Radar Waveform Optimization for Cooperative Radar and Communications Joint Receiver

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Abstract—We develop and present radar waveform design methods that jointly maximize performance of a joint radar-communications system that shares spectrum. Specifically, we present a novel waveform design technique that optimizes the spectral shape of the radar waveform for multiaccess communications and radar receiver. This technique also ensures that the radar waveform is unimodular. We also present an extension to an existing waveform design technique that has additional optimization constraints imposed and also utilizes more efficient optimization methods which significantly reduce the computation time. This technique does not guarantee unimodular radar waveforms. We perform a numerical study to compare the performance of these two techniques. We employ successive interference cancellation at the receiver to mitigate an in-band performance of these two techniques. We employ successive interference cancellation at the receiver to mitigate an in-band interference. We present a novel waveform design technique that optimizes the radar waveform in the presence of each other [2]–[4]. This is in stark contrast to the non-cooperative traditional approach to spectrum sharing of keeping all systems isolated (either in time, frequency or space).

In order to investigate the fundamental limits on performance for in-band cooperative radar and communications systems, a novel parameterization of radar information analogous to the communications data rate, the estimation rate was developed in References [3], [5]–[7]. The estimation rate and data rate were used to derive inner bounds on performance for a joint radar-communications system. However, these bounds were developed considering only local radar estimation errors (high SNR regime), and therefore simplifying assumptions about the radar waveform were made. In Reference [8], the estimation rate was modified to incorporate global radar estimation errors and new performance bounds were developed that included the formulation of an optimal radar waveform for both global radar estimation rate performance and communications performance for in-band communications users forced to mitigate radar returns. In this paper, we present and compare radar waveform design methods which optimize the shape of the radar waveform spectrum to maximize joint radar-communications performance. The problem scenario considered in this paper is given by Figure 1. We present a novel radar waveform design method that involves selecting the phase parameters of a nonlinear chirp waveform to maximize joint performance. We also present results in this paper that are an extension on the work presented in [8]. More specifically, we reformulate the original spectral-mask shaping algorithm as a nonlinear programming (NLP) problem and we also employ more efficient optimization algorithms to generate the optimal radar waveform in the previously derived spectral-mask shaping algorithm [8], significantly reducing the computation time. We also introduce additional constraints on the autocorrelation peak sidelobe to mainlobe ratio and spectral leakage for the waveform design problem. The new waveform design method described in this paper designs radar waveforms that are constant modulus or unimodular, which is...
not necessarily true for the spectral-mask shaping method. This key feature is highlighted in Figure 2.

In this paper, the global estimation rate and date rate capture radar and communications performance respectively.

A. Contributions

In this paper, we present radar waveform design techniques for joint radar-communications systems that maximize the performance of the system (in terms of data rate for communications and estimation rate for radar) by selecting the optimal radar waveform spectrum. As we can generate infinitely many possible waveforms, formulating this problem without parameterizing the shape of the waveform makes the problem intractable. Thus, we choose a parametric representation for the waveforms, and optimize the parameters to maximize the joint performance. The main contributions of this paper are

- Extend previously derived spectral-mask shaping algorithm to make it computationally more efficient
- Introduce constraint on autocorrelation peak sidelobe to mainlobe ratio and spectral leakage for the spectral-mask shaping waveform design method
- Develop novel polynomial chirp optimization waveform design method which ensures that the waveform has constant modulus in the time-domain
- Perform numerical study for a performance comparison between the above two methods with respect to various performance measures

B. Background

The performance bounds presented in References [3], [5]–[7], which only considered local estimation errors, were shown to be dependent primarily on the root mean square (RMS) bandwidth of the radar waveform. In Reference [8], the estimation rate was extended to account for global errors, and a closed form expression for the RMS bandwidth of a spectrally masked linear frequency modulated chirp was derived. An evolutionary optimization algorithm was applied to find the optimal spectral mask that maximizes both radar and communications performance (estimation and data rate respectively).

With the spectral congestion problem rapidly becoming a reality, researchers have searched diverse areas for potential solutions. However, certain methods are gaining more traction than others.

Waveform design has become a dominant research thread in joint radar-communications phenomenology. Researchers have considered a variety of waveform options including orthogonal frequency-division multiplexing (OFDM) [9]–[16]. Similar to OFDM, spread spectrum waveforms have been proposed for their attractive, noise-like autocorrelation properties [17]–[19].

