Planar Maximum Coverage Location Problem with Partial Coverage and General Spatial Representation of Demand and Service Zones

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Abstract. We introduce a new generalization of the classical planar maximum coverage location problem (PMCLP) in which demand zones and service zone of each facility are represented by spatial objects such as circles, polygons, etc., and are allowed to be located anywhere in a continuous plane. In addition, we allow partial coverage in its true sense, i.e., covering only part of a demand zone is allowed and the coverage accrued in the objective function as a result of this is proportional to the demand of the covered area only, and denote this generalization by PMCLP-PC-G. We present a greedy algorithm and a pseudo-greedy algorithm for it, and showcase that the solution value corresponding to the greedy (or pseudo-greedy) solution is within a factor of $1 - 1/e$ (or $1 - 1/e^{\eta}$) of the optimal solution value where e is the base of natural logarithm and $\eta \leq 1$. These algorithmic and theoretical results generalize the similar results of Cornuéjols et al. [Management Science, 23(8): 789-810, 1977] and Hochbaum and Pathria [NRL, 45(6): 615-627, 1998] for special cases of PMCLP-PC-G, where demand zones are represented by points and partial coverage is not allowed, respectively, and the locations of facilities belong to a finite set of pre-specified candidate positions.

Keywords. planar maximum coverage location problem; greedy approach; pseudo-greedy algorithm; partial coverage; spatial demand representation

1. Introduction

The Maximum Coverage Location Problem (MCLP) is one of the well-known facility location problems in which p service facilities with known service range are to be located such that the total covered demand is maximized [7]. The classical forms of MCLP and several succeeding studies consider a finite set of pre-specified candidate positions for the facilities; whereas its generalization, referred to as the planar MCLP, allows the facilities to be located anywhere in the continuous plane. In most of the studies on the (planar) MCLP, point or aggregated representation of demand has been used, which is obtained by dividing a region into demand zones and aggregating the demand of each zone at a single representative point. As a result, the concept of coverage in this representation is assumed to be "binary", i.e., the demand zone is assumed to be either totally covered (if its representative point is covered by a service zone) or not covered at all (if its representative point is not covered by any service zone). This type of coverage is referred to as "binary coverage" or "all or nothing coverage". Furthermore, even when demand in the planar MCLP is represented by line segments or polygons in [14], binary coverage is still an assumption, i.e., a linear/polygonal demand zone is either completely covered or not covered at all. The motivation for this assumption is to make the problems manageable by modeling them as linear binary integer programs (LBIP).

1.1 Pressing issues in the state-of-the-art facility location planning

Although binary coverage is a simplifying assumption that makes (planar) MCLP manageable using LBIP, it is oftentimes an assumption far from reality, and thereby producing suboptimal location patterns which lead to poor decision making. This happens because in using the binary coverage assumption (even when it is applied to spatial objects such as lines or polygons), the solutions in which a demand zone is only partially covered are ignored and essentially approximated by solutions in which all demands zones are either covered completely or not covered at all. There is a substantial body of research demonstrating the impacts of the approximation errors, popularly known as region misrepresentation coverage errors which arise due to the binary coverage assumption and/or point representation of demand, on the stability of location model solutions, and therefore, researchers have been consistently calling for better approaches to represent demand [13, 14, 16]. Note that, assuming the range of coverage of a facility is d, the service zone of the facility, using Euclidean distance, is a circle with a radius of d centered at the facility, and using the rectilinear distance, is a diamond (a square rotated 45 degrees) with a diagonal of $2d$ centered at the facility. Church [5, 6] and Murray [12] accentuated that in the true optimal solution to a real world application, service zone of a facility and demand zones are represented by different spatial objects in which the demand zones may only be partially covered by the service facilities (refer to Figure 1).

1.2 Novel facility location modeling framework

Though attempts have been made to separately address non-point representation of demand and partial coverage, but to our knowledge no study has tackled them together. More specifically, few studies have considered MCLP with partial/gradual coverage and point representation of demand zones [2, 3, 8, 10], and planar MCLP with polygon representation of demand zones and binary coverage [14]. (See section 2 for more details.) Whereas so far, no study has considered planar MCLP problem with general representation of demand and service zones using two-dimensional spatial objects along with "partial coverage" in its true sense, i.e., when covering only part of a demand zone is allowed and the coverage accrued in the objective function as a result of this is proportional to the demand of the covered area only. Only Bansal and Kianfar [1] and Song et al. [15] developed exact algorithms for the planar MCLP where partial coverage is allowed, service and demand zones are defined by axis-parallel rectangles, and $p \geq 1$ (multiple facilities) and $p = 1$ (single facility), respectively. In this paper, we introduce a new generalization of the planar MCLP where partial coverage is allowed, and service and demand zones can be represented by any two-dimensional spatial object such as circle, polygon, etc. We denote this generalization of the planar MCLP by PMCLP-PC-G (see Figure 1(b)), and a special case of the PMCLP-PC-G with axis-parallel rectangular service and demand zones by PMCLP-PC-R (studied in [1]).

