An Inexact Regularized Newton Framework with a Worst-Case Iteration Complexity of $O(\epsilon^{-3/2})$ for Nonconvex Optimization

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An Inexact Regularized Newton Framework with a Worst-Case Iteration Complexity of $O(\epsilon^{-3/2})$ for Nonconvex Optimization*

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Abstract

An algorithm for solving smooth nonconvex optimization problems is proposed that, in the worst-case, takes $O(\epsilon^{-3/2})$ iterations to drive the norm of the gradient of the objective function below a prescribed positive real number $\epsilon$ and can take $O(\epsilon^{-3})$ iterations to drive the leftmost eigenvalue of the Hessian of the objective above $-\epsilon$. The proposed algorithm is a general framework that covers a wide range of techniques including quadratically and cubically regularized Newton methods, such as the Adaptive Regularisation using Cubics (ARC) method and the recently proposed Trust-Region Algorithm with Contractions and Expansions (TRACE). The generality of our method is achieved through the introduction of generic conditions that each trial step is required to satisfy, which in particular allow for inexact regularized Newton steps to be used. These conditions center around a new subproblem that can be approximately solved to obtain trial steps that satisfy the conditions. A new instance of the framework, distinct from ARC and TRACE, is described that may be viewed as a hybrid between quadratically and cubically regularized Newton methods. Numerical results demonstrate that our hybrid algorithm outperforms a cubically regularized Newton method. Unconstrained optimization, nonlinear optimization, nonconvex optimization, inexact Newton methods, worst-case iteration complexity, worst-case evaluation complexity

1 Introduction

This paper proposes an algorithm for solving

$$\min_{x \in \mathbb{R}^n} f(x),$$

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where the (possibly nonconvex) objective function $f : \mathbb{R}^n \to \mathbb{R}$ is assumed to be twice-continuously differentiable. The optimization problem (1) has been widely studied, as evidenced by its appearance as the focal point of numerous textbooks; e.g., see [1], [2], [10], [20], [23], and [24].

For many years, the most popular methods for solving (1) were in classes known as line search and trust region methods. Recently, however, cubic regularization methods have become popular, which are based on the pioneering work by [19] and [21]. Their rise in popularity is due to increased interest in algorithms with improved complexity properties, which stems from the impact of so-called optimal algorithms for solving convex optimization problems. For problem (1), by complexity properties, we mean a guaranteed bound on the number of iterations (or function evaluations or derivative evaluations) needed by an algorithm before the norm of the gradient of the objective must fall below a positive threshold $\epsilon > 0$. In other words, if $x_k$ denotes the $k$th iteration of an algorithm, one seeks a bound on the number of iterations until it is guaranteed that

$$\|\nabla f(x_k)\| \leq \epsilon.$$

The complexity of a traditional trust region method (e.g., see Algorithm 6.1.1 in [10]) is $O(\epsilon^{-2})$ (see [5]), which falls short of the $O(\epsilon^{-3/2})$ complexity for cubic regularization methods (e.g., see the ARC method by [6, 7]). This latter complexity is optimal among a certain broad class of second-order methods when employed to minimize a broad class of objective functions; see [8]. That said, one can obtain even better complexity properties if higher-order derivatives are used; see [4] and [9].

The better complexity properties of regularization methods such as ARC have been a major point of motivation for discovering other methods that attain the same worst-case iteration complexity bounds. For example, the recently introduced (nontraditional) trust region method known as TRACE (see [11]) has the same optimal $O(\epsilon^{-3/2})$ complexity, while at the same time allowing traditional trust region trial steps to be computed and used. A key aspect of the TRACE framework is that a solution to an implicit trust region problem is obtained by varying a regularization parameter instead of a trust region radius. This key idea has been adopted and advanced further by [3]; in particular, they propose an algorithm that has optimal iteration complexity by solving quadratic subproblems that have a carefully chosen quadratic regularization parameter.

Contributions The main contributions of this paper relate to advancing the understanding of optimal complexity algorithms for solving the smooth optimization problem (1). Our proposed framework is intentionally very general; it is not a trust region method, a quadratic regularization method, or a cubic regularization method. Rather, we propose a generic set of conditions that each trial step must satisfy that still allow us to establish an optimal first-order complexity result as well as a second-order complexity bound similar to the methods above. Our framework contains as special cases other optimal complexity algorithms such as ARC and TRACE. To highlight this generality of our contribution, we describe one particular instance of our framework that appears to be new to the literature.

During the final preparation of this article, we came across the work in [13] and [14]. This work shares certain commonalities with our own and appears to have been developed at the same time. Although there are numerous differences, we shall only point out three of them. First, the precise conditions that they require for each trial step are different from ours. In particular, the condition stated as (3.1c) in [14] requires that regularization is used to compute every trial step, a property not shared by our method (which can employ Newton steps). Second, they do not consider second-order convergence or complexity properties, although they might be able to do so by incorporating second-order conditions similar to ours. Third, they focus on strategies for identifying an appropriate value for the regularization parameter. An implementation of our method might consider their proposals, but could employ other strategies as well. In any case, overall, we believe that our papers are quite distinct, and in some ways are complementary.

Organization In §2 we present our general framework that is formally stated as Algorithm 1. In §3 we prove that our framework enjoys first-order convergence (see §3.1), an optimal first-order complexity (see §3.2), and certain second-order convergence and complexity guarantees (see §3.3). In §4 we show that ARC and TRACE can be viewed as special cases of our framework, and present yet another instance that is distinct
from these methods. In §5 we present details of implementations of a cubic regularization method and our newly proposed instance of our framework, and provide the results of numerical experiments with both. Finally, in §6 we present final comments.

**Notation** We use \( \mathbb{R}_+ \) to denote the set of nonnegative scalars, \( \mathbb{R}_{++} \) to denote the set of positive scalars, and \( \mathbb{N}_+ \) to denote the set of nonnegative integers. Given a real symmetric matrix \( A \), we write \( A \succeq 0 \) (respectively, \( A \succeq 0 \)) to indicate that \( A \) is positive semidefinite (respectively, positive definite). Given a pair of scalars \( (a, b) \in \mathbb{R} \times \mathbb{R} \), we write \( a \perp b \) to indicate that \( ab = 0 \). Similarly, given such a pair, we denote their maximum as \( \max\{a, b\} \) and their minimum as \( \min\{a, b\} \). Given a vector \( v \), we denote its (Euclidean) \( \ell_2 \)-norm as \( \|v\| \).

Finally, given a discrete set \( S \), we denote its cardinality by \( |S| \).

Corresponding to the objective \( f : \mathbb{R}^n \to \mathbb{R} \), we define the gradient function \( g := \nabla f : \mathbb{R}^n \to \mathbb{R}^n \) and the Hessian function \( H := \nabla^2 f : \mathbb{R}^n \to \mathbb{R}^{n \times n} \). Given an iterate \( x_k \) in an algorithm for solving (1), we define \( f_k := f(x_k) \), \( g_k := g(x_k) := \nabla f(x_k) \), and \( H_k := H(x_k) := \nabla^2 f(x_k) \). Similarly, we apply a subscript to other algorithmic quantities whose definition depends on the iteration number \( k \).

## 2 Algorithm Description

Our algorithm involves generic conditions that a trial step toward solving problem (1) must satisfy. One can obtain a step satisfying these conditions by computing—for appropriate positive lower and upper bounds \( \sigma_k \) and \( \sigma_k^U \), respectively, on the ratio between a regularization variable \( \lambda \geq 0 \) and the norm of the trial step—an approximate solution of the subproblem

\[
P_k(\sigma_k^l, \sigma_k^U) := \min_{(s, \lambda) \in \mathbb{R}^n \times \mathbb{R}_+^n} f_k + g_k^T s + \frac{1}{2} s^T (H_k + \lambda I) s
\]

\[\text{s.t. } (\sigma_k^U)^2 \|s\|^2 \leq \lambda^2 \leq (\sigma_k^l)^2 \|s\|^2.\]

(2)

For a given value of the regularization variable \( \lambda \), this problem involves a quadratic objective function and an upper bound on the norm of the trial step, just as in a trust region method. However, it also includes a lower bound on the norm of the trial step, and, in general, with \( \lambda \) as a variable, it encapsulates other types of subproblems as well, including those present in a cubic regularization framework. For additional details on the properties of this subproblem and its solutions, see Appendices A and B.

The conditions that the \( k \)th trial step and regularization pair, i.e., \((s_k, \lambda_k)\), must satisfy are stated in Assumption 2.1 below, wherein we invoke the following (unregularized) quadratic model of \( f \) at \( x_k \):

\[ q_k(s) := f_k + g_k^T s + \frac{1}{2} s^T H_k s. \]

**Assumption 2.1.** The pair \((s_k, \lambda_k)\) is computed such that it is feasible for problem (2) and, with

\[
\Delta_k(s_k, \lambda_k) := \begin{cases} 
\|s_k\| & \text{if } \lambda_k = 0 \\
\frac{1}{\sqrt{\lambda_k}} \sqrt{\|q_k\| s_k} & \text{if } \lambda_k > 0
\end{cases}
\]

(3)

and constants \((\kappa_1, \kappa_2, \kappa_3) \in \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \), the following hold:

\[
f_k - q_k(s_k) \geq \frac{\|g_k\|}{6 \sqrt{2}} \min \left\{ \frac{\|g_k\|}{1 + \|H_k\|}, \Delta_k(s_k, \lambda_k) \right\}; \quad \text{(4a)}
\]

\[
s_k^T (g_k + (H_k + \lambda_k I) s_k) \leq \min \{ \kappa_1 \|s_k\|^2, \frac{1}{2} s_k^T (H_k + \lambda_k I) s_k + \frac{1}{2} \kappa_2 \|s_k\|^2 \}; \quad \text{and} \quad \text{(4b)}
\]

\[
\|g_k + (H_k + \lambda_k I) s_k\| \leq \lambda_k \|s_k\| + \kappa_3 \|s_k\|^2.
\]

(4c)

To see that Assumption 2.1 is well-posed and consistent with problem (2), we refer the reader to Theorem B.3 in Appendix B, wherein we prove that any solution of problem (2) with \( s \) restricted to a sufficiently large dimensional subspace of \( \mathbb{R}^n \) satisfies all of the conditions in Assumption 2.1. We also claim that one can obtain a pair satisfying Assumption 2.1 in either of the following two ways:
Choose $\sigma \in [\sigma_L^k, \sigma_U^k]$, compute $s_k$ by minimizing the cubic function
\[
c_k(s; \sigma) := q_k(s) + \frac{1}{2} \sigma \|s\|^3 = f_k + g_k^T s + \frac{1}{2} s^T H_k s + \frac{1}{2} \sigma \|s\|^3
\] (5)
over a sufficiently large dimensional subspace of $\mathbb{R}^n$ (assuming, when $\sigma = \sigma_L^k = 0$, that this function is not unbounded below), then set $\lambda_k \leftarrow \sigma \|s_k\|$. This is essentially the strategy employed in cubic regularization methods such as ARC.

Choose $\lambda_k \geq 0$, then compute $s_k$ by minimizing the objective of (2) with $\lambda = \lambda_k$ over a sufficiently large dimensional subspace of $\mathbb{R}^n$ (assuming that the function is not unbounded below). The resulting pair $(s_k, \lambda_k)$ satisfies Assumption 2.1 as long as it is feasible for (2). This is essentially the strategy employed in [3] and partly employed in trace.

One can imagine other approaches as well. Overall, we state problem (2) as a guide for various techniques for computing the pair $(s_k, \lambda_k)$. Our theory simply relies on the fact that any such computed pair satisfies the conditions in Assumption 2.1.

Our algorithm, stated as Algorithm 1, employs the following ratio (also employed, e.g., in trace) to determine whether a given trial step is accepted or rejected:
\[
\rho_k := \frac{f_k - f(x_k + s_k)}{\|s_k\|^3}.
\]
One potential drawback of employing this ratio is that the ratio is not invariant to scaling of the objective function. However, the use of this ratio can still be justified. For example, if one were to compute $s_k$ by minimizing the cubic model (5) for some $\sigma > 0$, then the reduction in this model yielded by $s_k$ is bounded below by a fraction of $\sigma \|s_k\|$ (see [7, Lemma 4.2]), meaning that $\rho_k \geq \eta$ holds when $\sigma \geq \eta$ and the actual reduction in $f$ is proportional to the reduction in the cubic model. For further justification for this choice—such as how it allows the algorithm to accept Newton steps when the norm of the trial step is small (and, indeed, the norms of accepted steps vanish asymptotically as shown in Lemma 3.7 later on)—we refer the reader to [3] and [11].