Information is well known in communications phenomenology, but less so in radar. Perhaps surprisingly, radars were looked at in the context of information theory soon after Shannon’s seminal work [20] by Woodward [21]. Interest resurged many years later with Bell’s work on waveform design using information for statistical scattering targets [22]. Recent results have found connections between information theory and estimation theory, equating estimation information and the integrated minimum mean-squared error (MMSE) [23].

Other researchers looked at spatial mitigation as a means to improve spectral interoperability [24]–[26]. Joint coding techniques, such as robust codes for communications that have desirable radar ambiguity properties, as well as codes that trade data rate and channel estimation error have been investigated as co-design solutions [27]–[30].

C. Problem Set-up

We consider a simple scenario involving a radar and communications user attempting to use the same spectrum-space-time as shown in Figure 1. This scenario is instructional, and can easily be scaled to more complicated scenarios by using it as a building block to construct real world examples. We consider the joint radar-communications receiver to be a radar transmitter/receiver that can act as a communications receiver. The joint receiver can simultaneously estimate the radar target parameters from the radar return and decode a received communications signal.

The key assumptions made in this work for the scenario described in Figure 1 are as follows

- Radar and communications operate in the same frequency allocation simultaneously
- Joint radar-communications receiver is capable of simultaneously decoding a communications signal and estimating a target parameter
- Radar detection and track acquisition have already taken place
- Radar system is an active, single-input single-output (SISO), mono-static, and pulsed system
• Radar system operates without any maximum unambiguous range
• A single SISO communications transmitter is present
• Only one radar target is present
• Target range or delay is the only parameter of interest
• Target cross-section is well estimated
• Communications signal is received through an antenna sidelobe; and antenna gains are not identical

It should be noted that the performance bounds and results presented in this paper are very closely tied to the receiver model we employ. We employ successive interference cancellation (SIC) mitigation technique at the receiver, which causes the communications data rate at the receiver to become dependent on the RMS bandwidth [5], a parameter that is determined by the shape of the spectral mask. Employing different mitigation techniques and changing the receiver model will result in a set of performance bounds that are different from the ones presented in this paper.

The paper is arranged as follows. Section II talks about the SIC mitigation techniques employed at the receiver. In Section III, we present a brief description of the joint radar-communications performance optimization problem. We also discuss the trade-off involved in optimizing the communications performance and the radar performance. In Section IV, we present the spectral-mask shaping waveform design method that involves optimizing a spectral mask that is used to shape the radar waveform spectrum. The results presented in this section are an extension of the work presented in [8]. In Section V, we present a novel waveform design method, the polynomial chirp optimization method, in which we optimize the phase terms of a non-linear chirp using nonlinear programming (NLP). In Section VI, we present examples of the waveform design techniques discussed in this paper for an example parameter set. We also discuss the optimization performance of all waveform design methods discussed in this paper. Finally, we provide concluding remarks in Section VII.

II. SUCCESSIVE INTERFERENCE CANCELLATION COMMUNICATIONS DATA RATE

At the receiver, we employ a process called SIC [3], an algorithm that takes advantage of the target tracking ability of the joint radar-communications system to ensure that communications signal decoding and radar detection can be done cooperatively. Using the available target information, we generate a predicted radar return and subtract it from the joint radar-communications received signal. After suppressing the radar return, the receiver then decodes and removes the communications signal from the observed waveform to obtain a radar return signal free of communications interference. The block diagram of the joint radar-communications system considered in this system is shown in Figure 3.

When applying SIC, the interference plus noise variance \( \sigma_{\text{int+n}}^2 \), from the communications receiver’s perspective, is given by [3], [5]

\[
\sigma_{\text{int+n}}^2 = P_{\text{rad}} \| a \|^2 B_{\text{rms}}^2 \sigma_{\text{proc}}^2 + k_B T_{\text{temp}} B,
\]

where \( P_{\text{rad}} \) is the radar transmit power, \( a \) is the radar gain-propagation-cross-section product, \( \tau \) is the time-delay (or range) of the target, \( B_{\text{rms}} \) is the RMS bandwidth, \( k_B \) is the Boltzmann’s constant, and \( T_{\text{temp}} \) is the absolute temperature of the receiver. The corresponding communications rate is given by

\[
R_{\text{com}} \leq B \log_2 \left( 1 + \frac{\| b \|^2 P_{\text{com}}}{\sigma_{\text{int+n}}^2} \right),
\]

where \( b \) is the communications channel gain.

