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Axial symmetry indices for convex cones: axiomatic formalism and applications

Alberto Seeger[1] and Mounir Torki[2]

Abstract. We address the issue of measuring the degree of axial symmetry of a convex cone. By following an axiomatic approach, we introduce and explore the concept of axial symmetry index. This concept is illustrated with the help of several interesting examples. By way of application, we establish a conic version of the Blekherman inequality concerning the quality of the approximation of a convex body by its inscribed ellipsoid.

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1 Introduction

A convex body is a convex compact set with nonempty interior. Central symmetry, also called point symmetry, is a geometric concept that plays a key role in various results pertaining to the theory of convex bodies; think for instance of the Blaschke-Santaló inequality or the John ellipsoid theorem. Similarly, the notion of axial symmetry has a bearing in a number of results concerning proper cones. Recall that a closed convex cone is proper if it has nonempty interior and it is pointed in the sense that it contains no line. Let Π_n be the collection of proper cones in the Euclidean space $\mathbb{R}^n$. We say that $K \in \Pi_n$ is symmetric with respect to a linear subspace L if $R_L(K) \subseteq K$, where R_L stands for the reflection matrix onto L. Axial symmetry means symmetry with respect to a one-dimensional linear subspace. In other words, $K \in \Pi_n$ is axially symmetric if there exists a vector $c \in \mathbb{S}_n$ such that

$$R_{\vec{c}}(K) \subseteq K, \tag{1}$$

where $\mathbb{S}_n$ is the unit sphere of $\mathbb{R}^n$ and $\vec{c}$ denotes the line generated by c. The reflection matrix onto $\vec{c}$ admits the explicit formula

$$R_{\vec{c}} = 2\, cc^\top - I_n,$$

[1]Université d'Avignon, Département de Mathématiques, 33 rue Louis Pasteur, 84000 Avignon, France (alberto.seeger@univ-avignon.fr)
[2]Université d'Avignon, CERI, 339 chemin de Meinajaries, 84911, Avignon, France (mounir.torki@univ-avignon.fr)

where I_n is the identity matrix of order n and the superscript "$\top$" stands for transposition. The axial symmetry center of an axially symmetric proper cone K is denoted by c_K and it is defined as the unique vector c in $K \cap \mathbb{S}_n$ that satisfies the stability condition (1). The line generated by c_K is called the symmetry axis of K. The concept of axial symmetry center is useful for geometric and analytic purposes. For instance, properties like rotundity and smoothness in an axially symmetric proper cone K can be characterized in terms of corresponding properties in the "transversal" section

$$\mathrm{T}(K) := \{x \in K : \langle c_K, x\rangle = 1\}.$$

The compact convex set $\mathrm{T}(K)$ can be seen as a convex body in the hyperplane whose normal direction is the unit vector c_K. The volume of an axially symmetric proper cone K is defined by

$$\mathrm{Vol}(K) := \frac{\mathrm{vol}_{n-1}(\mathrm{T}(K))}{n}\,, \tag{2}$$

where vol_d stands for the d-dimensional Lebesgue measure. Unfortunately, a vast majority of proper cones arising in applications are not axially symmetric; think for instance of the nonnegative orthant in $\mathbb{R}^n$ when $n \geq 3$. By theoretical and practical reasons, we need then to consider the following two natural questions:

Q_1) How to define a substitute for the axial symmetry center in case the proper cone under consideration is not axially symmetric?

Q_2) Let K_1 and K_2 be two proper cones that are not axially symmetric. What does it mean that one cone is less axially symmetric (or more axially asymmetric) than the other? Is there a suitable coefficient that measures the degree of axial symmetry of a proper cone?

Partial answers to the above questions have been proposed in Seeger and Torki [15]. The purpose of the present paper is to extend our previous work to a higher degree of abstraction. In particular, we wish to explore the next axiomatic formalism for the concept of axial symmetry index.

Definition 1.1. *Let $n \geq 3$. An axial symmetry index on Π_n is a continuous function $\Phi : \Pi_n \to \mathbb{R}$ satisfying the following axioms:*

A_1) $0 \leq \Phi(K) \leq 1$ *for all* $K \in \Pi_n$.

A_2) $\Phi(K) = 1$ *if and only if* K *is axially symmetric.*

A_3) $\Phi(U(K)) = \Phi(K)$ *for all* $K \in \Pi_n$ *and* $U \in \mathbb{O}(n)$.

Here, Π_n is equipped with the truncated Pompeiu-Hausdorff metric and $\mathbb{O}(n)$ denotes the set of orthogonal matrices of order n.

The definition of the truncated Pompeiu-Hausdorff metric is recalled in a moment. The interpretation of the axiom A_3 is clear: the degree of axial symmetry of a proper cone should not change if the proper cone undergoes an orthogonal transformation. In Sections 3 to 7 we explain how to construct an axial symmetry index. Some of our examples are quite natural and straightforward, some others are more sophisticated and require a good dose of critical reasoning.

