\|_{(x,y)}$ is continuous, this family of norms is a well-defined Finsler structure. Denoting a linear functional on $\mathbb{R}^2$ by $\ell = (\ell_1, \ell_2)$, the dual norm is

$$\|(\ell_1, \ell_2)\|_{(x,y)} = \max \left\{ \frac{|\ell_1|}{\sqrt{1 + |h(y)|}}, \frac{|\ell_2|}{\sqrt{1 + |g(x)|}} \right\}.$$
Algorithm 3.1: Duality structure gradient descent (DSGD)

1. input: Initial point \( w(1) \in \mathbb{R}^n \) and step-size sequence \( \epsilon(t) \).
2. for \( t = 1, 2, \ldots \) do
   3. Compute the search direction: \( \Delta(t) = \rho_{w(t)}(\partial f / \partial w)(w(t)) \).
   4. Update the parameter: \( w(t + 1) = w(t) - \epsilon(t) \Delta(t) \).
5. end

This can be established from Proposition 4.6. A duality map is

\[
\rho_{(x,y)}((\ell_1, \ell_2)) = \begin{cases} 
\left( \frac{\ell_1}{1 + |h(y)|}, 0 \right) & \text{if } \frac{|\ell_1|}{\sqrt{1 + |h(y)|}} \geq \frac{|\ell_2|}{\sqrt{1 + |g(x)|}} \\
0, & \text{else.}
\end{cases}
\]

This is also a consequence of Proposition 4.6. This concludes the example.

We now describe the DSGD algorithm (Algorithm 3.1) and its corresponding convergence guarantee. In order to run this algorithm, the user specifies an objective function \( f : \mathbb{R}^n \to \mathbb{R} \), a duality structure \( \rho_w \), an initial point \( w(1) \) and a step-size sequence \( \epsilon(t) \). Note that as the example of the norm \( \| \cdot \|_\infty \) shows, there may be multiple duality structures that can be chosen for a given Finsler structure. If this is the case, then any one of them can be chosen when running the algorithms, without affecting the convergence bounds. There are two main steps in the algorithm. In the first step (Line 3), we compute the duality map on the derivative at \( w(t) \), resulting in an update direction \( \Delta(t) \). Then in second step (Line 4) the next parameter \( w(t + 1) \) is obtained by taking a step in this direction.

The conditions on the objective \( f \) are that the function is differentiable and obeys a quadratic bound along each ray specified by the duality map.

Assumption 3.4. The function \( f : \mathbb{R}^n \to \mathbb{R} \) is continuously differentiable, bounded from below by \( f^* \in \mathbb{R} \), and there is an \( L \geq 0 \) such that, for all \( w \in \mathbb{R}^n \), all \( \eta \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}) \), and all \( \epsilon \in \mathbb{R} \),

\[
\left| f(w + \epsilon \rho_{w}(\eta)) - f(w) - \epsilon \frac{\partial f}{\partial w}(w) \cdot \rho_{w}(\eta) \right| \leq \frac{L}{2}\epsilon^2 \| \eta \|^2_w.
\]

This assumption is related to condition A3 of [7], except it concerns the simple search space of \( \mathbb{R}^n \), and it is adapted to use a Finsler structure as instead of a Riemannian structure.

The analysis concerns the convergence of the gradients in terms of the local-norms \( \| \cdot \|_{w(t)} \). This is criteria for gradient convergence in the manifold setting, is also used in Theorem 4 of [7], and Theorem 2 of [34].

Theorem 3.5. Let Assumption 3.4 hold. Starting from \( w(1) \in \mathbb{R}^n \), consider the sequence \( w(t) \) generated by Algorithm 3.1 using a constant stepsize \( \epsilon(t) := \epsilon \in (0, 2/L) \). Then the sequence \( w(t) \) satisfies one of these two conditions:

i. There is a \( T \) such that \( f(w(t)) < f(w(t - 1)) \) for \( 1 < t \leq T \), and \( \frac{\partial f}{\partial w}(w(T)) = 0 \).
ii. \( f(w(t)) < f(w(t - 1)) \) for all \( t \), \( \lim_{t \to \infty} \| \frac{\partial f}{\partial w}(w(t)) \|_{w(t)} = 0 \), and any accumulation point \( w^* \) of the algorithm is a stationary point, meaning \( \frac{\partial f}{\partial w}(w^*) = 0 \).

Furthermore, a non-asymptotic performance guarantee holds: For any \( \delta > 0 \), if \( T \geq \frac{1}{\delta^2} \frac{2(f(w(1)) - f^*)}{\epsilon(2 - L\epsilon)} \) then \( \min_{1 \leq t \leq T} \| \frac{\partial f}{\partial w}(w(t)) \|_{w(t)} \leq \delta \).
Proof. For \( t \geq 0 \), set \( \eta(t) = \frac{\partial f}{\partial w}(w(t)) \). Then Assumption 3.1 implies

\[
\begin{aligned}
f(w(t+1)) &\leq f(w(t)) + \epsilon \left( \left\| \frac{\partial f}{\partial w}(w(t)) \right\|_{w(t)}^2 + \frac{L}{2} \epsilon^2 \left\| \frac{\partial f}{\partial w}(w(t)) \right\|_{w(t)}^2 \right) + \frac{L}{2} \epsilon^2 \left\| \frac{\partial f}{\partial w}(w(t)) \right\|_{w(t)}^2 .
\end{aligned}
\]

Invoking the duality map properties \((9a)\) and \((9b)\),

\[
\begin{aligned}
&\leq f(w(t)) - \epsilon \left\| \frac{\partial f}{\partial w}(w(t)) \right\|_{w(t)}^2 + \frac{L}{2} \epsilon^2 \left\| \frac{\partial f}{\partial w}(w(t)) \right\|_{w(t)}^2 \\
&= f(w(t)) + \epsilon \left( \frac{L}{2} \epsilon - 1 \right) \left\| \frac{\partial f}{\partial w}(w(t)) \right\|_{w(t)}^2 .
\end{aligned}
\]

From the last inequality it is clear that the function decreases at iteration \( t \) unless \( \frac{\partial f}{\partial w}(w(t)) = 0 \). Summing our inequality over \( t = 1, 2, \ldots, T \) yields

\[
f(w(T)) \leq f(w(1)) + \epsilon \left( \frac{L}{2} \epsilon - 1 \right) \sum_{t=1}^{T} \left\| \frac{\partial f}{\partial w}(w(t)) \right\|_{w(t)}^2 .
\]

Upon rearranging terms and using that \( f(w(T)) > f^* \), we find that

\[
\sum_{t=1}^{T} \left\| \frac{\partial f}{\partial w}(w(t)) \right\|_{w(t)}^2 \leq \frac{2(f(w(1)) - f^*)}{\epsilon(2 - L \epsilon)} .
\]

As this inequality holds for arbitrary \( T \), it must be that \( \lim_{t \to \infty} \left\| \frac{\partial f}{\partial w}(w(t)) \right\|_{w(t)} = 0 \).

Let \( w^* \) be an accumulation point of the algorithm; this is defined as a point such that for any \( \gamma > 0 \) the ball \( \{ w \in \mathbb{R}^n : \left\| w - w^* \right\| < \gamma \} \) is entered infinitely often by the sequence \( w(t) \) (any norm \( \| \cdot \| \) can be used to define the ball.) Then there is a subsequence of iterates \( w(m(1)), w(m(2)), \ldots \) with \( m(k) < m(k+1) \) such that \( w(m(k)) \to w^* \). We know from \((12)\) that \( \left\| \frac{\partial f}{\partial w}(w(t)) \right\|_{w(t)} \to 0 \), and the same must hold for any subsequence. Hence \( \left\| \frac{\partial f}{\partial w}(w(m(k))) \right\|_{w(m(k))} \to 0 \). As the map \( (w, \ell) \to \| \ell \|_w \) is continuous on \( \mathbb{R}^n \times L(\mathbb{R}^n, \mathbb{R}) \) (see Lemma A.1 in the appendix), it must be that \( \left\| \frac{\partial f}{\partial w}(w^*) \right\|_{w^*} = 0 \). Since \( \| \cdot \|_{w^*} \) is a norm, then \( \frac{\partial f}{\partial w}(w^*) = 0 \).

For the rate of the gradient convergence, it follows from \((12)\) that

\[
\min_{1 \leq t \leq T} \left\| \frac{\partial f}{\partial w}(w(t)) \right\|_{w(t)}^2 \leq \frac{1}{T} \frac{2(f(w(1)) - f^*)}{\epsilon(2 - L \epsilon)} .
\]

If \( T \geq \frac{1}{\epsilon^2} \frac{2(f(w(1)) - f^*)}{\epsilon(2 - L \epsilon)} \), then the right hand side of this equation is at most \( \delta^2 \).

Note that if the Finsler structure simply assigns the Euclidean norm to each point in the space, expressed as \( \| \cdot \|_{w} = \| \cdot \|_2 \) for all \( w \), then the duality map is trivial: \( \rho_w(\ell) = \ell \) for all \( \ell \). In this case Algorithm 3.1 reduces to standard gradient descent, and the convergence result we just proved is essentially equivalent to that of Proposition 1.1. In the general case, the non-asymptotic performance guarantee concerns the quantities \( \left\| \frac{\partial f}{\partial w}(w(t)) \right\|_{w(t)} \), where the gradient magnitude is measured relative to the local norms \( w(t) \). We leave to future work the interesting question of under what conditions a relation can be established between the convergence of \( \left\| \frac{\partial f}{\partial w}(w(t)) \right\|_{w(t)} \) and the convergence of \( \left\| \frac{\partial f}{\partial w}(w(t)) \right\|_{w} \), where the norm is fixed.

Let us consider an example of an optimization problem where this theory applies.

**Example 3.6.** Let \( n = 2 \) and suppose that \( f : \mathbb{R}^2 \to \mathbb{R} \) is \( f(x, y) = g(x)h(y) \), where \( g : \mathbb{R} \to \mathbb{R} \) and \( h : \mathbb{R} \to \mathbb{R} \) are functions that have bounded second derivatives. For simplicity, assume that \( \| g'' \|_\infty \leq 1 \)
and \( \|h''\|_\infty \leq 1 \). Furthermore, assume that \( \sup_{(x,y) \in \mathbb{R}^2} g(x)h(y) \geq f^* \) for some \( f^* \in \mathbb{R} \) (for instance, this occurs if \( g \) and \( h \) are non-negative). The function \( f \) need not have a Lipschitz gradient, as the example of \( g(x) = x^2 \) and \( h(y) = y^2 \) demonstrates.

Let us denote pairs in \( \mathbb{R}^2 \) by \( w = (x, y) \). Define the Finsler structure

\[
\|(\delta x, \delta y)\|_w = \sqrt{1 + |h(y)||\delta x|} + \sqrt{1 + |g(x)||\delta y|}.
\]

The dual norm and duality map are as previously defined in Example 3.3. Let us show that the conditions of Assumption 3.4 are satisfied. Let \( \eta = (\eta_1, \eta_2) \) be any vector.

If \( \frac{m_1}{\sqrt{1 + |h(y)|}} \geq \frac{m_2}{\sqrt{1 + |g(x)|}} \), then \( \|\eta\|_w = \frac{|\eta_1|}{\sqrt{1 + |h(y)|}} \) and \( \rho(\eta) = \left( \frac{m_1 - \sqrt{1 + |h(y)|}}, 0 \right) \). Then, since the function \( x \mapsto f(x, y) \) is a function whose second derivative is bounded by \( |h(y)| \), we can apply a standard quadratic bound (see Lemma 4.4) to conclude that

\[
\left| f(w + \epsilon \rho_w(\eta)) - f(w) - \epsilon \frac{\partial f}{\partial w}(w) \cdot \rho_w(\eta) \right| = \left| f \left( \left( x + \epsilon \frac{\eta_1}{1 + |h(y)|} \right), y \right) \right| - f(w) - \epsilon \frac{1}{1 + |h(y)|} \frac{\partial f}{\partial x}(w) \eta_1 \\
\leq \epsilon^2 \frac{1}{(1 + |h(y)|)^2} \frac{1}{2} |h(y)| \|\eta_1\|^2 \\
= \epsilon^2 \frac{1}{1 + |h(y)|} \frac{1}{2} \|\eta\|^2_w \\
\leq \frac{\epsilon^2}{2} \|\eta\|^2_w.
\]

The case \( \frac{m_1}{\sqrt{1 + |h(y)|}} < \frac{m_2}{\sqrt{1 + |g(x)|}} \), is similar. This shows that Assumption 3.4 holds with \( L = 1 \). According to Theorem 3.5, convergence will be guaranteed in Algorithm 3.1 for any \( \epsilon \in (0, 2) \).

Let us consider the steps of Algorithm 3.1. In the first step (Line 3) the algorithm computes the derivative on the update direction. If

\[
\left| \frac{\partial f}{\partial x}(x(t), y(t)) \right| \leq \frac{1}{\sqrt{1 + |h(y(t))|}}, \quad \left| \frac{\partial f}{\partial y}(x(t), y(t)) \right| \leq \frac{1}{\sqrt{1 + |g(x(t))|}} \tag{13}
\]

then the update direction is \( \Delta(t) = \left( \frac{\partial f}{\partial x}(x(t), y(t)) \frac{1}{\sqrt{1 + |h(y(t))|}}, 0 \right) \). In this case, at the next step (Line 4) the next point is computed by keeping \( y \) the same \((y(t + 1) = y(t))\) and updating \( x \) as \( x(t + 1) = x(t) - \epsilon \frac{\partial f}{\partial x}(x(t), y(t)) \frac{1}{\sqrt{1 + |g(x(t))|}} \). If (13) does not hold, then \( y \) is updated instead: \( x(t + 1) = x(t) \) and \( y(t + 1) = y(t) - \epsilon \frac{\partial f}{\partial y}(x(t), y(t)) \frac{1}{\sqrt{1 + |g(x(t))|}} \). The resulting convergence guarantee associated with the algorithm is that

\[
\max \left\{ \left| \frac{\partial f}{\partial x}(x(t), y(t)) \right| \left| \frac{\partial f}{\partial y}(x(t), y(t)) \right| \right\} \to 0 \quad \text{as} \quad t \to \infty.
\]

Note that this example could be extended without difficult to the case where \( f \) is defined as the product of arbitrarily many functions that have bounded second derivatives.