III. JOINT WAVEFORM DESIGN PROBLEM

The spectral shape of the waveform (set by the parameters) determines whether the radar performance or the communications performance is maximized, or some weighting therein. The performance of communications is measured by the communications rate \( R_{\text{com}} \) (bits/sec), given by Equation (2).

We measure the performance of radar by the quality of the radar returns. To measure the quality of radar returns, a novel metric was introduced in [3], [5] called estimation rate (analogous to communications rate), which is also upper bounded as follows:

\[
R_{\text{est}} \leq \frac{\delta}{2T} \log_2 \left( 1 + \frac{\sigma_{\text{proc}}^2}{\sigma_{\text{est}}^2} \right),
\]

where \( \delta \) is the radar duty cycle, \( T \) is the radar pulse duration, and \( \sigma_{\text{est}}^2 \) is the range estimation noise variance. A detailed description of this metric can be found in [7].

Given an amount of spectral allocation, forcing the radar spectrum to be more impulse-like (more energy at the center) minimizes the RMS bandwidth, thereby maximizing data rate. Conversely, radar waveforms with more energy towards the edges of the spectral allocation maximize the RMS bandwidth and as a result, the estimation rate. The radar waveform spectrum shape is optimized such that the resultant estimation and data rates are jointly maximized.

The shape of the radar waveform spectrum also impacts radar estimation performance. Radar waveforms with high RMS bandwidth (more spectral energy towards the edges) will have better estimation performance (a smaller Cramér - Rao lower bound). However, having spectrum with more energy at the edges introduces ambiguity in radar estimation (non-local errors), thereby increasing the SNR threshold at which non-local estimation errors do not occur (local SNR regime). Similarly, radar waveforms with small RMS bandwidth (more spectral energy towards the center) have decreased estimation
performance, but a lower threshold for the local SNR regime. This relationship between the spectral mask and estimation performance is shown in Figure 4.

Thus, the shape of the radar spectrum poses a trade-off not only in terms of radar performance vs. communications performance (through the RMS bandwidth), but also in terms of improved estimation performance vs. an increased threshold for achieving the Cramér-Rao bound.

Fig. 4. The relationship between the spectral mask and radar estimation performance. The purple line shows the estimation performance when there is no spectral mask. The estimation performance when a communications optimal spectral mask (waveforms with small RMS bandwidth or more spectral energy towards the center) is used is shown in red. Estimation performance when a radar optimal spectral mask (waveforms with high RMS bandwidth or more spectral energy towards the edges) is used is shown in blue. The Cramér - Rao bound is shown in black.

The objective is to optimize the spectral shape of the radar waveform such that the performance with respect to radar and communications is jointly maximized. We introduce additional constraints to the waveform optimization problem originally defined in [8] in order to obtain optimal radar waveforms that not only ensure optimal joint radar-communications performance, but also satisfy additional real-world properties of that a traditional radar waveform would. The new constraints that are introduced are as follows

- **Peak Sidelobe to Mainlobe Ratio (Constraint C₁):** In Reference [8], the estimation rate was extended to consider global estimation errors (errors occurring when the radar waveform autocorrelation sidelobe is confused for the mainlobe) using the method of interval errors [31]. However, the method of interval errors only considers the errors occurring due to the peak sidelobe and ignores the rest. If the other sidelobes are high enough, they can still have a significant contribution to the global estimation error. By limiting the peak sidelobe to mainlobe ratio to be below a certain threshold, we can reduce the effect the peak sidelobe as well as any other high sidelobes will have on the global estimation error.

- **Spectral Leakage (Constraint C₂):** Since the system can only receive signals whose spectrum lies within the system’s bandwidth, any electromagnetic radio frequency (RF) energy that leaks outside of the bandwidth will be lost. To minimize this loss of RF energy, we introduce a constraint on the amount of energy present in the radar spectrum at frequencies out of the system bandwidth range. We enforce this constraint by having the radar spectrum be below a thresholding spectral mask such as the one seen in Figure 5.

