A revisit to a reverse-order law for generalized inverses of a matrix product and its variations

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Abstract. For a pair of complex matrices $m \times n$ matrix $A$ and $n \times p$ matrix $B$, the general form of reverse-order laws for generalized inverses of the product $AB$ can be written as $(AB)^{(i\ldots,j)} = B^{(i\ldots,j)} A^{(i\ldots,j)}$, where $A^{(i\ldots,j)}$, $B^{(i\ldots,j)}$, and $(AB)^{(i\ldots,j)}$ denote $(i\ldots,j)$-inverses of $A$, $B$, and $AB$ respectively. There are all $15^3 = 3375$ possible formulations for different choices of generalized inverses of $A$, $B$, and $AB$, respectively. This paper deals with a specific reverse-order law $(AB)^{(1)} = B^\dagger A^\dagger$, where $A^\dagger$ and $B^\dagger$ are the Moore–Penrose inverses of $A$ and $B$, respectively. We will collect and derive many known and novel equivalent statements for the reverse-order law and its variations to hold by using the methodology of ranks and ranges of matrices.

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1 Introduction

Throughout this paper, $\mathbb{R}^{m \times n}$ and $\mathbb{C}^{m \times n}$ stands for the collections of all $m \times n$ complex matrices. The symbols $r(A)$, $\mathcal{R}(A)$ and $\mathcal{N}(A)$ stand for the rank, range (column space) and kernel (null space) of a matrix $A \in \mathbb{C}^{m \times n}$, respectively. $I_m$ denotes the identity matrix of order $m$; $[A, B]$ denotes a row block matrix consisting of $A$ and $B$. The Moore–Penrose inverse of $A \in \mathbb{C}^{m \times n}$, denoted by $A^\dagger$, is the unique matrix $X \in \mathbb{C}^{n \times m}$ satisfying the four Penrose equations

\begin{align}
(\text{i}) \quad AXA &= A, \quad (\text{ii}) \quad XAX = X, \quad (\text{iii}) \quad (AX)^* = AX, \quad (\text{iv}) \quad (XA)^* = XA.
\end{align}

Further let $E_A = I_m - AA^\dagger$ and $F_A = I_n - A^\dagger A$ stand for two projectors induced by $A$. Moreover, a matrix $X$ is called an $(i\ldots,j)$-inverse of $A$, denoted by $A^{(i\ldots,j)}$, if it satisfies the $i$th, $\ldots$, $j$th equations. The collection of all $(i\ldots,j)$-inverses of $A$ is denoted by $\{A^{(i\ldots,j)}\}$. There are all 15 types of $(i\ldots,j)$-inverse under (1.1), but the eight-frequently used generalized inverses of $A$ are $A^1$, $A^{(1,3,4)}$, $A^{(1,2,4)}$, $A^{(1,2,3)}$, $A^{(1,4)}$, $A^{(1,3)}$, $A^{(1,2)}$, and $A^{(1)}$. In matrix theory, a fundamental matrix operation is to find the inverse of a square matrix when it is nonsingular, or find its generalized inverses when it is singular.

Assume that $A \in \mathbb{C}^{m \times n}$ and $B \in \mathbb{C}^{n \times p}$ are a pair of matrices. Then $A^{(i\ldots,j)}$, $B^{(i\ldots,j)}$, and $(AB)^{(i\ldots,j)}$ always exist. In this case, one of the fundamental equalities in the theory of generalized inverses is

\begin{align}
(AB)^{(i\ldots,j)} = B^{(i\ldots,j)} A^{(i\ldots,j)},
\end{align}

This equality is a direct extension of the well-known identity $(AB)^{-1} = B^{-1} A^{-1}$ for two invertible matrices, and is usually called the reverse-order law for generalized inverses of the matrix product $AB$. It is obvious that (1.2) includes $15^3 = 3375$ formulations for all possible choices of $A^{(i\ldots,j)}$, $B^{(i\ldots,j)}$, and $(AB)^{(i\ldots,j)}$ defined above.

We next present an illustrative example in statistical analysis where reverse-order laws of generalized inverses of matrix products may occur. Consider a linear random-effects model defined by

\begin{align}
\mathcal{M} : \quad y = A\alpha + \epsilon, \quad \alpha = B\beta + \gamma,
\end{align}

where in the first-stage model, $y \in \mathbb{R}^{n \times 1}$ is a vector of observable response variables, $A \in \mathbb{R}^{n \times p}$ is a known matrix of arbitrary rank, $\alpha \in \mathbb{R}^{p \times 1}$ is a vector of unobservable random variables, $\epsilon \in \mathbb{R}^{n \times 1}$ is a vector of randomly distributed error terms, and in the second-stage model, $B \in \mathbb{R}^{p \times k}$ is a known matrix of arbitrary rank, $\beta \in \mathbb{R}^{k \times 1}$ is a vector of fixed but unknown parameters, $\gamma \in \mathbb{R}^{p \times 1}$ is a vector of unobservable random variables. Substituting the second equation into the first equation in (1.3) yields

\begin{align}
\mathcal{N} : \quad y = AB\beta + A\gamma + \epsilon.
\end{align}

There exist two methods for deriving ordinary least-squares estimators (OLSEs) of the fixed but unknown parameter vector $v\beta$ in (1.3) and (1.4). A direct method is to find a $\beta$ that satisfies

\begin{align}
\|y - AB\beta\|^2 = (y - AB\beta)^T(y - AB\beta) = \min
\end{align}

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in the context of (1.4), which leads to the general expression of the OLSE of $\beta$ as follows

$$\text{OLSE}_\nu(\beta) = [(AB)\dagger + F_{AB}U]y = (AB)^{(1.3)}_\nu y,$$

(1.6)

where $U$ is an arbitrary matrix. On the other hand, solving $||y - Ao||^2 = \text{min}$ under (1.3) leads to the OLSE of $\alpha$ as follows

$$\text{OLSE}_\mu(\alpha) = (A\dagger + F_A U_1)y = A^{(1.3)}_\mu y,$$

(1.7)

where $U_1$ is an arbitrary matrix, and then substituting it into the second equation in (1.3) to yield

$$A^{(1.3)}_\mu y = B\beta + \gamma.$$

(1.8)

In this case, solving $||A^{(1.3)}_\mu y - B\beta||^2 = \text{min}$ under (1.8) leads to

$$\text{OLSE}_\nu(\beta) = (B\dagger + F_B U_2)A^{(1.3)}_\mu y = B^{(1.3)}_\mu A^{(1.3)}_\mu y,$$

(1.9)

where $U_2$ is an arbitrary matrix. Eqs. (1.6) and (1.9) are not necessarily equal, but comparing the coefficient matrices of $y$ in (1.6) and (1.9) and their special cases $(AB)\dagger$ and $B\dagger A\dagger$ leads to the following reverse-order laws

$$(AB)^{(1.3)} = B^{(1.3)} A^{(1.3)}, \quad (AB)\dagger = B\dagger A\dagger.$$ 

(1.10)

Because $AA^{(i,\ldots,j)}$, $A^{(i,\ldots,j)}A$, $BB^{(i,\ldots,j)}$, and $B^{(i,\ldots,j)}B$ are not necessarily identity matrices, (1.2) does not hold in general, namely, the reverse product $B^{(i,\ldots,j)}A^{(i,\ldots,j)}$ does not necessarily satisfy the four equations for the Moore–Penrose inverse of $AB$. In other words, (1.2) holds if and only if both $A$ and $B$ satisfy certain conditions. Characterization of reverse-order laws in (1.2) was greatly facilitated by the development of the matrix rank methodology. Reverse-order laws for generalized inverses have widely been investigated in the literature since 1960s due to their applications in simplifying matrix expressions involving generalized inverses. Reverse-order laws for generalized inverses of matrix products have been a core topic in the theory of generalized inverses since Penrose defined in 1950s the four equations in (1.1), which attracted much attention since 1960s (see, e.g., [1, 2, 5, 6]) and the research on the reverse-order laws made to some essential progress of the theory of generalized inverses, and led profound influence on development of methodology in matrix analysis.

Because both $A\dagger$ and $B\dagger$ are unique, thus reverse product $B\dagger A\dagger$ of $A\dagger$ and $B\dagger$ is unique also well and the product plays an important role in the characterization of various reverse-order laws. It is obvious that for the special choice $B\dagger A\dagger$ on the right-hand side of (1.2), there are 15 possible choices of $(AB)^{(i,\ldots,j)}$ on the left-hand side, but people are more interested in the following four specific situations

$$(AB)^{(1)} = B\dagger A\dagger, \quad (AB)^{(1.3)} = B\dagger A\dagger, \quad (AB)^{(1.4)} = B\dagger A\dagger, \quad (AB)\dagger = B\dagger A\dagger.$$ 

(1.11)

Much effort has been paid to the four reverse-order laws in the literature, while many necessary and sufficient conditions are derived for these reverse-orders to hold. In this paper, the present author reconsiders the first situation in (1.11) and its variations. By definition, the following equivalence

$$(AB)^{(1)} = B\dagger A\dagger \Leftrightarrow ABB\dagger A\dagger AB = AB.$$

(1.12)

holds. The second fundamental matrix equality in (1.12) and its variations occur widely in the theory of generalized inverses, and has been studied by many authors since 1960s, in particular, some rank formulas associated the matrix equality were established in [9, 10]. This paper aims at collecting and proving many known and new results for the reverse-order law in (1.12) to hold by using the methodology of ranks and ranges of matrices. In Section 2, we present various rank and range formulas of matrices that will be used in the proofs of the main results. In Section 3, we derive several groups of closed-form formulas for calculating the maximum ranks of $AB - ABB\dagger A\dagger AB$ and $ABB\dagger A\dagger AB - ABB\dagger A\dagger AB$. The main results and their proofs are presented in Section 4.