3 Convergence Analysis

In this section, we prove global convergence guarantees for Algorithm 1. In particular, we prove under common assumptions that, from remote starting points, the algorithm converges to first-order stationarity, has a worst-case iteration complexity to approximate first-order stationarity that is on par with the methods in [7, 11, and 3], and—at least in a subspace determined by the search path of the algorithm—converges to second-order stationarity with a complexity on par with the methods in [7] and [11].

3.1 First-Order Global Convergence

Our goal in this subsection is to prove that the sequence of objective gradients vanishes. We make the following assumption about the objective function, which is assumed to hold throughout this section.

Assumption 3.1. The objective function $f : \mathbb{R}^n \to \mathbb{R}$ is twice continuously differentiable and bounded below by a scalar $f_{\inf} \in \mathbb{R}$ on $\mathbb{R}^n$.

We also make the following assumption related to the sequence of iterates.

Assumption 3.2. The gradient function $g : \mathbb{R}^n \to \mathbb{R}^n$ is Lipschitz continuous with Lipschitz constant $g_{\text{Lip}} \in \mathbb{R}_{++}$ in an open convex set containing the sequences $\{x_k\}$ and $\{x_k + s_k\}$. Furthermore, the gradient sequence $\{g_k\}$ has $g_k \neq 0$ for all $k \in \mathbb{N}_+$ and is bounded in that there exists a scalar constant $g_{\max} \in \mathbb{R}_{++}$ such that $\|g_k\| \leq g_{\max}$ for all $k \in \mathbb{N}_+$.
Algorithm 1 Inexact Regularized Newton Framework

Require: an acceptance constant \(\eta \in \mathbb{R}_{++}\) with \(0 < \eta < 1\)

Require: bound update constants \(\{\gamma_1, \gamma_2\} \subset \mathbb{R}_{++}\) with \(1 < \gamma_1 \leq \gamma_2\)

Require: ratio lower and upper bound constants \(\{\sigma, \bar{\sigma}\} \subset \mathbb{R}_{++}\) such that \(\sigma \geq \sigma\)

1: procedure Inexact Regularized Newton
2: \hspace{1em} set \(x_0 \in \mathbb{R}^n\)
3: \hspace{1em} set \(\sigma_l^0 \leftarrow 0\) and \(\sigma_u^0 \in [\sigma, \bar{\sigma}]\)
4: \hspace{1em} for \(k \in \mathbb{N}_+\) do
5: \hspace{2em} set \((s_k, \lambda_k)\) satisfying Assumption 2.1
6: \hspace{2em} if \(\rho_k \geq \eta\) then \hspace{1em} [accept step]
7: \hspace{3em} set \(x_{k+1} \leftarrow x_k + s_k\)
8: \hspace{3em} set \(\sigma_{k+1}^l \leftarrow 0\) and \(\sigma_{k+1}^u \leftarrow \sigma_{k+1}^l\)
9: \hspace{2em} else (i.e., \(\rho_k < \eta\)) \hspace{1em} [reject step]
10: \hspace{3em} set \(x_{k+1} \leftarrow x_k\)
11: \hspace{2em} if \(\lambda_k < \sigma \|s_k\|\) then
12: \hspace{3em} set \(\sigma_{k+1}^l \in [\sigma, \bar{\sigma}]\) and \(\sigma_{k+1}^u \in [\sigma_{k+1}^l, \bar{\sigma}]\)
13: \hspace{2em} else
14: \hspace{3em} set \(\sigma_{k+1}^l \leftarrow \gamma_1 \frac{\lambda_k}{\|s_k\|}\) and \(\sigma_{k+1}^u \leftarrow \gamma_2 \frac{\lambda_k}{\|s_k\|}\)
15: \hspace{2em} end if
16: \hspace{2em} end if
17: \hspace{1em} end for
18: end procedure

It is worthwhile to note in passing that our complexity bounds for first- and second-order stationarity remain true even if one were to consider the possibility that \(g_k = 0\) for some \(k \in \mathbb{N}_+\), in which case one would have the algorithm terminate finitely or, if \(H_k \not\succeq 0\), compute an improving direction of negative curvature for \(H_k\). However, allowing this possibility—which is typically unlikely ever to occur in practice—would only serve to obscure certain aspects of our analysis. We refer the reader, e.g., to [7] (specifically, to the discussions at the ends of §2.1, §4, and §5 in that work) for commentary about why zero gradient values do not ruin complexity guarantees such as we present.

We begin with two lemmas each revealing an important consequence of Assumptions 3.1 and 3.2.

Lemma 3.1. For all \(k \in \mathbb{N}_+\), it follows that \(s_k \neq 0\).

Proof. The result follows by combining that \(g_k \neq 0\) for all \(k \in \mathbb{N}_+\) (see Assumption 3.2) with (4c).

Lemma 3.2. The Hessian sequence \(\{H_k\}\) is bounded in norm in that there exists a scalar constant \(H_{\max} \in \mathbb{R}_{++}\) such that \(\|H_k\| \leq H_{\max}\) for all \(k \in \mathbb{N}_+\).

Proof. The result follows by Assumption 3.1, the Lipschitz continuity of \(g\) in Assumption 3.2, and Lemma 1.2.2 in [22].

In our next lemma, we prove an upper bound for the regularization variable \(\lambda_k\).

Lemma 3.3. For all \(k \in \mathbb{N}_+\), the pair \((s_k, \lambda_k)\) satisfies

\[
\lambda_k \leq 2 \frac{\|g_k\|}{\|s_k\|} + \frac{3}{2} H_{\max} + \kappa_1.
\]
Proof. Since (4a) ensures $q_k(s_k) - f_k \leq 0$, it follows with (4b) and Lemma 3.2 that
\begin{align*}
0 \geq q_k(s_k) - f_k &= q_k^T s_k + \frac{1}{2}s_k^T H_k s_k \\
&\geq q_k^T s_k + \frac{1}{2}s_k^T H_k s_k + s_k^T (g_k + (H_k + \lambda_k I) s_k) - \kappa_1 ||s_k||^2 \\
&= 2g_k^T s_k + \frac{\lambda_k}{2} s_k^T H_k s_k + \lambda_k ||s_k||^2 - \kappa_1 ||s_k||^2 \\
&\geq -2||g_k|| ||s_k||^2 - \frac{3}{2} H_{max} ||s_k||^2 + \lambda_k ||s_k||^2 - \kappa_1 ||s_k||^2.
\end{align*}

After rearrangement and dividing by $||s_k||^2 \neq 0$ (see Lemma 3.1), the desired result follows.

Using Lemma 3.3 we now prove a lower bound for the reduction in $q_k$ yielded by $s_k$.

**Lemma 3.4.** For all $k \in \mathbb{N}_+$, the step $s_k$ satisfies
\begin{align*}
f_k - q_k(s_k) &\geq \frac{\|g_k\|}{6\sqrt{2}} \min \left\{ \frac{\|g_k\|}{1 + \|H_k\|}, \frac{\|s_k\|}{\sqrt{6}} \right\} \frac{\|g_k\|}{2\|g_k\| + \|s_k\| (\frac{3}{2} H_{max} + \kappa_1)}.
\end{align*}

Proof. If $\lambda_k = 0$, then by (4a) and Lemma 3.2 it follows that
\begin{align*}
f_k - q_k(s_k) &\geq \frac{\|g_k\|}{6\sqrt{2}} \min \left\{ \frac{\|g_k\|}{1 + \|H_k\|}, \frac{\|s_k\|}{\sqrt{6}} \right\} \geq \frac{\|g_k\|}{6\sqrt{2}} \min \left\{ \frac{\|g_k\|}{1 + \|H_{max}\|}, \frac{\|s_k\|}{\sqrt{6}} \right\}.
\end{align*}

On the other hand, if $\lambda_k > 0$, then (4a), Lemma 3.2 and Lemma 3.3 imply that
\begin{align*}
f_k - q_k(s_k) &\geq \frac{\|g_k\|}{6\sqrt{2}} \min \left\{ \frac{\|g_k\|}{1 + \|H_k\|}, \frac{\|s_k\|}{\sqrt{6}} \right\} \frac{1}{\lambda_k} \frac{\|g_k\| \|s_k\|}{\|g_k\| + \|s_k\| (\frac{3}{2} H_{max} + \kappa_1)} \\
&\geq \frac{\|g_k\|}{6\sqrt{2}} \min \left\{ \frac{\|g_k\|}{1 + \|H_{max}\|}, \frac{\|s_k\|}{\sqrt{6}} \right\} \frac{\|g_k\|}{2\|g_k\| + \|s_k\| (\frac{3}{2} H_{max} + \kappa_1)}.
\end{align*}

Combining the inequalities from these two cases, the desired result follows.

Going forward, for ease of reference, we respectively define sets of indices corresponding to accepted and rejected steps throughout a run of the algorithm as
\begin{align*}
A := \{ k \in \mathbb{N}_+ : \rho_k \geq \eta \} \quad \text{and} \quad R := \{ k \in \mathbb{N}_+ : \rho_k < \eta \}.
\end{align*}

We now show that if the algorithm were only to compute rejected steps from some iteration onward, then the sequence $\{ \lambda_k/\|s_k\| \}$ diverges to infinity.

**Lemma 3.5.** If $k \in R$ for all sufficiently large $k \in \mathbb{N}_+$, then $\{ \lambda_k/\|s_k\| \} \to \infty$.

Proof. Without loss of generality, assume that $R = \mathbb{N}_+$. We now prove that the condition in Step 11 cannot be true more than once. Suppose, in iteration $k \in \mathbb{N}_+$, Step 12 is reached, which means that $\lambda_{k+1} ||s_{k+1}|| \geq \sigma$ since $(s_{k+1}, l_{k+1})$ is required to be feasible for $P_{k+1}(\sigma_{k+1}^1, \sigma_{k+1}^2)$ in Step 5 where $\sigma_{k+1}^1 \geq \sigma$. Therefore, the condition in Step 11 tests false in iteration $(k + 1)$. Then, from Step 5, Step 14 and the fact that $\gamma_1 > 1$, it follows that $\{ \lambda_k/\|s_k\| \}$ is monotonically increasing for all $k \geq k$. Therefore, the condition in Step 11 cannot test true for any $k \geq k + 1$. Now, to see that the sequence diverges, notice from this fact, Step 5 and Step 14 it follows that for all $k \geq k + 1$ we have $\lambda_{k+1}/||s_{k+1}|| \geq \gamma_1 (\lambda_k/\|s_k\|)$ where $\gamma_1 > 1$. Thus, $\{ \lambda_k/\|s_k\| \} \to \infty$, as claimed.

We now prove that if the gradients are bounded away from zero and the sequence of ratios $\{ \lambda_k/\|s_k\| \}$ diverges, then $\rho_k \geq \eta$ for all sufficiently large $k \in \mathbb{N}_+$, meaning that the steps are accepted.
Lemma 3.6. Suppose that $I \subseteq \mathbb{N}_+$ is an infinite index set such that for $\epsilon \in \mathbb{R}^+$ independent of $k$, one finds that $\|g_k\| \geq \epsilon$ for all $k \in I$ and $\{\lambda_k/\|s_k\|\}_{k \in I} \to \infty$. Then, for all sufficiently large $k \in I$, it follows that $\rho_k \geq \eta$, meaning $k \in A$.