We define the PMCLP-PC-G as follows. Let $\mathcal{D} = \{d_i, i = 1, \ldots, n\}$ be a set of n spatial objects, referred to as demand zones (DZs), on the two-dimensional plane such that dimensions, shape, orientation, and location of each DZ are known. Also, there is a demand rate $v_i \in \mathbb{R}_+$ associated with each DZ d_i, i.e. demand per area-unit of d_i, $i \in \{1, \ldots, n\}$. Now, consider another set of spatial objects that provide coverage, referred to as service zones (SZs), and denote it by $\mathcal{S} := \{s_j, j = 1, \ldots, p\}$. We assume that the dimensions, shape, and orientation of each SZ are known but their locations are not known. For $j = 1, \ldots, p$, we identify the location of s_j by the coordinates of either center or a corner (if exists) of the SZ and denote it by $\left(x_{s_j}, y_{s_j}\right) \in \mathbb{R}^2$. In addition, we denote the collective set of positions of $s_1, \ldots, s_p$ by $(x_s, y_s) = (x_{s_1}, \ldots, x_{s_p}, y_{s_1}, \ldots, y_{s_p}) \in \mathbb{R}^{2p}$, and define function $f_i(x_s, y_s) : \mathbb{R}^{2p} \to \mathbb{R}_+$ that returns the covered demand of d_i if SZs are positioned at (x_s, y_s). More specifically, $f_i(x_s, y_s) = v_i A\left(d_i \cap \left(\cup_{j=1}^{p} s_j\right)\right)$ where function $A(.)$ returns the area of its argument, and $\cup$ and $\cap$ denote union and intersection operations on spatial objects. Therefore, the total demand covered by SZs positioned at (x_s, y_s) is given by $f(x_s, y_s) = \sum_{i=1}^{n} f_i(x_s, y_s)$. The PMCLP-PC-G is the problem of finding the location of SZs, (x_s, y_s), for which the total covered demand, i.e. $f(x_s, y_s)$, is maximized.

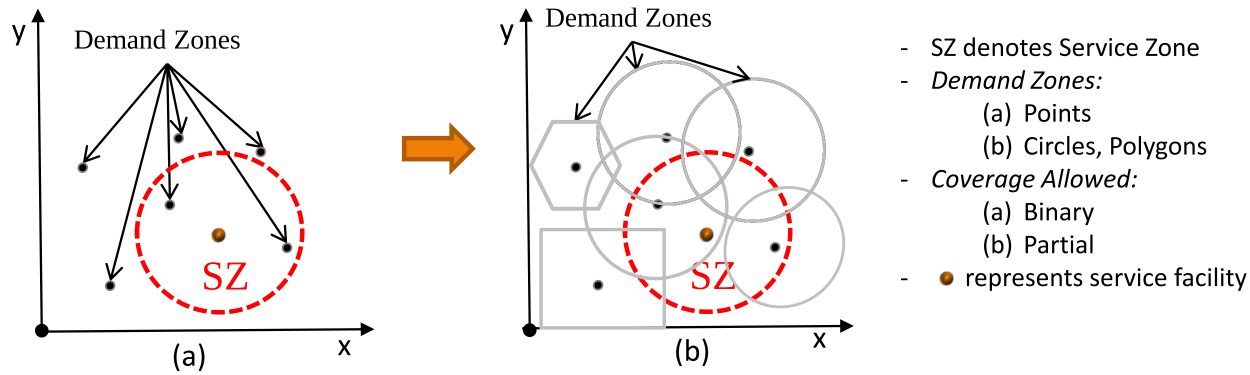

Figure 1: (a) Classical Planar MCLP [4]; (b) Planar MCLP with partial coverage with general representation of demand and service zones (PMCLP-PC-G). Observe that because of the binary coverage in 1(a), only two demand zones are completely covered by the SZ, whereas others are not at all covered. This causes errors because in reality, as shown in 1(b), five demand zones (represented by spatial objects) are partially covered by the SZ.