Fig. 5. Spectral Leakage Mask used constrain the amount of energy in the radar spectrum leaking out at frequencies out of the system bandwidth range. The spectral leakage constraint is enforced by having the radar spectrum be below this thresholding spectral leakage mask.

### IV. Spectral-Mask Shaping Method

We present a method to parameterize the shape of the radar waveform, and then optimize the parameters to maximize joint radar-communications performance. First, as seen in [8], we consider the radar waveform to be a linear frequency modulated chirp signal, which has been passed through a parameterized spectral mask. The shape of the spectral mask (set by the parameters) determines whether the radar waveform maximizes the estimation rate, communications rate, or some weighting therein. The spectral mask parameters are then optimized such that the resultant estimation and data rates are jointly maximized.

Specifically, we begin with a standard chirp signal (linear frequency modulated) given by \( x(t) = e^{j(\pi B/2) t^2} \). We control the spectral shape of this chirp signal to maximize joint performance. To achieve this, we first sample the chirp signal, and collect \( N \) samples in the frequency domain. Let \( u = (u_1, \ldots, u_N)^T \) be an array of spectral weights, where \( u_i \in [0, 1], \forall i \). We control the spectral shape of the chirp signal by multiplying the signal with the spectral weights in the frequency domain.

Our goal is to choose the spectral weights \( u \) such that the resulting waveform jointly maximizes the performance with respect to both radar and communications. Specifically, we maximize the geometric mean of the communications rate and the estimation rate. Additionally, the optimized waveform needs to satisfy the constraints discussed in the previous section: 1) spectrum amplitude at each frequency sample stays below a certain threshold, and 2) side-lobe to main-lobe ratio of the autocorrelation function is less than a threshold (fraction). The first constraint allows for a limited (and controlled) spectral leakage outside the frequency band.
The second constraint decreases the chance of mistaking side-lobes with the main-lobe (non-local estimation errors), which may happen when side-lobe and main-lobe amplitudes are comparable and when signal-to-noise ratio fluctuates (happens when the channel conditions vary).

We now pose the joint waveform design problem as a nonlinear program (NLP). As NLPs are typically hard to solve exactly, we present numerical methods to achieve suboptimal solutions. The joint waveform design problem can be stated as follows:

\[
\begin{align*}
\text{maximize} & \quad [R_{\text{com}}(u)]^\alpha [R_{\text{est}}(u)]^{1-\alpha}, \\
\text{subject to} & \quad r(u) \leq p \quad (C_1) \\
& \quad 1_A(u) = 1 \quad (C_2)
\end{align*}
\]

where \([0, 1]^N\) represents the unit interval, and \(0 \leq \alpha \leq 1\). Here, \(r(u)\) represents the fraction of the side-lobe to the main-lobe amplitude, the constraint \(C_1\) keeps this fraction below a threshold value \(p\). The constraint \(C_2\) constrains the weights \(u\) such that the resulting spectrum of the waveform stays below a certain masking threshold, which is represented by an indicator function, where \(A\) is the set of all spectral weights such that their resulting spectrum is below the masking threshold.

In Section VI, we discuss the numerical solutions to the above NLP, and discuss the performance trade-offs. In the following discussion, we derive an expression for the time-domain waveform after a known spectral mask (details discussed below) is applied on the standard chirp signal.

### A. Time Domain Form of Spectrally Masked Chirp

Given a linear frequency modulated chirp

\[
x(t) = e^{j \frac{2\pi}{T} t^2},
\]

with spectrum \(X(f)\), we apply the following spectral mask

\[
W(f) = a + b f^2,
\]

where \(f\) represents frequency, \(a\) and \(b\) are shaping parameters. The resultant spectrally masked chirp has the following spectrum

\[
Y(f) = a X(f) + b f^2 X(f).
\]

Using the linearity and time derivative properties of the inverse Fourier transform, the spectrally masked chirp in the time domain is given as follows

\[
y(t) = a x(t) + \frac{b}{4 \pi^2} x''(t) = a e^{j \frac{2\pi}{T} t^2} + \left[ \frac{b B^2 T^2}{2 \pi T} - \frac{i b B}{2 \pi T} \right] e^{j \frac{2\pi}{T} t^2}.
\]