Proof. From the Mean Value Theorem, there exists $\bar{x}_k \in [x_k, x_k + s_k]$ such that

$$ q_k(s_k) - f(x_k + s_k) = (g_k - g(\bar{x}_k))^T s_k + \frac{1}{2} s_k^T H_k s_k $$

$$ \geq -\|g_k - g(\bar{x}_k)\|\|s_k\| - \frac{1}{2}\|H_k\|\|s_k\|^2. \quad (6) $$

From this, Lemma 3.4 and Assumption 3.2 it follows that, for all $k \in I$,

$$ f_k - f(x_k + s_k) = f_k - q_k(s_k) + q_k(s_k) - f(x_k + s_k) $$

$$ \geq \frac{\epsilon}{6\sqrt{2}} \min \left\{ \frac{\|g_k\|}{1 + H_{\text{max}}} \sqrt{\frac{\|g_k\|}{2\|g_k\| + \|s_k\|((\frac{3}{2}H_{\text{max}} + \kappa_1)} \right\} - (g_{\text{Lip}} + \frac{1}{2}H_{\text{max}})\|s_k\|^2 $$

$$ \geq \frac{\epsilon}{6\sqrt{2}} \min \left\{ \frac{\|s_k\|}{1 + H_{\text{max}}} \sqrt{\frac{\epsilon}{2g_{\text{max}} + \|s_k\|((\frac{3}{2}H_{\text{max}} + \kappa_1)} \right\} - (g_{\text{Lip}} + \frac{1}{2}H_{\text{max}})\|s_k\|^2. $$

This shows that there exists a threshold $s_{\text{thresh}} > 0$ such that

$$ f_k - f(x_k + s_k) \geq \eta\|s_k\|^2 $$

whenever $k \in I$ and $\|s_k\| \leq s_{\text{thresh}}$.

We now claim that $\|s_k\|_{k \in I} \to 0$. To prove this claim, suppose by contradiction that there exists an infinite subsequence $I_s \subseteq I$ and scalar $\epsilon_s \in \mathbb{R}^+$ such that $\|s_k\| \geq \epsilon_s$ for all $k \in I_s$. It then follows from the boundedness of $\{\|g_k\|\}$ (see Assumption 3.2) and Lemma 3.3 that $\{\lambda_k\}_{k \in I_s}$ is bounded. This allows us to conclude that $\{\lambda_k/\|s_k\|\}_{k \in I_s}$ is bounded, which contradicts the assumptions of the lemma. Thus, $\|s_k\|_{k \in I} \to 0$. Hence, there exists $k_s \in I$ such that for all $k \in I$ with $k \geq k_s$ one finds $\|s_k\| \leq s_{\text{thresh}}$. Therefore, for all $k \in I$ with $k \geq k_s$, it follows that $\rho_k \geq \eta$, as claimed.

Next, we prove that the algorithm produces infinitely many accepted steps.

Lemma 3.7. It holds that $|A| = \infty$ and $\{s_k\}_{k \in A} \to 0$.

Proof. To derive a contradiction, suppose that $|A| < \infty$. This implies that there exists $k_0$ such that, for all $k \geq k_0$, one has $k \in \mathcal{R}$ and $(x_k, g_k, H_k) = (x_{k_0}, g_{k_0}, H_{k_0})$. From this fact and Assumption 3.2 it follows that $\|g_k\| \geq \epsilon$ for all $k \geq k_0$ for some $\epsilon \in \mathbb{R}_{++}^+$. From the fact that $k \in \mathcal{R}$ for all $k \geq k_0$ and Lemma 3.3 it follows that $\{\lambda_k/\|s_k\|\} \to \infty$. This fact and $\|g_k\| \geq \epsilon$ for all $k \geq k_0$ imply that all the conditions of Lemma 3.6 are satisfied for $I := \{k \in \mathbb{N}_+ : k \geq k_0\}$; therefore, Lemma 3.6 implies that for all sufficiently large $k \in I$, one finds $\rho_k \geq \eta$ so that $k \in A$, a contradiction.

To complete the proof, notice that the objective function values are monotonically decreasing. Combining this with the condition in Step 6, the fact that $f$ is bounded below by $f_{\inf}$ (see Assumption 3.1), and $|A| = \infty$, one deduces that $\{s_k\}_{k \in A} \to 0$, as claimed.

We now prove that there exists an infinite subsequence of iterates such that the sequence of gradients computed at those points converges to zero.

Lemma 3.8. It holds that

$$ \lim_{k \in \mathbb{N}_+, k \to \infty} \|g_k\| = 0. $$

Proof. To derive a contradiction, suppose that $\liminf_{k \in \mathbb{N}_+, k \to \infty} \|g_k\| > 0$, which along with the fact that $g_{k+1} = g_k$ for any $k \in \mathbb{N}_+ \setminus A$ means $\liminf_{k \in A, k \to \infty} \|g_k\| > 0$. Thus, there exists $\epsilon \in \mathbb{R}_{++}$ such that

$$ \|g_k\| \geq \epsilon $$

for all sufficiently large $k \in A$. \hfill (7)
Under (7), let us prove that \( \{\lambda_k\}_{k \in \mathcal{A}} \to \infty \). To derive a contradiction, suppose there exists an infinite \( \mathcal{A}_\lambda \subseteq \mathcal{A} \) such that \( \lambda_k \leq \lambda_{\text{max}} \) for some \( \lambda_{\text{max}} \in \mathbb{R}^+ \). On the other hand, by \( \{s_k\}_{k \in \mathcal{A}} \to 0 \) (see Lemma 3.7) and (10), it follows that \( \{g_k + (H_k + \lambda_k I)s_k\}_{k \in \mathcal{A}} \to 0 \). Combining the upper bound on \( \{\lambda_k\}_{k \in \mathcal{A}} \), the fact that \( \{s_k\}_{k \in \mathcal{A}} \to 0 \), and \( \|H_k\| \leq H_{\text{max}} \) (see Lemma 3.2), it follows that \( \{g_k\}_{k \in \mathcal{A}} \to 0 \), which violates (7). Therefore, \( \{\lambda_k\}_{k \in \mathcal{A}} \to \infty \).

Our next goal is to prove, still under (7), that \( k \in \mathcal{A} \) for all sufficiently large \( k \in \mathbb{N}_+ \). To prove this, our strategy is to show that the sets of iterations involving a rejected step followed by an accepted step are finite. In particular, let us define the index sets

\[
\mathcal{R}_1 := \{ k \in \mathcal{R} : \text{the condition in Step 11 tests true and } (k + 1) \in \mathcal{A} \} \quad \text{and} \quad \mathcal{R}_2 := \{ k \in \mathcal{R} : \text{the condition in Step 11 tests false and } (k + 1) \in \mathcal{A} \}.
\]

We aim to prove that these are finite. First, consider \( \mathcal{R}_1 \). To derive a contradiction, suppose that \( |\mathcal{R}_1| = \infty \). By definition, for all \( k \in \mathcal{R}_1 \), the condition in Step 11 tests true, meaning \( (s_{k+1}, \lambda_{k+1}) \) is found in Step 5 satisfying \( \lambda_{k+1}/\|s_{k+1}\| \leq \sigma \). On the other hand, since \( (k + 1) \in \mathcal{A} \) for all \( k \in \mathcal{R}_1 \), it follows from Lemma 3.7 that \( \{s_{k+1}\}_{k \in \mathcal{R}_1} \to 0 \). Combining the conclusions of these last two sentences shows that \( \{\lambda_{k+1}\}_{k \in \mathcal{R}_1} \to 0 \). This contradicts the fact that \( \{\lambda_k\}_{k \in \mathcal{A}} \to \infty \). Hence, we may conclude that \( |\mathcal{R}_1| < \infty \). Now consider \( \mathcal{R}_2 \). To derive a contradiction, suppose that \( |\mathcal{R}_2| = \infty \). The fact that the condition in Step 11 tests false for \( k \in \mathcal{R}_2 \) implies that \( (s_{k+1}, \lambda_{k+1}) \) is found in Step 5 satisfying \( \lambda_{k+1}/\|s_{k+1}\| \leq 2\sigma \). However, since \( \{s_{k+1}\}_{k \in \mathcal{R}_2} \to 0 \) (see Lemma 3.7), it follows that \( \{\lambda_{k+1}\}_{k \in \mathcal{R}_2} \to \infty \) (established in the previous paragraph), which combined with the previously established inequality \( \lambda_{k+1}/\|s_{k+1}\| \leq 2\sigma \) shows that \( \{\lambda_k/\|s_k\|\}_{k \in \mathcal{R}_2} \to \infty \). Therefore, with (7), the conditions in Lemma 3.6 hold for \( I = \mathcal{R}_2 \), meaning that, for all sufficiently large \( k \in \mathcal{R}_2 \), the inequality \( g_k \geq \eta \) holds. This contradicts the fact that \( \mathcal{R}_2 \subseteq \mathcal{R} \); hence, we conclude that \( \mathcal{R}_2 \) is finite. Since \( \mathcal{R}_1 \) and \( \mathcal{R}_2 \) are finite, it follows from the logic of Algorithm 1 that either \( k \in \mathcal{A} \) for all sufficiently large \( k \) or \( k \in \mathcal{R} \) for all sufficiently large \( k \). By Lemma 3.7, it follows that \( k \in \mathcal{A} \) for all sufficiently large \( k \).

Thus far, we have proved under (7) that \( \{\lambda_k\}_{k \in \mathcal{A}} \to \infty \) and that \( k \in \mathcal{A} \) for all large \( k \in \mathbb{N}_+ \). From this latter fact, it follows that there exists \( k_\sigma \) such that \( \sigma_{k_\sigma} = \sigma_{k_\sigma}^t \in \mathbb{R}^+ \) for all \( k \geq k_\sigma \). In addition, from Step 4 it follows that for \( k \geq k_\sigma \) one finds \( \lambda_k/\|s_k\| \leq \sigma_{k_\sigma}^t < \infty \). However, this leads to a contradiction to the facts that \( \{\lambda_k\}_{k \in \mathcal{A}} \to \infty \) and \( \{s_k\}_{k \in \mathcal{A}} \to 0 \) (see Lemma 3.7). Overall, we have shown that (7) cannot be true, which proves the desired result.

We close with our main global convergence result of this subsection, the proof of which borrows much from that of Theorem 3.14 in [11].

**Theorem 3.3.** Under Assumptions 2.1, 3.1, and 3.2, it follows that

\[
\lim_{k \in \mathbb{N}_+; k \rightarrow \infty} \|g_k\| = 0. \tag{8}
\]

**Proof.** For the purpose of reaching a contradiction, suppose that (8) does not hold. Combining this with the fact that \( |\mathcal{A}| = \infty \) (see Lemma 3.7), it follows that there exists an infinite subsequence \( \{t_i\} \subseteq \mathcal{A} \) (indexed over \( i \in \mathbb{N}_+ \)) such that, for all \( i \in \mathbb{N}_+ \), one finds \( \|g_{t_i}\| \geq 2\epsilon > 0 \). Also, the fact that \( |\mathcal{A}| = \infty \) and Lemma 3.8 imply that there exists an infinite subsequence \( \{t_i\} \subseteq \mathcal{A} \) (indexed over \( i \in \mathbb{N}_+ \)) such that, for all \( i \in \mathbb{N}_+ \) and \( k \in \mathbb{N}_+ \) with \( t_i \leq k < t_i \), one finds

\[
\|g_k\| \geq \epsilon \quad \text{and} \quad \|g_{t_i}\| < \epsilon. \tag{9}
\]

Let us now restrict our attention to indices in the infinite index set

\[
\mathcal{K} := \{ k \in \mathcal{A} : t_i \leq k < t_i \text{ for some } i \in \mathbb{N}_+ \}.
\]

Observe from (9) that, for all \( k \in \mathcal{K} \), it follows that \( \|g_k\| \geq \epsilon \). Also, from the definition of \( \mathcal{A} \),

\[
f_k - f_{k+1} \geq \eta \|s_k\|^3 \quad \text{for all } k \in \mathcal{K} \subseteq \mathcal{A}. \tag{10}
\]
Since \( \{f_k\} \) is monotonically decreasing and bounded below, one finds that \( \{f_k\} \to f \) for some \( f \in \mathbb{R} \), which when combined with (10) shows that
\[
\lim_{k \in \mathcal{K}, k \to \infty} \|s_k\| = 0. \tag{11}
\]