### 3.1 Stochastic Duality Structure Gradient Descent

In this section we analyze a Finslerian version of stochastic gradient descent, as presented in Algorithm 3.2. Each iteration of this algorithm uses an estimate \( g(t) \) of the derivative. The algorithm computes the duality map on this estimate, and the result serves as the update direction. A step-size \( \epsilon(t) \) determines how far to go in this direction.
The success criteria we use for our SGD algorithm is that an approximate stationary point is reached, just as in deterministic GD. Due to randomness, this cannot be guaranteed to occur within any fixed, finite number of steps. However, we can consider the expected amount of time until an approximate stationary point is generated. That is, we fix a $\gamma > 0$ and ask what is the expected amount of time until a point $w(t)$ is generated such that $\| \frac{\partial L}{\partial w}(w(t)) \|^2_{w(t)} \leq \gamma$. More formally, we consider the stopping time $\tau$, defined as

$$\tau = \inf \left\{ t \geq 1 \left| \left\| \frac{\partial L}{\partial w}(w(t)) \right\|^2_{w(t)} \leq \gamma \right. \right\},$$

and the goal of our analysis is to find an upper bound for $\mathbb{E}[\tau]$.

To state our assumptions, define the filtration $\{\mathcal{F}(t)\}_{t=0,1,\ldots}$, where $\mathcal{F}(0) = \sigma(w(1))$ and for $t \geq 1$, $\mathcal{F}(t) = \sigma(w(1), g(1), g(2), \ldots, g(t))$. We assume that the derivative estimates $g(t)$ are unbiased:

**Assumption 3.7.** The derivative estimates in Algorithm 3.2 are unbiased: For $t = 1, 2, \ldots$,

$$\mathbb{E}[g(t) | \mathcal{F}(t-1)] = \frac{\partial f}{\partial w}(w(t)).$$

We also assume the derivative estimates have bounded variance:

**Assumption 3.8.** For $t = 1, 2, \ldots$, define $\delta(t)$ as

$$\delta(t) = g(t) - \frac{\partial f}{\partial w}(w(t)).$$

It holds for all $t \geq 1$ that

$$\mathbb{E}\left[ \left\| \delta(t) \right\|^2_{w(t)} | \mathcal{F}(t-1) \right] \leq \sigma^2 < \infty.$$

For the purposes of analysis, we will need a theorem which allows us to bound the expected values of random walks that are stopped at random times.

**Proposition 3.9.** Let $\tau$ be a stopping time with respect to a filtration $\{\mathcal{F}(t)\}_{t=0,1,\ldots}$. Suppose there is a number $c < \infty$ such that $\tau \leq c$ with probability one. Let $x_1, x_2, \ldots$ be any sequence of random variables such that $\mathbb{E}[\|x_i\|] < \infty$ for all $i$. Then

$$\mathbb{E}\left[ \sum_{i=1}^{\tau} x_i \right] = \mathbb{E}\left[ \sum_{i=1}^{\tau} \mathbb{E}[x_i | \mathcal{F}(i-1)] \right].$$

**Proof.** This is a consequence of the optional stopping theorem from the theory of martingales. This is a consequence of the optional stopping theorem from the theory of martingales. 

**Theorem 10.10.** Define $S_n = \sum_{i=1}^{n} x_i - \mathbb{E}[x_i | \mathcal{F}(i-1)]$. Then $S_1, S_2, \ldots$ is a martingale with respect to the filtration $\{\mathcal{F}(t)\}_{t=0,1,\ldots}$ and the optional stopping theorem implies $\mathbb{E}[S_\tau] = \mathbb{E}[S_1]$. But $\mathbb{E}[S_1] = 0$, so $\mathbb{E}[S_\tau] = 0$, which is exactly equivalent to Equation 16. \qed
Compared to the analysis of SGD in the Euclidean case (e.g., that of [14]) the situation appears slightly more complicated due to the involvement of the duality map, which may be a nonlinear function. This means that even if \( g(t) \) is unbiased in the sense of Assumption 3.7 it may not be the case that \( \mathbb{E}[\rho_w(t)g(t)] = \rho_w(t)(\frac{\partial f}{\partial w}(w(t))) \), and the resulting update directions may be biased. To address this we appeal to some basic properties of duality maps to show that small variance in the derivative estimates lead to only a small bias in the update directions. This is expressed in the following Lemma.

**Lemma 3.10.** Let \( \| \cdot \| \) be any norm on \( \mathbb{R}^n \) and let \( \rho \) be a \( \mathbb{R}^n \)-valued random variable, and assume that \( \mathbb{E}[\delta] = 0 \). Then for any \( k > 0 \),

\[
\mathbb{E}[\ell(\rho(\ell + \delta))] \geq \left( 1 - \frac{k}{2} \right) \| \ell \|^2 - \left( 1 + \frac{1}{2k} \right) \mathbb{E}[\| \delta \|^2].
\]

The proof of this lemma is in the appendix. Using this result, we can proceed to our analysis of SGD. This theorem gives some conditions on \( \epsilon \) and \( \gamma \) that guarantee finiteness of the expected amount of time to reach a \( \gamma \)-approximate stationary point.

**Lemma 3.11.** Let Assumptions 3.4, 3.7 and 3.8 hold. Consider the sequence \( w(1), w(2), \ldots \) generated by Algorithm 3.2 using constant step-sizes \( \epsilon(t) := \epsilon > 0 \). Choose the step-size \( \epsilon \) and positive constants \( k_1, k_2 \) such that

\[
\left( 1 - \frac{k_1}{2} - \frac{L}{2}\epsilon(1 + k_2) \right) \gamma - \left( \frac{L}{2}\epsilon(1 + \frac{1}{k_2}) + 1 + \frac{1}{2k_1} \right) \sigma^2 > 0.
\]

Define the initial optimality gap \( G \) as \( G = f(w(1)) - f^* \). Then if \( \tau \) is defined as in Equation (19),

\[
\mathbb{E}[\tau] \leq \frac{G + \epsilon(1 - \frac{k_1}{2} - \frac{L}{2}\epsilon(1 + k_2)) \gamma - \epsilon(\frac{L}{2}\epsilon(1 + \frac{1}{k_2}) + 1 + \frac{1}{2k_1}) \sigma^2}{\epsilon(1 - \frac{k_1}{2} - \frac{L}{2}\epsilon(1 + k_2)) \gamma - \epsilon(\frac{L}{2}\epsilon(1 + \frac{1}{k_2}) + 1 + \frac{1}{2k_1}) \sigma^2}.
\]

**Proof.** By Assumption 3.4 we know that

\[
f(w(t + 1)) \leq f(w(t)) - \epsilon \frac{\partial f}{\partial w}(w(t)) \cdot \rho_w(t) g(t) + \frac{L}{2} \epsilon^2 \| g(t) \|^2_{w(t)}.
\]

Using the definition of \( \delta(t) \) given in Equation (15), this is equivalent to

\[
f(w(t + 1)) \leq f(w(t)) - \epsilon \frac{\partial f}{\partial w}(w(t)) \cdot \rho_w(t) \left( \frac{\partial f}{\partial w}(w(t)) + \delta(t) \right) + \frac{L}{2} \epsilon^2 \left\| \frac{\partial f}{\partial w}(w(t)) + \delta(t) \right\|^2_{w(t)}.
\]

Summing (19) over \( t = 1, 2, \ldots, N \) yields

\[
f(w(N + 1)) \leq f(w(1)) - \sum_{t=1}^{N} \epsilon \frac{\partial f}{\partial w}(w(t)) \cdot \rho_w(t) \left( \frac{\partial f}{\partial w}(w(t)) + \delta(t) \right)
\]

\[
+ \frac{L}{2} \sum_{t=1}^{N} \epsilon^2 \left\| \frac{\partial f}{\partial w}(w(t)) + \delta(t) \right\|^2_{w(t)}.
\]

Rearranging terms, and noting that \( f(w(N + 1)) \geq f^* \),

\[
\sum_{t=1}^{N} \epsilon \frac{\partial f}{\partial w}(w(t)) \cdot \rho_w(t) \left( \frac{\partial f}{\partial w}(w(t)) + \delta(t) \right) \leq G + \frac{L}{2} \sum_{t=1}^{N} \epsilon^2 \left\| \frac{\partial f}{\partial w}(w(t)) + \delta(t) \right\|^2_{w(t)}.
\]
According to Lemma 3.10 in the appendix, for any $k_1 > 0$ and for all $t$,  
\[
E \left[ \frac{\partial f}{\partial w}(w(t)) \cdot \rho_{w(t)} \left( \frac{\partial f}{\partial w}(w(t)) + \delta(t) \right) \mid F(t-1) \right] \geq \left( 1 - \frac{k_1}{2} \right) \left\| \frac{\partial f}{\partial w}(w(t)) \right\|_{w(t)}^2 - \left( 1 + \frac{1}{2k_1} \right) \sigma^2. \tag{22}
\]

Next, pick any $k_2 > 0$. By Young’s inequality, for all numbers $a, b$, it holds that $|a + b|^2 \leq (1 + k_2)a^2 + (1 + \frac{1}{k_2})b^2$, and in particular,
\[
E \left[ \left\| \frac{\partial f}{\partial w}(w(t))(w(t)) + \delta(t) \right\|_{w(t)}^2 \mid F(t-1) \right] \leq (1 + k_2) \left\| \frac{\partial f}{\partial w}(w(t)) \right\|_{w(t)}^2 + \left( 1 + \frac{1}{k_2} \right) \sigma^2. \tag{23}
\]

For $n \geq 1$ define the stopping time $\tau \wedge n$ to be the minimum of $\tau$ and the constant value $n$. Applying Proposition 3.9 and inequality (22), we have
\[
E \left[ \sum_{t=1}^{\tau \wedge n} \frac{\partial f}{\partial w}(w(t)) \cdot \rho_{w(t)} \left( \frac{\partial f}{\partial w}(w(t)) + \delta(t) \right) \right] \geq \left( 1 - \frac{k_1}{2} \right) \left\| \frac{\partial f}{\partial w}(w(t)) \right\|_{w(t)}^2 - \left( 1 + \frac{1}{2k_1} \right) \sigma^2 \right]. \tag{24}
\]

Applying Proposition 3.9 a second time, in this case to inequality (23), we see that
\[
E \left[ \sum_{t=1}^{\tau \wedge n} \frac{\partial f}{\partial w}(w(t)) + \delta(t) \right] \leq \left( 1 + k_2 \right) \left\| \frac{\partial f}{\partial w}(w(t)) \right\|_{w(t)}^2 + \left( 1 + \frac{1}{k_2} \right) \sigma^2 \right]. \tag{25}
\]

Combining (21) with (24) and (25) and rearranging terms,
\[
E \left[ \sum_{t=1}^{\tau \wedge n} \epsilon \left( 1 - \frac{k_1}{2} - \frac{L}{2} \epsilon(1 + k_2) \right) \left\| \frac{\partial f}{\partial w}(w(t)) \right\|_{w(t)}^2 \right] \leq G + \left( 1 + k_2 \right) \left\| \frac{\partial f}{\partial w}(w(t)) \right\|_{w(t)}^2 + \left( 1 + \frac{1}{k_2} \right) \sigma^2 \right]. \tag{26}
\]

Next, note that
\[
\sum_{t=1}^{\tau \wedge n} \epsilon \left( 1 - \frac{k_1}{2} - \frac{L}{2} \epsilon(1 + k_2) \right) \left\| \frac{\partial f}{\partial w}(w(t)) \right\|_{w(t)}^2 \geq \sum_{t=1}^{\tau \wedge n-1} \epsilon \left( 1 - \frac{k_1}{2} - \frac{L}{2} \epsilon(1 + k_2) \right) \left\| \frac{\partial f}{\partial w}(w(t)) \right\|_{w(t)}^2 \tag{27}
\]
\[
= \sum_{t=1}^{\tau \wedge n-1} \epsilon \left( 1 - \frac{k_1}{2} - \frac{L}{2} \epsilon(1 + k_2) \right) \gamma.
\]

Combining (26) with (27) yields
\[
\epsilon \left( 1 - \frac{k_1}{2} - \frac{L}{2} \epsilon(1 + k_2) \right) \gamma E[(\tau \wedge n) - 1] \leq G + \epsilon \left( \frac{L}{2} \epsilon(1 + \frac{1}{k_2}) + 1 + \frac{1}{2k_1} \right) E[\tau \wedge n] \sigma^2.
\]

Under our assumption on $\epsilon, k_1, k_2$, this can be rearranged into
\[
E[\tau \wedge n] \leq \frac{G + \epsilon \left( 1 - \frac{k_1}{2} - \frac{L}{2} \epsilon(1 + k_2) \right) \gamma}{\epsilon \left( 1 - \frac{k_1}{2} - \frac{L}{2} \epsilon(1 + k_2) \right) \gamma - \epsilon \left( \frac{L}{2} \epsilon(1 + \frac{1}{k_2}) + 1 + \frac{1}{2k_1} \right) \sigma^2}.
\]

Since the right-hand side of this equation is independent of $n$, the claimed inequality (18) follows by the monotone convergence theorem.

Next, we obtain a more concrete bound using some specific choices of $k_1, k_2$. 

Corollary 3.12. Let Assumptions 3.4, 3.7, and 3.8 hold, and let the step-sizes be constant, of the form \( \epsilon = \alpha \frac{2}{L} \) for some \( \alpha \in (0, 1) \). Then for any \( \gamma \) such that \( \gamma > \frac{13}{(1-\alpha)^2} \sigma^2 \), the expected time to reach a \( \gamma \)-approximate stationary point is bounded by

\[
\mathbb{E}[\tau] < \frac{4LG + \gamma}{4\alpha(1-\alpha)\left[\gamma - \frac{13}{(1-\alpha)^2} \sigma^2\right]}.
\]

Proof. The claimed inequality follows from Theorem 3.5, by setting the constants \( k_1, k_2 \) both equal to the value \( k = \frac{1-\alpha}{1+2\alpha} \). See the appendix for details.

4 Application to Neural Networks with Multiple Layers

In any application of the DSGD methodology there are three tasks:

1. Define the Finsler structure for the space,
2. Identify a duality structure to use,
3. Verify the Lipschitz-like condition of Assumption 3.4.

In this section we carry out these steps in the context of a neural network with multiple layers.

We first define the parameter space and the objective function. The network consists of an input layer and \( K \) non-input layers. We are going to consider the case that each layer is fully connected to the previous one and uses the same activation function, but this is only for ease of exposition. Networks with heterogeneous layer types (consisting for instance of convolutional layers, smooth types of pooling, softmax layers, etc.) and networks with biases at each layer can also be accommodated in our theory.