2 Preliminary material

Throughout this work we assume that $n \geq 3$. The truncated Pompeiu-Hausdorff metric on Π_n is given by

$$\delta(P,Q) := \max\left\{ \max_{z\in P\cap \mathbb{S}_n} \operatorname{dist}(z,Q), \max_{z\in Q\cap \mathbb{S}_n} \operatorname{dist}(z,P)\right\}.$$

Convergence with respect to the truncated Pompeiu-Hausdorff metric is known to be equivalent to convergence in the Painlevé-Kuratowski sense. In a number of occasions we shall rely on the Walkup-Wets isometry theorem, cf. [17, Theorem 1], which asserts that

$$\delta(P^*,Q^*) = \delta(P,Q) \quad \text{for all } P,Q \in \Pi_n, \tag{3}$$

where K^* stands for the dual cone of K. The next definition applies to an arbitrary real-valued function on Π_n, be it an axial symmetry index or not.

Definition 2.1. *A function $\Phi : \Pi_n \to \mathbb{R}$ is Lipschitzian if there exists a constant κ such that*

$$|\Phi(P) - \Phi(Q)| \leq \kappa\, \delta(P,Q) \quad \textit{for all } P,Q \in \Pi_n. \tag{4}$$

We say that Φ is dual-invariant if $\Phi(K^*) = \Phi(K)$ for all $K \in \Pi_n$.

If the inequality (4) holds with $\kappa = 1$, then we say that Φ is nonexpansive. The next proposition provides a lower estimate for the best Lispschitz constant

$$\operatorname{lip}_\delta(\Phi) := \sup_{\substack{P,Q\in\Pi_n\\ P\neq Q}} \frac{|\Phi(P) - \Phi(Q)|}{\delta(P,Q)}$$

of a Lipschitzian axial symmetry index Φ.

Proposition 2.1. *Let Φ be a Lipschitzian axial symmetry index on Π_n. Then* $\operatorname{lip}_\delta(\Phi) \geq 1 - \mu(\Phi)$, *where* $\mu(\Phi) := \inf_{K\in\Pi_n} \Phi(K)$.

Proof. Let $K \in \Pi_n$ be axially symmetric. For each positive real ε, there exists $K_\varepsilon \in \Pi_n$ such that $\Phi(K_\varepsilon) \leq \mu(\Phi) + \varepsilon$. Since $\delta(K, K_\varepsilon) \leq 1$, we get

$$\operatorname{lip}_\delta(\Phi) \;\geq\; \frac{\Phi(K) - \Phi(K_\varepsilon)}{\delta(K,K_\varepsilon)} \;\geq\; 1 - \mu(\Phi) - \varepsilon.$$

It suffices now to let ε go to zero. □

A Lipschitzian axial symmetry index can always be converted into a nonexpansive axial symmetry index. This can be done for instance with the help of a simple affine transformation. The details are explained in the next proposition, whose proof is trivial and therefore omitted.

Proposition 2.2. *Let Φ be an axial symmetry index on Π_n satisfying the Lipschitz condition* (4) *with $\kappa \geq 1$. Then*

$$(N_\kappa\Phi)(K) := 1 - \frac{1}{\kappa} + \frac{1}{\kappa}\, \Phi(K)$$

is a nonexpansive axial symmetry index on Π_n.

The property of dual-invariance mentioned in Definition 2.1 is not an axiom for qualifying to the name of axial symmetry index, but such a property is quite natural in the present context. If a proper cone K is axially symmetric, then its dual cone K^* is axially symmetric as well. So, the term $\Phi(K^*)$ should be related to $\Phi(K)$. As explained in the next proposition, an arbitrary axial symmetry index $\Phi : \Pi_n \to \mathbb{R}$ can be rendered dual-invariant by mixing $\Phi(K)$ and $\Phi(K^*)$ in a suitable way.

Proposition 2.3. *Let Φ be an axial symmetry index on Π_n. Then $K \mapsto \Phi^*(K) := \Phi(K^*)$ is also an axial symmetry index on Π_n. Furthermore,*

$$\frac{\Phi + \Phi^*}{2}\,, \quad \sqrt{\Phi\,\Phi^*}\,, \quad \min\{\Phi, \Phi^*\}\,, \quad \max\{\Phi, \Phi^*\} \tag{5}$$

are dual-invariant axial symmetry indices on Π_n.

Proof. As mentioned in (3), the function $K \mapsto K^*$ is an isometry on the metric space (Π_n, δ). Hence, $\Phi^* : \Pi_n \to \mathbb{R}$ is continuous, being a composition of two continuous functions. It is clear that Φ^* satisfies the axioms A_1 and A_2. On the other hand, we have

$$\Phi^*(U(K)) = \Phi([U(K)]^*) = \Phi(U(K^*)) = \Phi(K^*) = \Phi^*(K)$$

for all $K \in \Pi_n$ and $U \in \mathbb{O}(n)$. In other words, Φ^* satisfies the axiom A_3. The functions listed in (5) are of the form

$$K \in \Pi_n \mapsto \zeta(\Phi(K), \Phi^*(K)), \tag{6}$$

where $\zeta : [0, 1] \times [0, 1] \to [0, 1]$ is a continuous function with a special structure. For any of the following choices

$$\begin{aligned} &\zeta(t, s) = (1/2)(t + s)\,, \quad &&\zeta(t, s) = \sqrt{ts}\,, \\ &\zeta(t, s) = \min\{t, s\}\,, &&\zeta(t, s) = \max\{t, s\}\,, \end{aligned}$$

we readily see that (6) is continuous, dual-invariant, and satisfies the axioms A_1, A_2, and A_3. $\square$

If an axial symmetry index Φ is already dual-invariant, then all the functions listed in (5) coincide with Φ itself.