Using this fact, Lemma 3.4, Assumption 3.2 and the Mean Value Theorem (as it is used in the proof of Lemma 3.6) to yield (6), it follows that for all sufficiently large \( k \in \mathcal{K} \) one has
\[
f_k - f_{k+1} = f_k - q_k(s_k) + q_k(s_k) - f(x_k + s_k)
\geq \frac{\|g_k\|}{6\sqrt{2}} \min \left\{ \frac{\|g_k\|}{1 + H_{\max}}, \frac{\|s_k\|}{\sqrt{6}} \right\} \sqrt{2\|g_k\| + \|s_k\| (\frac{3}{2}H_{\max} + \kappa_1)} - (g_{\text{Lip}} + \frac{1}{2}H_{\max})\|s_k\|^2
\geq \frac{\epsilon}{6\sqrt{2}} \frac{\|s_k\|}{\sqrt{6}} \sqrt{2\|g_k\| + \|s_k\| (\frac{3}{2}H_{\max} + \kappa_1)} - (g_{\text{Lip}} + \frac{1}{2}H_{\max})\|s_k\|^2.
\]

It now follows from (9) and (11) that, as \( k \to \infty \) over \( k \in \mathcal{K} \), the square root term in the previous inequality converges to \( 1/\sqrt{2} \). Since the second term in the previous inequality is of order \( \|s_k\|^2 \), the first term is of order \( \|s_k\| \), and \( 1/\sqrt{2} > 1/\sqrt{3} \), one can thus conclude that \( f_k - f_{k+1} \geq \epsilon\|s_k\|/36 \) for all sufficiently large \( k \in \mathcal{K} \). Consequently, it follows that for all sufficiently large \( i \in \mathbb{N}_+ \) one finds
\[
\|x_{t_i} - x_{t_0}\| \leq \sum_{k \in \mathcal{K}, k = t_i}^{t_i-1} \|x_k - x_{k+1}\|
= \sum_{k \in \mathcal{K}, k = t_i}^{t_i-1} \|s_k\| \leq \sum_{k \in \mathcal{K}, k = t_i}^{t_i-1} \frac{36}{\epsilon} (f_k - f_{k+1}) = \frac{36}{\epsilon} (f_{t_i} - f_{t_0}).
\]

Since \( \{f_{t_i} - f_{t_0}\} \to 0 \) (recall that \( \{f_k\} \to f \) monotonically) this implies that \( \{\|x_{t_i} - x_{t_0}\|\} \to 0 \), which, in turn, implies that \( \{|g_{t_i} - g_{t_0}|\} \to 0 \) because of the continuity of \( g \). However, this is a contradiction since, for any \( i \in \mathbb{N}_+ \), we have \( |g_{t_i} - g_{t_0}| \geq \epsilon \) by the definitions of \( \{t_i\} \) and \( \{\ell_i\} \). Overall, we conclude that our initial supposition must be false, implying that (8) holds. \( \square \)

### 3.2 First-Order Complexity

Our next goal is to prove, with respect to a prescribed positive threshold, a worst-case upper bound on the number of iterations required for our algorithm to reduce the norm of the gradient below the threshold. In this subsection, along with Assumptions 2.1, 3.1, and 3.2, we add the following.

**Assumption 3.4.** The Hessian function \( H \) is Lipschitz continuous on a path defined by the sequence of iterates and trial steps; in particular, it is Lipschitz continuous with a scalar Lipschitz constant \( H_{\text{Lip}} > 0 \) on the set \( \{x_k + \tau s_k : k \in \mathbb{N}_+, \tau \in [0,1]\} \).

We begin our analysis in this subsection by providing a lemma that shows that successful steps always result if \( \lambda_k \) is sufficiently large relative to the size of the step.

**Lemma 3.9.** For any \( k \in \mathbb{N}_+ \), if the pair \((s_k, \lambda_k)\) satisfies
\[
\lambda_k \geq (H_{\text{Lip}} + \kappa_2 + 2\eta)\|s_k\|, \tag{12}
\]
then \( \rho_k \geq \eta \).

**Proof.** It follows from Assumption 3.4 and Taylor’s expansion with Lagrange remainder that there exists \( \bar{x}_k \) on the line segment \([x_k, x_k + s_k]\) such that
\[
g_k(s_k) - f(x_k + s_k) = \frac{1}{2} s_k^T (H_k - H(\bar{x}_k)) s_k \geq -\frac{1}{2} H_{\text{Lip}}\|s_k\|^3. \tag{13}
\]
Also, it follows from (14) that
\[
\begin{align*}
f_k - q_k(s_k) &= -g_k^T s_k - \frac{1}{2} s_k^TH_k s_k \\
&= -s_k^T(g_k + (H_k + \lambda_k I)s_k) + \frac{1}{2} \lambda_k \|s_k\|^2 + \frac{1}{2} s_k^T(H_k + \lambda_k I)s_k \\
&\geq -\frac{1}{2} s_k^T(H_k + \lambda_k I)s_k - \frac{3}{2} \kappa_2 \|s_k\|^3 + \frac{1}{2} \lambda_k \|s_k\|^2 + \frac{1}{2} s_k^T(H_k + \lambda_k I)s_k \\
&= -\frac{1}{2} \kappa_2 \|s_k\|^3 + \frac{1}{2} \lambda_k \|s_k\|^2.
\end{align*}
\]

From (13) and (14), it follows that
\[
f_k - f(x_k + s_k) = f_k - q_k(s_k) + q_k(s) - f(x_k + s_k) \\
\geq \frac{1}{2} \lambda_k \|s_k\|^2 - \frac{3}{2} \kappa_2 \|s_k\|^3 - \frac{1}{2} H_{lip} \|s_k\|^3,
\]
which together with (12) implies that \(\rho_k \geq \eta\), as claimed.

We now prove that the sequence \(\{\sigma^v_k\}\) is bounded above.

**Lemma 3.10.** There exists a scalar constant \(\sigma_{max} \in \mathbb{R}_{++}\) such that, for all \(k \in \mathbb{N}_+\),
\[
\sigma^v_k \leq \sigma_{max}.
\]

**Proof.** Consider any \(k \in \mathbb{N}_+\). If \(s_k\) is accepted (i.e., \(k \in \mathcal{A}\)), then \(\sigma^v_{k+1} \leftarrow \sigma^v_k\). On the other hand, if \(s_k\) is rejected (i.e., \(k \in \mathcal{R}\)), then it follows from Step 12 and Step 14 that \(\sigma^v_{k+1} \leq \max\{\sigma, \gamma_2 \lambda_k / \|s_k\|\}\).

Moreover, since \(k \in \mathcal{R}\), meaning that \(\rho_k < \eta\), it follows from Lemma 3.9 that \(\lambda_k / \|s_k\|\) is bounded above by \((H_{lip} + \kappa_2 + 2\eta)\). Thus, it follows that \(\sigma^v_{k+1} \leq \max\{\sigma, \gamma_2 (H_{lip} + \kappa_2 + 2\eta)\}\) for all \(k \in \mathcal{R}\). Overall, the desired result follows for any \(\sigma_{max} \geq \max\{\sigma, \gamma_2 (H_{lip} + \kappa_2 + 2\eta)\}\).

We now establish a lower bound on the norm of any accepted trial step.

**Lemma 3.11.** For all \(k \in \mathcal{A}\), it follows that
\[
\|s_k\| \geq \left(\frac{1}{2} H_{lip} + 2\sigma_{max} + \kappa_3\right)^{-1/2} \|g_{k+1}\|^{1/2}.
\]

**Proof.** Let \(k \in \mathcal{A}\). It follows that
\[
\|g_{k+1}\| \leq \|g_{k+1} - (g_k + (H_k + \lambda_k I)s_k)\| + \|g_k + (H_k + \lambda_k I)s_k\| \\
\leq \|g_{k+1} - (g_k + H_k s_k)\| + \|g_k + (H_k + \lambda_k I)s_k\| + \|g_k + (H_k + \lambda_k I)s_k\|.
\]

By Taylor’s theorem and Assumption 3.4, the first term on the right-hand side of this inequality satisfies
\[
\|g_{k+1} - (g_k + H_k s_k)\| \leq \left\| \int_0^1 (H(x_k + \tau s_k) - H_k)s_k d\tau \right\| \\
\leq \int_0^1 \|H(x_k + \tau s_k) - H_k\| d\tau \cdot \|s_k\| \\
\leq \int_0^1 \tau d\tau \cdot H_{lip} \|s_k\|^2 = \frac{1}{2} H_{lip} \|s_k\|^2.
\]

Combining this with (15) and observing Step 5 (4c), and Lemma 3.10 it follows that
\[
\|g_{k+1}\| \leq \frac{1}{2} H_{lip} \|s_k\|^2 + 2 \lambda_k \|s_k\|^2 + \kappa_3 \|s_k\|^2 \\
\leq \frac{1}{2} H_{lip} \|s_k\|^2 + 2 \sigma_{max} \|s_k\|^2 + \kappa_3 \|s_k\|^2,
\]
which, after rearrangement, completes the proof. \(\square\)
We are now prepared to prove a worst-case upper bound on the total number of accepted steps that may occur for iterations in which the norm of the gradient of the objective is above a positive threshold.

**Lemma 3.12.** For any $\epsilon \in \mathbb{R}_{++}$, the total number of elements in the index set

$$K_\epsilon := \{ k \in \mathbb{N}_+ : k \geq 1, (k-1) \in A, \|g_k\| > \epsilon \}$$

is at most

$$\left( \frac{f_0 - f_{\inf}}{\eta(\frac{1}{2}H_{Lip} + 2\sigma_{max} + \kappa_3)^{-3/2}} \right) \epsilon^{-3/2} =: N_A(\epsilon) \geq 0. \quad (16)$$

**Proof.** The proof follows in a similar manner as that of Lemma 3.20 in [11]. By Lemma 3.11, it follows that, for all $k \in K_\epsilon$, one finds

$$f_{k-1} - f_k \geq \eta\|s_{k-1}\|^3 \geq \eta(\frac{1}{2}H_{Lip} + 2\sigma_{max} + \kappa_3)^{-3/2}\|g_k\|^3/2 \geq \eta(\frac{1}{2}H_{Lip} + 2\sigma_{max} + \kappa_3)^{-3/2}\epsilon^{3/2}.$$ 

Rearranging this inequality to yield an upper bound for $|K_\epsilon|$ and using the fact that $f_0 - f_{\inf} \geq f_0 - f_{\hat{k}}$, one obtains the desired result.

In order to prove a result similar to Lemma 3.12 for the total number of iterations with $\|g_k\| > \epsilon$, we require an upper bound on the total number of trial steps that may be rejected between accepted steps. To this end, let us define, for a given $\hat{k} \in A \cup \{0\}$, the iteration number and corresponding set

$$k_A(\hat{k}) := \min\{k \in A : k > \hat{k}\}$$

and

$$I(\hat{k}) := \{ k \in \mathbb{N}_+ : \hat{k} < k < k_A(\hat{k}) \},$$

i.e, we let $k_A(\hat{k})$ be the smallest of all iteration numbers in $A$ that is strictly larger than $\hat{k}$, and we let $I(\hat{k})$ be the set of iteration numbers between $\hat{k}$ and $k_A(\hat{k})$.

We now show that the number of rejected steps between the first iteration and the first accepted step, or between consecutive accepted steps, is bounded above.

**Lemma 3.13.** For any $\hat{k} \in A \cup \{0\}$, it follows that

$$|I(\hat{k})| \leq 1 + \left[ \frac{1}{\log(\gamma_1)} \log \left( \frac{\sigma_{max}}{\sigma} \right) \right] =: N_R \geq 0.$$ 

**Proof.** The proof follows in a similar manner as for Lemma 3.24 in [11]. First, the result holds trivially if $|I(\hat{k})| = 0$. Thus, we may assume that $|I(\hat{k})| \geq 1$. Since $(\hat{k} + 1) \in R$ by construction, it follows from Steps 11 and Step 5 that $\lambda_{k+2}/\|s_{k+2}\| \geq \sigma$, which, due to the lower bound on $\lambda_{k+1}/\|s_{k+1}\|$ in Step 14, leads to

$$\lambda_{k_A(\hat{k})} \geq \sigma (\gamma_1)^{k_A(\hat{k})-k-2} \|s_{k_A(\hat{k})}\|.$$ 

Combining this with Step 5 and Lemma 3.10 shows that

$$\sigma_{max} \geq \sigma^{u}_{k_A(\hat{k})} \geq \lambda_{k_A(\hat{k})}/\|s_{k_A(\hat{k})}\| \geq \sigma (\gamma_1)^{k_A(\hat{k})-k-2}.$$
After rearrangement, it now follows that

$$k_A(\hat{k}) - \hat{k} - 2 \leq \frac{1}{\log(\gamma_1)} \log \left( \frac{\sigma_{\max}}{\sigma} \right).$$

The desired result follows from this inequality since $|I(\hat{k})| = k_A(\hat{k}) - \hat{k} - 1$. \qed

We are now prepared to prove our main complexity result of this subsection.