Let the input to the network be of dimensionality \( n_0 \), and let \( n_1, \ldots, n_K \) specify the number of nodes in each of \( K \) non-input layers. For \( k = 1, \ldots, K \) define \( W_k = \mathbb{R}^{n_k \times n_{k-1}} \) to be the space of \( n_k \times n_{k-1} \) matrices; a matrix in \( w_k \in W_k \) specifies weights from nodes in layer \( k-1 \) to nodes in layer \( k \). The overall parameter space is then \( W = W_1 \times \ldots \times W_K \). We define the output of the network as follows. For an input \( y \in \mathbb{R}^{n_0} \), and weights \( w = (w_1, \ldots, w_K) \in W \), the output is \( x^K(w; y) \in \mathbb{R}^{n_K} \) where \( x^0(w; y) = y \) and for \( 1 \leq k \leq K \),

\[
x^k_i(w; y) = \sigma\left(\sum_{j=1}^{n_{k-1}} w_{k,i,j} x^{k-1}_j(w; y)\right), \quad i = 1, 2, \ldots, n_k.
\]

Given \( m \) input/output pairs \((y_1, z_1), (y_2, z_2), \ldots, (y_m, z_m)\), where \((y_n, z_n) \in \mathbb{R}^{n_0} \times \mathbb{R}^{n_K}\), we seek to minimize the empirical error

\[
f(w) = \frac{1}{m} \sum_{i=1}^{m} f_i(w)
\]

where the \( f_i \) for \( i = 1, \ldots, m \) are

\[
f_i(w) = \|x^K(w; y_i) - z_i\|_2^2.
\]

and \( \| \cdot \|_2 \) is the Euclidean norm.
4.1 Layer-wise Lipschitz-gradient property for Neural Networks

Our assumptions on the nonlinearity $\sigma$, the inputs $y_i$, and the targets $z_i$, are as follows:

**Assumption 4.1.**

i. (Nonlinearity bounds) $|\sigma|_\infty \leq 1$, $|\sigma'|_\infty < \infty$, and $|\sigma''|_\infty < \infty$.

ii. (Input/Target bounds) $|y_i|_\infty \leq 1$ and $|z_i|_\infty \leq 1$ for $i = 1, 2, \ldots, m$.

The first assumption means that the activation function only takes values between $-1$ and $1$, and its first and second derivatives are also globally bounded. For example, this is satisfied by the sigmoid function $\sigma(u) = \frac{1}{1 + e^{-u}}$ and also the hyperbolic tangent function $\sigma(u) = \frac{e^u - e^{-u}}{e^u + e^{-u}} - 1$. The second assumption is that the components of the inputs and targets are between $-1$ and $1$.

In Proposition 4.3 we establish that, under these assumptions, the restriction of the objective function to the weights in any particular layer is a function with a bounded second derivative. To confirm the boundedness of the second derivative, any norm on the weight matrices can be used, because on finite dimensional spaces all norms are strongly equivalent. However, different norms will lead to different specific bounds. For the purposes of gradient descent, each pair of a norm and corresponding Lipschitz bound implies a different quadratic upper bound on the objective, and a potentially different update step, leading to a different algorithm. Our construction considers the induced matrix norms corresponding to the norms $\| \cdot \|_q$ for $q \in \{2, \infty\}$.

**Assumption 4.2.** Choose a $q \in \{2, \infty\}$. For each $i = 1, 2, \ldots, K$, the space $\mathbb{R}^n_i$ has the norm $\| \cdot \|_q$.

We are going to be working with the matrix norm induced by the given choice of $q$. For an $r \times c$ matrix $A$, the possibilities are

$$\|A\|_q = \begin{cases} \max_{1 \leq i \leq \min\{r,c\}} \sigma_i(A) & \text{if } q = 2, \\ \max_{1 \leq i \leq r} \sum_{j=1}^{c} |A_{i,j}| & \text{if } q = \infty, \end{cases}$$

where for a matrix $A$, we define $\sigma(A) = (\sigma_1(A), \ldots, \sigma_{\min\{r,c\}}(A))$ to be the vector of singular values of $m$. That is, when $q = 2$ the norm is the largest singular value, also known as the spectral norm, and when $q = \infty$ the norm is the largest absolute row sum [10].

For ease of notation, in the following proposition and throughout this section, we will assume that all the layers have the same number of nodes. Formally, this means $n_i = n_K$ for $i = 0, \ldots, K$. The general case can be handled in a very similar manner and is left to the reader. Also, we use the following notation for the composition of a bilinear map with a pair of linear maps: if $B : U \times U \to V$ is a bilinear map then $B(A_1 \oplus A_2)$ is the bilinear map which sends $(z_1, z_2)$ to $B(A_1z_1, A_2z_2)$. Recall that if $B : U \times U \to V$ is a bilinear map, then for any $(u_1, u_2) \in U \times U$ the inequality $\|B(u_1, u_2)\|_V \leq \|B\|_q \|u_1\|_U \|u_2\|_U$ holds. It follows that if $A_1 : Z \to U$ and $A_2 : Z \to U$ are any linear maps, then

$$\|B(A_1 \oplus A_2)\|_q \leq \|B\|_q \|A_1\|_q \|A_2\|_q. \quad (30)$$

**Proposition 4.3.** Let Assumptions 4.1 and 4.2 hold, and let $q$ be the constant chosen in Assumption 4.2. Let the spaces $W_1, \ldots, W_K$ have the norm induced by $\| \cdot \|_q$ and define functions $p_i$ as follows: Let $r_0 = 1$, and for $1 \leq n \leq K - 1$ the function $r_n$ is

$$r_n(z_1, \ldots, z_n) = |\sigma'|_\infty^n \prod_{i=1}^{n} z_i.$$

Then define $v_n$ recursively, with $v_0 = 0$, $v_1(z_1) = |\sigma''|_\infty z_1^2$, and for $2 \leq n \leq K - 1$, the function $v_n$ is

$$v_n(z_1, \ldots, z_n) = |\sigma''|_\infty z_n^2 |\sigma'|_\infty^{(n-1)} \prod_{i=1}^{n-1} z_i^2 + |\sigma''|_\infty z_n v_{n-1}(z_1, \ldots, z_{n-1}).$$
Define constants $d_{q,1}$, $d_{q,2}$ and $c_q$ as in Table 1. Then for $0 \leq n \leq K - 1$ the function $s_n$ is

$$s_n(z_1, \ldots, z_i) = d_{q,2}c_q^2\|\sigma'\|^2\|z_1^2 + \cdots + z_i^2\|v_i(z_1, \ldots, z_i) + d_{q,1}c_q^2\|\sigma''\|\|r_i(z_1, \ldots, z_i)

The $p_1, \ldots, p_K$ are then

$$p_i(w) = \sqrt{s_{K-1}(\|w_{i+1}\|_q, \ldots, \|w_K\|_q) + 1.}$$

Let $f$ be defined as in [28]. Then for all $w \in W$ and $1 \leq i \leq K$, the bound $\|\frac{\partial^2 f}{\partial x^2}(w)\|_q \leq p_i(w)^2$ holds.

Proof. It suffices to consider the case of a single input/output pair $(y, z) \in \mathbb{R}^n \times \mathbb{R}^n$. In this case, we can express the function $f$ as

$$f(w) = J(x^K(y, w))$$

where $J(x) = \|x - z\|^2$ is the squared distance of a state $x$ to the target $z$ and $x^K(y, w)$ is the output of a $K$-layer neural network with input $y$. The output $x^K$ is defined recursively as

$$x^i(y, w) = \begin{cases} h(x^{i-1}(y, w_{i-1}), w_i) & \text{if } 2 \leq i \leq K, \\ h(y, w_1) & \text{if } i = 1, \end{cases}$$

where the function $h(x, w)$ represents the computation performed by a single layer in the network:

$$h_k(x, w) = \sigma \left( \sum_{j=1}^n w_{k,j}x_j \right), \quad k = 1, 2, \ldots, n.$$ (34)

Taking the second derivative of [28] with respect to the weights $w_i$ for $1 \leq i \leq K$, we find that

$$\frac{\partial^2 f}{\partial w_i^2} = \frac{\partial^2 J}{\partial x^2}(x^K(y, w)) \left( \frac{\partial x^K}{\partial w_i}(y, w) + \frac{\partial x^K}{\partial w_{i-1}}(y, w) \right) + \frac{\partial J}{\partial x}(x^K(y, w)) \frac{\partial^2 x^K}{\partial w_i^2}(y, w).$$ (35)

To find formulas for bounds on these terms we will use the following identity: for $0 \leq k \leq i$,

$$x^i(y, w_{1:i}) = x^{i-k}(x^K(y, w_{1:K}), w_{K+1:i})$$

(36)

with the convention that $x^0(y) = y$. Differentiating Equation (36), with respect to $w_i$ for $1 \leq i \leq K$ gives

$$\frac{\partial x^K}{\partial w_i}(y, w_{1:K}) = \frac{\partial x^{K-i}}{\partial y}(x^i(y, w_{1:i}), w_{i+1:K}) \frac{\partial h}{\partial w}(x^{i-1}(y, w_{1:i-1}), w_i)$$

(37)

and differentiating a second time yields

$$\frac{\partial^2 x^K}{\partial w_i^2}(y, w_{1:K}) = \left( \frac{\partial^2 x^{K-i}}{\partial y^2}(x^i(y, w_{1:i}), w_{i+1:K}) \left( \frac{\partial h}{\partial w}(x^{i-1}(y, w_{1:i-1}), w_i) \right) + \frac{\partial x^{K-i}}{\partial y}(x^i(y, w_{1:i}), w_{i+1:K}) \frac{\partial^2 h}{\partial w^2}(x^{i-1}(y, w_{1:i-1}), w_i) \right)$$

(38)
Next, we consider the terms $\frac{\partial x^n}{\partial y}$ and $\frac{\partial^2 x^n}{\partial y^2}$ appearing in the two preceding equations (37), (38). By differentiating equation (33) with respect to the input parameter, we have, for any input $u$ and parameters $a_1, a_2, \ldots, a_n$,

$$\frac{\partial x^n}{\partial y}(u, a_{1:n}) = \frac{\partial h}{\partial x}(x^{n-1}(u, a_{1:n-1}), a_n) \frac{\partial x^{n-1}}{\partial y}(u, a_{1:n-1}),$$  \hspace{1cm} (39)

and upon differentiating a second time,

$$\frac{\partial^2 x^n}{\partial y^2}(u, a_{1:n}) = \frac{\partial^2 h}{\partial x^2}(x^{n-1}(u, a_{1:n}), a_n) \left( \frac{\partial x^{n-1}}{\partial y}(u, a_{1:n-1}) \odot \frac{\partial x^{n-1}}{\partial y}(u, a_{1:n-1}) \right) + \frac{\partial h}{\partial x}(x^{n-1}(u, a_{1:n-1}), a_n) \frac{\partial^2 x^{n-1}}{\partial y^2}(u, a_{1:n-1}).$$  \hspace{1cm} (40)

We will use some bounds on $h$ in terms of the norm $\|\cdot\|_q$. It follows from Lemma A.3 in the appendix that the following bounds hold for any $q \in \{2, \infty\}$:

$$\left\| \frac{\partial h}{\partial x}(x, w) \right\|_q \leq \|\sigma\|_\infty \|w\|_q, \hspace{1cm} \left\| \frac{\partial h}{\partial w}(x, w) \right\|_q \leq \|\sigma\|_\infty \|x\|_q,$$

$$\left\| \frac{\partial^2 h}{\partial x^2}(x, w) \right\|_q \leq \|\sigma''\|_\infty \|w\|_q^2, \hspace{1cm} \left\| \frac{\partial^2 h}{\partial y^2}(x, w) \right\|_q \leq \|\sigma''\|_\infty \|x\|_q^2.$$  \hspace{1cm} (41)

Combining (39) with (41) we obtain the following inequalities: For $n > 1$,

$$\left\| \frac{\partial x^n}{\partial y}(u, a_{1:n}) \right\|_q \leq \left\{ \begin{array}{ll} \|\sigma\|_\infty \|a_n\|_q \left\| \frac{\partial x^{n-1}}{\partial y}(u, a_{1:n-1}) \right\|_q & \text{if } n > 1, \\
\|\sigma\|_\infty \|a_1\|_q & \text{if } n = 1. 
\end{array} \right.$$  \hspace{1cm} (42)

Combining the two cases in inequality (42), and using the definition of $r_n$ we find that, for $n \geq 1$,

$$\left\| \frac{\partial x^n}{\partial y}(u, a_{1:n}) \right\|_q \leq r_n(\|a_1\|_q, \ldots, \|a_n\|_q).$$  \hspace{1cm} (43)

Now we turn to the second derivative $\frac{\partial^2 x^n}{\partial y^2}$. Taking norms in Equation (40), and applying (41), (43), and (38), we obtain the following inequalities:

$$\left\| \frac{\partial^2 x^n}{\partial y^2}(u, a_{1:n}) \right\|_q \leq \left\{ \begin{array}{ll} \|\sigma''\|_\infty \|a_n\|_q^2 \|\sigma''\|_\infty^{2(n-1)} \prod_{i=1}^{n-1} \|a_i\|_q^2 + \|\sigma\|_\infty \|a_n\|_q \left\| \frac{\partial^2 x^{n-1}}{\partial y^2}(u, a_{1:n-1}) \right\|_q & \text{if } n > 1, \\
\|\sigma''\|_\infty \|a_1\|_q^2 & \text{if } n = 1. 
\end{array} \right.$$  \hspace{1cm} (44)

By definition of $v_n$, then, for all $n > 0$,

$$\left\| \frac{\partial^2 x^n}{\partial y^2}(u, a_{1:n}) \right\|_q \leq v_n(\|a_1\|_q, \ldots, \|a_n\|_q).$$  \hspace{1cm} (45)
where the number \( c_q \), defined in Table \( \text{1} \), is the \( q \)-norm of the \( n_k \)-dimensional vectors of \( 1 \)'s \( (1, 1, \ldots , 1) \).