2 Some preliminaries

The scope of this section is to introduce the mathematical foundations of generalized inverses, followed by presenting various matrix rank formulas and their usefulness in the theory of generalized inverses. These initial preparations serve as tools for solving the problems that are described in Section 1. Note from the definitions of generalized inverses of a matrix that they are in fact are defined to be (common) solutions of some matrix equations. Thus analytical expressions of generalized inverses of matrix can be written as certain matrix-valued functions with one or more variable matrices. In fact, analytical formulas of generalized inverses of matrices and their functions are important issues and tools in the theory of generalized inverses of matrices, which can be found in most textbooks on generalized inverses of matrices. For instance, the basic formulas in the following lemma can be found, e.g., in [3, 4, 8].
Lemma 2.1. Let \( A \in \mathbb{C}^{m \times n} \). Then, the following results hold.

(a) The general expressions of the last seven generalized inverses of \( A \) in (1.2) can be written in the following parametric forms

\[
A^{(1,3,4)} = A^\dagger + F_A V E_A, \tag{2.1}
\]

\[
A^{(1,2,4)} = A^\dagger + A^\dagger A W E_A, \tag{2.2}
\]

\[
A^{(1,2,3)} = A^\dagger + F_A V A A^\dagger, \tag{2.3}
\]

\[
A^{(1,4)} = A^\dagger + W E_A, \tag{2.4}
\]

\[
A^{(1,3)} = A^\dagger + FAV, \tag{2.5}
\]

\[
A^{(1,2)} = (A^\dagger + FA V) A (A^\dagger + W E_A), \tag{2.6}
\]

\[
A^{(1)} = A^\dagger + F_A V + W E_A, \tag{2.7}
\]

where the two matrices \( V, W \in \mathbb{K}^{n \times m} \) are arbitrary.

(b) The following matrix equalities hold

\[
AA^{(1,3,4)} = AA^{(1,2,3)} = AA^{(1,3)} = AA^\dagger, \tag{2.8}
\]

\[
A^{(1,3,4)} A = A^{(1,2,4)} A = A^{(1,4)} A = A A^\dagger, \tag{2.9}
\]

\[
AA^{(1,2,4)} = AA^{(1,4)} = AA^{(1,2)} = AA^{(1)} = AA^\dagger + WE_A, \tag{2.10}
\]

\[
A^{(1,2,3)} A = A^{(1,3)} A = A^{(1,2)} A = A^{(1)} A = A A^\dagger + FA VA, \tag{2.11}
\]

where the two matrices \( V \) and \( W \) are arbitrary.

Lemma 2.2 ([10]). Let \( A \in \mathbb{C}^{m \times n} \) and \( B \in \mathbb{C}^{n \times p} \) be given. Then the product \( B^\dagger N A^\dagger \) can be written as

\[
B^\dagger A^\dagger = -[B^*, \ 0] \begin{bmatrix} 0 & A^* A^* \n \ B^* B^* & B^* A^* \n \end{bmatrix} A^* = -P J^\dagger Q, \tag{2.12}
\]

where the block matrices \( P, J, \) and \( Q \) satisfy

\[
r(J) = r(A) + r(B), \quad \mathcal{R}(Q) \subseteq \mathcal{R}(J), \quad \mathcal{R}(P^*) \subseteq \mathcal{R}(J^*). \tag{2.13}
\]

A powerful tool for approaching relations between matrix expressions and matrix equalities is using various expansion formulas for calculating ranks of matrices (also called the matrix rank methodology). Recall that \( A = 0 \) if and only if \( r(A) = 0 \), so that for two matrices \( A \) and \( B \) of the same size

\[
A = B \Leftrightarrow r(A - B) = 0. \tag{2.14}
\]

Furthermore, for two sets \( S_1 \) and \( S_2 \) consisting of matrices of the same size, the following assertions hold

\[
S_1 \cap S_2 \neq \emptyset \Leftrightarrow \min_{A \in S_1, \ B \in S_2} r(A - B) = 0; \tag{2.15}
\]

\[
S_1 \subseteq S_2 \Leftrightarrow \max_{A \in S_1, \ B \in S_2} r(A - B) = 0. \tag{2.16}
\]

These implications provide a highly flexible framework for characterizing equalities of matrices via ranks of matrices. If some expansion formulas for the rank of \( A - B \) are established, we can use the formulas to characterize relations between two matrices \( A \) and \( B \), and to obtain many valuable consequences. For instance, (1.2) holds if and only if

\[
\min_{(AB)^{(i,\ldots,j)}, \ A^{(i,\ldots,j)}, \ B^{(i,\ldots,j)}} r\left[ (AB)^{(i,\ldots,j)} - B^{(i,\ldots,j)} A^{(i,\ldots,j)} \right] = 0.
\]

This idea was first introduced by the present author in early 1990s when characterizing reverse-order laws for Moore–Penrose inverses of matrix products. In the past thirty years, the present author established thousands of closed-form formulas for calculating ranks of matrices, and used them to derive a huge amount of results on matrix equalities, matrix equations, matrix set inclusions, matrix-valued functions, etc., while matrix rank methodology has naturally become one of the most important and widely used tools in characterizing matrix equalities that involve generalized inverses of matrices. In order to establish and simplify various matrix equalities composed by generalized inverses of matrices, we need the following well-known rank formulas for matrices to make the paper self-contained.
Lemma 2.3 ([7]). Let $A \in \mathbb{C}^{m \times n}, B \in \mathbb{C}^{m \times k}, C \in \mathbb{C}^{l \times n},$ and $D \in \mathbb{C}^{l \times k}$. Then

\begin{align*}
    r[A, B] &= r(A) + r(E_{AB}) = r(B) + r(E_{BA}), \quad (2.17) \\
    r[A \mid C] &= r(A) + r(CF_A) = r(C) + r(AF_C), \quad (2.18) \\
    r[A \mid B \mid C, 0] &= r(B) + r(C) + r(E_{BA}CF_C), \quad (2.19) \\
    r[A \mid B \mid C \mid D] &= r(A) + r \left[ 0 \mid CF_A \mid D \mid CA_1B \right]. \quad (2.20)
\end{align*}

If $\mathcal{R}(B) \subseteq \mathcal{R}(A)$ and $\mathcal{R}(C^*) \subseteq \mathcal{R}(A^*)$, then

\begin{equation}
    r[A \mid B] = r(A) + r(D - CA_1B). \quad (2.21)
\end{equation}

Furthermore, the following results hold.

(a) $r[A, B] = r(A) \Leftrightarrow \mathcal{R}(B) \subseteq \mathcal{R}(A) \Leftrightarrow AA^\dagger B = B \Leftrightarrow E_{AB} = 0$.

(b) $r[A \mid C] = r(A) \Leftrightarrow \mathcal{R}(C^*) \subseteq \mathcal{R}(A^*) \Leftrightarrow CA_1A = C \Leftrightarrow CF_A = 0$.

(c) $r[A, B] = r(A) + r(B) \Leftrightarrow \mathcal{R}(A) \cap \mathcal{R}(B) = \{0\} \Leftrightarrow \mathcal{R}([E_{AB}]^*) = \mathcal{R}(B^*) \Leftrightarrow \mathcal{R}([E_{BA}]^*) = \mathcal{R}(A^*)$.

(d) $r[A \mid C] = r(A) + r(C) \Leftrightarrow \mathcal{R}(A^*) \cap \mathcal{R}(C^*) = \{0\} \Leftrightarrow \mathcal{R}(CF_A) = \mathcal{R}(C) \Leftrightarrow \mathcal{R}(AF_C) = \mathcal{R}(A)$.

Lemma 2.4 ([10]). Let $A \in \mathbb{C}^{m \times n}, B \in \mathbb{C}^{m \times k}, C \in \mathbb{C}^{l \times n},$ and $D \in \mathbb{C}^{l \times k}$. Then

\begin{equation}
    r(D - CA_1B) = r \left[ \begin{array}{cc}
        A^*AA^* & A^*B \\
        CA^* & D
    \end{array} \right] - r(A). \quad (2.22)
\end{equation}

In particular,

\begin{equation}
    r(A^* - A_1^\dagger) = r(AA^*A - A). \quad (2.23)
\end{equation}

The following result is well known.