1.3 Significance of the PMCLP-PC-G

The impacts of the PMCLP-PC-G extend well beyond the confines of facility location problems as it also has direct applications in variety of emerging domains:

(a) *Telerobotics.* The advent of network-based telerobotic camera systems enable multiple participants or researchers in space exploration, health-care, and distance learning, to interact with a remote physical environment using shared resources. This system of p networked robotic cameras receives rectangular subregions as DZs from multiple users for monitoring [15, 17, 18]. Each subregion has an associated reward rate (weight per-unit area) that may depend on the priority of the user or the importance level associated with monitoring that subregion. The goal is to select the best view frame for the cameras (rectangular SZs) to maximize the total reward from the captured parts of the requested subregions. Interestingly, this problem is same as the PMCLP-PC-R. In the literature, Song et al. [15] study the PMCLP-PC-R for single camera, i.e. $p = 1$, and Xu et al. [17, 18] consider the PMCLP with rectangular DZs and binary coverage where the rectangular SZs are not allowed to overlap. The PMCLP-PC-G subsumes the aforementioned problems and their variants with circular or polygonal requested subregions.

(b) *Disaster Management.* In the events of natural disasters such as earthquake and hurricane, and human-made disasters such as oil spill in the oceans, satellite imaging is utilized to gather information about the highly impacted regions for the purpose of the disaster emergency planning [14, 15]. In this spatial optimization problem, the disaster regions (or DZs) are represented by polygons with per area-unit intensity of destruction (demand) associated to them, and the objective is to select the position of the best view frame of cameras (rectangular SZs) to identify the most destructed regions. The PMCLP-PC-G subsumes the foregoing problem along with its variants which utilizes varieties of satellite sensors (i.e. SZs of different shapes).

1.4 Organization and contribution of this paper

In Section 2, we discuss challenges in solving PMCLP-PC-G using well-known approaches for solving (planar) MCLP and its variants. We present a greedy algorithm and a pseudo-greedy algorithm for solving the PMCLP-PC-G (Section 3), and showcase that the solution value corresponding to the greedy solution is within a factor of $1-1/e$ of the optimal solution value where e is the base of natural logarithm. It is important to note that the greedy (or pseudo-greedy) approach utilizes an exact (or an η-approximate where $\eta \leq 1$) algorithm for the PMCLP-PC-G with $p = 1$ (single facility), thereby providing approximate solutions for the PMCLP-PC-G. Till date no such exact or η-approximate algorithm is known, except for PMCLP-PC-R with $p = 1$ [1, 15]. Nevertheless, whenever an exact or η-approximate algorithm will be developed in future for the single facility PMCLP-PC-G or its special cases, it can be automatically embedded within the greedy (or pseudo-greedy) algorithm presented in this paper to provide solutions for the multiple (p) facility version of the corresponding problem. This demonstrates the significance of these two algorithms. More importantly, as we will prove in Section 4, for a given $p \geq 1$, the greedy and pseudo-greedy algorithms provide solutions whose values are at least $1 - [(p-1)/p]^p$ and $1 - [(p-\eta)/p]^p$, respectively, times the optimal value for the PMCLP-PC-G instance. This extends the similar results of Cornuéjols et al. [9] and Hochbaum and Pathria [11] for special cases of PMCLP-PC-G (see Section 2.2 for details).

2. Challenges in Solving PMCLP-PC-G

In this section, we present two well-known approaches for solving (planar) MCLP and its variants, and discuss how they cannot be directly utilized for solving the PMCLP-PC-G.

2.1 Mixed integer programming

The motivation for the binary coverage assumption in the (planar) MCLP (where demand is represented by points, line segments, or polygons) is to make the problem manageable by readily formulating them as an LBIP. This is easy to do for the MCLP because of the discrete nature of candidate locations for service facilities (as per the definition). Moreover, even in studies considering a planar setting, i.e., allowing the facilities to be located anywhere in the continuous plane, coverage is still assumed to be binary [14] because this helps to show that a finite number of potential facility locations, called the circle intersection point set (CIPS) [4] and polygon intersection point set (PIPS) [14], exist which contain an optimal solution to this problem. Thereby resulting in the following well-known LBIP for the (planar) MCLP:

$$\max \left\{ \sum\nolimits_i v_i x_i : \sum\nolimits_j a_{ij} y_j \geq x_i \text{ for all } i, \sum\nolimits_j y_j = p, x_i, y_j \in \{0,1\} \right\}, \tag{1}$$

where $x_i = 1$ if demand zone i is covered, $y_j = 1$ if a service facility is sited at the candidate/PIP/CIP point j, v_i is the given total demand of demand zone i, and a_{ij} is the given binary value which is 1 if demand zone i is covered by locating a facility at candidate/CIP/PIP point j, i.e. distance between point i is no greater than known service range of a facility (denoted by r) located at point j or $dist(i,j) \leq r$.