Combining \( \text{38}, \text{41}, \text{43}, \) and \( \text{44} \),

\[
\left\| \frac{\partial ^2 f}{\partial w_i^2}(y, w_{1:K}) \right\|_q \leq \left\| \frac{\partial ^2 f}{\partial x^2}(x^i(y, w_{1:K})) \right\|_q + d_{q,1} \left\| \frac{\partial ^2 f}{\partial w_i^2}(y, w) \right\|_q
\]

Now we arrive at bounding the derivatives of the function \( f \). As shown in Lemma \( \text{A.6} \) in the appendix, the following inequalities hold:

\[
\sup_{w,y} \left\| \frac{\partial J}{\partial x}(x^i(y, w)) \right\|_q \leq d_{q,1}, \quad (47a)
\]

\[
\sup_{w,y} \left\| \frac{\partial ^2 J}{\partial x^2}(x^i(y, w)) \right\|_q = d_{q,2}. \quad (47b)
\]

where \( d_{q,1} \) and \( d_{q,2} \) are as in Table \( \text{1} \). Combining \( \text{35}, \text{45}, \text{46}, \text{47a} \) and \( \text{47b} \), it holds that for \( i = 1, \ldots , K \),

\[
\left\| \frac{\partial ^2 f}{\partial w_i^2}(y, w_{1:K}) \right\|_q \leq d_{q,2} \left\| \frac{\partial f}{\partial x}(x^i(y, w)) \right\|_q^2 + d_{q,1} \left\| \frac{\partial ^2 f}{\partial w_i^2}(y, w) \right\|_q
\]

\[
\leq d_{q,2} c_q^2 \|\sigma\|_\infty r_0^2 (\|w_i+1\|_q, \ldots , \|w_K\|_q)
+ d_{q,1} c_q^2 \|\sigma\|_\infty r_0 (\|w_i+1\|_q, \ldots , \|w_K\|_q)
+ d_{q,1} c_q^2 \|\sigma''\|_\infty r_0 (\|w_i+1\|_q, \ldots , \|w_K\|_q)
= s_{K-i}(\|w_i+1\|_q, \ldots , \|w_K\|_q)
\]

\[
< p_i(w)^2.
\]

For example, in a network with one hidden layer \( (K = 2) \), the two polynomials \( p_1, p_2 \) are

\[
p_1(w) = \sqrt{s_1(\|w_2\|_q) + 1}
= \sqrt{d_{q,2} c_q^2 \|\sigma\|_\infty r_1^2 (\|w_2\|_q)^2 + d_{q,1} c_q^2 \|\sigma\|_\infty v_1 (\|w_2\|_q) + d_{q,1} c_q^2 r_1 (\|w_2\|_q) + 1} \quad (48a)
\]

\[
p_2(w) = \sqrt{s_0 + 1}
= \sqrt{d_{q,2} c_q^2 \|\sigma\|_\infty r_0^2 + d_{q,1} c_q^2 \|\sigma\|_\infty v_0 + d_{q,1} c_q^2 \|\sigma''\|_\infty r_0 + 1}
\quad (48b)
\]

\[
= \sqrt{c_q^2 (d_{q,2} \|\sigma\|_\infty + d_{q,1} \|\sigma''\|_\infty + 1)}.
\]

This theorem enables us to analyze algorithms that update only one layer at a time. Specifically, if we update any one of the layers in the direction of the image of the gradient under the duality map, then using a small enough step-size guarantees improvement in the objective function. This is a consequence of the following Lemma:
**Lemma 4.4.** Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function with continuous derivatives up to 2nd order. Let $\|\cdot\|$ be an arbitrary norm on $\mathbb{R}^n$ and let $\rho$ be a duality map for this norm. Suppose that 
\[ \sup_{w}, \sup_{\|u_1\|=\|u_2\|=1} \left\| \frac{\partial^2 f}{\partial w^2}(w) \cdot (u_1, u_2) \right\| \leq L. \]
Then for any $\epsilon > 0$ and any $\Delta \in \mathbb{R}^n$, $f(w - \epsilon \Delta) \leq f(w) - \epsilon \frac{\partial f}{\partial w}(w) \cdot \Delta + \epsilon^2 \frac{L}{2} \|\Delta\|^2$. In particular, $f \left( w - \epsilon \rho \left( \frac{\partial f}{\partial w}(w) \right) \right) \leq f(w) - \epsilon \left( 1 - \frac{\epsilon}{2} \right) \left\| \frac{\partial f}{\partial w}(w) \right\|^2.$

**Proof.** Let for any $w \in \mathbb{R}^n$, $\Delta \in \mathbb{R}^n$ and let $\epsilon > 0$. Applying the fundamental theorem of calculus, first on the function $f$ and then on its derivative, we have

\[
 f(w - \epsilon \Delta) = f(w) - \epsilon \int_0^1 \frac{\partial f}{\partial w}(w - \lambda \epsilon \Delta) \cdot \Delta \, d\lambda 
= f(w) - \epsilon \int_0^1 \left[ \frac{\partial f}{\partial w}(w) \cdot \Delta - \epsilon \int_0^\lambda \frac{\partial^2 f}{\partial w^2}(w - u \epsilon \Delta) \cdot (\Delta, \Delta) \, du \right] \, d\lambda 
= f(w) - \epsilon \frac{\partial f}{\partial w}(w) \cdot \Delta + \epsilon^2 \int_0^\lambda \int_0^\lambda \frac{\partial^2 f}{\partial w^2}(w + u \epsilon \Delta) \cdot (\Delta, \Delta) \, du \, d\lambda. 
\]

Using our assumption on the second derivative,

\[
 \leq f(w) - \epsilon \frac{\partial f}{\partial w}(w) \cdot \Delta + \epsilon^2 \frac{L}{2} \|\Delta\|^2. 
\]

Letting $\Delta = \rho(\frac{\partial f}{\partial w}(w))$, and using the two defining equations of duality maps (9a, 9b), we see that

\[
 \leq f(w) - \epsilon \left\| \frac{\partial f}{\partial w}(w) \right\|^2 + \epsilon^2 \frac{L}{2} \left\| \frac{\partial f}{\partial w}(w) \right\|^2. 
\]

Combining the terms yields the result. \hfill \square

With this lemma in mind, consider the following greedy algorithm. Identify a layer $i^*$ such that $i^* = \arg \max_{1 \leq i \leq K} \frac{1}{p_i(w)} \left\| \frac{\partial f}{\partial w_i}(w) \right\|$ and then make an update of parameter $w_{i^*}$, using a step-size $\frac{1}{p_{i^*}(w)^{2}}$ in the direction $\rho(\frac{\partial f}{\partial w_i}(w))$. Then as a consequence of Proposition 4.3 and Lemma 4.4 this update will lead to a decrease in the objective of at least $\frac{1}{2p_{i^*}(w)^{2}} \|\frac{\partial f}{\partial w_{i^*}}(w)\|^2$. This greedy algorithm is depicted (in a slightly generalized form) in Algorithm 4. In the remainder of this section, we will show how this sequence of operations can be explained with a particular Finsler duality structure on $\mathbb{R}^n$, in order to apply the convergence theorems of Section 3.

### 4.2 Finsler structure and duality structure

In this section we define a Finsler structure and an associated duality structure. The Finsler structure is defined in terms of the functions $p_i$ from (31) as follows. For any $w = (w_1, \ldots, w_K) \in W$ and any $(\delta w_1, \ldots, \delta w_K) \in W$, define $\| (\delta w_1, \ldots, \delta w_K) \|_w$ as

\[
\| (\delta w_1, \ldots, \delta w_K) \|_w = p_1(w) \| \delta w_1 \|_q + \ldots + p_K(w) \| \delta w_K \|_q. 
\]

Note that the Finsler structure and the polynomials $p$ depend on the user-supplied parameter $q$ from Assumption 4.2, although we omit this from the notation.

To obtain the duality structure, we derive duality maps for matrices with the norm $\|\cdot\|_q$, and then use a general construction for product spaces. The first part is summarized in the following Proposition. Note that when we use the arg max to find the index of the largest entry of a vector, any tie-breaking rule can be used in case there are multiple maxima. For instance, the arg max may be defined to return the smallest such index.
Proposition 4.5. Let $\ell \in \mathcal{L}(\mathbb{R}^{r \times c}, \mathbb{R})$ be a linear functional defined on a space of matrices with the norm $\|\cdot\|_q$ for $q \in \{2, \infty\}$. Then the dual norm is

$$\|\ell\|_q = \begin{cases} \min \left\{ r, c \right\} \sum_{i=1}^{\min \{r, c\}} \sigma_i(\ell) & \text{if } q = 2, \\ \sum_{i=1}^{r} \max_{1 \leq j \leq c} |\ell_{i,j}| & \text{if } q = \infty. \end{cases} \quad (50)$$

That is, for $q = 2$ the dual norm is the sum of the singular values, and when $q = \infty$ it is the sum of the largest entries in each row.

Possible choices for duality maps are as follows. For $q = 2$, a duality map is $\rho_2$, which replaces all the singular values of $\ell$ with 1: If $\ell = U\Sigma V^T$ is the singular value decomposition of $\ell$, written in terms of column vectors as $U = [u_1, \ldots, u_c], V = [v_1, \ldots, v_c]$, and denoting the rank of the matrix $\ell$ by $\text{rank} \, \ell$, then

$$\rho(\ell)_2 = \|\ell\|_2 \sum_{i=1}^{\text{rank} \, \ell} u_i v_i^T. \quad (51)$$

For $q = \infty$, a duality map is $\rho_\infty$, which sends $\ell$ to a matrix that picks out a maximum in each row:

$$\rho_\infty(\ell) = \|\ell\|_\infty m \text{ where } m \text{ is the } r \times c \text{ matrix } m_{i,j} = \begin{cases} \text{sgn}(\ell_{i,j}) & \text{if } j = \arg \max_{1 \leq k \leq c} |\ell_{i,k}|, \\ 0 & \text{otherwise.} \end{cases} \quad (52)$$

The proof of this proposition is in the appendix.

Next, we construct a duality map for a product space from duality maps on the components. Recall that in a product vector space $Z = X_1 \times \ldots \times X_K$ each linear functional $\ell \in \mathcal{L}(Z, \mathbb{R})$ uniquely decomposes as $\ell = (\ell_1, \ldots, \ell_K) \in \mathcal{L}(X_1, \mathbb{R}) \times \ldots \times \mathcal{L}(X_K, \mathbb{R})$.

Proposition 4.6. If $X_1, \ldots, X_K$ are normed spaces, carrying duality maps $\rho_{X_1}, \ldots, \rho_{X_K}$ respectively, and the product $Z = X_1 \times \ldots \times X_K$ has norm $\| (x_1, \ldots, x_K) \|_Z = p_1 \| x_1 \|_{X_1} + \ldots + p_K \| x_K \|_{X_K}$, for some positive coefficients $p_1, \ldots, p_K$, then the dual norm for $Z$ is

$$\| (\ell_1, \ldots, \ell_K) \|_Z = \max \left\{ \frac{1}{p_1} \| \ell_1 \|_{X_1}, \ldots, \frac{1}{p_K} \| \ell_K \|_{X_K} \right\} \quad (53)$$

and a duality map for $Z$ is given by

$$\rho_Z((\ell_1, \ldots, \ell_K)) = \left( (0, \ldots, \frac{1}{(p_{i_r})^*} \rho_{X_i}((\ell_{i_r}), \ldots, 0) \right) \text{ where } i_r^* = \arg \max_{1 \leq i \leq K} \left\{ \frac{1}{p_i} \| \ell_i \|_{X_i} \right\}. \quad (54)$$

See the appendix for a proof of Proposition 4.6. Based on Proposition 4.6 and the definition of the Finsler structure from [49], the dual norm at a point $w \in W = W_1 \times \ldots \times W_K$ is

$$\| (\ell_1, \ldots, \ell_K) \|_W = \max_{1 \leq i \leq K} \frac{1}{p_i(w)} \| \ell_i \|_q. \quad (55)$$

We define the Finsler duality structure on the neural net parameter space as follows:

1. Each space $W_1, \ldots, W_K$ has the duality map $\rho_q(\cdot)$, defined according to [52].
2. The duality map at each point $w$ is defined according to Proposition 4.6.

$$\rho_w((\ell_1, \ldots, \ell_K)) = \left( (0, \ldots, \frac{1}{(p_{i_r}(w))^*} \rho_q(\ell_{i_r}), \ldots, 0) \right) \text{ where } i_r^* = \arg \max_{1 \leq i \leq K} \left\{ \frac{1}{p_i(w)} \| \ell_i \|_q \right\}. \quad (56)$$
4.3 Lipschitz condition

Next, we must show that the Finsler structure is compatible with our objective function, by verifying the Lipschitz-like condition of Assumption 3.4.

**Proposition 4.7.** Let Assumptions 3.1 and 3.2 hold, and let \( q \) be the constant chosen in Assumption 4.2. Then Assumption 3.4 is satisfied with \( L = 1 \).

**Proof.** Let \( w \in W \) and \( \eta \in L(W, \mathbb{R}) \) be arbitrary. Let \( i^* = \arg \max_{1 \leq i \leq K} \left\{ \frac{\lambda}{p_i(w)} \| \eta_i \|_q \right\} \). Then \( \rho_w(\eta) \) is of the form \( \rho_w(\eta) = (0, \ldots, \Delta_i, \ldots, 0) \), where \( \Delta_i \in W_i \) is \( \Delta_i = \frac{1}{p_i(w)} p_i(\eta_i) \). Applying Taylor’s theorem, it holds that

\[
\begin{align*}
\frac{\partial f}{\partial w}(w + \epsilon \rho_w(\eta)) \cdot \rho_w(\eta) + \epsilon^2 \int_0^1 \frac{\partial^2 f}{\partial w^2} (w + \epsilon \rho_w(\eta)) \cdot (\rho_w(\eta), \rho_w(\eta)) \, du \, d\lambda.
\end{align*}
\]

The only components of \( \rho_w(\eta) \) that are potentially non-zero are those corresponding to layer \( i^* \). Then

\[
\begin{align*}
\frac{\partial^2 f}{\partial w_i^2} (w + \epsilon \rho_w(\eta)) \cdot (\Delta_i, \Delta_i) \leq p_{i^*} (w + \epsilon \rho_w(\eta))^2 \| \Delta_i \|_q^2.
\end{align*}
\]

Since the function \( p_{i^*} \) only depends on the weights in layers \((i^* + 1), (i^* + 2), \ldots, K\),

\[
p_{i^*} (w + \epsilon \rho_w(\eta)) = p_{i^*} (w).
\]

By the definition of the dual norm \( \| \cdot \|_q \),

\[
p_{i^*} (w^*) \| \Delta_i \|_q = \| \eta \|_w.
\]

By combining Equations (57) - (61), then,

\[
\left| f(w + \epsilon \rho_w(\eta)) - f(w) + \epsilon \frac{\partial f}{\partial w} (w) \cdot \rho_w(\eta) \right| \leq \epsilon^2 \frac{1}{2} \| \eta \|_w^2.
\]

Now that Assumption 3.4 has been established, we can proceed to the analysis of batch gradient descent.