3 Distance to the set of axially symmetric cones

In this and the next four sections we provide a battery of examples of axial symmetry indices. We start with the first example that probably comes to the mind of the reader. Let $\mathcal{A}_n$ denote the set of axially symmetric proper cones in $\mathbb{R}^n$. The distance from a given $K \in \Pi_n$ to the set $\mathcal{A}_n$ is defined by

$$\mathrm{dist}(K, \mathcal{A}_n) := \inf_{Q \in \mathcal{A}_n} \delta(K, Q). \tag{7}$$

Since $\mathcal{A}_n$ is a closed set in the metric space (Π_n, δ), the expression (7) is equal to zero exactly when K is axially symmetric. This observation leads to the construction of an axial symmetry index of the composite form

$$\Phi^{[\psi]}(K) := \psi(\mathrm{dist}(K, \mathcal{A}_n)), \tag{8}$$

where ψ is a suitable scaling function.

Theorem 3.1. *Let $\psi : [0,1] \to [0,1]$ be decreasing, continuous, and such that $\psi(0) = 1$. Then $\Phi^{[\psi]}$ is a dual-invariant axial symmetry index on Π_n. In particular,*

$$\Phi_{\text{met}}(K) := 1 - \text{dist}(K, \mathcal{A}_n)$$

is a dual-invariant axial symmetry index on Π_n. Furthermore,

(a) Φ_{met} is nonexpansive. In fact, $\text{lip}_\delta(\Phi_{\text{met}}) = 1$.

(b) Φ_{met} is pointwise smallest among all nonexpansive axial symmetry indices on Π_n.

Proof. The composition $\Phi^{[\psi]} = \psi \circ \text{dist}(\,\cdot, \mathcal{A}_n)$ is continuous and clearly satisfies the axioms A_1 and A_2. The axiom A_3 is also in force because

$$\begin{aligned} \text{dist}(U(K), \mathcal{A}_n) &= \inf_{P \in \mathcal{A}_n} \delta(U(K), P) \;=\; \inf_{Q \in \mathcal{A}_n} \delta(U(K), U(Q)) \\ &= \inf_{Q \in \mathcal{A}_n} \delta(K, Q) \;=\; \text{dist}(K, \mathcal{A}_n) \end{aligned}$$

for all $U \in \mathbb{O}(n)$. For proving dual-invariance we observe that

$$\begin{aligned} \text{dist}(K^*, \mathcal{A}_n) &= \inf_{Q \in \mathcal{A}_n} \delta(K^*, Q) \;=\; \inf_{Q \in \mathcal{A}_n} \delta(K, Q^*) \\ &= \inf_{P \in \mathcal{A}_n} \delta(K, P) \;=\; \text{dist}(K, \mathcal{A}_n). \end{aligned}$$

Consider now the choice $\psi(t) = 1 - t$. The function Φ_{met} is nonexpansive and $\text{lip}_\delta(\Phi_{\text{met}}) = 1$, because the distance function $\text{dist}(\,\cdot, \mathcal{A}_n)$ is nonexpansive and has 1 as smallest Lipschitz constant. Let $\Phi : \Pi_n \to \mathbb{R}$ be another nonexpansive axial symmetry index and let $K \in \Pi_n$. By passing to the infimum with respect to $Q \in \mathcal{A}_n$ on the right-hand side of

$$1 - \Phi(K) \leq 1 - \Phi(Q) + \delta(K, Q)$$

and using the axiom A_2, we obtain $1 - \Phi(K) \leq \text{dist}(K, \mathcal{A}_n)$, i.e., $\Phi_{\text{met}}(K) \leq \Phi(K)$. □

A small change in the proof of Theorem 3.1 (b) yields the next easy result.

Proposition 3.1. *If an axial symmetry index $\Phi : \Pi_n \to \mathbb{R}$ is Lipschitzian, then it is metrically consistent in the sense that*

$$\lim_{\nu \to \infty} \text{dist}(K_\nu, \mathcal{A}_n) = 0 \quad \Rightarrow \quad \lim_{\nu \to \infty} \Phi(K_\nu) = 1$$

for any sequence $\{K_\nu\}_{\nu \in \mathbb{N}}$ in Π_n.