Lemma 2.5. If $A \in \mathbb{C}^{n \times n}$, then

\begin{equation}
    r(A - A_2^2) = r(I_n - A) + r(A) - n. \quad (2.24)
\end{equation}

Lemma 2.6 ([15]). Let $P, Q \in \mathbb{C}^{m \times m}$ be two orthogonal projectors. Then

\begin{equation}
    r(PQ - QP) = 2r[P, Q] + 2r(PQ) - 2r(P) - 2r(Q). \quad (2.25)
\end{equation}

In addition, we use the following simple properties (see [3, 4, 8]) when simplifying various matrix rank and range equalities

\begin{align*}
    \mathcal{R}(A) \subseteq \mathcal{R}(B) \quad \text{and} \quad r(A) = r(B) \Rightarrow \mathcal{R}(A) = \mathcal{R}(B), \quad (2.26) \\
    \mathcal{R}(A) \subseteq \mathcal{R}(B) \Rightarrow \mathcal{R}(PA) \subseteq \mathcal{R}(PB), \quad (2.27) \\
    \mathcal{R}(A) = \mathcal{R}(B) \Rightarrow \mathcal{R}(PA) = \mathcal{R}(PB), \quad (2.28) \\
    \mathcal{R}(A) = \mathcal{R}(AA^*) = \mathcal{R}(AA^*A) = \mathcal{R}(AA^\dagger) = \mathcal{R}([A^\dagger]^*], \quad (2.29) \\
    \mathcal{R}(A^*) = \mathcal{R}(A^*A) = \mathcal{R}(AA^*) = \mathcal{R}(A^\dagger) = \mathcal{R}(A^\dagger A), \quad (2.30) \\
    \mathcal{R}(AB^\dagger B) = \mathcal{R}(AB) = \mathcal{R}(AB^*B) = \mathcal{R}(AB^*), \quad (2.31) \\
    \mathcal{R}(A_1) = \mathcal{R}(A_2) \quad \text{and} \quad \mathcal{R}(B_1) = \mathcal{R}(B_2) \Rightarrow r[A_1, B_1] = r[A_2, B_2]. \quad (2.32)
\end{align*}
Lemma 2.7 ([12, 13]). Let $A \in \mathbb{C}^{m \times n}$ and $B \in \mathbb{C}^{n \times p}$ be given. Then the following rank identities hold

\[ r(A^*ABB^*) = r(A^*AB) = r(ABB^*) = r(AB), \]  
\[ r(BB^* A^* A) = r(BB^* A^*) = r(B^* A^*) = r(B^* A) = r(AB), \]  
\[ r(ABB^* A^*) = r(B^* A^* AB) = r(AB), \]  
\[ r((A^*)^1/2(BB^*)^1/2) = r((BB^*)^1/2(A^*)^1/2) = r(AB), \]  
\[ r(B^i A^i) = r(B^i A^*) = r(B^i A) = r(AB), \]  
\[ r((A^i)^* (B^i)^*) = r[(A^i)^* B] = r[A(B^i)^*] = r(AB), \]  
\[ r(BB^i A^i A) = r(BB^i A^*) = r(BB^i A) = r(AB), \]  
\[ r(A^i AB^i) = r(A^i AB^*) = r(A^i ABB^i) = r(AB), \]  
\[ r(AB^i A^i AB) = r(AB^i A^*) = r(AB^i A) = r(AB), \]  
\[ r(A^i AB^i AB) = r(A^i A^* AB) = r(A^i A^* A) = r(A^i A) = r(AB), \]  
\[ r((BB^*)^i (A^*)^i) = r[(BB^*)^i (A^*)^i] = r[(BB^*) (A^*)^i] = r(AB), \]  
\[ r(B^i (A^* A)^i) = r(B^i A^* A) = r(B^* (A^* A)^i) = r(AB), \]  
\[ r((BB^*)^i A^i) = r[(BB^*)^i A^*] = r[(BB^*)^i A] = r(AB). \]

Lemma 2.8 ([11, 14]). Let $A \in \mathbb{C}^{m \times n}$, $B \in \mathbb{C}^{m \times k}$, and $C \in \mathbb{C}^{k \times n}$. Then

\[ \max_{X \in \mathbb{C}^{k \times l}} r(A - BXC) = \min \left\{ r [A, B], r \left[ \begin{array}{c} A \\ C \end{array} \right] \right\}, \]  
\[ \min_{X \in \mathbb{C}^{k \times l}} r(A - BXC) = r [A, B] + r \left[ \begin{array}{c} A \\ C \end{array} \right] - r \left[ \begin{array}{c} A \\ C \end{array} \right], \]  
\[ r \left[ \begin{array}{c} A \\ C \end{array} \right] = 0. \]  

3 Rank formulas for some matrix expressions

By definition, $B^{(i_1,\ldots,i_p)} A^{(i_1,\ldots,i_p)} \in \{ (AB)^{(1)} \}$ if and only if $ABB^{(i_1,\ldots,i_p)} A^{(i_1,\ldots,i_p)} AB = AB$. Also by (2.14),

\[ ABB^{(i_1,\ldots,i_p)} A^{(i_1,\ldots,i_p)} AB = AB \iff r(AB - ABB^{(i_1,\ldots,i_p)} A^{(i_1,\ldots,i_p)} AB) = 0. \]  

In order to establish closed-form formulas for calculating the ranks of the differences, we need to rewrite

\[ AB - ABB^{(i_1,\ldots,i_p)} A^{(i_1,\ldots,i_p)} AB, \]  
\[ ABB^i A^i AB - ABB^{(i_1,\ldots,i_p)} A^{(i_1,\ldots,i_p)} AB. \]

as certain linear or nonlinear matrix-valued functions that involve one or more variable matrices. We then establish exact formulas for calculating the maximum ranks of the two expressions, and then use the formulas to derive identifying conditions for (1.12) to hold. The following expansion formulas follow directly from (2.1)–(2.11).

\[ AB - ABB^1 A^{(1,3,4)} AB = AB - ABB^1 A^{(1,2,4)} AB = AB - ABB^1 A^{(1,4)} AB = AB - ABB^1 A^{(1,2,3,4)} AB = AB - ABB^1 A^{(1,2,3)} AB = AB - ABB^1 A^{(1,3)} AB = AB - ABB^1 A^{(1,2)} AB = AB - ABB^1 A^{(1)} AB, \]

\[ \begin{array}{c}
\end{array} \]
Theorem 3.1. Let $A \in \mathbb{C}^{m \times n}$ and $B \in \mathbb{C}^{n \times p}$, and denote $M = [A^*, B]$.

(a) \cite{9, 10} The rank of $AB - ABB^\dagger A^\dagger AB$ satisfies the identity

$$r(AB - ABB^\dagger A^\dagger AB) = r(M) + r(AB) - r(A) - r(B).$$  \hspace{1cm} (3.9)
(b) The following rank formulas hold

\[
\begin{align*}
\max_{A^{(1,2,3)}} r(AB - ABB^\dagger A^{(1,2,3)}AB) &= \max_{A^{(2)}} r(AB - ABB^\dagger A^{(1,3)}AB) \\
= & \max_{A^{(1,2)}} r(AB - ABB^\dagger A^{(1,2)}AB) = \max_{B^{(1)}} r(AB - ABB^\dagger A^{(1)}AB) \\
= & \max_{B^{(1,2,3)}, A^{(1,2,3)}} r(AB - ABB^{(1,3,4)}A^{(1,2,3)}AB) = \max_{B^{(1,3,4)}, A^{(1,3)}} r(AB - ABB^{(1,3,4)}A^{(1,3)}AB) \\
= & \max_{B^{(1,2)}, A^{(1,2,3)}} r(AB - ABB^{(1,2,3)}A^{(1,2,3)}AB) = \max_{B^{(1,2)}, A^{(1,2,3)}} r(AB - ABB^{(1,2,3)}A^{(1,3)}AB) \\
= & \max_{B^{(1,2)}, A^{(1,3)}} r(AB - ABB^{(1,2)}A^{(1,2)}AB) = \max_{B^{(1,3)}, A^{(1,3)}} r(AB - ABB^{(1,2)}A^{(1)}AB) \\
= & \max_{B^{(1,3)}, A^{(1,3)}} r(AB - ABB^{(1,3)}A^{(1,3)}AB) = \max_{B^{(1,3)}, A^{(1)}} r(AB - ABB^{(1,3)}A^{(1)}AB) \\
= & \max_{B^{(1,3)}, A^{(1)}} r(AB - ABB^{(1,3)}A^{(1,2)}AB) = \max_{B^{(1), A^{(1)}}} r(AB - ABB^{(1,3)}A^{(1)}AB) \\
= & r(M) + r(AB) - r(A) - r(B).
\end{align*}
\] (3.10)

(c) The following rank formulas hold

\[
\begin{align*}
\max_{B^{(1,2,4)}} r(AB - ABB^{(1,2,4)}A^\dagger AB) &= \max_{B^{(1,2,4)}, A^{(1,3,4)}} r(AB - ABB^{(1,2,4)}A^{(1,3,4)}AB) \\
= & \max_{B^{(1,2,4)}, A^{(1,2,4)}} r(AB - ABB^{(1,2,4)}A^{(1,2,4)}AB) = \max_{B^{(1,2,3)}, A^{(1,4)}} r(AB - ABB^{(1,2,4)}A^{(1,4)}AB) \\
= & \max_{B^{(1,2)}, A^{(1,4)}} r(AB - ABB^{(1,4)}A^\dagger AB) = \max_{B^{(1,2)}, A^{(1,4)}} r(AB - ABB^{(1,4)}A^{(1,4)}AB) \\
= & \max_{B^{(1,2)}, A^{(1,4)}} r(AB - ABB^{(1,2)}A^{(1,2)}AB) = \max_{B^{(1,2)}, A^{(1,4)}} r(AB - ABB^{(1,2)}A^{(1,4)}AB) \\
= & \max_{B^{(1)}, A^{(1,4)}} r(AB - ABB^{(1)}A^\dagger AB) = \max_{B^{(1)}, A^{(1,4)}} r(AB - ABB^{(1)}A^{(1,4)}AB) \\
= & \max_{B^{(1)}, A^{(1,4)}} r(AB - ABB^{(1)}A^{(1,2)}AB) = \max_{B^{(1)}, A^{(1,4)}} r(AB - ABB^{(1)}A^{(1,4)}AB) \\
= & r(M) + r(AB) - r(A) - r(B).
\end{align*}
\] (3.11)