Furthermore in literature, LBIP formulations have also been used to tackle MCLP without the binary coverage assumption by considering so-called gradual covering location problem (GCLP). Similar to the MCLP, in GCLP, the demand zones are still represented by points but the coverage level depends on their distance from the facilities. So far LBIP formulations are known only for the GCLP defined over a finite set of pre-specified candidate positions for the facilities [2, 3, 8], and planar GCLP with single facility, i.e. $p = 1$ [10]. More specifically, given a set of n demand points and a set $\mathcal{Y}$ of finite number of positions where p facilities can be placed, the GCLP can be stated as follows: $\max_{F \subset \mathcal{Y}} \left\{ \sum_{i=1}^n v_i g_i(\Delta_i(F)) : |F| = p \right\}$ where F is a set of locations where facilities are located, $\Delta_i(F) = \min_{j \in F}\{dist(i,j)\}$ is the minimum distance between demand point

i and any facility location in F, and $g_i(.) \in [0, 1]$ is a pre-defined coverage function. The following LBIP formulations have been derived for the GCLP with linear decay coverage function [3, 10], i.e. $g_i(\Delta) = 1 - \beta\Delta$ where $\beta > 0$ is a constant, or step-coverage function [2, 8], i.e. $g_i(\Delta) = \delta_k$ if $\Delta \in (r^i_{k-1}, r^i_k]$ for $k = 1, \ldots, K$ where $\delta_1 = 1 > \delta_2 > \ldots > \delta_K = 0$ (coverage levels) and $r^i_0 = 0 < r^i_1 = r < r^i_2 < \ldots < r^i_K$ (coverage radii):

$$\max\left\{\sum_{i=1}^{n}\sum_{j\in\mathcal{Y}} c_{ij}x_{ij} : a_{ij}y_j \geq x_{ij} \text{ for all } i,j; \sum_{j} a_{ij}x_{ij} \leq 1 \text{ for } i; \sum\nolimits_{j} y_j = p, x_{ij}, y_j \in \{0,1\}\right\} \quad (2)$$

where $c_{ij} = v_i g_i(dist(i, j))$, $x_{ij} = 1$ if facility located at point j covers the demand point i, and a_{ij} is the given binary value which is 1 if demand at point i can be partially/completely covered by locating a facility at candidate point j, i.e. $dist(i, j) < r^i_K$. Notice that if $g_i(\Delta) = 1$ for $\Delta \leq r$ and $g_i(\Delta) = 0$ for $\Delta > r$, then the GCLP reduces to the MCLP.

It is important to note that the PMCLP-PC-G with its features of partial coverage and general spatial representation of DZs and SZs is significantly harder to solve compared to the (planar) MCLP problem with binary coverage, even when a demand zone is represented by a line segment or polygon, and the GCLP. Using LBIPs to solve the PMCLP-PC-G is no more feasible: To use LBIP one must be able to (a) describe an optimum-containing discrete set of candidate locations for SZs, and (b) capture the covered demand from each demand zone in the objective function with a univariate function. Although (a) is possible for a special case of the PMCLP-PC-G, i.e. PMCLP-PC-R [1], the set of candidate locations for a SZ not only depends on where DZs are located but also on where the other SZs are sited, making the structure of this set much more complicated than the case of binary coverage. The foregoing results for the PMCLP-PC-G (even with euclidean distances, i.e. circular SZs and DZs) are still not known. However, even if one explicitly derive this optimum-containing set, it would be still impossible to do (b) because in calculating the coverage obtained from each DZ by each SZ, one needs to eliminate the parts of the DZ that are covered by other SZs. Therefore, just a univariate function for each DZ fails to capture the interaction between SZs, i.e., $A\left(d_i \cap \left(\cup_{j=1}^{p} s_j\right)\right)$. If one wants to formulate a LBIP, introduction of many multi-variate nonlinear functions, auxiliary binary variables, and constraints to capture these interactions will be necessary. This makes integer programming formulation approach extremely challenging.