**Theorem 3.5.** Under Assumptions 2.1, 3.1, 3.2, and 3.4, for a scalar $\epsilon \in \mathbb{R}^+$, the total number of elements in the index set \( \{ k \in \mathbb{N}^+ : \|g_k\| > \epsilon \} \) is at most

$$N(\epsilon) := 1 + N_R N_A(\epsilon),$$

(17)

where $N_A(\epsilon)$ and $N_R$ are defined in Lemmas 3.12 and 3.13, respectively. Consequently, for any $\epsilon \in \mathbb{R}^+$, it follows that $N(\epsilon) = O(\epsilon^{-3/2})$ for all $\epsilon \in (0, \tau]$.

**Proof.** Without loss of generality, we may assume that at least one iteration is performed. Lemma 3.12 guarantees that the total number of elements in the index set \( \{ k \in A : k \geq 1, \|g_k\| > \epsilon \} \) is at most $N_A(\epsilon)$, where, immediately prior to each of the corresponding accepted steps, Lemma 3.13 guarantees that at most $N_R$ trial steps are rejected. Accounting for the first iteration, the desired result follows. \qed

### 3.3 Second-Order Global Convergence and Complexity

Our goal in this subsection is to prove results showing that, in some sense, the algorithm converges to second-order stationarity and does so with a worst-case iteration complexity on par with the methods in [7] and [11]. In particular, our results show that if the algorithm computes each search direction to satisfy a curvature condition over a subspace, then second-order stationarity is reached in a manner that depends on the subspaces.

In this subsection, we make the following additional assumption about the subproblem solver.

**Assumption 3.6.** For all $k \in \mathbb{N}^+$, let $L_k \subseteq \mathbb{R}^n$ denote a subspace with an orthonormal basis formed from the columns of a matrix $R_k$. The step $s_k$ satisfies

$$\xi(R_k^T H_k R_k) \geq -\kappa_4 \|s_k\|$$

(18)

for some $\kappa_4 \in \mathbb{R}^+$, where $\xi(R_k^T H_k R_k)$ indicates the smallest eigenvalue of $R_k^T H_k R_k$.

This assumption is reasonable, e.g., in cases when $s_k$ is computed by solving problem 2 with the component $s$ restricted to a subspace of $\mathbb{R}^n$. We refer the reader to Theorem B.3 for a proof of this fact, which also reveals that this assumption is congruous with Assumption 2.1.

Under this assumption, we have the following second-order convergence result.

**Theorem 3.7.** Suppose Assumptions 2.1, 3.1, 3.2, 3.4, and 3.6 hold. It follows that

$$\liminf_{k \in A, k \to \infty} \xi(R_k^T H_k R_k) \geq 0.$$

**Proof.** The result follows from (18) since $\{s_k\}_{k \in A} \to 0$ (see Lemma 3.7). \qed

As a consequence of Theorem 3.7, if the sequence $\{R_k\}_{k \in A}$ tends toward full-dimensionality as $k \to \infty$, then any limit point $x_*$ of $\{x_k\}$ must have $H(x_*) \succeq 0$.

Our next goal is to prove a worst-case iteration complexity result for achieving second-order stationarity in a sense similar to that in Theorem 3.7. Toward this end, we first prove the following lemma, which is similar to Lemma 3.12.
Lemma 3.14. For any $\epsilon \in \mathbb{R}^{++}$, the total number of elements in the index set

$$K_{\epsilon, \xi} := \{k \in \mathbb{N}_+ : k \geq 1, (k-1) \in A, \xi(R_k^T H_k R_k) < -\epsilon\}$$

is at most

$$\left\lfloor \left( \frac{f_0 - f_{\text{inf}}}{\eta \kappa_4^{-3}} \right) \epsilon^{-3} \right\rfloor =: N_{A, \xi}(\epsilon) \geq 0. \quad (19)$$

Proof. Under Assumption 3.6, it follows that, for all $k \in K_{\epsilon, \xi}$, one finds

$$f_{k-1} - f_k \geq \eta \|s_{k-1}\|^3 \geq \eta \left( \frac{-\xi(R_k^T H_k R_k)}{\kappa_4} \right)^3 \geq \eta \kappa_4^{-3} \epsilon^3.$$ 

It follows from this inequality, the fact that $f$ is monotonically decreasing over the sequence of iterates, and Assumption 3.1 that

$$f_0 - f_{\text{inf}} \geq \sum_{k \in K_{\epsilon, \xi}} (f_{k-1} - f_k) \geq |K_{\epsilon, \xi}| \eta \kappa_4^{-3} \epsilon^3.$$ 

Rearranging this inequality to yield an upper bound for $|K_{\epsilon, \xi}|$ gives the result. \qed

We close with the following second-order complexity result.

Theorem 3.8. Under Assumptions 2.1, 3.1, 3.2, 3.4, and 3.6, for any pair of scalars $(\epsilon_1, \epsilon_2) \in \mathbb{R}^{++} \times \mathbb{R}^{++}$, the number of elements in the index set

$$\{k \in \mathbb{N}_+ : \|g_k\| > \epsilon_1 \lor \xi(R_k^T H_k R_k) < -\epsilon_2\}$$

is at most

$$N(\epsilon_1, \epsilon_2) := 1 + N_{R} \max\{N_A(\epsilon_1), N_{A, \xi}(\epsilon_2)\}, \quad (20)$$

where $N_A(\cdot)$, $N_R$, and $N_{A, \xi}(\cdot)$ are defined in Lemmas 3.12, 3.13, and 3.14, respectively. Consequently, for any pair of scalars $(\bar{\epsilon}_1, \bar{\epsilon}_2) \in \mathbb{R}^{++} \times \mathbb{R}^{++}$, it follows that

$$N(\epsilon_1, \epsilon_2) = O(\max\{\epsilon_1^{-3/2}, \epsilon_2^{-3}\}) \text{ for all } (\epsilon_1, \epsilon_2) \in (0, \bar{\epsilon}_1] \times (0, \bar{\epsilon}_2].$$

Proof. The proof follows in a similar manner as that of Theorem 3.5 by additionally incorporating the bound proved in Lemma 3.14. \qed

4 Algorithm Instances

Algorithm 1 is a broad framework containing, amongst other algorithms, ARC and TRACE. Indeed, the proposed framework and its supporting analyses cover a wide range of algorithms as long as the pairs in the sequence $\{(s_k, \lambda_k)\}$ satisfy Assumption 2.1.

In this section, we show that ARC and TRACE are special cases of our proposed framework in that the steps these algorithms accept would also be acceptable for our framework, and that the procedures followed by these methods after a step is rejected are consistent with our framework. We then introduce an instance of our framework that is new to the literature. (If desired for the guarantees in §3.3, one could also mind whether the elements in the sequence $\{(s_k, \lambda_k)\}$ satisfy Assumption 3.6. However, for brevity in this section, let us suppose that one is interested only in Assumption 2.1.)
4.1 ARC as a Special Case

The ARC method, which was inspired by the work in [19] and [21], was first proposed and analyzed in [6] [7]. In these papers, various sets of step computation conditions are considered involving exact and inexact subproblem solutions yielding different types of convergence and worst-case complexity guarantees. For our purposes here, we consider the more recent variant of ARC stated and analyzed as “ARP” with $p = 2$ in [4]. (For ease of comparison, we consider this algorithm when their regularization parameter update—see Step 4 in their algorithm—uses $\eta_1 = \eta_2$. Our algorithm is easily extended to employ a two-tier acceptance condition, involving two thresholds $\eta_1$ and $\eta_2$, as is used in [4] and [6] [7].)

Suppose that a trial step $s_k$ is computed by this version of ARC. In particular, let us make the reasonable assumption that the subproblem for which $s_k$ is an approximate solution is defined by some regularization value $\sigma_k \in [\sigma_k^1, \sigma_k^3]$ (with $\sigma_k^1 \geq \sigma_{\text{min}}$ since ARC ensures that $\sigma_k \geq \sigma_{\text{min}} \in \mathbb{R}_{++}$ for all $k \in \mathbb{N}$) and that this subproblem is minimized over a subspace $L_k$ such that $g_k \in L_k$ (see Appendix [B]). As is shown using a similar argument as in the proof of our Theorem [B.3] [b], one can show under these conditions that $(s_k, \lambda_k)$ with \[ \lambda_k = \sigma_k \| s_k \| \] satisfies (4a). In addition, considering the algorithm statement in [4], one requires to have

\[ s_k^T H_k s_k + \frac{1}{2} s_k^T H_k s_k + \lambda_k \| s_k \|^2 < 0 \quad \text{and} \quad \| g_k + (H_k + \lambda_k I) s_k \| \leq \theta \| s_k \|^2 \quad \text{for some} \quad \theta \in \mathbb{R}_{++}. \]

It is easily seen that $(s_k, \lambda_k)$ satisfying these conditions also satisfies (4b)–(4c) for any $(\kappa_1, \kappa_2, \kappa_3)$ such that $\kappa_1 \geq \frac{1}{2} H_{\text{max}}$ and $\kappa_3 \geq \theta$. Overall, we have shown that a trial step $s_k$ computed by this version of ARC satisfies Assumption 2.1 meaning that it satisfies the condition in Step 5 in Algorithm 1. If this trial step is accepted by ARC, then this means that $f_k - f(x_k + s_k) \geq \eta_1 (f_k - q_k(s_k))$. Along with [4] Lemma 2.1], this implies that $f_k - f(x_k + s_k) \geq \frac{1}{4} \eta_1 \| s_k \|^3$, meaning that $\rho_k \geq \frac{1}{4} \eta_1 \sigma_{\text{min}}$. Hence, this trial step would also be accepted in Algorithm 1 under the assumption that $\eta \in (0, \frac{1}{4} \eta_1 \sigma_{\text{min}}]$.

Finally, if a trial step is rejected in this version of ARC, then $\sigma_{k+1}$ is set to a positive multiple of $\sigma_k$. This is consistent with the procedure after a step rejection in Algorithm 1 where it is clear that, with appropriate parameter choices, one would find $\sigma_{k+1} \in [\sigma_k^{l+1}, \sigma_k^{r+1}]$.

4.2 TRACE as a Special Case

TRACE is proposed and analyzed in [11]. Our goal in this subsection is to show that, with certain parameter settings, a trial step that is computed and accepted by TRACE could also be one that is computed and accepted by Algorithm 1 and that the procedures for rejecting a step in TRACE are consistent with those in Algorithm 1. Amongst other procedures, TRACE involves dynamic updates for two sequences, $\{ \delta_k \}$ and $\{ \Delta_k \}$. The elements of $\{ \delta_k \}$ are the trust region radii while $\{ \Delta_k \}$ is a monotonically nondecreasing sequence of upper bounds for the trust region radii; consequently, $\| s_k \| \leq \delta_k \leq \Delta_k$ with $\Delta_{k+1} \geq \Delta_k$ for all $k \in \mathbb{N}$. For simplicity in our discussion here, let us assume that $\| s_k \| < \Delta_k$ for all $k \in \mathbb{N}$. This is a fair assumption since, as shown in [11] Lemma 3.11], the manner in which $\{ \Delta_k \}$ is set ensures that $\| s_k \| = \Delta_k$ only a finite number of times in any run.