Proof. Suppose that the Lipschitz condition (4) is in force. Let $K \in \Pi_n$. By passing to the infimum with respect to $Q \in \mathcal{A}_n$ on the right-hand side of

$$1 - \Phi(K) \leq 1 - \Phi(Q) + \kappa\, \delta(K, Q),$$

we get $\Phi(K) \geq 1 - \kappa\, \text{dist}(K, \mathcal{A}_n)$. The latter inequality implies that Φ is metrically consistent. □

The geometric meaning of metric consistency is clear: if a proper cone is near the set of axially symmetric cones, then its degree of axial symmetry should be high. Every axial symmetry index of the composite form (8) is metrically consistent. Note that metric consistency does not imply Lipschitzness. For instance, the axial symmetry index

$$\Phi_{\bullet}(K) = 1 - [\mathrm{dist}(K, \mathcal{A}_n)]^{1/2}$$

is metrically consistent, but not Lipschitzian. A bothering aspect of the axial symmetry index Φ_{met} is that $\mathrm{dist}(K, \mathcal{A}_n)$ is almost impossible to evaluate in practice, even when the proper K under consideration has a simple structure. The reason is that the minimization problem (7) takes place in a metric space and not in a vector space. For finding a solution to (7) we cannot rely on such things as necessary or sufficient optimality conditions involving derivatives.

4 Use of the reflection gap function

The stability condition (1) amounts to saying that $R_{\vec{c}}\, x \in K$ for all $x \in K \cap \mathbb{S}_n$. In view of this observation it is natural to introduce the expression

$$\mathrm{rg}(c, K) := \max_{x \in K \cap \mathbb{S}_n} \mathrm{dist}(R_{\vec{c}}\, x, K), \tag{9}$$

which is a number in $[0, 1]$ that measures the degree of violation of the stability condition (1). We call (9) the reflection gap of K with respect to the line $\vec{c}$. For subsequent use, we state the following Lipschitz continuity result.

Lemma 4.1. *The function* $\mathrm{rg} : \mathbb{S}_n \times \Pi_n \to \mathbb{R}$ *satisfies the Lipschitz condition*

$$|\mathrm{rg}(c, P) - \mathrm{rg}(b, Q)| \;\leq\; 4\|c - b\| + 2\delta(P, Q) \tag{10}$$

for all $c, b \in \mathbb{S}_n$ *and* $P, Q \in \Pi_n$*.*

Proof. We claim that, for all $K \in \Pi_n$ and $c \in \mathbb{S}_n$, we have

$$\mathrm{rg}(c, K) = \delta(R_{\vec{c}}(K), K). \tag{11}$$

Such a formula provides an alternative characterization of the reflection gap function. By using the identity $R_{\vec{c}}(K \cap \mathbb{S}_n) = R_{\vec{c}}(K) \cap \mathbb{S}_n$ we readily get

$$\mathrm{rg}(c, K) = \max_{z \in R_{\vec{c}}(K) \cap \mathbb{S}_n} \mathrm{dist}(z, K). \tag{12}$$

On the other hand, the identity $\mathrm{dist}(R_{\vec{c}}\, x, K) = \mathrm{dist}(x, R_{\vec{c}}(K))$ yields

$$\mathrm{rg}(c, K) = \max_{x \in K \cap \mathbb{S}_n} \mathrm{dist}(x, R_{\vec{c}}(K)). \tag{13}$$

Formula (11) is obtained by combining (12) and (13). We claim next that, for each $K \in \Pi_n$, the function $\mathrm{rg}(\cdot, K)$ satisfies the Lipschitz condition

$$|\mathrm{rg}(c, K) - \mathrm{rg}(b, K)| \;\leq\; 4\|c - b\| \quad \text{for all } \; c, b \in \mathbb{S}_n. \tag{14}$$

To prove this claim, we start by observing that

$$\mathrm{dist}(R_{\vec{c}}\, x, K) \;\leq\; \mathrm{dist}(R_{\vec{b}}\, x, K) \;+\; \|R_{\vec{c}}\, x - R_{\vec{b}}\, x\| \tag{15}$$

for all $x \in \mathbb{R}^n$. Note that

$$\begin{aligned} R_{\vec{c}}x - R_{\vec{b}}x &= (2\langle c, x\rangle c - x) - (2\langle b, x\rangle b - x) \\ &= 2(\langle c, x\rangle - \langle b, x\rangle)c + 2\langle b, x\rangle(c - b). \end{aligned}$$

By using the Cauchy-Schwarz inequality, we get

$$\|R_{\vec{c}}x - R_{\vec{b}}x\| \leq 4\,\|c - b\|\,\|x\|. \tag{16}$$

The combination of (15) and (16) leads directly to (14). Finally, we claim that, for each $c \in \mathbb{S}_n$, the function $\mathrm{rg}(c, \cdot\,)$ satisfies the Lipschitz condition

$$|\mathrm{rg}(c, P) - \mathrm{rg}(c, Q)| \;\leq\; 2\delta(P, Q) \quad \text{for all } \; P, Q \in \Pi_n. \tag{17}$$

Since $R_{\vec{c}}$ is an orthogonal matrix, we have $\delta\left(R_{\vec{c}}(P), R_{\vec{c}}(Q)\right) = \delta\left(P, Q\right)$. Hence,

$$\begin{aligned} \mathrm{rg}(c, P) &= \delta\left(R_{\vec{c}}(P), P\right) \\ &\leq \delta\left(R_{\vec{c}}(P), R_{\vec{c}}(Q)\right) + \delta\left(R_{\vec{c}}(Q), Q\right) + \delta\left(Q, P\right) \\ &= \mathrm{rg}(c, Q) + 2\delta\left(P, Q\right). \end{aligned}$$

This shows (17) and completes the proof of the lemma. Indeed, (10) is obtained by combining (14) and (17). □

We now are ready to introduce our second example of Lipschitzian axial symmetry index.