2.2 Greedy-based algorithms

Since even MCLP is an NP-hard problem, approximation algorithms have also been developed in the literature for solving (planar) MCLP. Among them, greedy approximation algorithm is a well-known approach because it requires solving single facility problem for multiple (p) times. In this direction, Cornuéjols et al. [9] and Hochbaum and Pathria [11] provide greedy algorithms for solving variants of MCLP along with their approximation ratios. More specifically, in a seminal paper on locating bank accounts (or facilities) in at most p out of m known cities to cover n clients (or DZs represented by points), Cornuéjols et al. [9] consider a variant of MCLP (or GCLP) in which fixed cost of locating accounts is also deducted in the objective function and it is assumed that the coverage function $g_i(dist(i, j)) = \phi_{ij}$ is constant for each pair of client i and facility location j. They derive an LBIP formulation which is same as (2) when the fixed costs are zero, and also present a greedy approach which provides a solution whose value is within a factor of $1 - 1/e$ of the optimal solution value. Clearly with zero fixed costs, this problem is a special case of PMCLP-PC-G.

Thereafter, Hochbaum and Pathria [11] extend the results in [9] by considering a so-called maximum p-coverage problem (MCP) which is defined as follows: Given a universal set of elements U where each element $i \in U$ has weight w_i associated to it and a class $\mathcal{V}$ of subsets of U, the goal is to select p members (or subsets of U) from the class $\mathcal{V}$ such that the selected subsets do not overlap and the sum of the weights of the elements in the union of these subsets is maximum. We can build its correspondence with facility location problem, in particular PMCLP-PC-G, by considering each element of the set U as a DZ and each member of the class $\mathcal{V}$ as a set of DZs that are completely covered by a SZ located at a pre-specified candidate position. Observe that the foregoing problem does not allow partial coverage of elements/DZs and restricts selected members to belong to the class $\mathcal{V}$, which is equivalent to SZs to be located at pre-specified candidate positions. This implies that the MCP is a special case of PMCLP-PC-G. Hochbaum and Pathria [11] provide a greedy algorithm and a pseudo-greedy algorithm for MCP, and showcase that the solution value corresponding to the greedy (or pseudo-greedy) solution is within a factor of $1 - 1/e$ (or $1 - 1/e^{\eta}$) of the optimal solution value where e is the base of natural logarithm and $\eta \leq 1$. In this paper, we further extend their algorithmic and theoretical results for PMCLP-PC-G. Note that because of the binary coverage assumption of MCP and discrete nature of the bank account location problem, the proofs in [11] and [9], respectively, cannot be directly utilized for the PMCLP-PC-G.

3. Greedy and Pseudo-Greedy Algorithms for PMCLP-PC-G

In this section, we first present greedy and pseudo-greedy algorithms for solving the PMCLP-PC-G.

Algorithm 1 Greedy Algorithm for PMCLP-PC-G

```
1: function GreedyAlgorithm(𝒟, 𝒮)
2:     𝒟_1^g := 𝒟; ψ_g := 0;
3:     for j = 1, ..., p do
4:         (ψ_g^j, x_{s_j}^g, y_{s_j}^g) := SingleSZProblem(𝒟_j^g, s_j);
5:         ψ_g ← ψ_g + ψ_g^j;
6:         for d ∈ 𝒟_j^g do
7:             𝒟_{j+1}^g ← {𝒟_j^g \ d} ∪ TrimOut(d, s_j, x_{s_j}^g, y_{s_j}^g);
8:         end for
9:     end for
10:    return (ψ_g, x_{s_1}^g, ..., x_{s_p}^g, y_{s_1}^g, ..., y_{s_p}^g)
11: end function
```

Greedy Algorithm. Assuming that there exists an exact algorithm for solving the PMCLP-PC-G with $p = 1$, referred to as Single SZ Problem (SSP), we solve multiple SSPs in our greedy-based algorithm for the PMCLP-PC-G. The pseudocode is presented in Algorithm 1, where for a set of DZs $\mathcal{D}_j^g$, the function $\texttt{SingleSZProblem}\Big(\mathcal{D}_j^g, s_j\Big)$, $j \in \{1, \ldots, p\}$, returns the maximum demand, ψ_g^j, covered by the SZ s_j along with its optimal position $(x_{s_j}^g, y_{s_j}^g)$ (Line 4). We initialize the algorithm in Line 2 by setting $\mathcal{D}_1^g = \mathcal{D}$ (the original set of all given DZs). We also use the function $\texttt{TrimOut}\big(d, s_j, x_{s_j}^g, y_{s_j}^g\big)$ to eliminate the parts of DZ d that are covered by SZ s_j positioned at $(x_{s_j}^g, y_{s_j}^g)$. In Lines 6-8, we create set $\mathcal{D}_{j+1}^g$ for the next iteration by replacing each DZ d in the set $\mathcal{D}_j^g$ with trimmed DZs. We denote the set of trimmed DZs, that replaces DZ d, by T_d. See Figure 2 for different output possibilities after every application of $\texttt{TrimOut}$ when the DZ d and SZ s_j are axis-parallel rectangles. The summation of the maximum covered demand by calling $\texttt{SingleSZProblem}\Big(\mathcal{D}_j^g, s_j\Big)$ over $j \in \{1, \ldots, p\}$ gives a feasible solution and a lower bound on the optimal objective value of the PMCLP-PC-G.