In TRACE, during iteration $k \in \mathbb{N}$, a trust region radius $\delta_k \in \mathbb{R}_{++}$ is given and a trial step $s_k$ and regularization value $\lambda_k$ are computed satisfying the standard trust region subproblem optimality conditions

\[ g_k + (H_k + \lambda_k I) s_k = 0, \quad H_k + \lambda_k I \succeq 0, \quad \text{and} \quad \lambda_k (\delta_k - \| s_k \|) = 0, \quad \text{where} \quad (\lambda_k, \delta_k - \| s_k \|) \geq 0. \]

By the first of these conditions, the pair $(s_k, \lambda_k)$ clearly satisfies (4b)–(4c). In addition, one can use standard trust region theory, in particular related to Cauchy decrease (see [10] or [23]), to show that the pair also satisfies (4a). Overall, assuming that the pair $(\sigma_k^1, \sigma_k^3)$ is set such that $\lambda_k / \| s_k \| \in [\sigma_k^1, \sigma_k^3]$, it follows that Assumption 2.1 is satisfied, meaning that TRACE offers the condition in Step 3 in Algorithm 1. If the trial step $s_k$ is subsequently accepted by TRACE, then it would also be accepted by Algorithm 1 since both algorithms use the same step acceptance condition.

Now suppose that a trial step is not accepted in TRACE. This can occur in two circumstances. It can occur if $\rho_k \geq \eta$ while $\lambda_k > \sigma_k \| s_k \|$, in which case the trust region radius is expanded and a new subproblem
is solved. By the proof of [11] Lemma 3.7, the solution of this new subproblem yields (in iteration $k + 1$ in TRACE) the relationship that $\lambda_{k+1}/\|s_{k+1}\| \leq \sigma_{k+1} = \sigma_k$. Hence, under the same assumption as above that the pair $(\sigma_k^i, \sigma_k^j)$ is set such that $\lambda_k/\|s_k\| \in [\sigma_k^i, \sigma_k^j]$, this shows that the procedure in TRACE involving an expansion of the trust region radius and the computation of the subsequent trial step yields a trial step that would be offered in a single iteration in Algorithm [1]. The other circumstance in which a trial step is rejected in TRACE is when $\rho_k < \eta$, in which case the trust region radius is contracted. In this case, one can see that the outcome of the CONTRACT subroutine in TRACE is consistent with Steps 11–14 of Algorithm [1] in the sense that the solution of the subsequent subproblem in TRACE will have $\lambda_{k+1}/\|s_{k+1}\| \in [\sigma, \sigma]$ (if $\lambda_k < \sigma/\|s_k\|$) or $\lambda_{k+1}/\|s_{k+1}\|$ within a range defined by positive multiples of $\lambda_k/\|s_k\|$; see Lemmas 3.17 and 3.23 in [11].

4.3 A Hybrid Algorithm

The primary distinguishing feature of our algorithm instance is the manner in which we compute the pair $(s_k, \lambda_k)$ in Step 5 of Algorithm [1]. Our newly proposed hybrid algorithm considers two cases.

Case 1: $\sigma_k > 0$. In this case, we find a pair $(s_k, \lambda_k)$ by solving problem (A.3) over a sequence of increasingly higher dimensional Krylov subspaces as described in [6] until [4] and [18] are satisfied. The reason we know that [4] and [18] will eventually be satisfied can be seen as follows. Solving problem (A.3) over a Krylov subspace is equivalent to solving problem (B.3) with an appropriate choice of $R_k$ as a basis for that Krylov subspace, then setting $s_k = R_kv_k$. Then, it follows from Theorem (B.2) that solving (B.3) is equivalent to solving (B.2), which in turn is equivalent to solving (B.1) in the sense that if $(v_k, \lambda_k, \beta_k^i, \beta_k^j, \beta_k^m)$ is a first-order primal-dual solution of problem (B.2), then $(s_k, \lambda_k, \beta_k^i, \beta_k^j, \beta_k^m)$ with $s_k = R_kv_k$ is a solution of problem (B.1). Finally, we need only note from Theorem (B.3) that solutions to problem (B.1) satisfy (4a) for all Krylov subspaces $L_k$ (recall that $g_k$ is contained in all Krylov subspaces), (4b) for all Krylov subspaces, (4c) if the Krylov subspace $L_k$ includes enough of the space (in the worst case, $L_k = \mathbb{R}^n$), and (4d) for all Krylov subspaces.

Case 2: $\sigma_k = 0$. In this case, we begin by applying the linear CG method in an attempt to solve the linear system $H_k s = -g_k$, which iteratively solves

$$\min_{s \in \mathbb{R}^n} q_k(s)$$

over a sequence of expanding Krylov subspaces. One of two outcomes is possible. First, the CG algorithm may ultimately identify a vector $s_k$ such that $(s_k, \lambda_k)$ with $\lambda_k = 0$ satisfies (4) and (18). Second, the CG algorithm may never identify a vector $s_k$ such that $(s_k, \lambda_k)$ with $\lambda_k = 0$ satisfies (4) and (18). Indeed, this might occur if CG encounters a direction of negative curvature—in which case we terminate CG immediately—or if CG solves (21) accurately or reaches an iteration limit, and yet at least one condition in (4) and (18) is not satisfied. In such a case, we choose to reset $\sigma_k^i \in (0, \sigma_k^j]$, then solve problem (A.3) over a sequence of expanding Krylov subspaces as described in Case 1. In this manner, we are guaranteed to identify a pair $(s_k, \lambda_k)$ satisfying (4) and (18) as required.

5 Implementation and Numerical Results

We implemented two algorithms in MATLAB, one following the strategy in [4.3] and, for comparison purposes, one following the ARC algorithm in [7] with ideas from [4]. We refer to our implementation of the former as iR_newton, for inexact Regularized Newton, and to our implementation of the latter as iARC, for inexact ARC. In this section, we describe our approach for computing the pairs $(s_k, \lambda_k)$ in iR_newton and iARC, as well as other implementation details, and discuss the results of numerical experiments on a standard set of nonlinear optimization test problems.
5.1 Implementation Details

Let us begin by noting that the implemented algorithms terminate in iteration $k \in \mathbb{N}_+$ if
\[ \|g_k\|_\infty \leq 10^{-6} \max\{\|g_0\|_\infty, 1\}. \]

We chose not to employ a termination test based on a second-order stationarity condition. Correspondingly, neither of the algorithms check a second-order condition when computing a trial step; e.g., in \texttt{iRNewton}, we are satisfied with a step satisfying \([4]\) and do not check \([13]\). In addition, for practical purposes, we set an maximum iteration limit of $10^6$, a time limit of four hours, and a minimum step norm limit of $10^{-20}$. For reference, the input parameter values we used are given in Table 1. We chose these values as ones that worked well on our test set for both implemented algorithms.

| $\eta_1$ | 1.0e-16 |
| $\eta_2$ | 1.0e-01 |
| $\gamma_0$ | 2.0e-01 |
| $\gamma_1$ | 1.0e-01 |
| $\gamma_2$ | 2.0e+02 |
| $\kappa_3$ | 1.0e+10 |

Table 1: Input parameters for \texttt{iARC} and \texttt{iRNewton}

For both implemented algorithms, we employ a sequence $\{\sigma_k\}$ that is updated dynamically. In \texttt{iARC}, this sequence is handled as described in \([7]\), namely,
\[
\sigma_{k+1} \leftarrow \begin{cases} 
\max\{\sigma, \gamma_0 \sigma_k\} & \text{if } f_k - f(x_k + s_k) \geq \eta_2 \\
\sigma_k & \text{if } f_k - f(x_k + s_k) \in [\eta_1, \eta_2] \\
\gamma_1 \sigma_k & \text{if } f_k - f(x_k + s_k) < \eta_1
\end{cases}
\]

The value $\sigma_k$ is used in defining $c_k(\cdot; \sigma_k)$ (recall \([5]\)) that is minimized approximately to compute the trial step $s_k$ for all $k \in \mathbb{N}_+$. In particular, the implementation iteratively constructs Krylov subspaces of increasing dimension using the Lanczos process, where for each subspace we employ the RQS function from the GALAHAD software library (see \([17]\) and \([18]\)) to minimize $c_k(\cdot; \sigma_k)$ over the subspace. If the subspace is full-dimensional or the resulting step $s_k$ satisfies
\[
\|g_k + (H_k + \sigma_k s_k) I s_k\| \leq \kappa_3 \|s_k\|^2,
\]
then it is used as the trial step. Otherwise, the process continues with a larger subspace. We remark that condition (22) is more restrictive than our condition (4c), but we use it since it is one that has been proposed for cubic regularization methods; e.g., see (2.13) in \([11]\).

One could employ more sophisticated techniques for setting the elements of the sequence $\{\sigma_k\}$ in \texttt{iARC} that attempt to reduce the number of rejected steps; e.g., see \([10]\). Such improvements might aid \texttt{iRNewton} as well. However, for simplicity and to avoid the need for additional parameter tuning, we did not include such enhancements in our implemented algorithms.

As for \texttt{iRNewton}, for consistency between the two implementations, we do not explicitly compute the sequence $\{\lambda_k\}$, but rather employ $\{\sigma_k \|s_k\|\}$ in its place. For example, whenever an acceptable step is computed with $\sigma_k = 0$, then, as described in Case 2 in \([4,6]\), we effectively use $\lambda_k = 0$. On the other hand, when $\sigma_k > 0$, we employ the same iterative approach as used for \texttt{iARC} to compute the trial step $s_k$ as an approximate minimizer of $c_k(\cdot; \sigma_k)$, where in place of $\lambda_k$ in \([4]\), we employ $\sigma_k \|s_k\|$. Then, in either case, in the remainder of iteration $k \in \mathbb{N}_+$, specifically for setting $\sigma_k$ and $\sigma_k$, we use $\sigma_k \|s_k\|$ in place of $\lambda_k$ in Steps \([11]\) and \([14]\). We also define an auxiliary sequence $\{\sigma_k\}$ using the update
\[
\sigma_{k+1} \leftarrow \begin{cases}
\max\{\sigma, \gamma_0 \sigma_k\} & \text{if } \rho_k \geq \eta_1 \text{ and } \sigma_k > 0 \\
\sigma_k & \text{if } \sigma_k = 0 \\
\min\{\gamma_1 \sigma_k, \sigma\} & \text{if } \rho_k < \eta_1 \text{ and } \sigma_k > 0.
\end{cases}
\]

As for \texttt{iRNewton}, for consistency between the two implementations, we do not explicitly compute the sequence $\{\lambda_k\}$, but rather employ $\{\sigma_k \|s_k\|\}$ in its place. For example, whenever an acceptable step is computed with $\sigma_k = 0$, then, as described in Case 2 in \([4,6]\), we effectively use $\lambda_k = 0$. On the other hand, when $\sigma_k > 0$, we employ the same iterative approach as used for \texttt{iARC} to compute the trial step $s_k$ as an approximate minimizer of $c_k(\cdot; \sigma_k)$, where in place of $\lambda_k$ in \([4]\), we employ $\sigma_k \|s_k\|$. Then, in either case, in the remainder of iteration $k \in \mathbb{N}_+$, specifically for setting $\sigma_k$ and $\sigma_k$, we use $\sigma_k \|s_k\|$ in place of $\lambda_k$ in Steps \([11]\) and \([14]\). We also define an auxiliary sequence $\{\sigma_k\}$ using the update
\[
\sigma_{k+1} \leftarrow \begin{cases}
\max\{\sigma, \gamma_0 \sigma_k\} & \text{if } \rho_k \geq \eta_1 \text{ and } \sigma_k > 0 \\
\sigma_k & \text{if } \sigma_k = 0 \\
\min\{\gamma_1 \sigma_k, \sigma\} & \text{if } \rho_k < \eta_1 \text{ and } \sigma_k > 0.
\end{cases}
\]

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This update is similar to the one employed for iARC with the added assurance that $\{\sigma_k\} \subset [\sigma, \bar{\sigma}]$. The elements of this sequence are used in two circumstances. First, if, as described in Case 2 in §4.3, CG fails to produce a trial step $s_k$ satisfying (4) (with $\lambda_k = 0$), then we reset $\sigma^k \leftarrow \sigma_k$ and revert to the same scheme as above to compute the trial step when $\sigma^k > 0$. Second, if a step is rejected and $\sigma^k < \sigma$ (equivalently, $\lambda_k < \sigma\|s_k\|_2$ as in Step 12 in Algorithm 1), then we set $\sigma^{k+1} \leftarrow \sigma_{k+1}$. Lastly, we note that if CG ever performs $n$ iterations and the resulting solution (due to numerical error) does not satisfy (4) and no negative curvature is detected, then the resulting approximate solution $s_k$ is used as the trial step.