4.4 Batch gradient descent

The pseudocode for the optimization procedure is shown in Algorithm 4.1. Each iteration starts on Line 4 by computing the derivatives of the objective function. This is a standard back-propagation step. Next, on Line 8 for each layer \( i \) the polynomials \( p_i \) and the \( q \)-norms of the derivatives \( g_i \) are computed. Note that for any \( i < K \), computing \( p_i \) will involve computing the matrix norms \( ||w_{i+1}||_q, \ldots, ||w_K||_q \). In Line 9 we identify which layer \( i \) has the largest value of \( ||g_i(t)||_q/p_i(w(t)) \). This is equivalent to maximizing \( ||g_i(t)||_q^2/2p_i(w(t))^2 \), which is exactly the lower bound guaranteed by Lemma 4.4. Having chosen the layer, in Lines 10 through 13 we perform the update of layer \( i^* \), and keep all others fixed.

The convergence analysis of the algorithm, formalized in Proposition 4.8, is based on the abstract results from the previous section. The idea is that the update performed in the algorithm is exactly equivalent to taking a step in the direction of the duality map \( \rho \) as applied to the derivative of \( f \), so the algorithm is simply a special case of Algorithm 3.1.
Algorithm 4.1: Duality structure gradient descent for a multi-layer neural network

1. **input**: Parameter $q \in (2, \infty)$, training data $(y_i, z_i)$ for $1 \leq i \leq m$, initial point $w(1) \in W$, step-size $\epsilon$, selection of mode Batch or Stochastic, and batch-size $b$ (only required for Stochastic mode.)

2. **for** $t = 1, 2, \ldots$ **do**

3. **if** Mode = Batch **then**

4. Compute full derivative $g(t) = \frac{\partial f}{\partial w}(w(t))$.

5. **else if** Mode = Stochastic **then**

6. Compute mini-batch derivative $g(t) = \frac{1}{b} \sum_{j \in B(t)} \frac{\partial f}{\partial w}(w(t))$.

7. **end**

8. Compute dual norms $\frac{1}{p_1(w(t))} \|g_1(t)\|_q, \ldots, \frac{1}{p_K(w(t))} \|g_K(t)\|_q$. (Using (51) and (50))

9. Select layer to update: $i^* = \arg\max_{1 \leq i \leq K} \frac{1}{p_i(w(t))} \|g_i(t)\|_q$.

10. Update $w(t + 1)_i = w(t)_i - \epsilon \frac{1}{p_{i^*}(w(t))} \rho_q(g_i(t))$. (Using (51) or (52))

11. **for** $i \in \{1, 2, \ldots, K\} \setminus \{i^*\}$ **do**

12. Copy previous parameter: $w(t + 1)_i = w(t)_i$.

13. **end**

14. **end**

Proposition 4.8. Let the function $f$ be defined as in (28), let Assumptions 4.1 and 4.2 hold, and let $q$ be the constant chosen in Assumption 4.2. Give $W$ the Finsler structure (49) and duality structure (50). Starting from an initial point $w(1) \in W$, consider the sequence $w(t)$ generated by Algorithm 4.1 running in batch mode, using constant step-sizes $\epsilon(t) := \epsilon \in (0, 2)$. Then the sequence $w(t)$ is guaranteed to satisfy one of the conditions 3.5 or 3.5. In particular, $\min_{1 \leq t \leq T} \|\frac{\partial f}{\partial w}(w(t))\|_{w(t)} \leq \delta$ when $T \geq \frac{1}{\sqrt{\epsilon}} \frac{2}{2 - \epsilon} f(1)$.

Proof. The update performed in Algorithm 4.1 running in batch mode can be expressed in the form $w(t + 1) = w(t) - \epsilon \Delta(t)$, where $\epsilon = 1$ and $\Delta(t) = \rho_w(t) \left( \frac{\partial f}{\partial w}(w(t)) \right)$, for $\rho_w$ the duality structure (50) corresponding to Finsler structure (49). Hence it is a special case of Algorithm 3.1. We have established Assumption 3.4 in Section 4.3, and the result follows by Theorem 3.5 using $L = 1$ and $f^* = 0$.

To get some intuition for this convergence bound, note that the local derivative norm can be lower bounded as

$$\left\| \frac{\partial f}{\partial w}(w(t)) \right\|_{w(t)} \geq \max_{1 \leq i \leq K} \frac{\left\| \frac{\partial f}{\partial w}(w(t)) \right\|_q}{p_i(w(t))} \geq \frac{\sum_{i=1}^{K} \left\| \frac{\partial f}{\partial w}(w(t)) \right\|_q}{K \sum_{j=1}^{K} p_j(w(t))}.$$ 

Therefore, if we choose a step-size $\epsilon = 1$, then as consequence of the convergence bound, if $T \geq \frac{1}{\sqrt{\epsilon}} \frac{2}{2 - \epsilon} f(1)$.
where
\[ \frac{2}{\tau^2} f(w(1)) \] then
\[ \frac{\sum_{i=1}^{K} \left\| \frac{\partial f}{\partial w_i}(w(t)) \right\|_q}{K \sum_{j=1}^{K} p_j(w(t))} \leq \delta. \]

In this inequality, the term on the left-hand side is the magnitude of the gradient relevant to a fixed norm independent of the weights \( w \), divided by a term that is an increasing function of the weight norms \( \|w(t)\| \).

**Remark 4.9.** Note that in our analysis of DSGD for neural networks, it is important that the abstract theory is not constrained to update schemes based on inner-product norms, as is the case in Riemannian gradient descent. In our case, the Finsler structure on the parameter space is defined so that the corresponding duality structure generates updates that are confined to a single layer. This feature will not be present in the duality map for any inner product norm, since the duality map for an inner product norm is always a linear function (see Equation 10.) More explicitly, suppose that \( \ell_1 \) and \( \ell_2 \) are linear functionals and \( \rho \) is a duality map for an inner product norm. If \( \rho(\ell_1) \) has non-zero components in only the first layer, and \( \rho(\ell_2) \) only has non-zero components in the second layer, then, due to linearity, \( \rho(\ell_1 + \ell_2) = \rho(\ell_1) + \rho(\ell_2) \) has non-zero components in both layers.

### 4.5 Stochastic gradient descent

Let us consider the setting of mini-batch stochastic gradient descent. This corresponds to executing the steps of Algorithm 4.1 and instead of computing the full derivative at each iteration, as is done in batch mode, approximate derivatives are calculated by averaging the gradient of our loss function over some number of randomly selected instances in our training set. Formally, this is expressed in Line 6 of the Algorithm 4.1. We represent \( b \) randomly chosen instances as a random subset \( B(t) \subseteq \{1, \ldots, m\}^b \) and the gradient estimate \( g(t) \) is
\[ g(t) = \frac{1}{b} \sum_{j \in B(t)} \frac{\partial f}{\partial w}(w(t)). \] (62)

We first show that this gradient estimate has a uniformly bounded variance relative to our Finsler structure.

**Proposition 4.10.** Let Assumptions 4.1 and 4.2 hold and let \( q \) be the constant chosen in Assumption 4.2. Let \( g(t) \) be as in (62) and define \( \delta(t) = \frac{\partial f}{\partial w}(w(t)) - g(t) \). Then the variance of \( g(t) \) is bounded as
\[ E \left[ \|\delta(t)\|_w^2 \right] \leq \frac{c}{b}. \] (63)

where
\[ c = 32\nu_1 K^2. \] (64)

**Proof.** Define the norm \( \|\cdot\|_{1,q} \) on \( \mathcal{L}(W, \mathbb{R}) = \mathcal{L}(W_1 \times \ldots \times W_K, \mathbb{R}) \) as
\[ \|(\ell_1, \ldots, \ell_K)\|_{1,q} = \max_{1 \leq i \leq K} \|\ell_i\|_q. \] (65)

For each \( w \in W \) there is a linear map \( A(w(t)) \) on \( W \) such that the norm \( \|\cdot\|_{w(t)} \) on the dual space \( \mathcal{L}(W, \mathbb{R}) \) can be represented as
\[ \|\ell\|_{w(t)} = \|A(w(t))\ell\|_{1,q}. \] (66)

This can be deduced from inspecting the formula (55). Although not material for our further arguments, \( A(w(t)) \) is a block-structured matrix, with coefficients \( A(w(t))_{i,j} = 0 \) whenever \( i, j \) correspond to parameters in separate layers, and \( A(w(t))_{i,j} = \frac{1}{p_{k}(w)} \) when \( i, j \) are weights in layer \( k \).
In general, if \( q \in \{2, \infty\} \) and \( A \) is an \( n_K \times n_K \) matrix, then
\[
\frac{1}{\sqrt{n_K}} \|A\|_2 \leq \|A\|_q \leq \sqrt{n_K} \|A\|_2. \tag{67}
\]
Also, the Frobenius norm on \( n_K \times n_K \) matrices satisfies
\[
\|A\|_2 \leq \|A\|_F \leq \sqrt{n_K} \|A\|_2. \tag{68}
\]
Combining (67) and (68), then,
\[
\frac{1}{n_K} \|A\|_F \leq \|A\|_q \leq n_K \|A\|_F. \tag{69}
\]
It follows from (69) and Proposition A.3 in the appendix that for any linear functional \( \ell \in L(\mathbb{R}^{n_K \times n_K}, \mathbb{R}) \),
\[
\frac{1}{n_k} \|\ell\|_F \leq \|\ell\|_q \leq n_K \|\ell\|_F. \tag{70}
\]
Inequality (70), together with the definition (65), means that for any \( \ell \in L(W, \mathbb{R}) \),
\[
\frac{1}{n_K} \max_{1 \leq i \leq K} \|\ell_i\|_F \leq \|\ell\|_{1,q} \leq n_K \max_{1 \leq i \leq K} \|\ell_i\|_F. \tag{71}
\]
For any vector \( u \) in \( \mathbb{R}^K \) we have,
\[
\frac{1}{\sqrt{K}} \|u\|_2 \leq \|u\|_\infty \leq \sqrt{K} \|u\|_2. \tag{72}
\]
Combining (71) and (72) implies that for all \( \ell \in L(W, \mathbb{R}) \),
\[
\frac{1}{n_K \sqrt{K}} \|\ell\|_2 \leq \|\ell\|_{1,q} \leq n_K \sqrt{K} \|\ell\|_2. \tag{73}
\]
Let \( k_3 = n_K \sqrt{K} \). Then
\[
\mathbb{E} \left[ \| \delta(t) \|_{w(t)}^2 | \mathcal{F}(t-1) \right] = \mathbb{E} \left[ \| A(w(t)) \delta(t) \|_{1,q}^2 | \mathcal{F}(t-1) \right] \quad \text{(by (66))}
\leq k_3^2 \mathbb{E} \left[ \| A(w(t)) \delta(t) \|_2^2 | \mathcal{F}(t-1) \right] \quad \text{(by (73))}
\leq \frac{1}{b^3} k_3^3 \mathbb{E} \left[ \left\| A(w(t)) \left( \frac{\partial f}{\partial w}(w(t)) - \frac{\partial f_i}{\partial w}(w(t)) \right) \right\|_2^2 | \mathcal{F}(t-1) \right]
\leq \frac{1}{b^3} k_3^3 \mathbb{E} \left[ \left\| A(w(t)) \left( \frac{\partial f_i}{\partial w}(w(t)) - \frac{\partial f_i}{\partial w}(w(t)) \right) \right\|_{1,q}^2 | \mathcal{F}(t-1) \right]. \quad \text{(by (73))}
\]
In the third step, we used the fact that \( b \) items in a mini-batch reduces the (Euclidean) variance by a factor of \( b \) compared to using a single instance, which we have represented with the random index \( i \in \{1, 2, \ldots, m\} \).

Applying the Equation (66) once more, this yields
\[
\mathbb{E} \left[ \| \delta(t) \|_{w(t)}^2 | \mathcal{F}(t-1) \right] \leq \frac{1}{b^3} k_3^3 \mathbb{E} \left[ \left\| \frac{\partial f_i}{\partial w}(w(t)) - \frac{\partial f_i}{\partial w}(w(t)) \right\|_{w(t)}^2 | \mathcal{F}(t-1) \right]. \tag{74}
\]
Next, observe that for any pair \( i, j \in \{1, 2, \ldots, m\} \),
\[
\left\| \frac{\partial f_j}{\partial w}(w(t)) - \frac{\partial f_i}{\partial w}(w(t)) \right\|_{w(t)}^2 \leq 2 \left( \left\| \frac{\partial f_j}{\partial w}(w(t)) \right\|_{w(t)}^2 + \left\| \frac{\partial f_i}{\partial w}(w(t)) \right\|_{w(t)}^2 \right). \tag{75}
\]
Applying the chain rule to the function \(f_i\) as defined in Equation (29), and using Inequalities (47a), (45), we see that for all \(w, i, k\),

\[
\left\| \frac{\partial f_i}{\partial w_k}(w) \right\|_q \leq d_{q,1} \left\| \frac{\partial \bar{f}_K}{\partial w_k}(y_i; w) \right\|_q \\
\leq d_{q,1}r_{K-k}(\|w_{k+1}\|_q, \ldots, \|w_K\|_q)\|\sigma'||\|\|_\infty \ c_q
\]

(76)

Using the definition of \(p_k\) from (31), then for any \(w, i, k\),

\[
\left\| \frac{\partial f_i}{\partial w_k}(w) \right\|_q \leq \frac{d_{q,1}}{\sqrt{d_{q,2}}} p_k(w) = \sqrt{8nK} p_k(w).
\]

(77)

Note that the equality in (77) follows from the definitions in Table 1; for each \(q \in \{2, \infty\}\), it holds that \(d_{q,1}/\sqrt{d_{q,2}} = \sqrt{8nK}\). Combining (77) with the definition of the dual norm at \(w\) (55)

\[
\left\| \frac{\partial f_i}{\partial w}(w(t)) \right\|_{w(t)} \leq \sqrt{8nK}
\]

(78)

Using (74) and (75) together with (78),

\[
\mathbb{E} \left[ \|\delta(t)\|_{w(t)}^2 | \mathcal{F}(t-1) \right] \leq \frac{4}{b} k_3^4 8nK = \frac{32}{b} n_K^5 K^2.
\]

This confirms Equation (63).