Theorem 4.1. *The function* $\Phi_{\mathrm{rg}} : \Pi_n \to \mathbb{R}$ *given by*

$$\Phi_{\mathrm{rg}}(K) := 1 - \min_{c \in \mathbb{S}_n} \mathrm{rg}(c, K) \tag{18}$$

is an axial symmetry index. Furthermore, Φ_{rg} *is dual-invariant and satisfies the Lipschitz condition*

$$|\Phi_{\mathrm{rg}}(P) - \Phi_{\mathrm{rg}}(Q)| \;\leq\; 2\delta(P, Q) \quad \textit{for all } \; P, Q \in \Pi_n. \tag{19}$$

Proof. The minimum on the right-hand side of (18) is attained because such a problem amounts to minimizing a continuous function on a compact set. Note that

$$\begin{aligned} \Phi_{\mathrm{rg}}(K) = 1 \quad &\Leftrightarrow \quad \mathrm{rg}(c, K) = 0 \text{ for some } c \in \mathbb{S}_n \\ &\Leftrightarrow \quad K \text{ is axially symmetric.} \end{aligned}$$

Observe also that $R_{\vec{Uc}} = U R_{\vec{c}} U^\top$ for all $U \in \mathbb{O}(n)$. Hence,

$$\begin{aligned} \mathrm{rg}(Uc, U(K)) &= \delta(R_{\vec{Uc}}(U(K)), U(K)) \;=\; \delta((UR_{\vec{c}})(K), U(K)) \\ &= \delta(R_{\vec{c}}(K), K) \;=\; \mathrm{rg}(c, K) \end{aligned}$$

and, a posteriori,

$$\min_{c \in \mathbb{S}_n} \mathrm{rg}(c, U(K)) = \min_{c \in \mathbb{S}_n} \mathrm{rg}(Uc, U(K)) = \min_{c \in \mathbb{S}_n} \mathrm{rg}(c, K)\,.$$

This takes care of the axiom A_3. As a consequence of (17), we get

$$1 - \mathrm{rg}(c, P) \;\leq\; 1 - \mathrm{rg}(c, Q) + 2\delta(P, Q)$$

for all $c \in \mathbb{S}_n$. By passing to the maximum with respect to $c \in \mathbb{S}_n$, we obtain

$$\Phi_{\rm rg}(P) \leq \Phi_{\rm rg}(Q) + 2\delta(P,Q).$$

This takes care of the Lipschitz condition (19). For proving dual-invariance we write

$$\begin{aligned} {\rm rg}(c,K^*) &= \delta(R_{\vec{c}}(K^*),K^*) = \delta([R_{\vec{c}}(K)]^*,K^*) \\ &= \delta(R_{\vec{c}}(K),K) = {\rm rg}(c,K), \end{aligned}$$

and then we pass to the minimum with respect $c \in \mathbb{S}_n$. □

It is not clear to us whether the factor 2 in (19) is the smallest possible. As shown in the next proposition, the best Lipschitz constant of $\Phi_{\rm rg}$ is bigger than $\sqrt{2}$. In particular, $\Phi_{\rm rg}$ is not nonexpansive.

Proposition 4.1. *We have*

$${\rm lip}_\delta(\Phi_{\rm rg}) \geq \frac{\sqrt{2n}}{\sqrt{n-1}-1}\left(1-\frac{2}{n}\right) > \sqrt{2}\,.$$

Proof. The proposition is obtained by working out the right-hand side of the inequality

$${\rm lip}_\delta(\Phi_{\rm rg}) \geq \frac{\Phi_{\rm rg}(\Gamma_n) - \Phi_{\rm rg}(\mathbb{R}^n_+)}{\delta(\Gamma_n,\mathbb{R}^n_+)}\,,$$

where

$$\Gamma_n := \left\{x \in \mathbb{R}^n : (1/\sqrt{2}\,)\|x\| \leq \langle \widehat{1}_n, x\rangle\right\}$$

is the self-dual revolution cone whose symmetry axis is generated by the unit vector

$$\widehat{1}_n := (1/\sqrt{n}\,)\,(1,\ldots,1)^\top.$$

We have $\Phi_{\rm rg}(\Gamma_n) = 1$ because Γ_n is axially symmetric. A matter of computation shows that

$$\delta(\Gamma_n,\mathbb{R}^n_+) = \frac{1}{\sqrt{2n}}\left(\sqrt{n-1}-1\right)$$

and that $\Phi_{\rm rg}(\mathbb{R}^n_+) = 2/n$. □

5 Use of generalized reflection matrices

Perhaps the most popular way of measuring the degree of central symmetry of a convex body C in $\mathbb{R}^n$ is by using the Minkowski coefficient

$$\pi(C) := \max_{z\in\mathbb{R}^n} {\rm sym}(z,C), \tag{20}$$

where

$${\rm sym}(z,C) := \max\{\beta \in \mathbb{R} : \beta(z-C) \subseteq C - z\} \tag{21}$$