Algorithm 1 generalizes the greedy-based polynomial-time heuristic of Bansal and Kianfar [1] for solving the PMCLP-PC-R, i.e. PMCLP-PC-G where DZs d_i, $i = 1, \ldots, n$, and SZs s_j, $j = 1, \ldots, p$, are axis-parallel rectangles. Interestingly, for PMCLP-PC-R, the function $\texttt{SingleSZProblem}\Big(\mathcal{D}_j^g, s_j\Big)$,

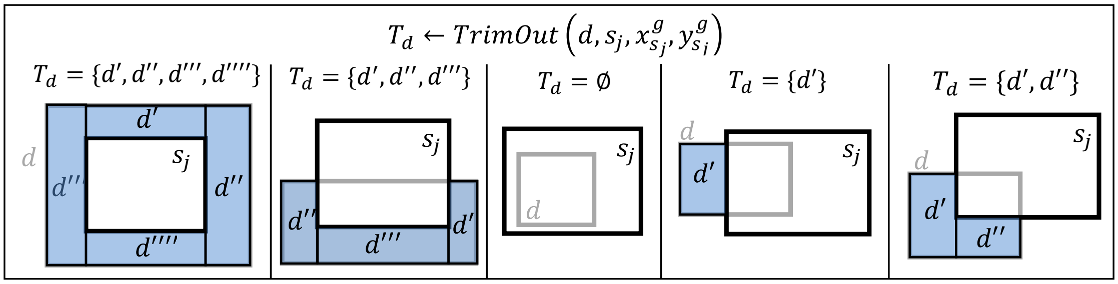

Figure 2: Possible outputs of `TrimOut` for axis-parallel rectangular DZ and SZ

$j \in \{1, \ldots, p\}$, takes $O(n_j^2)$ time where n_j is the number of rectangular DZs in the set $\mathcal{D}^g_j$, and returns the optimal demand covered by the rectangular SZ s_j along with its optimal position $(x^g_{s_j}, y^g_{s_j})$, i.e. the coordinates of its lower left corner [1, 15]. Moreover, for PMCLP-PC-R, each application of `TrimOut` replaces the rectangular DZ d with at most 4 trimmed rectangular DZs, thereby adding at most 3 rectangular DZs in $\mathcal{D}^g_{j+1}$. Figure 2 shows that when one, two, or four corners of a rectangular SZ lie(s) within a DZ, one, two, or three, respectively, new rectangular trimmed DZs are created. Notice that since the trimmed DZs in T_d do not overlap, any corner of another rectangular SZ cannot lie in more than one trimmed DZ. As a result, in the next iteration of the outer for loop, the application of the function `TrimOut` on the trimmed rectangular DZs in T_d will add at most 4 rectangular (further) trimmed DZs in $\mathcal{D}^g_{j+2}$. The same argument applies to all following iterations and all $d \in \mathcal{D}$. This implies that in each iteration of outer loop at most $4n$ number of rectangular DZs are added in $\mathcal{D}^g_j$, making $n_{j+1} \leq 4nj$ for $j \geq 1$. Since the function `TrimOut` is a constant time operation for this problem, the complexity of the greedy algorithm for PMCLP-PC-R is $O\left(\sum_{j=1}^{p} \left((4nj)^2 + 4nj\right)\right) = O(n^2p^3)$.

Observation 1. *In iteration $j \in \{1, \ldots, p\}$ of the greedy algorithm (Algorithm 1), we exactly solve a SSP for $\mathcal{D}^g_j$ set of DZs. Observe that summing the demand covered in the iteration, i.e., ψ^j_g, for p times provides an upper bound on the optimal solution value of the PMCLP-PC-G for $\mathcal{D}^g_j$ set of DZs. This is because the summation does not consider overlapping of the SZs.*