(d) The following rank formulas hold

\[
\begin{align*}
\max_{A^{(1,2,3)}} r(ABB^\dagger A^\dagger AB - ABB^\dagger A^{(1,2,3)}AB) &= \max_{A^{(1,3)}} r(ABB^\dagger A^\dagger AB - ABB^\dagger A^{(1,3)}AB) \\
= & \max_{A^{(1,2)}} r(ABB^\dagger A^\dagger AB - ABB^\dagger A^{(1,2)}AB) = \max_{B^{(1)}} r(ABB^\dagger A^\dagger AB - ABB^\dagger A^{(1)}AB) \\
= & \max_{B^{(1,2,3)}, A^{(1,2,3)}} r(ABB^\dagger A^\dagger AB - ABB^{(1,3,4)}A^{(1,2,3)}AB) = \max_{B^{(1,3,4)}, A^{(1,3)}} r(ABB^\dagger A^\dagger AB - ABB^{(1,3,4)}A^{(1,3)}AB) \\
= & \max_{B^{(1,3,4)}, A^{(1,3)}} r(ABB^\dagger A^\dagger AB - ABB^{(1,3,4)}A^{(1,3)}AB) = \max_{B^{(1,3,4)}, A^{(1,3)}} r(ABB^\dagger A^\dagger AB - ABB^{(1,3,4)}A^{(1,2)}AB) \\
= & \max_{B^{(1,3,4)}, A^{(1,3)}} r(ABB^\dagger A^\dagger AB - ABB^{(1,3,4)}A^{(1,2)}AB).
\end{align*}
\]
The following rank formulas hold

\[
\begin{align*}
\max_{B^{(1,3,4)}, A^{(1)}} & \quad r(ABB^t A^t AB - ABB^{(1,3,4)} A^t AB) \\
= & \max_{B^{(1,2,3)}, A^{(1,2,3)}} r(ABB^t A^t AB - ABB^{(1,2,3)} A^{(1,2,3)} AB) \\
= & \max_{B^{(1,2,3)}, A^{(1,3)}} r(ABB^t A^t AB - ABB^{(1,2,3)} A^{(1,3)} AB) \\
= & \max_{B^{(1,2,3)}, A^{(1,2)}} r(ABB^t A^t AB - ABB^{(1,2,3)} A^{(1,2)} AB) \\
= & \max_{B^{(1,2,3), A^{(1)}}} r(ABB^t A^t AB - ABB^{(1,2,3)} A^{(1)} AB) \\
= & \max_{B^{(1,3)}, A^{(1,2,3)}} r(ABB^t A^t AB - ABB^{(1,3)} A^{(1,2,3)} AB) \\
= & \max_{B^{(1,3)}, A^{(1,3)}} r(ABB^t A^t AB - ABB^{(1,3)} A^{(1,3)} AB) \\
= & \max_{B^{(1,3)}, A^{(1,2)}} r(ABB^t A^t AB - ABB^{(1,3)} A^{(1,2)} AB) \\
= & \max_{B^{(1,3)}, A^{(1)}} r(ABB^t A^t AB - ABB^{(1,3)} A^{(1)} AB) \\
= & r(M) + r(AB) - r(A) - r(B). \\
\end{align*}
\]

(e) The following rank formulas hold

\[
\begin{align*}
\max_{B^{(1,2,4)}} & \quad r(ABB^t A^t AB - ABB^{(1,2,4)} A^t AB) \\
= & \max_{B^{(1,2,3), A^{(1,3,4)}}} r(ABB^t A^t AB - ABB^{(1,2,4)} A^{(1,3,4)} AB) \\
= & \max_{B^{(1,2,3), A^{(1,2,4)}}} r(ABB^t A^t AB - ABB^{(1,2,4)} A^{(1,2,4)} AB) \\
= & \max_{B^{(1,2,3), A^{(1,4)}}} r(ABB^t A^t AB - ABB^{(1,2,4)} A^{(1,4)} AB) \\
= & \max_{B^{(1,4), A^{(1,2,3)}}} r(ABB^t A^t AB - ABB^{(1,4)} A^{(1,3,4)} AB) \\
= & \max_{B^{(1,4), A^{(1,2,4)}}} r(ABB^t A^t AB - ABB^{(1,4)} A^{(1,2,4)} AB) \\
= & \max_{B^{(1,4), A^{(1,4)}}} r(ABB^t A^t AB - ABB^{(1,4)} A^{(1,4)} AB) \\
= & \max_{B^{(1,4), A^{(1,2)}}} r(ABB^t A^t AB - ABB^{(1,2)} A^{(1,4)} AB) \\
= & \max_{B^{(1,4), A^{(1)}}} r(ABB^t A^t AB - ABB^{(1,2)} A^{(1)} AB) \\
= & \max_{B^{(1,1), A^{(1,2,4)}}} r(ABB^t A^t AB - ABB^{(1)} A^{(1,3,4)} AB) \\
= & \max_{B^{(1,1), A^{(1,2,4)}}} r(ABB^t A^t AB - ABB^{(1)} A^{(1,2,4)} AB) \\
= & \max_{B^{(1,1), A^{(1,4)}}} r(ABB^t A^t AB - ABB^{(1)} A^{(1,4)} AB) \\
= & \max_{B^{(1,1), A^{(1)}}} r(ABB^t A^t AB - ABB^{(1)} A^{(1)} AB) \\
= & r(M) + r(AB) - r(A) - r(B). \\
\end{align*}
\]

(3.13)
Proof. Applying (2.21) to (2.12) and simplifying, we obtain

\[ r(AB - ABB^tA^tAB) = r(AB + ABPJ^tQAB) \]

\[ = r \begin{bmatrix} B^*A^* B^*BB^* \noalign{\hline} A^*AA^* & 0 & A^*AB \\ \noalign{\hline} 0 & ABB^* & -AB \end{bmatrix} - r(A) - r(B) \]

\[ = r \begin{bmatrix} B^*A^* B^*B \noalign{\hline} AA^* & AB \noalign{\hline} 0 & AB \end{bmatrix} - r(A) - r(B) \]

\[ = r \begin{bmatrix} B^*A^* B^*B \noalign{\hline} AA^* & AB \noalign{\hline} 0 & AB \end{bmatrix} + r(AB) - r(A) - r(B) \]

\[ = r(\begin{bmatrix} A \\ B^* \end{bmatrix} [A^*, B^*]) + r(AB) - r(A) - r(B) \]

\[ = r(M) + r(AB) - r(A) - r(B), \quad (3.14) \]

Thus establishing (3.9).

Applying (2.47) to (3.5) gives

\[ \max_W r(AB - ABB^tA^tAB - ABB^tF_AWAB) = \min \{ r(AB - ABB^tA^tAB, ABB^tF_A), r(AB) \}, \quad (3.15) \]

\[ \min_W r(AB - ABB^tA^tAB - ABB^tF_AWAB) = r(AB - ABB^tA^tAB, ABB^tF_A) - r(AB)^tF_A, \quad (3.16) \]

where by (2.17) and (2.18),

\[ r[AB - ABB^tA^tAB, ABB^tF_A] = r \begin{bmatrix} AB - ABB^tA^tAB & ABB^t \noalign{\hline} 0 & A \end{bmatrix} - r(A) \]

\[ = r \begin{bmatrix} AB & ABB^t \noalign{\hline} 0 & A \end{bmatrix} - r(A) = r \begin{bmatrix} AB & 0 \noalign{\hline} 0 & AE_B \end{bmatrix} - r(A) \]

\[ = r(AB) + r(E_BA^*) - r(A) \]

\[ = r(M) + r(AB) - r(A) - r(B), \quad (3.17) \]

and

\[ r(AB)^tF_A = r(M) + r(AB) - r(A) - r(B). \quad (3.18) \]

Substituting (3.17) and (3.18) into (3.15) and (3.16), and noticing that \( r(M) + r(AB) - r(A) - r(B) \leq r(AB) \), we obtain (3.10).

By a similar approach to (3.6), we have

\[ \max_W r(AB - ABB^tA^tAB - ABVE_BA^tAB) = r(M) + r(AB) - r(A) - r(B), \]

as required for (3.11).

Applying (2.47) and (3.18) to (3.7) and (3.8) gives

\[ \max_W r(AB^tF_AWAB) = r(AB)^tF_A = r(M) + r(AB) - r(A) - r(B), \quad (3.19) \]

\[ \max_V r(ABVE_BA^tAB) = r(E_BA^tAB) = r(M) + r(AB) - r(A) - r(B), \quad (3.20) \]

establishing (3.12) and (3.13). \[ \square \]

4 Main Results

Theorem 4.1. Let \( A \in \mathbb{C}^{m \times n} \) and \( B \in \mathbb{C}^{n \times p} \), and denote \( M = [A^*, B^*] \). Then, the following statements are equivalent:

1. \( B^tA^t \in \{(AB)^{1}\} \), namely, \( ABB^tA^tAB = AB \).
(2) $B^iA^i \in \{(AB)^{(2)}\}$, namely, $B^iA^iABB^iA^i = B^iA^i$.