5.2 Results on the CUTEst Test Set

We employed our implemented algorithms, iARC and iR_Newton, to solve unconstrained problems in the CUTEst test set; see [13]. Among 171 unconstrained problems in the set, one (FLETCBV2) was removed since the algorithms terminated at the initial point, five (ARGLINC, DECONVU, FLETCHBV, INDEFM, and POWER) were removed due to a function evaluation error or our memory limitation of 8GB, and nine (EIGENBLS, EIGENCLS, FMINSURF, NONMSQRT, SRBVBND, SCURLY10, SCURLY20, SCURLY30, and SSCOSINE) were removed since neither algorithm terminated within our time limit. In addition, four were removed since neither of the algorithms terminated successfully: for HIELOW, iARC reached our maximum iteration limit; for CURLY20 and SSCOSINE, iARC reached the time limit; for INDEF, iARC terminated due to a subproblem solver error; and for all of these four problems, iR_Newton terminated due to our minimum step norm limit. The remaining set consisted of 152 test problems with number of variables ranging from 2 to 100,000. For additional details on the problems used and their sizes, see Appendix C.

To compare the performance of the implemented algorithms, we generated performance profiles for the number of iterations and number of Hessian-vector products required before termination. These are shown in Figure 1. A performance profile graph of an algorithm at point $\alpha$ shows the fraction of the test set for which the algorithm is able to solve within a factor of $2^\alpha$ of the best algorithm for the given measure; see [12]. When generating the profiles, we did not include three of the test problems—CURLY10, CURLY30, and MODEBEALE—on which iARC was unsuccessful while iR_Newton was successful. (In particular, iARC reached the time limit for all problems.) We feel that this gives a fairer comparison with respect to the problems on which both algorithms were successful.

As seen in Figure 1, the algorithms performed relatively comparably when it came to the number of iterations required, though clearly iR_Newton had an edge in terms of requiring fewer iterations on various problems. The difference in terms of numbers of Hessian-vector products required was more drastic, and indeed we point to this as the main measure of improved performance for iR_Newton versus iARC. One reason for this discrepancy is that iR_Newton required fewer iterations on some problems. However, more significantly, the difference was due in part to iR_Newton’s ability to employ and accept inexact Newton steps (with $\lambda_k = 0$) on many iterations. This is due to the fact that, in CG, one is able to compute the Hessian-vector product $H_k s_k$, needed to check the termination conditions for the computation of $s_k$, by taking a linear combination of Hessian-vector products already computed in CG; i.e., if $\{p_{k,i}\}$ are the search directions computed in CG such that $s_k = \sum_i \alpha_{k,i} p_{k,i}$, then CG involves computing $H_k p_{k,i}$ for each $i$ and can compute $H_k s_k = \sum_i \alpha_{k,i} (H_k p_{k,i})$. By contrast, one is unable to retrieve this product via a linear combination when the step is computed from the minimization of a cubic function, as is needed in iARC and in iR_Newton whenever $\sigma^k > 0$. Overall, we claim that the primary strength of iR_Newton as compared to iARC is its ability to employ inexact Newton steps.

For further details of our numerical results, see Appendix C. In these results, we also indicate the number of tridiagonal factorizations required; at least one is needed involving a tridiagonal matrix of size $m \times m$ every time an algorithm solves a cubic subproblem over an $m$-dimensional subspace.

6 Conclusion

We have proposed a general framework for solving smooth nonconvex optimization problems and proceeded to prove worst-case iteration complexity bounds for it. In fact, for a certain class of second-order methods em-
ployed to minimize a certain class of nonconvex functions, our first-order complexity result for our method is known to be optimal; see [8]. Our framework is flexible enough to cover a wide range of popular algorithms, an achievement made possible by the use of generic conditions that each trial step is required to satisfy. The use of such conditions allows for the calculation of inexact Newton steps, for example by performing minimization over expanding Krylov subspaces. Although we have presented a particular instance of our framework motivated by subproblem (2), additional instances can easily be derived by applying other optimization strategies for solving (2). Numerical experiments with an instance of our algorithm showed that it can lead to improved performance on a broad test set as compared to an implementation of a straightforward cubic regularization approach.

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A Subproblem Solution Properties

In this appendix, we explore properties of any first-order stationary solution of problem $P_k(\sigma^l_k, \sigma^u_k)$ defined as (2). Let us define a Lagrangian function for (2) as

$$L(s, \lambda, \beta^l_k, \beta^u_k, \beta^n_k) = f_k + g_k^T s + \frac{1}{2} s^T (H_k + \lambda I) s - \frac{\beta^l_k}{2} (\lambda^2 - (\sigma^l_k)^2 \|s\|^2) - \frac{\beta^u_k}{2} (\lambda^2 - (\sigma^u_k)^2 \|s\|^2) - \beta^n_k \lambda,$$

where $(\beta^l, \beta^u) \in \mathbb{R}_+ \times \mathbb{R}_+$ are the dual variables associated with the left-hand and right-hand constraints on $\lambda$, respectively, and $\beta^n \in \mathbb{R}_+$ is the dual variable associated with the nonnegativity constraint on $\lambda$. The tuple $(s_k, \lambda_k, \beta^l_k, \beta^u_k, \beta^n_k)$ is a first-order primal-dual stationary solution of $P_k(\sigma^l_k, \sigma^u_k)$ if it satisfies the following conditions:

$$g_k + (H_k + \lambda_k I)s_k + \beta^l_k (\sigma^l_k)^2 s_k - \beta^u_k (\sigma^u_k)^2 s_k = 0, \quad (A.1a)$$

$$\frac{1}{2} \|s_k\|^2 - \lambda_k (\beta^l_k - \beta^u_k) - \beta^n_k = 0, \quad (A.1b)$$

$$0 \leq \beta^l_k \perp (\lambda^2_k - (\sigma^l_k)^2 \|s_k\|^2) \geq 0, \quad (A.1c)$$

$$0 \leq \beta^u_k \perp (\lambda^2_k - (\sigma^u_k)^2 \|s_k\|^2) \leq 0, \quad \text{and} \quad (A.1d)$$

$$0 \leq \beta^n_k \perp \lambda_k \geq 0. \quad (A.1e)$$

We make the following assumption throughout this appendix.
**Assumption A.1.** The vector $g_k$ is nonzero.

Under this assumption, the following lemma is a simple consequence of (A.1a).

**Lemma A.1.** Any solution of \((2)\) has $s_k \neq 0$.

We now establish conditions that must hold depending on the value of $\sigma_k^+ \in \mathbb{R}_+$.

**Lemma A.2.** The following hold true for any solution of \((A.1)\).

(i) If $\sigma_k^+ > 0$, then $\lambda_k > 0$, $\beta_k^u = 0$, $\beta_k^l > 0$, and $\lambda_k = \sigma_k^+ \|s_k\|$.

(ii) If $\sigma_k^+ = 0$, then $\lambda_k = 0$.

**Proof.** Consider part (i). For the sake of deriving a contradiction, suppose $\sigma_k^+ > 0$ and $\lambda_k = 0$. These, along with Lemma A.1, imply that $0 = \lambda_k^2 < (\sigma_k^+)^2 \|s_k\|^2$, which contradicts (A.1c). Hence, $\lambda_k > 0$, as claimed. Then, it follows from (A.1a) that $\beta_k^u = 0$, as claimed. Next, observe that from (A.1b), Lemma A.1 implies that $\beta_k^v = 0$, $\lambda_k > 0$, and $(\beta_k^l, \beta_k^v) \geq 0$, it follows that $\beta_k^l > 0$, as claimed. This, along with (A.1c), implies that $\lambda_k^2 = (\sigma_k^+)^2 \|s_k\|^2$. This implies that $\lambda_k = \pm (\sigma_k^+ \|s_k\|)$, which combined with $\lambda_k \in \mathbb{R}_+$ means that $\lambda_k = \sigma_k^+ \|s_k\|$, as claimed.

Now consider part (ii). For the sake of deriving a contradiction, suppose that $\sigma_k^+ = 0$ and $\lambda_k > 0$. Then, it follows from (A.1a) that $\beta_k^u = 0$. Moreover, combining $\sigma_k^+ = 0$ and $\lambda_k > 0$, it follows from (A.1a) that $\beta_k^l = 0$. It now follows from $\beta_k^l = 0$, $\beta_k^u = 0$, and (A.1b) that
\[
\frac{1}{2} \|s_k\|^2 = -\lambda_k \beta_k^v \leq 0,
\]
where the inequality follows from $\lambda_k > 0$ and $\beta_k^v \geq 0$. This contradicts Lemma A.1. \(\square\)

Our main result is the following. In part (i) with $\sigma_k^+ > 0$, we show that solving (2) is equivalent to solving what may be referred to as an arc subproblem \([6]\). In part (ii) with $\sigma_k^+ = 0$, we show that it is equivalent to minimizing a quadratic, if a minimizer exists.

**Theorem A.2.** The following hold true.

(i) Suppose $\sigma_k^+ > 0$. Then, (2) has a solution $(s_k, \lambda_k)$, which can be obtained as
\[
s_k \in \text{arg min}_{s \in \mathbb{R}^n} \left( f_k + g_k^T s + \frac{1}{2} s^T H_k s + \frac{1}{2} \sigma_k^+ \|s\|^2 \right),
\]

then setting $\lambda_k = \sigma_k^+ \|s_k\| > 0$.

(ii) If $\sigma_k^+ = 0$, then a solution of problem (2) exists if and only if $H_k \succeq 0$ and $g_k^T u = 0$ for all $u \in \text{Null}(H_k)$. In such cases, computing a solution $(s_k, \lambda_k)$ of problem (2) is equivalent to computing a solution $s_k$ of problem (21) and setting $\lambda_k = 0$.

**Proof.** Consider part (i). Since $\sigma_k^+ > 0$, it follows from Lemma A.2 that problem (2) is equivalent to
\[
\min_{(s, \lambda) \in \mathbb{R}^n \times \mathbb{R}_+} \left( f_k + g_k^T s + \frac{1}{2} s^T (H_k + \lambda I) s \right)
\]
s.t. $\sigma_k^+ \|s\| = \lambda$,

where, by Lemma A.1, it follows that the solution has $\lambda_k > 0$, as desired. Substituting the constraint of (A.4) into the objective of (A.4), one finds that solving it is equivalent to solving (A.3) for $s_k$, then setting $\lambda_k = \sigma_k^+ \|s_k\|$, as claimed. Since $\sigma_k^+ > 0$, a minimizer of problem (A.3) exists because it involves the minimization of a coercive function.

Now consider part (ii). Since $\sigma_k^+ = 0$, it follows from Lemma A.2 that $\lambda_k = 0$, meaning that problem (2) is equivalent to (21). This problem has a solution if and only if the objective is bounded below, which is the case if and only if $H_k \succeq 0$ and $g_k^T u = 0$ for all $u \in \text{Null}(H_k)$. \(\square\)
In this appendix, we explore properties of any first-order stationary solution (when one exists) of problem \( \mathcal{P}_k(\sigma_k^l, \sigma_k^u) \) defined as \( \mathcal{P}_k \) when the search space for \( s \) is restricted to a subspace of \( \mathbb{R}^n \). Specifically, for some \( m \)-dimensional subspace \( \mathcal{L}_k \subseteq \mathbb{R}^n \), consider the problem

\[
\min_{(s, \lambda) \in \mathcal{L}_k \times \mathbb{R}_+} f_k + g_k^T s + \frac{1}{2} s^T (H_k + \lambda I) s \\
\text{s.t. } (\sigma_k^l)^2 \|s\|^2 \leq \lambda^2 \leq (\sigma_k^u)^2 \|s\|^2.
\]  

(B.1)

Given an orthogonal basis \( R_k \) for \( \mathcal{L}_k \), a solution of (B.1) can be obtained from that of

\[
\min_{(v, \lambda) \in \mathbb{R}^m \times \mathbb{R}_+} f_k + g_k^T R_k v + \frac{1}{2} (R_k v)^T (H_k + \lambda I) R_k v \\
\text{s.t. } (\sigma_k^l)^2 \|v\|^2 \leq \lambda^2 \leq (\sigma_k^u)^2 \|v\|^2.
\]  

(B.2)

Specifically, if \((v_k, \lambda_k, \beta_k^l, \beta_k^u, \sigma_k^l, \sigma_k^u)\) is a first-order primal-dual solution of problem \( \text{(B.2)} \), then the tuple \((\sigma_k^l, \lambda_k, \beta_k^l, \beta_k^u, \sigma_k^l, \sigma_k^u)\) with \( s_k = R_k v_k \) is such a solution of problem \( \text{(B.1)} \).