Pseudo-Greedy Algorithm. In the above discussed greedy algorithm, we assume that an exact algorithm for solving the SSP is known. However, in case this assumption fails and only an η-approximate algorithm for solving the SSP is known, then we utilize a pseudo-greedy algorithm for solving the PMCLP-PC-G. More specifically, the pseudo-greedy algorithm is same as the greedy algorithm (Algorithm 1) except that the function `SingleSZProblem` is replaced by function η-`ApproxSSP` which returns an approximate demand, κ^j_r, covered by the SZ s_j that is within a known factor of η (≤ 1) of the optimal solution value returned by `SingleSZProblem`. The pseudocode is given in Algorithm 2. Note that for $\eta = 1$, the pseudo-greedy algorithm is exactly same as the greedy algorithm. Furthermore, for $\eta < 1$, the sets of DZs $\mathcal{D}^g_j$ and $\mathcal{D}^r_j$, $j = 2, \ldots, p$, in the greedy

Algorithm 2 Pseudo-Greedy Method

function PseudoGreedyAlgorithm($\mathcal{D}$, $\mathcal{S}$)
  $\mathcal{D}^r_1 := \mathcal{D}; \kappa_r := 0;$
  for $j = 1, \ldots, p$ **do**
    $(\kappa^j_r, x^r_{s_j}, y^r_{s_j}) :=$ η-`ApproxSSP`$(\mathcal{D}^r_j, s_j)$;
    $\kappa_s \leftarrow \kappa_r + \kappa^j_r$;
    for $d \in \mathcal{D}^r_j$ **do**
      $\mathcal{D}^r_{j+1} \leftarrow \{\mathcal{D}^r_j \setminus d\} \cup$ `TrimOut`$\left(d, s_j, x^r_{s_j}, y^r_{s_j}\right)$;
    end for
  end for
  return $\left(\kappa_r, x^r_{s_1}, \ldots, x^r_{s_p}, y^r_{s_1}, \ldots, y^r_{s_p}\right)$
end function

algorithm (Algorithm 1) and pseudo-greedy algorithm (Algorithm 2), respectively, are different.

Remark 1. It is important to note that the complexity of the greedy (or pseudo-greedy) algorithm depends on the complexity of the functions `SingleSZProblem` (or η-`ApproxSSP`) and `TrimOut`, and the number of trimmed DZs generated after each iteration. As mentioned before, till date these functions are not known, except for PMCLP-PC-R with $p = 1$ [1, 15]. However, whenever they will be developed in future, we can embed them within our greedy (or pseudo-greedy) algorithm to provide solutions for the PMCLP-PC-G along with their computational complexity.

4. Approximation Ratios

In the following theorem, we provide approximation ratios associated with the greedy and pseudo-greedy algorithms for the PMCLP-PC-G. Let $(x_s^*, y_s^*) \in \mathbb{R}^{2p}$ be the optimal solution and $(x_s^a, y_s^a) \in \mathbb{R}^{2p}$ be an approximate solution for the PMCLP-PC-G. Then the approximation ratio corresponding to the approximate solution (or algorithm) is defined by

$$\gamma_a = \frac{f(x_s^a, y_s^a)}{f(x_s^*, y_s^*)}. \tag{3}$$

Theorem 2. *Let the approximation ratios for the greedy algorithm (Algorithm 1) and the pseudo-greedy algorithm (Algorithm 2) be denoted by γ_g and γ_r, respectively. Then*

$$\gamma_g > 1 - \frac{1}{e}, \text{ and } \gamma_r > 1 - \frac{1}{e^\eta}, \tag{4}$$

where e is the base of natural logarithm.

Proof. Recall that the notations ψ_g^j in Algorithm 1 and κ_r^j in Algorithm 2 denote the covered demand returned by the functions $\texttt{SingleSZProblem}\left(\mathcal{D}_j^g, s_j\right)$ and $\eta\texttt{-ApproxSSP}\left(\mathcal{D}_j^r, s_j\right)$, respectively, for $j \in \{1, \ldots, p\}$. In other words, ψ_g^j is the maximum demand and κ_r^j is the η-approximate demand covered by the SZ s_j for the given set of DZs $\mathcal{D}_j^g$ and $\mathcal{D}_j^r$, respectively. This implies that $\kappa_r^j \geq \eta\psi_r^j$ where ψ_r^j is the maximum demand covered by the SZ s_j for $\mathcal{D}_j^r$ set of DZs. For the sake of convenience, in this proof, we denote the optimal solution value for the PMCLP-PC-G instance, i.e., $f(x_s^*, y_s^*)$, by f^*. Therefore, the approximation ratio for the greedy algorithm (Algorithm 1) and the pseudo-greedy algorithm (Algorithm 2) are given by