measures the degree of symmetry of C relative to a particular point $z \in \mathbb{R}^n$. A wealth of information concerning the function ${\rm sym}(\,\cdot\,,C)$ and the Minkowski coefficient (20) can be found in Grünbaum [6], see also Belloni and Freund [2]. We introduced in our previous work [15] a sort of Minkowski coefficient for measuring the degree of axial symmetry of a proper cone K. Instead of minimizing (21) with respect to the point $z \in \mathbb{R}^n$, we minimize the expression

$$f(c, K) := \max\{\beta \in \mathbb{R} : R(\vec{c}, \beta)(K) \subseteq K\} \tag{22}$$

with respect to the direction $c \in \mathbb{S}_n$. Here

$$R(\vec{c}, \beta) := cc^\top + \beta(cc^\top - I_n)$$

is a generalized reflection matrix onto the line $\vec{c}$ and β is a parameter called reflection factor.

Theorem 5.1. *The function $\Phi_\diamond : \Pi_n \to \mathbb{R}$ given by*

$$\Phi_\diamond(K) := \max_{c \in \mathbb{S}_n} f(c, K) \tag{23}$$

is an axial symmetry index. Furthermore, $\Phi_\diamond$ is dual-invariant, but not metrically consistent.

Proof. Except for the lack of metrical consistency, everything else has been proven in [15]. For each integer $\nu \geq 1$, let K_ν be the simplicial cone generated by the columns of the square matrix

$$A_\nu := \begin{bmatrix} \beta_\nu & \alpha_\nu & \cdots & \alpha_\nu \\ \alpha_\nu & \beta_\nu & \cdots & \alpha_\nu \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_\nu & \alpha_\nu & \cdots & \beta_\nu \end{bmatrix},$$

where $\beta_\nu := \left[1 - (n-1)\alpha_\nu^2\right]^{1/2}$ and

$$\alpha_\nu := \left(1 - \frac{1}{\nu}\right)\frac{1}{\sqrt{n}} + \frac{1}{\nu}\frac{1}{\sqrt{n-1}}.$$

The real β_ν is well defined because $(n-1)\alpha_\nu^2 \leq 1$. Furthermore, $\beta_\nu \neq \alpha_\nu$. The matrix A_ν is invertible because the determinant of $A_\nu^\top A_\nu$ is positive. Indeed, Lemma 3 in Iusem and Seeger [9] yields

$$\det(A_\nu^\top A_\nu) = (1 - \gamma_\nu)^{n-1}(1 + (n-1)\gamma_\nu)$$

with $\gamma_\nu := 2\alpha_\nu\beta_\nu + (n-2)\alpha_\nu^2$ belonging to $]0, 1[$. Since K_ν is simplicial, Theorem 4.3 in [15] yields $\Phi_\diamond(K_\nu) = (n-1)^{-1}$. Let Q_ν be the revolution cone circumscribed around K_ν, i.e., the revolution cone of smallest aperture angle that encloses K_ν. Since K_ν is an equiangular simplicial cone, it is not difficult to check that

$$Q_\nu = \{x \in \mathbb{R}^n : (\cos\theta_\nu)\|x\| \leq \langle \widehat{1}_n, x\rangle\},$$

where the half-aperture angle

$$\theta_\nu = \arccos\left[\frac{\beta_\nu + (n-1)\alpha_\nu}{\sqrt{n}}\right]$$

goes to 0 as ν goes to infinity. Since

$$\operatorname{dist}(K_\nu, \mathcal{A}_n) \leq \delta(K_\nu, Q_\nu) \leq \sin(2\theta_\nu),$$

we get $\lim_{\nu\to\infty} \operatorname{dist}(K_\nu, \mathcal{A}_n) = 0$. This proves that $\Phi_\diamond$ is not metrically consistent. □

The maximal value (23) can be written also in the form

$$\Phi_\diamond(K) = \max_{c \in K \cap \mathbb{S}_n} f(c, K) \tag{24}$$

and the optimization problem (24) has exactly one solution, cf. [15]. The unique solution to (24) is denoted by $\varrho_\diamond(K)$ and it is called the least asymmetry direction of K. Hence,

$$\Phi_\diamond(K) = f(\varrho_\diamond(K), K) \tag{25}$$

is obtained by evaluating the function $f(\cdot, K)$ at a special direction in K. We have $\varrho_\diamond(K) = c_K$ whenever K is axially symmetric. The least asymmetry direction is then a suitable substitute for the axial symmetry center in case the proper cone under consideration is not axially symmetric. This answers to question Q_1 raised in Section 1.

6 Use of a central axis selector

The function $\varrho_\diamond : \Pi_n \to \mathbb{S}_n$ enjoys a number of noteworthy properties. Besides being continuous, it satisfies

$$\begin{aligned} \varrho_\diamond(K^*) &= \varrho_\diamond(K), \\ \varrho_\diamond(U(K)) &= U\varrho_\diamond(K), \\ \varrho_\diamond(K) &\in \operatorname{int}(K \cap K^*) \end{aligned}$$

for all $K \in \Pi_n$ and $U \in \mathbb{O}(n)$. Hence, we may see $\varrho_\diamond$ as a particular instance of a dual-compatible central axis selector, cf. Definition 6.1.