In Appendix A, we proved properties of a solution (if one exists) of a problem of the form \( \text{(B.2)} \). Let us now translate the results of that appendix to the present setting, for which we require the following assumption on the reduced gradient \( R_k^T g_k \).

**Assumption B.1.** The vector \( R_k^T g_k \) is nonzero.

**Lemma B.1.** Any solution of \( \text{(B.2)} \) has \( v_k \neq 0 \).

**Lemma B.2.** The following hold for any first-order primal-dual solution of \( \text{(B.1)} \).

(i) If \( \sigma_k^l > 0 \), then \( \lambda_k > 0 \), \( \beta_k^l = 0 \), \( \beta_k^u > 0 \), and \( \lambda_k = \sigma_k^l \|v_k\| \).

(ii) If \( \sigma_k^l = 0 \), then \( \lambda_k = 0 \).

**Theorem B.2.** The following hold true.

(i) Suppose \( \sigma_k^l > 0 \). Then, \( \text{(B.2)} \) has a solution \((v_k, \lambda_k)\), which can be obtained as

\[
v_k \in \arg \min_{v \in \mathbb{R}^m} \left( f_k + g_k^T R_k v + \frac{1}{2} v^T R_k^T H_k R_k v + \frac{1}{2} \sigma_k^l \|v\|^3 \right),
\]

then setting \( \lambda_k = \sigma_k^l \|v_k\| > 0 \).

(ii) If \( \sigma_k^l = 0 \), then a solution of \( \text{(B.2)} \) exists if and only if \( R_k^T H_k R_k \geq 0 \) and \( g_k^T R_k v = 0 \) for all \( u \in \text{Null}(R_k^T H_k R_k) \). In such cases, computing a solution \((v_k, \lambda_k)\) of problem \( \text{(B.2)} \) is equivalent to computing a solution \( v_k \) of

\[
\min_{v \in \mathbb{R}^m} f_k + g_k^T R_k v + \frac{1}{2} v^T R_k^T H_k R_k v
\]

and setting \( \lambda_k = 0 \).

Considering problem \( \text{(B.3)} \), we obtain the following result from \( \text{[6], Lemma 3.2} \).

**Lemma B.3.** If \( \sigma_k^l > 0 \), then \( v_k \) from \( \text{(B.3)} \) satisfies

\[
g_k^T R_k v_k + v_k^T R_k^T H_k R_k v_k + \frac{3}{2} \sigma_k^l \|v_k\|^3 = 0 \quad \text{(B.5a)}
\]

\[
v_k^T R_k^T H_k R_k v_k + \frac{3}{2} \sigma_k^l \|v_k\|^3 \geq 0 \quad \text{(B.5b)}
\]

\[
R_k^T H_k R_k + \frac{3}{2} \sigma_k^l \|v_k\| I \geq 0 \quad \text{(B.5c)}
\]

We now show that, under certain reasonable assumptions, solutions of the primal-dual reduced-space subproblem \( \text{(B.1)} \) satisfy the conditions required by Assumptions \( \text{2.1 and 3.6} \).
Theorem B.3. The following hold true.

(a) Any solution of problem (\textit{B.1}) satisfies (4b).
(b) Any solution of problem (\textit{B.1}) satisfies (4a), provided \(g_k \in \mathcal{L}_k\).
(c) Any solution of problem (\textit{B.1}) satisfies (4c) provided \(\mathcal{L}_k = \mathbb{R}^n\).
(d) Any solution of problem (\textit{B.1}) satisfies (18) for any \(\kappa_4 \geq \frac{3}{2} \sup_{k \in \mathbb{N}_+} \{\sigma_k^4\}\).

Proof. Any first-order primal-dual solution \((s_k, \lambda_k, \beta_k^a, \beta_k^b, \beta_k^c)\) of problem (\textit{B.1}) corresponds to such a solution \((v_k, \lambda_k, \beta_k^a, \beta_k^b, \beta_k^c)\) of problem (\textit{B.2}) where \(s_k = R_k v_k\). Hence, throughout this proof, for any solution vector \(s_k\) for problem (\textit{B.1}), we may let \(s_k = R_k v_k\) where \(v_k\) satisfies the properties in Lemmas B.1–B.3.

First, suppose \(\sigma_k^4 > 0\), which by Theorem B.2(i) implies that problem (\textit{B.1}) has a solution. Then, it follows from (\textit{B.5a}), \(s_k = R_k v_k\), and Lemma B.2(i) that

\[
0 = g_k^T s_k + s_k^T H_k s_k + \frac{3}{2} \sigma_k^4 \|s_k\|^3 = g_k^T s_k + s_k^T H_k s_k + \frac{3}{2} \lambda_k \|s_k\|^2,
\]

which means that

\[
s_k^T (g_k + (H_k + \lambda_k I) s_k) = -\frac{1}{2} \lambda_k \|s_k\|^2. \quad \text{(B.6)}
\]

Meanwhile, from (\textit{B.5b}), \(s_k = R_k v_k\), and Lemma B.2(i), it follows that

\[
0 \leq s_k^T H_k s_k + \frac{3}{2} \lambda_k \|s_k\|^2 = s_k^T (H_k + \lambda_k I) s_k + \frac{1}{2} \lambda_k \|s_k\|^2,
\]

which means that

\[
-\frac{1}{2} \lambda_k \|s_k\|^2 \leq \frac{1}{2} s_k^T (H_k + \lambda_k I) s_k. \quad \text{(B.7)}
\]

It follows from (B.6), (B.7), \(\lambda_k > 0\) (by Lemma B.2(i)), and \((\kappa_1, \kappa_2) \in \mathbb{R}_{++} \times \mathbb{R}_{++}\) that

\[
s_k^T (g_k + (H_k + \lambda_k I) s_k) = -\frac{1}{2} \lambda_k \|s_k\|^2 \leq \min \{\frac{1}{2} \kappa_1 \|s_k\|^2, \frac{1}{2} s_k^T (H_k + \lambda_k I) s_k + \frac{1}{2} \lambda_k \|s_k\|^2\}
\]

\[
\leq \min \{\frac{1}{2} \kappa_1 \|s_k\|^2, \frac{1}{2} s_k^T (H_k + \lambda_k I) s_k + \frac{1}{2} \kappa_2 \|s_k\|^3\},
\]

which implies (4b). This establishes that part (a) is true. Now consider part (b). From Theorem B.2 (Lemma 2.1), and \(s_k = R_k v_k\), it follows that

\[
f_k - q_k(s_k) - \frac{1}{2} \sigma_k^2 \|s_k\|^3 \geq \frac{\|R_k^T g_k\|}{6 \sqrt{2}} \min \left\{ \frac{\|R_k^T g_k\|}{1 + \|R_k^T H_k R_k\|}, \frac{1}{\sqrt{6}} \sqrt{\frac{\|R_k^T g_k\|}{\sigma_k^4}} \right\}.
\]

Since, under assumption, \(g_k \in \mathcal{L}_k\) so that \(g_k = R_k y\) for some \(y \in \mathbb{R}^n\), it follows that

\[
\|R_k^T g_k\| = \|R_k^T R_k y\| = \|y\| = \|R_k y\| = \|g_k\|.
\]

Combining this with \(\|R_k^T H_k R_k\| \leq \|H_k\|\) and the previous displayed inequality shows

\[
f_k - q_k(s_k) - \frac{1}{2} \sigma_k^2 \|s_k\|^3 \geq \frac{\|g_k\|}{6 \sqrt{2}} \min \left\{ \frac{\|g_k\|}{1 + \|H_k\|}, \frac{1}{\sqrt{6}} \sqrt{\frac{\|g_k\|}{\sigma_k^4}} \right\}.
\]

This may now be combined with Theorem B.2 (specifically \(\lambda_k = \sigma_k^4 \|s_k\| > 0\)) to obtain

\[
f_k - q_k(s_k) \geq f_k - q_k(s_k) - \frac{1}{2} \sigma_k^2 \|s_k\|^3 \geq \frac{\|g_k\|}{6 \sqrt{2}} \min \left\{ \frac{\|g_k\|}{1 + \|H_k\|}, \frac{1}{\sqrt{6}} \sqrt{\frac{\|g_k\|}{\lambda_k}} \right\},
\]

which means that \((s_k, \lambda_k)\) satisfies (4a), proving part (b). Now consider part (c). It follows from Theorem A.2(i) and the optimality conditions for problem (A.3) that

\[
0 = g_k + H_k s_k + \frac{3}{2} \sigma_k^2 \|s_k\| = g_k + H_k s_k + \frac{3}{2} \lambda_k s_k = g_k + (H_k + \lambda_k I) s_k + \frac{1}{2} \lambda_k s_k.
\]
This and the fact that $\kappa_3 > 0$ imply that
\[ \|g_k + (H_k + \lambda_k I) s_k\| = \frac{1}{2} \lambda_k \|s_k\| \leq \lambda_k \|s_k\| + \kappa_3 \|s_k\|^2, \]
which completes the proof of part (c). Finally, consider part (d). From (B.5c), the fact that $\|s_k\| = \|v_k\|$, and $\kappa_4 \geq \frac{3}{2} \sup_{k \in \mathbb{N}_+} \{\sigma_k^4\}$, it follows that
\[ \xi(R_k^T H_k R_k) \geq \frac{3}{2} \sigma_k^4 \|s_k\| \geq -\kappa_4 \|s_k\|, \]
as desired to prove part (d).

Now suppose that $\sigma_k^2 = 0$. From Theorem [B.2 (ii)], a solution of problem (B.1) exists if and only if $R_k^T H_k R_k \geq 0$ and $g_k^T R_k u = 0$ for all $u \in \text{Null}(R_k^T H_k R_k)$. If this is not the case, then there is nothing left to prove; hence, let us assume that these conditions hold. From these conditions, Theorem [B.2 (ii)], the optimality conditions of problem (B.4), the fact that $\lambda_k = 0$, and $s_k = R_k v_k$, it follows that
\[ g_k^T s_k + s_k^T H_k s_k = 0 \quad \text{and} \quad s_k^T H_k s_k \geq 0. \]
This shows that (4b) holds, proving part (a) for this case. Next, since $v_k$ is given by the solution of problem (B.4), it follows that the reduction in the objective yielded by $v_k$ is at least as large as the reduction obtained by minimizing the objective over the span of $-R_k^T g_k$. Hence, from standard theory on Cauchy decrease (see [10] or [23]), one can conclude that
\[ f_k - q_k(s_k) \geq \frac{\|R_k^T g_k\|}{2} \min \left\{ \frac{\|R_k^T g_k\|}{1 + \|R_k^T H_k R_k\|}, \|s_k\| \right\}. \]
Thus, using the arguments in the previous paragraph under the assumption that $g_k \in \mathcal{L}_k$, one is led to the conclusion that (4a) holds, which proves part (b) for this case. Next, when $\mathcal{L}_k = \mathbb{R}^n$, the optimality conditions for problem (B.4) imply that $g_k + H_k s_k = 0$, which, since $\lambda_k = 0$, implies that (4c) holds, proving part (c). Finally, since $R_k^T H_k R_k \geq 0$, it follows that (18) holds, proving part (d).

C Detailed Numerical Results

Further details of the results of our numerical experiments are shown in Table 2. In the table, #Var indicates the number of variables, #Iter indicates the number of iterations required (with %Newton indicating the percentage that were inexact Newton steps with $\lambda_k = 0$), #Acc indicates the number of accepted steps (again with %Newton indicating the percentage that were inexact Newton steps), #Hv-prod indicates the number of Hessian-vector products required, and #T-fact indicates the number of tridiagonal matrix factorizations required.

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