(3) $B^iA^i \in \{(AB)^{(1,2)}\}$.

(4) $B^iA^* \in \\{([A^*]B)^{(1)}\}$.

(5) $B^*A^i \in \{[A(B^i)]^*\}$.

(6) $B^i(A^*A)^i \in \{(A^*AB)^{(1)}\}$ and/or $(BB^i)^iA^i \in \{(ABB^i)^{(1)}\}$.

(7) $(BB^*^i)^i(A^*A)^i \in \{(A^*ABB^*)^{(1)}\}$ and/or $(A^*A)^i(BB^*)^i \in \{(BB^*A^*A)^{(1)}\}$.

(8) $[(BB^*^i)^i(A^*A)^{(1)}]^{i^{i^{(1)}}} \in \{([A^*A]^{(1)}(BB^*^i)^{(1)}^{(2)}\}^{(1)}$ and/or $[(A^*A)^{(1)}]^{i^{(1)}}(BB^*)^{(1)}^{(2)}\}^{(1)} \in \{([BB^*]^{{(1)}}^{(1)}(A^*A)^{(1)}^{(2)}\}^{(1)}$.

(9) $(BB^*)^i((AA^*A)^i) \in \{(AA*ABB^*B)^{(1)}\}$.

(10) $B^iA^iA \in \{(A^*AB)^{(1)}\}$ and/or $BB^iA^i \in \{(ABB)^{(1)}\}$.

(11) $BB^iA^iA \in \{(A^*ABB)^{(1)}\}$ and/or $A^iABB^i \in \{(BB^iA^iA)^{(1)}\}$.

(12) $BB^iF_A \in \{(F_ABB)^{(1)}\}$ and/or $F_ABB^i \in \{(BB^iF_A)^{(1)}\}$.

(13) $E_BA^iA \in \{(A^iAE_B)^{(1)}\}$ and/or $A^iAE_B \in \{(E_BA^iA)^{(1)}\}$.

(14) $E_BF_A \in \{(F_EF_A)^{(1)}\}$ and/or $F_AF_E \in \{(E_AF)^{(1)}\}$.

(15) $(A^iAB)^{(1)} = BB^iA^iA$ and/or $(BB^iA^iA)^i = A^iABB^i$.

(16) $(BB^iF_A) = BB^iF_A$ and/or $(F_AF_B)^i = F_AF_B$.

(17) $(A^iAE_B) = A^iAE_B$ and/or $(E_BA^iA)^i = E_BA^iA$.

(18) $(F_AF_E)^i = F_AF_E$ and/or $(E_AF)^i = E_AF_E$.

(19) $A^iABB^i = BB^iA^iA$ and/or $F_ABB^i = BB^iF_A$, $A^iAE_B = E_BA^iA$, $F_AF_E = E_AF_E$.

(20) $(A^iABB)^{(2)} = A^iABB^i$ and/or $(BB^iA^iA)^2 = BB^iA^iA$.

(21) $(F_ABB)^2 = F_ABB^i$ and/or $(BB^iF_A)^2 = BB^iF_A$ and/or $(BB^iF_A)^2 = BB^iF_A$.

(22) $(A^iAE_B)^2 = A^iAE_B$ and/or $(E_BA^iA)^2 = E_BA^iA$.

(23) $(F_AF_E)^2 = F_AF_E$ and/or $(E_AF)^2 = E_AF_E$.

(24) $(ABB^iA^i)^2 = ABB^iA^i$ and/or $(B^iA^i)^2 = B^iA^iAB$.

(25) $(AB^iA^i)^2 = AB^iA^i$ and/or $(B^iF_A)^2 = B^iF_A$.

(26) $E_BA^iABB^i = 0$ and/or $BB^iA^iAE_B = 0$.

(27) $A^iABB^iF_A = 0$ and/or $F_ABB^iA^i = 0$.


(29) $[F_A, B][F_A, B]^i = FA + BB^i - F_ABB^i$ and/or $[F_A, B]^i[F_A, B]^i = FA + BB^i - BB^iF_A$.

(30) $[A^*, E_B][A^*, E_B]^i = A^iA + E_B - A^iAE_B$ and/or $[A^*, E_B][A^*, E_B]^i = A^iA + E_B - E_BA^iA$.

(31) $[F_A, E_B][F_A, E_B]^i = F_A + E_B - F_AE_B$ and/or $[F_A, E_B][F_A, E_B]^i = F_A + E_B - E_BF_A$.

(32) $\{B^iA^i\} \subseteq \{(AB)^{(1)}\}$.

(33) $\{B^iA^{(1,2)}\} \subseteq \{(AB)^{(1)}\}$.

(34) $\{B^iA^{(1,3)}\} \subseteq \{(AB)^{(1)}\}$.

(35) $\{B^iA^{(1,4)}\} \subseteq \{(AB)^{(1)}\}$.

(36) $\{B^iA^{(1,2,3)}\} \subseteq \{(AB)^{(1)}\}$.

(37) $\{B^iA^{(1,2,4)}\} \subseteq \{(AB)^{(1)}\}$.
\( \{ B^\dagger A^{(1,3,4)} \} \subseteq \{ (AB)^{(1)} \} \).
\( \{ B^{(1,3,4)} A^{(1)} \} \subseteq \{ (AB)^{(1)} \} \).
\( \{ B^{(1,3,4)} A^{(1,2)} \} \subseteq \{ (AB)^{(1)} \} \).
\( \{ B^{(1,3,4)} A^{(1,3)} \} \subseteq \{ (AB)^{(1)} \} \).
\( \{ B^{(1,3,4)} A^{(1,4)} \} \subseteq \{ (AB)^{(1)} \} \).
\( \{ B^{(1,3,4)} A^{(1,2,3)} \} \subseteq \{ (AB)^{(1)} \} \).
\( \{ B^{(1,3,4)} A^{(1,2,4)} \} \subseteq \{ (AB)^{(1)} \} \).
\( \{ B^{(1,3,4)} A^{(1,3,4)} \} \subseteq \{ (AB)^{(1)} \} \).
\( \{ B^{(1,3,4)} A^{(1)} \} \subseteq \{ (AB)^{(1)} \} \).
\( \{ B^{(1,2,4)} A^{(1,1)} \} \subseteq \{ (AB)^{(1)} \} \).
\( \{ B^{(1,2,4)} A^{(1,2,4)} \} \subseteq \{ (AB)^{(1)} \} \).
\( \{ B^{(1,2,4)} A^{(1,3,4)} \} \subseteq \{ (AB)^{(1)} \} \).
\( \{ B^{(1,2,4)} A^{(1)} \} \subseteq \{ (AB)^{(1)} \} \).
\( \{ B^{(1,2,3)} A^{(1)} \} \subseteq \{ (AB)^{(1)} \} \).
\( \{ B^{(1,2,3)} A^{(1,2)} \} \subseteq \{ (AB)^{(1)} \} \).
\( \{ B^{(1,2,3)} A^{(1,3)} \} \subseteq \{ (AB)^{(1)} \} \).
\( \{ B^{(1,2,3)} A^{(1,4)} \} \subseteq \{ (AB)^{(1)} \} \).
\( \{ B^{(1,2,3)} A^{(1,2,3)} \} \subseteq \{ (AB)^{(1)} \} \).
\( \{ B^{(1,2,3)} A^{(1,2,4)} \} \subseteq \{ (AB)^{(1)} \} \).
\( \{ B^{(1,2,3)} A^{(1,3,4)} \} \subseteq \{ (AB)^{(1)} \} \).
\( \{ B^{(1,2,3)} A^{(1)} \} \subseteq \{ (AB)^{(1)} \} \).
\( \{ B^{(1,4)} A^{(1)} \} \subseteq \{ (AB)^{(1)} \} \).
\( \{ B^{(1,4)} A^{(1,2)} \} \subseteq \{ (AB)^{(1)} \} \).
\( \{ B^{(1,4)} A^{(1,2,4)} \} \subseteq \{ (AB)^{(1)} \} \).
\( \{ B^{(1,4)} A^{(1,3,4)} \} \subseteq \{ (AB)^{(1)} \} \).
\( \{ B^{(1,4)} A^{(1)} \} \subseteq \{ (AB)^{(1)} \} \).
\( \{ B^{(1,3)} A^{(1)} \} \subseteq \{ (AB)^{(1)} \} \).
\( \{ B^{(1,3)} A^{(1,2)} \} \subseteq \{ (AB)^{(1)} \} \).
\( \{ B^{(1,3)} A^{(1,3)} \} \subseteq \{ (AB)^{(1)} \} \).
\( \{ B^{(1,3)} A^{(1,4)} \} \subseteq \{ (AB)^{(1)} \} \).
\( \{ B^{(1,3)} A^{(1,2,3)} \} \subseteq \{ (AB)^{(1)} \} \).
\( \{ B^{(1,3)} A^{(1,2,4)} \} \subseteq \{ (AB)^{(1)} \} \).
\( \{ B^{(1,3)} A^{(1,3,4)} \} \subseteq \{ (AB)^{(1)} \} \).
\( \{ B^{(1,3)} A^{(1)} \} \subseteq \{ (AB)^{(1)} \} \).
\( \{ B^{(1,2)} A^{(1)} \} \subseteq \{ (AB)^{(1)} \} \).
\( \{ B^{(1,2)} A^{(1,2,4)} \} \subseteq \{ (AB)^{(1)} \} \).
\( \{ B^{(1\cdot2)}A^{(1\cdot3\cdot4)} \} \subseteq \{(AB)^{(1)}\} \).