Definition 6.1. *A central axis selector on Π_n is a continuous function $\varrho : \Pi_n \to \mathbb{S}_n$ satisfying*

$$\varrho(K) \in K \tag{26}$$

$$\varrho(U(K)) = U\varrho(K) \tag{27}$$

for all $K \in \Pi_n$ and $U \in \mathbb{O}(n)$. A central axis selector ϱ is called dual-compatible if $\varrho(K) \in K^*$ for all $K \in \Pi_n$.

Several examples of dual-compatible central axis selectors will be given in a moment. For the sake of the exposition, we open a parenthesis and recall the concept of well-axed proper cone. For each $K \in \Pi_n$, the symbol $\mathcal{S}(K)$ denotes the intersection of all linear subspaces of $\mathbb{R}^n$ with respect to which K is symmetric. We say that K is well-axed if $\mathcal{S}(K)$ is a line. The central direction of a well-axed cone K is denoted by w_K and it is defined as the unique vector $w \in K \cap \mathbb{S}_n$ such that $\vec{w} = \mathcal{S}(K)$. If K is well-axed, then also K^* is well-axed and $w_{K^*} = w_K$. On the other hand, if K is axially symmetric, then K is well-axed and $w_K = c_K$. Note that well-axedness does not imply axial symmetry; think for instance of an equiangular simplicial cone. The next result is known and recalled here just for convenience, cf. Seeger and Torki [11, Theorem 2.4].

Proposition 6.1. *Let $\varrho : \Pi_n \to \mathbb{S}_n$ be any function satisfying* (26)-(27). *Then $\varrho(K) \in \mathcal{S}(K)$ for all $K \in \Pi_n$. In particular, $\varrho(K) = w_K$ whenever K is well-axed.*

The next result is the main motivation behind Definition 6.1. We may see Theorem 6.1 as a generalization of Theorem 5.1.

Theorem 6.1. *Let $\varrho : \Pi_n \to \mathbb{S}_n$ be a dual-compatible central axis selector. Then*

$$\Phi_\varrho(K) := f(\varrho(K), K) \tag{28}$$

is an axial symmetry index on Π_n. Furthermore,

(a) $\Phi_\varrho(K) \le \Phi_\diamond(K)$ for all $K \in \Pi_n$.

(b) Φ_ϱ is not metrically consistent.

Proof. Part (a) is obvious because $\varrho_\diamond(K)$ is the solution to the maximization problem (24), whereas $\varrho(K)$ is just a feasible point. It has been shown in [15] that

$$0 < f(c, K) \le 1 \qquad \text{if } \vec{c} \cap \operatorname{int}(K \cap K^*) \neq \emptyset, \tag{29}$$

$$f(c, K) = 0 \qquad \text{if } \vec{c} \cap \operatorname{bd}(K \cap K^*) \neq \{0\}, \tag{30}$$

$$f(c, K) = -1 \qquad \text{if } \vec{c} \cap K \cap K^* = \{0\}. \tag{31}$$

Since $\varrho(K) \in K \cap K^*$, the case (31) can be ruled out and (29)-(30) yields $0 \le f(\varrho(K), K) \le 1$ for all $K \in \Pi_n$. This takes care of the axiom A_1. Suppose that $K \in \Pi_n$ is axially symmetric. By applying Proposition 6.1 we get $\varrho(K) = \varrho_\diamond(K)$ and, a posteriori, $\Phi_\varrho(K) = \Phi_\diamond(K) = 1$. Conversely, suppose that $\Phi_\varrho(K) = 1$. Part (a) implies that $\Phi_\diamond(K) = 1$, in which case K is axially symmetric. In short, Φ_ϱ satisfies the axiom A_2. Let $U \in \mathbb{O}(n)$. The combination of (27) and the general identity

$$f(Uc, U(K)) = f(c, K)$$

yield

$$\begin{aligned} \Phi_\varrho(U(K)) &= f(\varrho(U(K)), U(K)) \;=\; f(U\varrho(K), U(K)) \\ &= f(\varrho(K), K) \;=\; \Phi_\varrho(K). \end{aligned}$$

This takes care of the axiom A_3. Proving that Φ_ϱ is continuous is a delicate matter because $f : \mathbb{S}_n \times \Pi_n \to \mathbb{R}$ is not continuous. We claim that f is continuous when restricted to the subset

$$\Omega := \{(c, K) \in \mathbb{S}_n \times \Pi_n : c \in K \cap K^*\}.$$

If the claim were true, then Φ_ϱ would be continuous, being a composition

$$\begin{array}{ccccc} \Pi_n & \to & \Omega & \to & \mathbb{R} \\ K & \mapsto & (\varrho(K), K) & \mapsto & f(\varrho(K), K) \end{array}$$

of continuous functions. By applying Berge's maximum theorem (cf. Aubin and Frankowska [1, Theorem 1.4.6]) to the parametric optimization problem (22), we see that $f : \mathbb{S}_n \times \Pi_n \to \mathbb{R}$ is upper-semicontinuous. In particular, f is upper-semicontinuous as function on Ω. It remains to check the lower-semicontinuity condition