Now that we have established a bound on the variance of the gradient estimates \(g(t)\), we can proceed to the performance guarantee for stochastic gradient descent.

**Proposition 4.11.** Let the function \(f\) be defined as in (28), let Assumptions 4.1 and 4.2 hold let \(q\) be the constant chosen in Assumption 4.2. Give \(W\) the Finsler structure (49) and duality structure (56). Given an initial point \(w(1) \in W\), consider the sequence \(w(t)\) generated by Algorithm 3.2 in stochastic mode, with a batch size \(b\) and constant step-sizes \(\epsilon = \alpha^2\) for some \(\alpha \in (0, 1)\). Set \(\sigma^2 = \frac{32}{b} n_K^5 K^2\). Then for any \(\gamma \geq \frac{13}{(1-\alpha)^2}\sigma^2\), it is the stopping time

\[
\tau = \inf \left\{ t \geq 1 \left| \left\| \frac{\partial f}{\partial w}(w(t)) \right\|_{w(t)}^2 \leq \gamma \right. \right\},
\]

(79)

it holds that

\[
\mathbb{E}[\tau] < \frac{4f(w(1)) + \gamma}{4\alpha(1-\alpha) \left\| \gamma - \frac{13}{(1-\alpha)^2}\sigma^2 \right.}
\]

Proof. Assumption 3.4 was established in Section 4.3. Assumption 3.7 is satisfied since we are using mini-batches to define the gradient estimates. Assumption 3.8 follows by Proposition 4.10. The result follows from Corollary 3.12 using \(L = 1\) and \(f^* = 0\).

In this section we established convergence guarantees for both batch training, in Proposition 4.8, and mini-batch training, in Proposition 4.11. In the next section we investigate the performance of the algorithm on several benchmark datasets.
5 Numerical Experiment

The previous section established convergence guarantees for DSGD and SDSGD (Theorems 4.8 and 4.11). In this section we investigate the practical efficiency of DSGD and SDSGD with numerical experiments on several machine learning benchmark problems. We compare three algorithms: Two DSGD variants, corresponding to using the 2-norms \( \| \cdot \|_\infty \) and standard Euclidean gradient descent (EGD). We also compare full batch and mini-batch variants of all the algorithms.

The problems considered were the minimization of the empirical error in the MNIST [22], SVHN [27], and CIFAR-10 [19] image classification tasks. The architecture of the networks, depicted in Figure 2, were as follows. In each case, the network had one hidden layer, meaning \( K = 2 \). The hidden layer had \( n_1 = 300 \) units, and the output layer had \( n_2 = 10 \) units (one for each class). For the MNIST experiment, the input size was \( n_0 = 784 \) (the dimensionality of a \( 28 \times 28 \) greyscale image), and, for the SVHN and CIFAR-10 experiments we had \( n_0 = 3072 \) (the input is a \( 32 \times 32 \) image with 3 input channels.) The nonlinearity used in all the experiments was the sigmoid function \( \sigma(x) = \frac{1}{1 + e^{-x}} \). For all three datasets, the objective function is defined as in Equation (28). The \( m \) in Eqn. (28) is defined as \( m = 60,000 \) for MNIST, \( m = 50,000 \) for CIFAR, and \( m = 73,257 \) in the SVHN experiment. In all cases, a training pair \((y_n, t_n)\) consists of an image and a 10 dimensional indicator vector that indicates the label for the image.

The details of the DSGD procedure are shown in Algorithm 4.1. When \( q = 2 \), computing the polynomials \( p_1 \) involves computing the spectral norm of the weight-matrix \( w_2 \), and computing the magnitude \( g_1, g_2 \) uses the norm dual to the spectral norm, as defined in the first case of Equation (50). When \( q = \infty \), the norm of \( w_2 \) is computed using the matrix \( \infty \)-norm, and computing the magnitude of \( g_1 \) and \( g_2 \) uses the dual norm as defined in the second case of (50).

For EGD, the step-sizes were determined experimentally. For each dataset, we picked 10,000 random samples. We then ran gradient descent on this random subset, for each choice of \( \epsilon \in \{0.001, 0.01, 0.1, 1, 10\} \), and evaluated the training error at the end of the optimization period (500 epochs for stochastic EGD, 20,000 updates for full-batch). The step-size that gave the smallest training error was used for the full experiments. This gives a set of 6 step-sizes, one for each dataset and training modality (batch or stochastic).

For all the algorithms, the network weights were initialized using uniformly distributed random variables. In the batch experiments, the algorithm ran for 20,000 weight updates. In the stochastic
Figure 3: These figures show how frequently each layer was updated in one run of the DSGD algorithm, for the case of batch training on the MNIST dataset. The graphs show the running count of the number of updates by layer. The graph on the left indicates that in the DSGD-2 algorithm, the first layer is updated more frequently than the second layer. The graph on the right shows that in the beginning stages of the DSGD-∞ algorithm, the updates are confined to the second layer.

The DSGD algorithm only updates a single layer at a time, depending on which layer $i$ has the largest value of $\| \frac{\partial L}{\partial w} \|_{q}/p_i(w)$. We have tracked the choice of layer across optimization, and this is depicted in Figure 3 for the particular case of batch training on the MNIST dataset. The result for the DSGD-2 algorithm is on the left of Figure 3 and the result for DSGD-∞ is on the right. The graphs show the running count of how many times each layer has been updated. For instance, the graph on the left suggests that each layer is updated at roughly constant rates, and that the frequency of first layer updates is larger than frequency of second layer updates. The graph on the right suggest a similar dynamic when the DSGD-∞ algorithm is used, except that at the early stage of the algorithm the second layer is updated more frequently. In the run from which this graph was generated, all of the updates during the first 3008 iterations were in the second layer.

Figure 4 shows the evolution of the training error (left column) and testing accuracy (right column) across the three different datasets, for one run of full-batch optimization. The same quantities are plotted for mini-batch optimization in Figure 5.

On the MNIST dataset, the 2-norm based DSGD performs similarly to EGD, in terms of both training and testing error. The ∞-norm based updates are much slower. In the SVHN dataset, the EGD algorithm resulted in both smaller training error and better accuracy. On the CIFAR dataset, The Euclidean GD resulted in a better training error, but we can clearly see overfitting in the testing curve, and the 2-norm based DSGD has a better test error.

The experiments were repeated 5 times, and the statistics of the final training and testing errors and accuracies are reported in Table 2 for the batch case and Table 3 for SGD. For each dataset and each criteria we have highlighted the best performing algorithm. Also highlighted are any other algorithms with results within one standard deviation of the optimum.

Surprisingly, the rigorous DSGD methods led to results that were competitive with the EGD in some cases. This suggest that DSGD may be a promising approach for rigorous training of more sophisticated architectures.
6 Discussion

This work was motivated by the desire for an optimization method for neural networks with theoretical guarantees. Our starting point was the observation that the empirical error function for a multilayer network has a Layer-wise Lipschitz property. We showed how a greedy algorithm that updates one layer at a time can be explained with a geometric interpretation involving Finsler structures. Different variants of the algorithm can be generated by varying the underlying norm on the state-space, and the choice of norm can have a significant impact on the practical efficiency.

Our abstract algorithmic framework, Finsler Gradient Descent, can in some cases provide non-asymptotic performance guarantees while making less restrictive assumptions compared to vanilla gradient descent. In particular, the analysis does not assume that the objective function has a Lipschitz gradient. The class of functions that the method applies to includes neural networks with
Figure 5: These figures show a comparison of stochastic DSGD with stochastic Euclidean gradient descent with constant step sizes, during 110,000 steps of optimization. Each figure plots the value of the empirical error Eqn. (28) for the four different optimization algorithms.

Although it was expected that the method would yield step sizes that were too conservative to be competitive with vanilla gradient descent, this turned out not to be the case. We believe this is because our framework is better able to integrate problem structure as compared to naive Euclidean gradient descent. When designing our Finsler structure, a good deal of problem information was used, such as the hierarchical structure of the network, bounds on various derivatives, and bounds on the input.

The theoretical guarantees we obtained concerned the objective over the training data. However, in practice the quantity that is of interest is the test error. Extending this framework to address generalization would be of great interest. Secondly, our framework required bounds on the first and
Table 3: Results of mini-batch training experiments on several datasets. Each entry is of the form $\mu \pm \sigma$, where $\mu$ and $\sigma$ the mean and standard deviation, respectively, of the statistic computed over 5 repetitions of the experiment.

<table>
<thead>
<tr>
<th>Dataset</th>
<th>Algorithm</th>
<th>Training error $\mu$</th>
<th>Training accuracy $\mu$</th>
<th>Test error $\sigma$</th>
<th>Test accuracy $\sigma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>MNIST</td>
<td>EGD</td>
<td>0.073 ± 0.0002</td>
<td>0.959 ± 0.0003</td>
<td>0.084 ± 0.0004</td>
<td>0.951 ± 0.0007</td>
</tr>
<tr>
<td></td>
<td>DSGD-2</td>
<td>0.059 ± 0.0003</td>
<td>0.973 ± 0.0001</td>
<td>0.084 ± 0.0002</td>
<td>0.955 ± 0.0005</td>
</tr>
<tr>
<td></td>
<td>DSGD-∞</td>
<td>0.508 ± 0.0060</td>
<td>0.800 ± 0.0058</td>
<td>0.501 ± 0.0063</td>
<td>0.809 ± 0.0055</td>
</tr>
<tr>
<td>SVHN</td>
<td>EGD</td>
<td>0.588 ± 0.0408</td>
<td>0.459 ± 0.0455</td>
<td>0.684 ± 0.0436</td>
<td>0.397 ± 0.0439</td>
</tr>
<tr>
<td></td>
<td>DSGD-2</td>
<td>0.670 ± 0.0024</td>
<td>0.484 ± 0.0020</td>
<td>0.745 ± 0.0046</td>
<td>0.421 ± 0.0078</td>
</tr>
<tr>
<td></td>
<td>DSGD-∞</td>
<td>0.850 ± 0.0014</td>
<td>0.337 ± 0.0044</td>
<td>0.869 ± 0.0023</td>
<td>0.305 ± 0.0073</td>
</tr>
<tr>
<td>CIFAR</td>
<td>EGD</td>
<td>0.255 ± 0.1326</td>
<td>0.806 ± 0.0895</td>
<td>0.871 ± 0.0344</td>
<td>0.403 ± 0.0352</td>
</tr>
<tr>
<td></td>
<td>DSGD-2</td>
<td>0.507 ± 0.0010</td>
<td>0.669 ± 0.0011</td>
<td>0.675 ± 0.0018</td>
<td>0.481 ± 0.0038</td>
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<tr>
<td></td>
<td>DSGD-∞</td>
<td>0.771 ± 0.0012</td>
<td>0.416 ± 0.0017</td>
<td>0.777 ± 0.0009</td>
<td>0.403 ± 0.0026</td>
</tr>
</tbody>
</table>

Table 2: Results of full batch training experiments on several datasets. Each entry is of the form $\mu \pm \sigma$, where $\mu$ and $\sigma$ the mean and standard deviation, respectively, of the statistic computed over 5 repetitions of the experiment.

second derivatives of the activation function. This assumption would not hold for certain activation functions, including the Rectified Linear function $x \mapsto \max\{0, x\}$, and smooth variants thereof. Hence relaxing the assumption on the derivatives of the activation function would also be of interest.

Appendix

**Lemma A.1.** Let $\|\cdot\|_w$ be a Finsler structure on $\mathbb{R}^n$ and define the dual norm at each point $w$ as in [8]. Then the map $(w, \ell) \mapsto \|\ell\|_w$ is also a continuous function on $\mathbb{R}^n \times \mathcal{L}(\mathbb{R}^n, \mathbb{R})$.

Two auxiliary results will be used to prove Lemma A.1

**Proposition A.2.** Let $\|\cdot\|_w$ be a Finsler structure on $\mathbb{R}^n$, and let $\|\cdot\|$ be an arbitrary norm. Then for any $w_0 \in \mathbb{R}^n$ and any $\epsilon > 0$ we can find a $\delta(w_0, \epsilon) > 0$ such that whenever $\|w_1 - w_0\| < \delta(w_0, \epsilon)$, then the norms $\|\cdot\|_w$ and $\|\cdot\|_{w_0}$ satisfy

$$
\frac{1}{1 + \epsilon} \|w_1\| \leq \|w_0\| \leq (1 + \epsilon) \|w_1\|.
$$

(80)
Proof. Fix a \(\gamma \in (0, 1)\). Since the function \((w, u) \mapsto \|u\|_w\) is continuous, for each \(u \in \mathbb{R}^n\) such that \(\|u\|_{\|u\|_w} \leq 1\) we can find an neighborhood around \((w_0, u)\) of the form \(W_u \times U_u\) where \(W_u \subseteq \mathbb{R}^n, U_u \subseteq \mathbb{R}^n\) are open subsets, such that \(\|u'\|_w - \|u\|_w \leq \gamma\) for all \((w', u') \in W_u \times B_u\).

Note that the collection of open sets \(\{u \in \mathbb{R}^n \mid \|u\|_{\|u\|_w} \leq 1\}\) forms an open cover of the unit ball \(\{u \in \mathbb{R}^n \mid \|u\|_w \leq 1\}\); hence we can extract a finite subcover of the unit ball \(U_{u_1}, U_{u_2}, \ldots, U_{u_n}\). Let \(B \subseteq \mathbb{R}^n\) be the open set \(B = \cap_{i=1}^{n} U_{u_i}\); this is an open neighborhood of \(w_0\) since the intersection is finite. Therefore \(B\) must contain a ball of radius \(\delta' > 0\) around \(w_0\). Set \(\delta(w_0, \gamma)\) to be this radius \(\delta'\).