(74) \( \{ B^{(1\cdot2)}A^{(1)} \} \subseteq \{(AB)^{(1)}\} \).

(75) \( \{ B^{(1\cdot2\cdot4)} \} \subseteq \{(AB)^{(1)}\} \).

(76) \( \{ B^{(1\cdot2\cdot4)} \} \subseteq \{(AB)^{(1)}\} \).

(77) \( \{ B^{(1\cdot2\cdot4)} \} \subseteq \{(AB)^{(1)}\} \).

(78) \( \{ B^{(1\cdot2\cdot4)} \} \subseteq \{(AB)^{(1)}\} \).

(79) \( ABB^{(1\cdot3)}A^{(1)}AB \) is invariant with respect to the choice of \( A^{(1)} \) and \( B^{(1\cdot3)} \).

(80) \( ABB^{(1\cdot4)}A^{(1\cdot4)}AB \) is invariant with respect to the choice of \( A^{(1\cdot4)} \) and \( B^{(1\cdot3)} \).

(81) \( r(A*, B) = r(A) + r(B) - r(AB) \).

(82) \( r(F_A, B) = r(F_A) + r(B) - r(F_A B) \).

(83) \( r(A, E_B) = r(A) + r(E_B) - r(AE_B) \).

(84) \( r(F_A, E_B) = r(F_A) + r(E_B) - r(F_A E_B) \).

(85) \( r(I_n - A^t ABB^{(1)}) = n - r(A^t ABB^{(1)}) \) and/or \( r(I_n - B^{(1)} A^t A) = n - r(B^{(1)} A^t A) \).

(86) \( r(I_n - F_A BB^{(1)}) = n - r(F_A BB^{(1)}) \) and/or \( r(I_n - B^{(1)} F_A) = n - r(B^{(1)} F_A) \).

(87) \( r(I_n - A^t A E_B) = n - r(A^t A E_B) \) and/or \( r(I_n - E_B A^t A) = n - r(E_B A^t A) \).

(88) \( r(I_n - F_A E_B) = n - r(F_A E_B) \) and/or \( r(I_n - E_B F_A) = n - r(E_B F_A) \).

(89) \( r(I_m - ABB^{(1)} A^t) = m - r(AB^{(1)} A^t) \) and/or \( r(I_p - B^{(1)} A^t AB) = p - r(B^{(1)} A^t AB) \).

(90) \( r(I_m - A E_B A^t) = m - r(A E_B A^t) \) and/or \( r(I_p - B^{(1)} F_A) = p - r(B^{(1)} F_A) \).

(91) \( \dim[\mathcal{A}(A^*) \cap \mathcal{R}(B)] = r(AB) \) and/or \( \dim[\mathcal{R}(F_A) \cap \mathcal{R}(B)] = r(F_A B) \), \( \dim[\mathcal{A}(A^*) \cap \mathcal{R}(E_B)] = r(A E_B) \), \( \dim[\mathcal{R}(F_A) \cap \mathcal{R}(E_B)] = r(F_A E_B) \).

(92) \( \mathcal{R}(A^t ABB^{(1)}) \subseteq \mathcal{R}(BB^{(1)}) \) and/or \( \mathcal{R}(BB^{(1)} A^t A) \subseteq \mathcal{R}(A^t A) \).

(93) \( \mathcal{R}(F_A BB^{(1)}) \subseteq \mathcal{R}(BB^{(1)}) \) and/or \( \mathcal{R}(BB^{(1)} F_A) \subseteq \mathcal{R}(F_A) \).

(94) \( \mathcal{R}(A^t A E_B) \subseteq \mathcal{R}(E_B) \) and/or \( \mathcal{R}(E_B A^t A) \subseteq \mathcal{R}(A^t A) \).

(95) \( \mathcal{R}(F_A E_B) \subseteq \mathcal{R}(E_B) \) and/or \( \mathcal{R}(E_B F_A) \subseteq \mathcal{R}(F_A) \).

(96) \( \mathcal{R}(A^t ABB^{(1)}) = \mathcal{R}(A^t A) \cap \mathcal{R}(BB^{(1)}) \) and/or \( \mathcal{R}(BB^{(1)} A^t A) = \mathcal{R}(A^t A) \cap \mathcal{R}(BB^{(1)}) \).

(97) \( \mathcal{R}(F_A BB^{(1)}) = \mathcal{R}(F_A) \cap \mathcal{R}(BB^{(1)}) \) and/or \( \mathcal{R}(BB^{(1)} F_A) = \mathcal{R}(F_A) \cap \mathcal{R}(BB^{(1)}) \).

(98) \( \mathcal{R}(A^t A E_B) = \mathcal{R}(A^t A) \cap \mathcal{R}(E_B) \) and/or \( \mathcal{R}(E_B A^t A) = \mathcal{R}(A^t A) \cap \mathcal{R}(E_B) \).

(99) \( \mathcal{R}(F_A E_B) = \mathcal{R}(F_A) \cap \mathcal{R}(E_B) \) and/or \( \mathcal{R}(E_B F_A) = \mathcal{R}(F_A) \cap \mathcal{R}(E_B) \).

(100) \( \mathcal{R}(A^t ABB^{(1)}) = \mathcal{R}(BB^{(1)} A^t A) \) and/or \( \mathcal{R}(A^t A E_B) = \mathcal{R}(E_B A^t A) \), \( \mathcal{R}(F_A BB^{(1)}) = \mathcal{R}(BB^{(1)} F_A) \), \( \mathcal{R}(F_A E_B) \). \( \mathcal{R}(E_B F_A) = \mathcal{R}(E_B F_A) \).

(101) \( \mathcal{A}(AB) \cap \mathcal{A}(E_B) = \{0\} \) and/or \( \mathcal{A}(F_A B) \cap \mathcal{A}(F_A E_B) = \{0\} \), \( \mathcal{R}(B^* A^*) \cap \mathcal{R}(B^* F_A) = \{0\} \), \( \mathcal{R}(E_B A^*) \cap \mathcal{R}(E_B F_A) = \{0\} \).

**Proof.** Setting both sides of (3.9) equal to zero leads to the equivalence of (1) and (81). By (2.17),

\[
\begin{align*}
    r(F_A) &= n - r(A), \quad r(F_A B) = r[A^*, B] - r(A^*), \\
    r(E_B) &= n - r(B), \quad r(A E_B) = r[A^*, B] - r(B), \\
    r(F_A, B) &= r(F_A) + r(A^t A B) = n - r(A) + r(AB), \\
    r(A^*, E_B) &= r(A B B^\dagger) + r(E_B) = n - r(B) + r(AB), \\
    r(F_A, E_B) &= r(F_A) + r(A^t A E_B) = n - r(A) - r(B) - r[A^*, B], \\
    r(F_A E_B) &= r[A^*, E_B] - r(A^* - n - r(A) - r(B) + r(AB). 
\end{align*}
\]
Substituting (4.1)–(4.6) into the three rank equalities in (82), (83), and (84) and simplifying, we obtain the equivalence of (81)–(84).

Applying (3.9) to $B^\dagger A^\dagger - B^\dagger A^\dagger ABB^\dagger A^\dagger$ and simplifying by (2.29), (2.30), (2.32), and (2.37), we obtain

\begin{equation}
\begin{aligned}
    r(B^\dagger A^\dagger - B^\dagger A^\dagger ABB^\dagger A^\dagger) &= r\left((B^\dagger)^* + r(B^\dagger A^\dagger) - r(A^\dagger)\right) \\
    &= r[A^*, B] + r(AB) - r(B) - r(A).
\end{aligned}
\end{equation}

Setting both side of (4.7) equal to zero, we obtain the equivalence of (2) and (81).

Both (1) and (2) imply (3). Conversely, (3) implies (1) and (2).