### V. POLYNOMIAL CHIRP OPTIMIZATION

Although optimizing the weights of the spectral mask, as described in the previous section, lets us achieve the desired shape of the waveform, the resulting optimized waveform may not be unimodular, i.e., the signal in the time-domain may have constant modulus. As it is desirable to have peak-to-average power ratio of the signal to be minimum, most radar systems now require the signal to be unimodular, which keeps the peak and the average power the same over any time period. We present the following method, which lets us achieve the desired unimodular waveform that maximizes the joint radar-communications performance. We begin with a polynomial chirp signal as follows:

\[
x(t) = e^{j2\pi \left( \sum_{i=1}^N p_i t^2 \right)},
\]

where \(N\) is a positive integer and \(p_i \in \mathbb{R}, \forall i\). We let the polynomial phase to have only even terms to ensure symmetry in the frequency domain. Clearly, the above signal is unimodular, and its spectral shape is determined by the parameters \(p_i, i = 1, \ldots, N\). In other words, the performance of the system with respect to the radar and the communications depends on the parameters \(p_1, \ldots, p_N\). Let \(\vec{p} = (p_1, \ldots, p_N)^T\). We reformulate the joint waveform design problem as follows:

\[
\begin{align*}
\text{maximize} & \quad [R_{\text{com}}(\vec{p})]^\alpha [R_{\text{est}}(\vec{p})]^{1-\alpha}, \\
\text{subject to} & \quad \vec{p} \in [L_1, U_1] \times \cdots \times [L_N, U_N] \quad (C_1) \\
& \quad r(\vec{p}) \leq b \quad (C_2) \\
& \quad 1_A(\vec{p}) = 1 \quad (C_2)
\end{align*}
\]

where \(\vec{p}\) is restricted to lie within the hypercube \([L_1, U_1] \times \cdots \times [L_N, U_N]\). As in the previous section, the constraint \(C_1\) limits the fraction of the side-lobe to the main-lobe amplitude to stay below a certain threshold \(b\). Similarly, the constraint \(C_2\) keeps the resulting spectrum of the waveform below the masking threshold, which is represented by an indicator function, where \(A = [L_1, U_1] \times \cdots \times [L_N, U_N]\).

The results from the numerical study of the above optimization problem are discussed in the next section.

### VI. SIMULATION RESULTS

In this section, we present examples of the two waveform design techniques discussed in this paper, the spectral-mask shaping method and the polynomial chirp optimization method for an example parameter set. The parameters used in the example are shown in Table I. We also perform a numerical study of the performance of both waveform design algorithms and compare the two methods.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Bandwidth</td>
<td>5 MHz</td>
</tr>
<tr>
<td>Center Frequency</td>
<td>3 GHz</td>
</tr>
<tr>
<td>Effective Temperature</td>
<td>1000 K</td>
</tr>
<tr>
<td>Communications Range</td>
<td>10 km</td>
</tr>
<tr>
<td>Communications Power</td>
<td>0.3 W</td>
</tr>
<tr>
<td>Communications Antenna Gain</td>
<td>0 dBi</td>
</tr>
<tr>
<td>Communications Receiver Side-lobe Gain</td>
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<tr>
<td>Radar Target Range</td>
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</tr>
<tr>
<td>Radar Antenna Gain</td>
<td>30 dBi</td>
</tr>
<tr>
<td>Radar Power</td>
<td>100 kW</td>
</tr>
<tr>
<td>Target Cross Section</td>
<td>10 m²</td>
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<tr>
<td>Time-Bandwidth Product</td>
<td>128</td>
</tr>
<tr>
<td>Radar Duty Factor</td>
<td>0.01</td>
</tr>
</tbody>
</table>
A. Spectral-Mask Shaping Method

We present a numerical study on the performance of the spectral-mask shaping method discussed in Section IV. Here, we also assess the impact of the constraints \( C_1 \) and \( C_2 \) (in Equation (3)) on the joint performance. We also perform a Monte-Carlo simulation to compare the performance of the system with three different optimization solvers - MATLAB’s \textit{fmincon} [32], \textit{Knitro} [33], and an evolutionary algorithm, which was earlier used in Reference [8] to solve the optimization problem in Equation (3) (except constraints \( C_1 \) and \( C_2 \) were not enforced). For constraint \( C_1 \), we set the threshold value \( p = -15 \)dB. In Figures 6 to 8, we plot the autocorrelation function of the optimized waveform (solved via \textit{fmincon}) for \( \alpha \in \{0, 0.5, 1\} \). We see that enforcing the constraint \( C_1 \) suppresses the side-lobe by more than \(-15\)dB with respect to the main-lobe as desired. Although, we can obtain better side-lobe to main-lobe ratio by further decreasing the value of \( p \), but this comes at the cost of reduction in the feasibility region of Equation (3) as with any constrained optimization problem, thus decreasing the optimal objective value, i.e., decreasing the joint performance.