Let \(w \in B\) and let \(u \in \mathbb{R}^n\) be a vector such that \(\|u\|_{\|u\|_w} \leq 1\). Using the open cover \(\{U_{u_i}\}_{i=1}^{n}\), there is some \(u_i\) such that \(u \in U_{u_i}\). Since \(w \in B_{u_i}\), it holds that \(\|u\|_{\|u\|_w} - \|u_i\|_{\|u\|_w} \leq \gamma\), which implies

\[
\|u\|_{\|u\|} \leq \|u_i\|_{\|u\|_w} + \gamma = 1 + \gamma.
\]

Then for any vector \(u \in \mathbb{R}^n\),

\[
\|u\| \leq (1 + \gamma)\|u\|_{\|u\|_w} \tag{81}
\]

By the same reasoning we find that for any \(u \in \mathbb{R}^n\),

\[
\|u\| \geq (1 - \gamma)\|u\|_{\|u\|_w} \tag{82}
\]

Combining (81) and (82), then,

\[
\frac{1}{1 + \gamma}\|u\|_{\|u\|} \leq \|u\|_{\|u\|_w} \leq \frac{1}{1 - \gamma}\|u\|.
\]

The claimed inequality (80) follows by setting \(\gamma = \frac{\epsilon}{1 + \epsilon}\).

\[\square\]

Proposition A.3. Let \(\|\cdot\|_A, \|\cdot\|_B\) be two norms on \(\mathbb{R}^n\), such that for all vectors \(u \in \mathbb{R}^n\) it holds that \(\|u\|_A \leq K\|u\|_B\). Then for any linear functional \(\ell \in \mathcal{L}(\mathbb{R}^n, \mathbb{R})\), it holds that \(\|\ell\|_B \leq K\|\ell\|_A\).

Proof. Given a norm \(\|\cdot\|\) on \(\mathbb{R}^n\), the corresponding dual norm on \(\mathcal{L}(\mathbb{R}^n, \mathbb{R})\) can be expressed as \(\|\ell\| = \sup_{u \neq 0} \frac{|\ell(u)|}{\|u\|\|u\|}\). Using this formula, and the assumption on \(\|\cdot\|_A, \|\cdot\|_B\), then,

\[
\|\ell\|_B = \sup_{u \neq 0} \frac{\ell(u)}{\|u\|_B} \leq \sup_{u \neq 0} \frac{\ell(u)}{\|u\|_A} K = K\sup_{u \neq 0} \frac{\ell(u)}{\|u\|_A} = K\|\ell\|_A.
\]

\[\square\]

Proof of Lemma A.2. Let \((w_0, \ell_0) \in \mathbb{R}^n \times \mathcal{L}(\mathbb{R}^n, \mathbb{R})\) and let \(\epsilon > 0\) be given. We will find numbers \(\delta_1 > 0, \delta_2 > 0\) so that whenever \((w_1, \ell_1)\) are such that \(\|w_1 - w_0\|_2 \leq \delta_1\) and \(\|\ell_1 - \ell\|_{\|u\|_w} \leq \delta_2\), then \(\|\ell_1\|_{\|w_1\|_w} - \|\ell_0\|_{\|w_0\|_w} \leq \epsilon\).

By Proposition A.2, for any \(\epsilon > 0\) we can find a \(\delta(w_0, \epsilon) > 0\) so that for all \(u \in \mathbb{R}^n\) and \(\|w_1 - w_0\|_2 \leq \delta(w_0, \epsilon)\),

\[
\frac{1}{1 + \epsilon}\|u\|_{\|w_1\|} \leq \|u\|_{\|w_0\|} \leq (1 + \epsilon)\|u\|_{\|w_1\|} \tag{83}
\]

Using Proposition A.3, we can derive from equation (83) a similar inequality for the dual norms: for all \(\ell \in \mathcal{L}(\mathbb{R}^n, \mathbb{R})\) and \(\|w_1 - w_0\|_2 \leq \delta(w_0, \epsilon)\),

\[
\frac{1}{1 + \epsilon}\|\ell\|_{\|w_1\|} \leq \|\ell\|_{\|w_0\|} \leq (1 + \epsilon)\|\ell\|_{\|w_1\|} \tag{84}
\]

Equation (84) implies

\[
\|\ell\|_{\|w_1\|} - \|\ell\|_{\|w_0\|} \leq (1 + \epsilon)\|\ell\|_{\|w_0\|} - \|\ell\|_{\|w_0\|} = \epsilon\|\ell\|_{\|w_0\|} \tag{85}
\]

and

\[
\|\ell\|_{\|w_1\|} - \|\ell\|_{\|w_0\|} \geq \frac{1}{(1 + \epsilon)}\|\ell\|_{\|w_0\|} - \|\ell\|_{\|w_0\|} = -\frac{\epsilon}{1 + \epsilon}\|\ell\|_{\|w_0\|} \tag{86}
\]

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Combining (85) and (86), we find that for all \( \ell \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}) \) and \( \|w_1 - w_0\|_2 \leq \delta(w_0, \epsilon) \),
\[
\|\|\ell\|_{w_0} - \|\ell\|_{w_1}\| \leq \max \left\{ \frac{\epsilon}{1 + \epsilon}, \epsilon \right\} \|\ell\|_{w_0} = \epsilon \|\ell\|_{w_0}
\]

Let \( \delta_1 = \delta \left( w_0 , \frac{\epsilon}{2\|w\|_{w_0}} \right) \). and set \( \delta_2 = \frac{\epsilon}{2(1 + \epsilon)} \). Then for all \( \ell_1, w_1 \) such that \( \|w_1 - w_0\|_2 \leq \delta_1 \), and \( \|\ell_1 - \ell_0\|_{w_0} \leq \delta_2 \),
\[
\|\ell_1\|_{w_1} - \|\ell_0\|_{w_0} \leq \|\ell_1 - \ell_0\|_{w_1} + \|\ell_0\|_{w_1} - \|\ell_0\|_{w_0}
\]
\[
\leq (1 + \epsilon) \|\ell_0 - \ell_1\|_{w_0} \leq \frac{\epsilon}{2\|\ell_0\|_{w_0}} \|\ell_0\|_{w_0}.
\]
\[
\leq (1 + \epsilon) \frac{\epsilon}{2(1 + \epsilon)} + \frac{\epsilon}{2}
\]
\[
= \epsilon.
\]

To prove Lemma 3.10 we will use the following auxiliary Lemma, that is essentially a special case of [9] Theorem 4.1] and we include a proof for completeness.

**Lemma A.4 (9).** Let \( \|\cdot\| \) be a norm on \( \mathbb{R}^n \) and let \( \rho \) be a corresponding duality map. Then for any \( \ell_1, \ell_2 \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}) \), it holds that \( \|\ell_1 + \ell_2\|^2 \geq \|\ell_1\|^2 + 2\ell_2(\rho(\ell_1)). \)

**Proof.**
\[
\|\ell_1\|^2 = \ell_1(\rho(\ell_1))
\]
\[
= (\ell_1 + \ell_2)(\rho(\ell_1)) - \ell_2(\rho(\ell_1))
\]
\[
\leq \|\ell_1 + \ell_2\|\|\ell_1\| - \ell_2(\rho(\ell_1))
\]
\[
\leq \frac{1}{2}\|\ell_1 + \ell_2\|^2 + \frac{1}{2}\|\ell_1\|^2 - \ell_2(\rho(\ell_1))
\]

Subtract \( \frac{1}{2}\|\ell_1\|^2 \) from each side of this inequality and multiply by two. This gives
\[
\|\ell_1\|^2 \leq \|\ell_1 + \ell_2\|^2 - 2\ell_2(\rho(\ell_1)).
\]
The result follows after adding \( 2\ell_2(\rho(\ell_1)) \) to each side of this inequality. \( \square \)

**Proof of Lemma 3.10.** First use the definition of duality maps:
\[
\ell(\rho(\ell + \delta)) = (\ell + \delta)(\rho(\ell + \delta)) - \delta(\rho(\ell + \delta)) = \|\ell + \delta\|^2 - \delta(\rho(\ell + \delta)) \tag{87}
\]
Next, according to Lemma A.4 for any norm and duality map, it holds that
\[
\|\ell + \delta\|^2 \geq \|\ell\|^2 + 2\delta(\rho(\ell)) \tag{88}
\]
Furthermore, using Peter-Paul inequality with any \( k > 0 \), we get
\[
|\delta(\rho(\ell + \delta))| \leq \|\delta\||\rho(\ell + \delta)||
\]
\[
= \|\delta\||\ell + \delta||
\]
\[
\leq \|\delta\||\ell||\delta||^2
\]
\[
\leq \frac{1}{2}[k||\ell||^2 + \frac{1}{k}||\delta||^2] + ||\delta||^2
\]

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Combining (87), (88), and (89) yields
\[ \ell (\rho (\ell + \delta)) \geq \|\ell\|^2 + 2\delta \rho (\ell) - k \frac{1}{2} \|\ell\|^2 - \frac{1}{2k} \|\delta\|^2 - 1 \|\delta\|^2 \]
\[ = (1 - \frac{k}{2}) \|\ell\|^2 + 2\delta (\rho (\ell)) - (\frac{1}{2k} + 1) \|\delta\|^2 \]

Taking expectations yields
\[ \mathbb{E} [\ell (\rho (\ell + \delta))] \geq (1 - \frac{k}{2}) \|\ell\|^2 -(1 + \frac{1}{2k}) \mathbb{E} [\|\delta\|^2] \]
where the term \(\delta (\rho (\ell))\) vanishes since we are dealing with an unbiased estimate.

**Proof of Corollary 3.12.** Set \(k_1 = k_2 = k\), where \(k = \frac{1 - \alpha}{1 + 2\alpha}\). Then

\[ \left(1 - \frac{k}{2} - \frac{L}{2} \epsilon(1 + k)\right) \gamma - \left(1 + \frac{1}{2k} + \frac{L}{2} \epsilon(1 + \frac{1}{k})\right) \sigma^2 \geq \frac{1}{2}(1 - \alpha)\gamma - \left(1 + \alpha + \frac{1}{2}(1 + 2\alpha)^2 \frac{1}{1 - \alpha}\right) \sigma^2 \]
\[ \geq \frac{1}{2}(1 - \alpha)\gamma - \left(2 + \frac{9}{2(1 - \alpha)}\right) \sigma^2 \]
\[ > \frac{13}{2(1 - \alpha)} \sigma^2 - \left(2 + \frac{9}{2(1 - \alpha)}\right) \sigma^2 \]
\[ = \frac{1}{2} \sigma^2 \left(\frac{4}{1 - \alpha} - 4\right) \geq 0. \]

Hence the condition (17) is satisfied and Lemma 3.11 applies. Next, observe that
\[ \epsilon \left(1 - \frac{k_1}{2} - \frac{L}{2} \epsilon(1 + k_2)\right) \gamma = \epsilon \frac{1}{2}(1 - \alpha)\gamma \]
and
\[ \epsilon \left(1 - \frac{k}{2} - \frac{L}{2} \epsilon(1 + k)\right) \gamma - \epsilon \left(1 + \frac{1}{2k} + \frac{L}{2} \epsilon(1 + \frac{1}{k})\right) \sigma^2 \geq \epsilon \left[\frac{1}{2}(1 - \alpha)\gamma - \left(2 + \frac{9}{2(1 - \alpha)}\right) \sigma^2\right] \]
\[ = \epsilon \left[\left(1 - \alpha\right)\gamma - \left(4 + \frac{9}{1 - \alpha}\right) \sigma^2\right] \]
\[ \geq \epsilon \frac{1}{2} \left[\left(1 - \alpha\right)\gamma - \frac{13}{1 - \alpha} \sigma^2\right] \]
\[ = \frac{1}{2}(1 - \alpha) \left[\gamma - \frac{13}{(1 - \alpha)^2} \sigma^2\right]. \]

According to the inequality (18), then,
\[ \mathbb{E}[\tau] \leq \frac{G}{\epsilon \frac{1}{2}(1 - \alpha)\gamma} \left[\gamma - \frac{13}{(1 - \alpha)^2} \sigma^2\right] \]
\[ = \frac{LG + \alpha(1 - \alpha)\gamma}{\alpha(1 - \alpha) \left[\gamma - \frac{13}{(1 - \alpha)^2} \sigma^2\right]} \]
\[ \leq \frac{4LG + \gamma}{4\alpha(1 - \alpha) \left[\gamma - \frac{13}{(1 - \alpha)^2} \sigma^2\right]}. \]

The second inequality follows from the definition of \(\epsilon\), and the last inequality is due to the fact that \(\alpha(1 - \alpha) < 1/4\).
Lemma A.5. Let $h : \mathbb{R}^n \times \mathbb{R}^{n \times n} \to \mathbb{R}^n$ be the function defined by Equation (34). Let $\mathbb{R}^n$ have the norm $\| \cdot \|_q$ for $q \in \{2, \infty\}$ and put the induced norm on the matrices $W$. Then the following inequalities hold:

$$
\left\| \frac{\partial h}{\partial x} (x,w) \right\|_q \leq \|\sigma'\|_\infty \|w\|_q, \\
\left\| \frac{\partial h}{\partial w} (x,w) \right\|_q \leq \|\sigma'\|_\infty \|x\|_q,
$$

$$
\left\| \frac{\partial^2 h}{\partial x^2} (x,w) \right\|_q \leq \|\sigma''\|_\infty \|w\|_2^2, \\
\left\| \frac{\partial^2 h}{\partial w^2} (x,w) \right\|_q \leq \|\sigma''\|_\infty \|x\|_2^2.
$$

Proof. Define the pointwise product of two vectors as $u \odot v = (u_1 v_1, \ldots, u_n v_n)$. We will rely on the following two properties that are shared by the norms $\| \cdot \|_q$ for $q \in \{2, \infty\}$. Firstly, for all vectors $u,v$,

$$
\|u \odot v\|_q \leq \|u\|_q \|v\|_q. \tag{90}
$$

Secondly, for any diagonal matrix $D$,

$$
\|D\|_q = \max_i |D_{i,i}|. \tag{91}
$$

Recall that the component functions of $h$ are $h_i(x,w) = \sigma \left( \sum_{k=1}^n w_{i,k} x_k \right)$. Then

$$
\frac{\partial h_i}{\partial x_j} (x,w) = \sigma' \left( \sum_{k=1}^n w_{i,k} x_k \right) w_{i,j}.
$$

Set $D(x,w)$ to be the diagonal matrix

$$
D(x,w)_{i,i} = \sigma' \left( \sum_{k=1}^n w_{i,k} x_k \right). \tag{92}
$$

Then $\frac{\partial h}{\partial x} (x,w) = D(x,w)w$. Hence

$$
\left\| \frac{\partial h}{\partial x} (x,w) \right\|_q = \|D(x,w)w\|_q \\
\leq \|D(x,w)\|_q \|w\|_q \\
= \sup_i \|D(x,w)_{i,i}\| \|w\|_q \\
\leq \|\sigma'\|_\infty \|w\|_q
$$