The following rank formulas can be established by a similar approach

\begin{equation}
\begin{aligned}
    r[(A^\dagger)^*B - (A^\dagger)^*BB^\dagger A^*(A^\dagger)^*B] &= r(M) + r(AB) - r(A) - r(B), \\
    r[A(B^\dagger)^* - A(B^\dagger)^*B^*A^\dagger(A^\dagger)^*B^*] &= r(M) + r(AB) - r(A) - r(B), \\
    r[(A^\dagger A)B - (A^\dagger A)BB^\dagger(A^\dagger A)B] &= r(M) + r(AB) - r(A) - r(B), \\
    r[A(BB^\dagger) - A(BB^\dagger)(BB^\dagger)^\dagger A^\dagger A(BB^\dagger)] &= r(M) + r(AB) - r(A) - r(B), \\
    r[(A^\dagger A)(BB^\dagger) - (A^\dagger A)(BB^\dagger)(BB^\dagger)^\dagger(A^\dagger A)(BB^\dagger)] &= r(M) + r(AB) - r(A) - r(B), \\
    r\{ (A^\dagger A)^{1/2}(BB^\dagger)^{1/2} - (A^\dagger A)^{1/2}(BB^\dagger)^{1/2}(BB^\dagger)^{1/2}[(A^\dagger A)^{1/2}(BB^\dagger)^{1/2}] \} &= r(M) + r(AB) - r(A) - r(B), \\
    r[(AA^\dagger A)(BB^\dagger B) - (AA^\dagger A)(BB^\dagger B)(BB^\dagger B)^\dagger(AA^\dagger A)(BB^\dagger B)] &= r(M) + r(AB) - r(A) - r(B), \\
    r[A^\dagger AB - (A^\dagger AB)B^\dagger A^\dagger A^\dagger A^\dagger A^\dagger A^\dagger A^\dagger A^\dagger A^\dagger] &= r(M) + r(AB) - r(A) - r(B), \\
    r[A^\dagger ABB^\dagger - (A^\dagger ABB^\dagger)BB^\dagger A^\dagger A(A^\dagger ABB^\dagger)] &= r(M) + r(AB) - r(A) - r(B).
\end{aligned}
\end{equation}

Setting all both sides of (4.8)–(4.17) equal to zero leads to the equivalence of (2)–(11) and (81).

The following rank formulas can be established by a similar approach

\begin{equation}
\begin{aligned}
    r[F_ABB^\dagger - (F_ABB^\dagger)BB^\dagger F_A(F_ABB^\dagger)] &= r[F_A, B] - r(F_A) - r(B) + r(F_A B), \\
    r[A^\dagger AE_B - (A^\dagger AE_B)E_B A^\dagger A(A^\dagger AE_B)] &= r[A, E_B] - r(A) - r(E_B) + r(AE_B), \\
    r[F_AE_B - (F_AE_B)E_B F_A(F_AE_B)] &= r[F_A, E_B] - r(F_A) - r(E_B) + r(F_A AE_B).
\end{aligned}
\end{equation}

Setting the both sides of (4.18)–(4.20) equal to zero leads to the equivalence of (12)–(14) and (82)–(84).
Applying (2.22) and simplifying by (2.17), we obtain

\[
\begin{align*}
 r(I_n - BB^\dagger A^\dagger A) &= r(I_n - A^\dagger ABB^\dagger) \\
 &= r\left[\begin{array}{c}
 AA^* & ABB^\dagger \\
 A^* & I_n
\end{array}\right] - r(A) \\
 &= r\left[\begin{array}{c}
 AA^* - ABB^\dagger A^* \\
 0 & I_n
\end{array}\right] - r(A) \\
 &= n + r( AA^* - ABB^\dagger A^* ) - r(A) \\
 &= n + r\left[\begin{array}{c}
 B^* B & B^* A^* \\
 AB & AA^*
\end{array}\right] - r(A) - r(B) \\
 &= n + r(\langle A, B^* \rangle [A, B^*] ) - r(A) - r(B) \\
 &= n + r(M) - r(A) - r(B),
\end{align*}
\]

(4.21)

\[
\begin{align*}
 r(I_m - ABB^\dagger A^\dagger) &= r\left[\begin{array}{c}
 B^* B & B^* A^\dagger \\
 AB & I_m
\end{array}\right] - r(B) \\
 &= r\left[\begin{array}{c}
 B^* B - B^* A^\dagger AB \\
 0 & I_m
\end{array}\right] - r(B) \\
 &= r( B^* F_A B ) - r(B) + m \\
 &= r(F_A B) - r(B) + m \\
 &= r(M) - r(A) - r(B) + m,
\end{align*}
\]

(4.22)

\[
\begin{align*}
 r(I_p - B^\dagger A^\dagger AB) &= r\left[\begin{array}{c}
 AA^* & AB \\
 B^\dagger A^* & I_p
\end{array}\right] - r(A) \\
 &= r\left[\begin{array}{c}
 AA^* - ABB^\dagger A^* \\
 0 & I_p
\end{array}\right] - r(A) \\
 &= r( AE_B A^* ) - r(A) + p \\
 &= r(E_B A^* ) - r(A) + p \\
 &= r(M) - r(A) - r(B) + p.
\end{align*}
\]

(4.23)

Combining (4.21)–(4.23) with (2.39)–(2.42) yields

\[
\begin{align*}
 r(I_n - BB^\dagger A^\dagger A) &= n + r( BB^\dagger A^\dagger A ) - r(M) - r(A) - r(B) + r(AB), \\
 r(I_m - ABB^\dagger A^\dagger) &= m + r( ABB^\dagger A^\dagger ) - r(M) - r(A) - r(B) + r(AB), \\
 r(I_p - B^\dagger A^\dagger AB) &= p + r( B^\dagger A^\dagger AB ) - r(M) - r(A) - r(B) + r(AB).
\end{align*}
\]

(4.24)–(4.26)

Setting both sides of (4.24)–(4.26) equal to zero leads to the equivalence of \(85\), \(89\), and \(81\).

Replacing \(A^\dagger A\) with \(F_A\) in \(85\) leads to the equivalence of \(86\) and \(82\). Replacing \(BB^\dagger\) with \(E_B\) in \(85\) leads to the equivalence of \(87\) and \(83\). Replacing \(A^\dagger A\) with \(F_A\) and \(BB^\dagger\) with \(E_B\) in \(85\) leads to the equivalence of \(88\) and \(84\).

Replacing \(A^\dagger A\) with \(F_A\) and \(BB^\dagger\) with \(E_B\) in \(89\) respectively leads to the equivalence of \(90\), \(82\), and \(83\).

Applying (2.24), (4.21)–(4.23) to \((A^\dagger ABB^\dagger)^2 - A^\dagger ABB^\dagger\), we obtain

\[
\begin{align*}
 r[(A^\dagger ABB^\dagger)^2 - A^\dagger ABB^\dagger] &= r(I_n - A^\dagger ABB^\dagger) + r(A^\dagger ABB^\dagger) - n \\
 &= r(M) - r(A) - r(B) + r(AB), \\
 r[(BB^\dagger A^\dagger A)^2 - BB^\dagger A^\dagger A] &= r(I_n - BB^\dagger A^\dagger A) + r(BB^\dagger A^\dagger) - n \\
 &= r(M) - r(A) - r(B) + r(AB), \\
 r[(ABB^\dagger A^\dagger )^2 - ABB^\dagger A^\dagger] &= r(I_m - ABB^\dagger A^\dagger ) + r(ABB^\dagger A^\dagger ) - m \\
 &= r(M) - r(A) - r(B) + r(AB), \\
 r[(B^\dagger A^\dagger AB)^2 - B^\dagger A^\dagger AB] &= r(I_p - B^\dagger A^\dagger AB) + r(B^\dagger A^\dagger AB) - p \\
 &= r(M) - r(A) - r(B) + r(AB).
\end{align*}
\]

(4.27)–(4.30)

Setting both sides of (4.27)–(4.30) equal to zero leads to the equivalence of \(20\), \(24\), and \(81\).
Applying (2.23) \((A^1ABB^1)^\dagger - BB^1A^1A\), we obtain
\[
\begin{align*}
r[(A^1ABB^1)^\dagger - BB^1A^1A] &= r[(A^1ABB^1)^\dagger - (A^1ABB^1)^*] \\
&= r[(A^1ABB^1) - (A^1ABB^1)(A^1ABB^1)^*(A^1ABB^1)] \\
&= r[(A^1ABB^1) - (A^1ABB^1)^2] \\
&= r[(A^1ABB^1 + r[I_n - A^1ABB^1] - n \\
&= r(M) + r(AB) - r(A) - r(B). \\
&= 2r(A^1A, BB^1) - 2r(A^1A) - 2r(BB^1) + 2r(A^1ABB^1) \\
&= 2r(M) - 2r(A) - 2r(AB) + 2r(AB).
\end{align*}
\]
(4.31)

Setting both sides of (4.31) equal to zero leads to the equivalence of \((15)\) and \((81)\).

Replacing \(A^1A\) with \(F_A\) in \((11)\) leads to the equivalence of \((12)\) and \((82)\). Replacing \(BB^1\) with \(E_B\) in \((11)\) leads to the equivalence of \((13)\) and \((83)\). Replacing \(A^1A\) with \(F_A\) and \(BB^1\) with \(E_B\) in \((11)\) leads to the equivalence of \((14)\) and \((84)\).

Replacing \(A^1A\) with \(F_A\) in \((15)\) leads to the equivalence of \((16)\) and \((82)\). Replacing \(BB^1\) with \(E_B\) in \((15)\) leads to the equivalence of \((17)\) and \((83)\). Replacing \(A^1A\) with \(F_A\) and \(BB^1\) with \(E_B\) in \((15)\) leads to the equivalence of \((18)\) and \((84)\).

Applying (2.25) to \(A^1ABB^1 - BB^1A^1A\) and simplifying by (2.32) and (2.40), we obtain
\[
\begin{align*}
r(A^1ABB^1 - BB^1A^1A) &= 2r[A^1A, BB^1] - 2r(A^1A) - 2r(BB^1) + 2r(A^1ABB^1) \\
&= 2r(M) - 2r(A) - 2r(AB) + 2r(AB).
\end{align*}
\]
(4.32)

Setting both sides of (4.32) equal to zero leads to the equivalence of \((19)\) and \((81)\).