As it is hard to solve the NLP (in Equation (3)) exactly, we choose local solvers to find a local optimum. In this regard, we compare the performance of three different local solvers to solve the waveform optimization problem (Equation (3)) - MATLAB’s \textit{fmincon}, \textit{knitro}, and an evolutionary algorithm [8]. We choose three performance measures to compare the solvers - final objective value, execution time, estimation rate vs. communications rate curve. The first measure is the final objective value obtained by the solver, while the second measure corresponds to the time it takes for the solver to solve the NLP (i.e., reach a local optimum). The third measure is a plot of the estimation rate vs. the communications rate for different values of \( \alpha \) (between 0 and 1) in Equation (3). Here, we collect the final radar estimation rates (\( R_{est} \)) and communications rates (\( R_{comm} \)) from each of the solvers for different values of \( \alpha \), and plot \( R_{est} \) vs. \( R_{comm} \). We solve the optimization problem in Equation (3) using the above solvers, and for 50 Monte-Carlo runs. In Figure 9, we compare the cumulative frequency of the final objective values from each of the solvers. These plots show that all three solvers display similar performance. However, Figure 10 shows that both \textit{fmincon} and \textit{knitro} achieve this performance much faster than the evolutionary algorithm (used in [8]). Figure 11 compares the above-mentioned \( R_{est} \) vs. \( R_{comm} \) plots for each solver, and we notice that all three solvers achieve similar performance, i.e., overlapping rate-rate curves. In the following subsection, we study the impact of the constraint \( C_2 \) on the joint performance, and also on the performance with respect to a different measure.
Performance Improvement With Constraint $C_2$: We now study the impact of constraint $C_2$ on the system performance. For the purpose of this study, we choose the solver *fmincon* to solve the NLP, as it is the fastest among the solvers discussed earlier. We choose a spectral mask as depicted by the dotted lines in Figures 12 and 13. The design of this spectral leakage mask is inspired from the masks that are typically used in 4G-LTE communications. From Figure 12, it is clear that without imposing $C_2$, a significant amount of energy gets leaked outside the desired frequency band. Figure 13 shows that with $C_2$, we can suppress the spectral leakage as desired and as determined by the spectral mask we choose, moreover, we achieve this without compromising the quality of the autocorrelation function, as can be seen in Figure 14.

To quantify the merit of imposing constraint $C_2$, we introduce a new performance metric - fraction of energy spilled (compared to the total energy of the waveform) outside the desired frequency band (frequency samples between -82 to 82 in Figure 13), which is denoted by $E_{spill}$. We now perform a Monte-Carlo study to assess the quantitative improvement the constraint $C_2$ brings with respect to $E_{spill}$. Figure 15 shows the plots of $E_{spill}$ vs. the sample index (50 Monte-Carlo runs) with and without the spectral leakage constraint $C_2$. This plot shows that the constraint $C_2$ significantly improves the performance with respect to $E_{spill}$, and does not compromise on the objective value significantly as can be seen in Figure 16.

B. Polynomial-Chirp Optimization

We now present numerical results from solving Equation (5) via the solver *fmincon*. For the rest of the study, we only
choose the solver fmincon to optimize the waveform, to stress more on the performance of the method rather than the solver. In Equation (5), we set $N = 3$, i.e., $ar{p} = (p_1, p_2, p_3)^T$. We set the minimum side-lobe to main-lobe ratio threshold $b = -1\, \text{dB}$, and we set the limits on the parameter values as $L_i = 0$ and $U_i = 50 \, \forall i$. Figure 17 shows the autocorrelation plots and the spectrum of the optimized waveforms against different values of $\alpha$. Figure 17 shows that the optimized waveforms satisfy the constraints for each $\alpha \in \{0, 0.5, 1\}$.