$$\gamma_g = \frac{1}{f^*}\left(\sum_{l=1}^{p} \psi_g^l\right) \text{ and } \gamma_r = \frac{1}{f^*}\left(\sum_{l=1}^{p} \kappa_r^l\right), \tag{5}$$

respectively. Let the optimal solution value of PMCLP-PC-G instance with $\mathcal{D}_j^g$ (or $\mathcal{D}_j^r$) as the input set of DZs be denoted by ζ_j^g (or ζ_j^r). Then, based on Observation 1,

$$p\psi_g^j \geq \zeta_j^g \text{ and } p\kappa_r^j \geq p\eta\psi_r^j \geq \eta\zeta_j^r \tag{6}$$

for $j = 1, \ldots, p$. Note that for $j = 1$, $\mathcal{D}_1^g = \mathcal{D}_1^r = \mathcal{D}$, and hence $\zeta_1^g = \zeta_1^r = f^*$. Also, since after each iteration $l \in \{1, \ldots, j-1\}$ of the greedy algorithm and pseudo-greedy algorithm, we "trim-out" DZs of total demand ψ_g^l and κ_r^l, respectively, we get

$$p\psi_g^j \geq \zeta_j^g \geq f^* - \sum_{l=1}^{j-1} \psi_g^l \text{ and } p\kappa_r^j \geq \eta\zeta_j^r \geq \eta\left(f^* - \sum_{l=1}^{j-1} \kappa_r^l\right), \tag{7}$$

where $\psi_g^0 = \kappa_r^0 = 0$. This implies that

$$p\sum_{l=1}^{j}\psi_g^l \geq f^* + (p-1)\sum_{l=1}^{j-1}\psi_g^l \geq f^*\left(1+\frac{p-1}{p}+\left(\frac{p-1}{p}\right)^2+\ldots+\left(\frac{p-1}{p}\right)^{j-1}\right) \quad (8)$$

and

$$p\sum_{l=1}^{j}\kappa_r^l \geq \eta f^* + (p-\eta)\sum_{l=1}^{j-1}\kappa_r^l \geq \eta f^*\left(1+\frac{p-\eta}{p}+\left(\frac{p-\eta}{p}\right)^2+\ldots+\left(\frac{p-\eta}{p}\right)^{j-1}\right). \quad (9)$$

For $j = p$, Inequalities (8) and (9) reduce to

$$\gamma_g = \frac{1}{f^*}\left(\sum_{l=1}^{p}\psi_g^l\right) \geq \left(1-\left(\frac{p-1}{p}\right)^p\right) \text{ and } \gamma_r = \frac{1}{f^*}\left(\sum_{l=1}^{p}\kappa_r^l\right) \geq \left(1-\left(\frac{p-\eta}{p}\right)^p\right), \quad (10)$$

respectively. Now because the functions in the right-hand sides of the last two inequalities are decreasing in p, we compute limit of these functions as p approaches infinity and get

$$\gamma_g > 1-\frac{1}{e} \text{ and } \gamma_r > 1-\frac{1}{e^\eta}. \quad (11)$$

This completes the proof. □

Corollary 3. *For a given $p \geq 1$, the greedy algorithm (Algorithm 1) and the pseudo-greedy algorithm (Algorithm 2) provide solutions whose values are at least $1-[(p-1)/p]^p$ and $1-[(p-\eta)/p]^p$, respectively, times the optimal value for the PMCLP-PC-G instance.*

5. Conclusion

We introduced a new generalization of the classical planar maximum coverage location problem (PMCLP) where demand zones (DZs) and service zone (SZ) of facilities are represented by two-dimensional spatial objects such as polygons, circles, etc., and the partial coverage is allowed in its true sense. We presented a greedy algorithm and a pseudo-greedy algorithm to provide approximate solutions for the PMCLP-PC-G by assuming that an exact algorithm and an η-approximate algorithm, respectively, for the single facility PMCLP-PC-G, i.e. $p = 1$, exists. We also proved that the solution value corresponding to the greedy (or pseudo-greedy) solution is within a factor of $1 - 1/e$ (or $1 - 1/e^\eta$) of the optimal solution value where e is the base of natural logarithm and $\eta \leq 1$. It is important note that till date no exact or η-approximate algorithm is known for PMCLP-PC-G with $p = 1$, except the so-called plateau vertex algorithm (PVT) [15] and improved PVT [1] exact algorithms for PMCLP-PC-R with $p = 1$. However, whenever such algorithms will be developed in future for other special cases of the SSP, they can be automatically embedded within the greedy (or pseudo-greedy) algorithm presented in this paper to provide solutions for the multi-SZ version of the corresponding problem.

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