Observe that

$$
\frac{\partial h_i}{\partial w_{j,i}} (x,w) = \begin{cases} 
\sigma' \left( \sum_{k=1}^n w_{i,k} x_k \right) x_l & \text{if } j = i, \\
0 & \text{else},
\end{cases}
$$

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Let $\Delta$ be a matrix such that $\|\Delta\|_q = 1$. Then $\frac{\partial h}{\partial w}(x, w) \cdot \Delta$ is a vector with $i$th component
\[
\left( \frac{\partial h}{\partial w}(x, w) \cdot \Delta \right)_i = \sum_{j,l} \frac{\partial h_i}{\partial w_{j,l}}(x, w) \Delta_{j,l}
\]
\[
= \sum_l \frac{\partial h_i}{\partial w_{i,l}}(x, w) \Delta_{i,l}
\]
\[
= \sum_l \sigma' \left( \sum_{k=1}^n w_{i,k} x_k \right) x_l \Delta_{i,l}
\]
\[
= \sigma' \left( \sum_{k=1}^n w_{i,k} x_k \right) \sum_l x_l \Delta_{i,l}
\]
\[
= \sigma' \left( \sum_{k=1}^n w_{i,k} x_k \right) \sum_l \Delta_{i,l} x_l
\]
\[
= (D(x, w) \Delta x)_i
\]
where $D$ is as in (92). Hence
\[
\left\| \frac{\partial h}{\partial w}(x, w) \cdot \Delta \right\|_q = \|D(x, w)\Delta x\|_q
\]
\[
\leq \|D(x, w)\|_q \|\Delta\|_q \|x\|_q \quad \text{(submultiplicative property of matrix norm)}
\]
\[
= \sup_i \|D(x, w)_{i,i}\| \|\Delta\|_q \|x\|_q \quad \text{(absolute norm)}
\]
and therefore
\[
\left\| \frac{\partial h}{\partial w}(x, w) \right\|_q \leq \|\sigma'\|_\infty \|x\|_q.
\]

Observe that
\[
\frac{\partial^2 h}{\partial x_j \partial x_\ell}(x, w) = \sigma'' \left( \sum_{k=1}^n w_{i,k} x_k \right) w_{i,j} w_{i,\ell}.
\]

That means $\frac{\partial^2 h}{\partial x_j \partial x_\ell}(x, w) \cdot (u, v)$ is a vector with components
\[
\left( \frac{\partial^2 h}{\partial x_j \partial x_\ell}(x, w) \cdot (u, v) \right)_i = \sum_{j=1}^n \sum_{\ell=1}^n \sigma'' \left( \sum_{k=1}^n w_{i,k} x_k \right) w_{i,j} w_{i,\ell} u_j v_\ell
\]
\[
= \sigma'' \left( \sum_{k=1}^n w_{i,k} x_k \right) \left( \sum_{j=1}^n w_{i,j} u_j \right) \left( \sum_{\ell=1}^n w_{i,\ell} v_\ell \right)
\]
\[
= E(x, w)_{i,i} (Wu)_i (Wv)_i
\]
where $E$ is the diagonal matrix
\[
E(x, w)_{i,i} = \sigma'' \left( \sum_{k=1}^n w_{i,k} x_k \right).
\]

Using the notation $\odot$ for the entry-wise product of vectors, then
\[
\frac{\partial^2 h}{\partial x^2}(x, w) \cdot (u, v) = E(x, w) \cdot ((wu) \odot (wv)).
\]
Then

\[ \left\| \frac{\partial^2 h}{\partial x^2}(x, w) \cdot (u, v) \right\|_q \leq \left\| E(x, w) \right\|_q \left\| \sigma'' \right\|_\infty \left\| x \right\|_q^2 \]

\[ \leq \left\| E(x, w) \right\|_q \left\| \sigma'' \right\|_\infty \left\| u \right\|_q \left\| v \right\|_q \]

Hence

\[ \left\| \frac{\partial^2 h}{\partial x^2}(x, w) \right\|_q \leq \left\| \sigma'' \right\|_\infty \left\| w \right\|_q^2 \]

Observe that

\[ \frac{\partial^2 h_i}{\partial w_j \partial w_{k,m}}(x, w) = \begin{cases} \sigma'' \left( \sum_{k=1}^n w_{i,k} x_k \right) x_l x_m & \text{if } j = i \text{ and } k = i, \\ 0 & \text{else.} \end{cases} \]

Hence for matrices \( u, v \), \( \frac{\partial^2 h}{\partial w^2}(x, w) \cdot (u, v) \) is a vector with entries

\[ \left( \frac{\partial^2 h_i}{\partial w^2}(x, w) \cdot (u, v) \right)_i = \sum_{j,l} \sum_{k,m} \frac{\partial^2 h_i}{\partial w_j \partial w_{k,m}}(x, w) u_{j,l} v_{k,m} \]

where \( E \) is as in (93). Hence

\[ \frac{\partial^2 h}{\partial w^2}(x, w) \cdot (u, v) = E(x, w) \cdot ((ux) \odot (vx)) \]

which means

\[ \left\| \frac{\partial^2 h}{\partial w^2}(x, w) \cdot (u, v) \right\|_q \leq \left\| E(x, w) \right\|_q \left\| ux \right\|_q \left\| vx \right\|_q \]

\[ \leq \left\| E(x, w) \right\|_q \left\| u \right\|_q \left\| v \right\|_q \left\| x \right\|_q^2 \]

Then

\[ \left\| \frac{\partial^2 h}{\partial w^2}(x, w) \cdot (u, v) \right\|_q \leq \left\| \sigma'' \right\|_\infty \left\| x \right\|_q^2. \]

\[ \square \]

**Lemma A.6.** Let \( z \in \mathbb{R}^n \) and define the function \( J : \mathbb{R}^n \to \mathbb{R} \) as

\[ J(x) = \sum_{i=1}^n (x_i - z_i)^2 \]
Then for all $x, z$ in $\mathbb{R}^n$,
\[
\left\| \frac{\partial J}{\partial x} (x) \right\|_2 = 2 \| x - z \|_2, \quad \left\| \frac{\partial J}{\partial x} (x) \right\|_\infty = 2 \| x - z \|_1,
\]
and
\[
\left\| \frac{\partial^2 J}{\partial x^2} (x) \right\|_2 = 2, \quad \left\| \frac{\partial^2 J}{\partial x^2} (x) \right\|_\infty = 2n.
\]

Proof. By direct calculation, the components of the derivative of $J$ are
\[
\frac{\partial J}{\partial x_i} (x) = 2(x_i - z_i).
\]
Since the dual of the two-norm is also the two-norm, then,
\[
\left\| \frac{\partial J}{\partial x} (x) \right\|_2 = 2 \| x - z \|_2,
\]
and since the dual of the $\infty$-norm is the one-norm,
\[
\left\| \frac{\partial J}{\partial x} (x) \right\|_\infty = 2 \| x - z \|_1.
\]
For the second derivative, the components are
\[
\frac{\partial^2 J}{\partial x_i \partial x_j} = \begin{cases} 
0 & \text{if } i \neq j, \\
1 & \text{if } i = j.
\end{cases}
\]
Then for any vectors $u, v$,
\[
\frac{\partial J^2}{\partial x^2} (x) \cdot (u, v) = \sum_{i=1}^n u_i v_i.
\]
Therefore if $\|u\|_2 = \|v\|_2 = 1$ then
\[
\left| \frac{\partial J^2}{\partial x^2} (x) \cdot (u, v) \right| \leq \|u\|_2 \|v\|_2 = 1
\] (94)
and equality is achieved in (94) by setting $u = v = \frac{1}{\sqrt{n}} (1, \ldots, 1)$. For the $\infty$-norm, note that in general, if $\|u\|_\infty = \|v\|_\infty = 1$, then
\[
\left| \frac{\partial J^2}{\partial x^2} (x) \cdot (u, v) \right| \leq n \|u\|_\infty \|v\|_\infty = n
\] (95)
and equality is achieved in (95) by setting $u = v = (1, \ldots, 1)$.

Proof of Proposition 4.5. Let $\ell$ be given and consider $q = \infty$. For any matrix $A$ with $\|A\|_\infty = 1$,
\[
\ell(A) = \sum_{i=1}^r \sum_{j=1}^c \ell_{i,j} A_{i,j} \leq \sum_{i=1}^r \max_{1 \leq j \leq c} |\ell_{i,j}|
\] (96)
since each row sum $\sum_{j=1}^c |A_{i,j}|$ is at most 1. Let $m$ be the matrix defined in equation (52). Clearly this matrix has maximum-absolute-row-sum 1. Furthermore,
\[
\ell(m) = \sum_{i=1}^r \max_{1 \leq j \leq c} |\ell_{i,j}|
\] (97)
Combining equation (97) with the inequality (96) confirms that the dual norm is \( \|\ell\|_\infty = \sum_{i=1}^{r} \max_{1 \leq k \leq c} |\ell_{i,k}|. \)

For \( q = 2 \), that the duality between the spectral norm is the sum of singular values, or trace norm, is well-known, and can be proved for instance as in the proof of Theorem 7.4.24 of [16].

For the duality maps, let the matrix \( \rho_\infty(\ell) \) be defined as in (52). Then \( \|\rho_\infty(\ell)\|_\infty = \|\ell\|_\infty \|m\|_\infty = \|\ell\|_\infty \) and \( \ell(\rho_\infty(\ell)) = \|\ell\|_\infty \ell(m) = \|\ell\|_\infty^2 \).

For \( q = 2 \), let the \( \ell = U\Sigma V^T \) be the singular value decomposition of \( \ell \), and let the matrix \( \rho_2(\ell) \) be defined as in (51). Then \( \ell(\rho_2(\ell)) = \|\ell\|_2 \ell(A) \) where \( A = \sum_{i=1}^{\text{rank } \ell} u_i v_i^T \). It remains to show that \( \ell(A) = \sum_{i=1}^{\text{rank } \ell} \sigma_i(\ell) \):

\[
\ell(A) = \mathrm{tr} \left( (U\Sigma V^T)^T A \right) = \mathrm{tr} \left( \Sigma \Sigma^T U^T U V V^T \right) = \mathrm{tr} \left( \sum_{i=1}^{\text{rank } \ell} \sigma_i(\ell) v_i u_i^T \right) \left( \sum_{j=1}^{\text{rank } \ell} u_j v_j^T \right) = \sum_{i=1}^{\text{rank } \ell} \sigma_i(\ell) v_i^T u_i.
\]

In the second to last inequality we used the fact that the columns of \( U \) are orthogonal. In the last inequality we used the linearity of trace together with the fact that the columns of \( V \) are unit vectors (that is, \( \mathrm{tr}(v_i v_i^T) = 1 \)).

**Proof of Proposition 4.6.** First we compute the dual norm on \( Z \). For any \( u = (u_1, \ldots, u_K) \in X_1 \times \ldots X_K \) with \( \|(u_1, \ldots, u_K)\|_Z = 1 \), we have

\[
\ell(u) = \ell_1(u_1) + \ldots + \ell_K(u_K) \leq \frac{p_1}{p_i} \|\ell_1\|_{X_i} \|u_1\|_{X_i} + \ldots + \frac{p_K}{p_K} \|\ell_K\|_{X_K} \|u_K\|_{X_K} \leq \left( \frac{p_1}{p_i} \|u_1\|_{X_i} + \ldots + \frac{p_K}{p_K} \|u_K\|_{X_K} \right) \max_{1 \leq i \leq K} \left\{ \frac{1}{p_i} \|\ell_i\|_{X_i} \right\} = \max_{1 \leq i \leq K} \left\{ \frac{1}{p_i} \|\ell_i\|_{X_i} \right\}. 
\]

Therefore,

\[
\|\ell\|_Z \leq \max_{1 \leq i \leq K} \left\{ \frac{1}{p_i} \|\ell_i\|_{X_i} \right\}. \tag{98}
\]

Define \( i^* \) as

\[
i^* = \arg \max_{1 \leq i \leq K} \left\{ \frac{1}{p_i} \|\ell_i\|_{X_i} \right\}.
\]
To show that (98) is in fact an equality, consider the vector \( u \) defined as
\[
u = (u_1, \ldots, u_K) = \left(0, \ldots, \frac{1}{\|\ell_i\|_{\ell_i^*}} \rho_{X_i^*}(\ell_i^*), \ldots, 0\right).
\]
The norm of this vector is
\[
\|u\|_Z = p_i^* \frac{1}{\|\ell_i\|_{\ell_i^*}} \|\rho_{X_i^*}(\ell_i^*)\| = \frac{1}{\|\ell_i\|_{\ell_i^*}} \|\ell_i^*\| = 1,
\]
and
\[
\ell(u) = \frac{1}{\|\ell_i\|_{\ell_i^*}} \ell_i^* \left(\rho_{X_i^*}(\ell_i^*), \ldots, 0\right) = \frac{1}{\|\ell_i\|_{\ell_i^*}} \|\ell_i^*\|_2 = \frac{\|\ell_i^*\|_{X_i^*}}{p_i^*}.
\]
Therefore the dual norm is given by (53).

Next, we show that the function \( \rho_Z \) defined in (54) is a duality map, by verifying the conditions (9a) and (9b).

\[
\ell(\rho_Z(\ell)) = \ell \left(0, \ldots, \frac{1}{(p_i^*)^2} \rho_{X_i^*}(\ell_i^*), \ldots, 0\right) = \frac{1}{(p_i^*)^2} \ell_i^* \left(\rho_{X_i^*}(\ell_i^*)\right)
\]
\[
= \frac{1}{(p_i^*)^2} \|\ell_i^*\|_{X_i^*}^2
\]
\[
= \|\ell\|_Z^2.
\]
This shows that (9b) holds. It remains to show \( \|\rho_Z(\ell)\|_Z = \|\ell\|_Z \). By definition of \( i^* \), we have
\[
\rho_Z(\ell) = \left(0, \ldots, \frac{1}{(p_i^*)^2} \rho_{X_i^*}(\ell_i^*), \ldots, 0\right)
\]
so
\[
\|\rho_Z(\ell)\|_Z = p_i^* \frac{1}{(p_i^*)^2} \|\rho_{X_i^*}(\ell_i^*)\|_{X_i^*} = \frac{1}{p_i^*} \|\ell_i^*\|_{X_i^*} = \|\ell\|_Z.
\]

References


\[ o(\epsilon^{-3/2}) \]