C. Performance Comparison of Waveform Design Algorithms

We run a Monte-Carlo study to compare the performance of the spectral-mask shaping and the polynomial chirp optimization methods. We implement the two methods with the same parameter settings, and solve the NLPs in Equations (3) and (5) via fmincon for $\alpha \in \{0, 0.5, 1\}$. Figure 18 shows the cumulative distribution frequency plots of execution times and the final objective values from the two proposed methods. Also, Table II shows the average (over the Monte-Carlo runs for $\alpha = 0.5$) of several other performance measures from the Monte-Carlo study such as the final objective values, estimation rates ($R_{\text{est}}$), communications rates ($R_{\text{comm}}$), and the peak-to-average-power-ratio values. Figure 18 shows that the spectral-shaping method outperforms the polynomial chirp optimization method in terms of the final objective value, which is expected as the latter method has the additional constraint that the waveform must be unimodular. Particularly, Table II shows that we can trade-off the constant modulus property of the waveforms for better performance with respect to the geometric mean of the estimation and the communications rates, and also with respect to the estimation rate.
Fig. 17. Comparing the radar waveform autocorrelation vs. $\alpha$ and the radar spectrum vs. $\alpha$ for the polynomial chirp optimization method for different $\alpha$. We see that the optimized waveforms satisfy the constraints for each $\alpha \in \{0, 0.5, 1\}$.

However, with constant modulus waveforms, we keep peak-to-average-power-ratio to zero dB (as opposed to a non-zero value with variable modulus waveforms), which is desirable for many radar systems. Moreover, the polynomial chirp optimization method outperforms the spectral-mask shaping method with respect to the execution time. Employing the spectral-mask shaping algorithm will result in better joint system performance but will take a longer time to optimize the radar waveform, which is evident from Figure 18. The opposite is true for the polynomial chirp optimization method. Additionally, the polynomial chirp optimization method has the advantage of generating constant modulus waveforms.

TABLE II
POLYNOMIAL CHIRP OPTIMIZATION (CONSTANT MODULUS) VS. SPECTRAL-MASK SHAPING (VARIABLE MODULUS) FOR $\alpha = 0.5$

<table>
<thead>
<tr>
<th>Performance metric (average values)</th>
<th>Constant Modulus</th>
<th>Variable Modulus</th>
</tr>
</thead>
<tbody>
<tr>
<td>Final Objective Value</td>
<td>$9.24 \times 10^4$</td>
<td>$9.29 \times 10^4$</td>
</tr>
<tr>
<td>$R_{est}$ (b/s)</td>
<td>945.75</td>
<td>1082.8</td>
</tr>
<tr>
<td>$R_{comm}$ (b/s)</td>
<td>$9.02 \times 10^6$</td>
<td>$7.97 \times 10^6$</td>
</tr>
<tr>
<td>peak-avg-power-ratio (dB)</td>
<td>0</td>
<td>$1.93 \times 10^{-15}$</td>
</tr>
</tbody>
</table>

VII. CONCLUSIONS

We presented novel radar waveform design techniques that maximize the performance of a spectrum sharing joint radar-communications system. One of the proposed techniques is an extension to a previously derived spectral-mask shaping waveform design method, introducing additional constraints on spectral leakage and radar autocorrelation peak sidelobe to mainlobe ratio for the waveform design problem and making the algorithm more computationally efficient. The global estimation rate, an extension on the estimation rate that takes into account non-local or global estimation errors, and the data rate are used to measure radar and communications performance respectively. The first method optimizes the spectral mask (to be applied on the standard chirp) such that the geometric mean of the communications rate and the estimation rate is maximized. The second method optimizes the polynomial coefficients of the chirp waveform in the time-domain, which ultimately optimizes the spectral-shape of the waveform to maximize the same performance metric as before, but ensures that the radar waveform is constant modulus, which was not necessarily true for the spectral-mask shaping method. We presented examples of the waveform design techniques discussed in this paper for an example parameter set and
also compared the performance of the two waveform design methods proposed here. We saw that the spectral-mask shaping method outperforms the polynomial chirp optimization method as the former method is less constrained compared to the latter, however the latter method outperforms the former with respect to execution time. Moreover, the polynomial chirp optimization method is preferable to most systems as the method ensures that the optimized waveform has constant modulus, which is a desired feature.

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