Replacing \(A^1A\) with \(F_A\) in \((20)\) leads to the equivalence of \((21)\) and \((82)\). Replacing \(BB^1\) with \(E_B\) in \((20)\) leads to the equivalence of \((22)\) and \((83)\). Replacing \(A^1A\) with \(F_A\) and \(BB^1\) with \(E_B\) in \((20)\) leads to the equivalence of \((23)\) and \((84)\).

Replacing \(A^1A\) with \(F_A\) and \(BB^1\) with \(E_B\) in \((24)\) respectively leads to the equivalence of \((25)\), \((82)\), and \((83)\).

Applying (2.17) to and simplifying by Lemma 2.3(c), we obtain
\[
\begin{align*}
r(E_B A^1ABB^1) &= r(BB^1A^1AE_B) = r[B, A^1ABB^1] - r(B) \\
&= r[B - A^1AB, A^1AB] - r(B) \\
&= r(B - A^1AB) + r(A^1AB) - r(B) \\
&= r(M) - r(A) - r(B) + r(AB),
\end{align*}
\]
(4.33)
\[
\begin{align*}
r(A^1ABB^1F_A) &= r(F_A B B^1A^1A) = r[A^*, BB^1A^1A] - r(A) \\
&= r[A^* - BB^1A^*, BB^1A^*] - r(A) \\
&= r(A^* - BB^1A^*) + r(BB^1A^*) - r(A) \\
&= r(M) - r(A) - r(B) + r(AB).
\end{align*}
\]
(4.34)

Setting all sides of (4.33) and (4.34) equal to zero leads to the equivalence of \((26)\), \((27)\), and \((81)\).

The following rank identities were proved in [16]
\[
\begin{align*}
r([A^*, B][A^*, B]^\dagger - A^1A - BB^1 + A^1ABB^1) &= r(M) - r(A) - r(B) + r(AB).
\end{align*}
\]

Setting three sides of these two identities equal to zero leads to the equivalence of \((29)\) and \((81)\). Replacing \(A^1A\) with \(F_A\) in \((29)\) leads to the equivalence of \((30)\) and \((82)\). Replacing \(BB^1\) with \(E_B\) in \((29)\) leads to the equivalence of \((31)\) and \((83)\). Replacing \(A^1A\) with \(F_A\) and \(BB^1\) with \(E_B\) in \((29)\) leads to the equivalence of \((32)\) and \((84)\).

Setting all sides of \((3.10)\) and \((3.11)\) equal to zero leads to the equivalence of \((32)\)–\((78)\), and \((81)\).

Setting all sides of \((3.12)\) and \((3.13)\) equal to zero leads to the equivalence of \((79)\), \((80)\), and \((81)\).

The equivalence of \((81)\)–\((84)\), and \((91)\) follows from the following well-known dimension formulas
\[
\begin{align*}
\dim[\mathcal{R}(A^*) \cap \mathcal{R}(B)] &= r(A) + r(B) - r[A^*, B], \\
\dim[\mathcal{R}(F_A) \cap \mathcal{R}(B)] &= r(F_A) + r(B) - r[F_A, B], \\
\dim[\mathcal{R}(A^*) \cap \mathcal{R}(E_B)] &= r(A) + r(E_B) - r[A^*, E_B], \\
\dim[\mathcal{R}(F_A) \cap \mathcal{R}(E_B)] &= r(F_A) + r(E_B) - r[F_A, E_B].
\end{align*}
\]

Applying Lemma 2.3(a) and (b), to \((26)\) and \((27)\) leads to the equivalence of \((26)\) and \((27)\), and \((92)\).
Replacing \( A^\dagger A \) with \( F_A \) in \( \langle 92 \rangle \) leads to the equivalence of \( \langle 93 \rangle \) and \( \langle 82 \rangle \). Replacing \( BB^\dagger \) with \( E_B \) in \( \langle 92 \rangle \) leads to the equivalence of \( \langle 94 \rangle \) and \( \langle 83 \rangle \). Replacing \( A^\dagger A \) with \( F_A \) and \( BB^\dagger \) with \( E_B \) in \( \langle 92 \rangle \) leads to the equivalence of \( \langle 95 \rangle \) and \( \langle 84 \rangle \).

It follows first from \( \langle 92 \rangle \) that

\[
\mathcal{R}(A^\dagger ABB^\dagger) \subseteq \mathcal{R}(A^\dagger A) \cap \mathcal{R}(BB^\dagger) \quad \text{and/or} \quad \mathcal{R}(BB^\dagger A^\dagger A) \subseteq \mathcal{R}(A^\dagger A) \cap \mathcal{R}(BB^\dagger). \tag{4.35}
\]

Furthermore, it follows first from \( \langle 81 \rangle \) that

\[
\dim[\mathcal{R}(A^\dagger A) \cap \mathcal{R}(BB^\dagger)] = r(A^\dagger A) + r(BB^\dagger) - r[A^\dagger A, BB^\dagger] = r(A^\dagger ABB^\dagger) = r(BB^\dagger A^\dagger A). \tag{4.36}
\]

Applying (2.26) to (4.35) and (4.36) leads to the range identities in \( \langle 96 \rangle \). Conversely, \( \langle 96 \rangle \) obviously implies \( \langle 92 \rangle \). The equivalence of \( \langle 93 \rangle \) and \( \langle 97 \rangle \), \( \langle 94 \rangle \) and \( \langle 98 \rangle \), and \( \langle 95 \rangle \) and \( \langle 99 \rangle \) can be established similarly.

By Lemma 2.3(c) and (2.19),

\[
\begin{align*}
r[A^\dagger ABB^\dagger, BB^\dagger A^\dagger A] &= r[(I_n - BB^\dagger)A^\dagger ABB^\dagger, BB^\dagger A^\dagger A] \\
&= r[(I_n - BB^\dagger)A^\dagger ABB^\dagger] + r(BB^\dagger A^\dagger A) \\
&= r[B, A^\dagger AB] - r(B) + r(AB) \\
&= r[(I_n - A^\dagger A)B, A^\dagger AB] - r(B) + r(AB) \\
&= r[(I_n - A^\dagger A)B] + r(A^\dagger AB) - r(B) + r(AB) \\
&= r(M) - r(A) - r(B) + 2r(AB). \tag{4.37}
\end{align*}
\]

Combining (4.37) with (2.39) and (2.40), we obtain

\[
r[A^\dagger ABB^\dagger, BB^\dagger A^\dagger A] - r(A^\dagger ABB^\dagger) = r[A^\dagger ABB^\dagger, BB^\dagger A^\dagger A] - r(BB^\dagger A^\dagger A) \\
= r(M) - r(A) - r(B) + r(AB). \tag{4.38}
\]

Setting both sides of (4.38) equal to zero, we obtain the equivalence of the first range equality in \( \langle 100 \rangle \) and \( \langle 81 \rangle \). Replacing \( A^\dagger A \) with \( F_A \) and \( BB^\dagger \) with \( E_B \) in the first range equality of \( \langle 100 \rangle \), respectively, leads to the equivalences of the second, third, and fourth range equalities in \( \langle 100 \rangle \) and those in \( \langle 82 \rangle \), \( \langle 83 \rangle \), and \( \langle 84 \rangle \).

Finally by (2.17),

\[
\dim[\mathcal{R}(AB) \cap \mathcal{R}(AE_B)] = r(AB) + r(AE_B) - r[AB, AE_B] = r[A^\dagger, B] - r(A) - r(B) + r(AB). \tag{4.39}
\]

Setting both sides of (4.39) equal to zero, we obtain the equivalence of the first range equality in \( \langle 101 \rangle \) and \( \langle 81 \rangle \). The equivalence of the third fact in \( \langle 101 \rangle \) with \( \langle 81 \rangle \) can be established similarly. Replacing \( A^\dagger A \) with \( F_A \) and \( BB^\dagger \) with \( E_B \) in the first and third facts of \( \langle 101 \rangle \), respectively, leads to the equivalences of the second and fourth facts in \( \langle 101 \rangle \), \( \langle 82 \rangle \), and \( \langle 83 \rangle \).

\[
\square
\]

## 5 Concluding remarks

We have collected/established, as a summery, hundreds of known/novel identifying conditions for the reverse-order law in (1.12) to hold by using various matrix rank/range formulas, which surprisingly link versatile results in matrix theory together, and thus provide us different views of the reverse-order laws and a solid foundation for further approach to (1.2). It is easy to use these results in various situations where generalized inverses of matrices are involved. Observe further that the last three reverse-order laws in (1.11) are special cases of the first in (1.11), it is possible to derive many novel/valuable identifying conditions for the last three reverse-order laws in (1.11) to hold based in the previous results.

The matrix rank methodology has been recognized as a direct and effective tool for establishing and characterizing various matrix equalities in matrix theory since 1970s, while this method was introduced in the establishments of reverse-order laws in 1990s by the present author (see [9, 10]). As mentioned in Section 1, it is a tremendous work to establish necessary and sufficient conditions for the thousands of possible equalities in (1.2) and their variations to hold. It is expected that more matrix rank formulas associated with (1.2) can be produced, and many new and unexpected necessary and sufficient conditions will be derived from the rank formulas for (1.2) to hold.

## References