$$f(c_\infty, K_\infty) \;\le\; \liminf_{\substack{(c,K)\to(c_\infty,K_\infty) \\ (c,K)\in\Omega}} f(c, K) \tag{32}$$

at a given point $(c_\infty, K_\infty) \in \Omega$. There are two cases for consideration. The first case occurs when $c_\infty \in \mathrm{bd}(K_\infty \cap K^*_\infty)$. In such a situation, $f(c_\infty, K_\infty) = 0$ and (32) holds trivially because f is nonnegative on Ω. The second case occurs when

$$c_\infty \in \mathrm{int}(K_\infty \cap K^*_\infty). \tag{33}$$

Let $\{(c_\nu, K_\nu)\}_{\nu\in\mathbb{N}}$ be a sequence in Ω converging to (c_∞, K_∞) and such that $\lim_{\nu\to\infty} f(c_\nu, K_\nu) = a$, where a denotes the lower limit on the right-hand side of (32). In view of (33) and a certain stability result for Painlevé-Kuratowski limits established in Rockafellar and Wets [10, Proposition 4.15], there is no loss of generality in assuming that each (c_ν, K_ν) belongs to

$$\widetilde{\Omega} := \{(c, K) \in \mathbb{S}_n \times \Pi_n : c \in \mathrm{int}(K \cap K^*)\}\,.$$

From Seeger and Torki [15, Lemma 2.9] we know that

$$f(c, K) = \frac{1}{g(c, K) - 1} \tag{34}$$

for all $(c, K) \in \widetilde{\Omega}$, where

$$g(c, K) \;:=\; \max_{\substack{x\in K\cap\mathbb{S}_n\\ y\in K^*\cap\mathbb{S}_n}} \frac{\langle x, y\rangle}{\langle c, x\rangle\langle c, y\rangle}$$

is finite and greater than 1. In view of the relation (34), what we have to prove is that

$$\lim_{\nu\to\infty} g(c_\nu, K_\nu) \leq g(c_\infty, K_\infty)\,. \tag{35}$$

For each $\nu \in \mathbb{N}$, pick $x_\nu \in K_\nu \cap \mathbb{S}_n$ and $y_\nu \in K^*_\nu \cap \mathbb{S}_n$ such that

$$g(c_\nu, K_\nu) = \frac{\langle x_\nu, y_\nu\rangle}{\langle c_\nu, x_\nu\rangle\langle c_\nu, y_\nu\rangle}\,.$$

Taking subsequences if necessary, we may assume that $x_\infty := \lim_{\nu\to\infty} x_\nu$ exists and belongs to $K_\infty \cap \mathbb{S}_n$ and, similarly, $y_\infty := \lim_{\nu\to\infty} y_\nu$ exists and belongs to $K^*_\infty \cap \mathbb{S}_n$. We get in this way

$$\lim_{\nu\to\infty} g(c_\nu, K_\nu) \;=\; \frac{\langle x_\infty, y_\infty\rangle}{\langle c_\infty, x_\infty\rangle\langle c_\infty, y_\infty\rangle} \;\leq\; g(c_\infty, K_\infty).$$

This shows (35) and completes the proof of the continuity of Φ_ϱ. Finally, we prove part (b). Consider the sequence $\{K_\nu\}_{\nu\geq 1}$ of equiangular simplicial cones introduced in the proof of Theorem 5.1. Such cones are well-axed and have $\widehat{1}_n$ as common central direction. As a consequence of Proposition 6.1, we get $\varrho(K_\nu) = \varrho_\diamond(K_\nu) = \widehat{1}_n$. In particular,

$$\Phi_\varrho(K_\nu) = \Phi_\diamond(K_\nu) = (n-1)^{-1},$$

and, as in the proof of Theorem 5.1, we see that Φ_ϱ is not metrically consistent. $\square$

In contrast to the special case $\Phi_\diamond$, the more general function Φ_ϱ is not necessarily dual-invariant. Besides the axial symmetry index (25) discussed in Section 5, other interesting particular cases of

(28) are

$$\begin{cases} \Phi_{\rm si}(K) &:=& f(\varrho_{\rm si}(K), K), \\ \Phi_{\rm sc}(K) &:=& f(\varrho_{\rm sc}(K), K), \\ \Phi_{\rm ei}(K) &:=& f(\varrho_{\rm ei}(K), K), \\ \Phi_{\rm ec}(K) &:=& f(\varrho_{\rm ec}(K), K), \\ \Phi_{\rm vol}(K) &:=& f(\varrho_{\rm vol}(K), K), \end{cases} \tag{36}$$

where the spherical incenter $\varrho_{\rm si}(K)$, the spherical circumcenter $\varrho_{\rm sc}(K)$, the elliptic incenter $\varrho_{\rm ei}(K)$, the elliptic circumcenter $\varrho_{\rm ec}(K)$, and the volumetric center $\varrho_{\rm vol}(K)$, are different concepts of central direction for an arbitrary proper cone K. By gathering information from various sources, cf. [7, 12, 14], we see that $\varrho_{\rm si}$, $\varrho_{\rm sc}$, $\varrho_{\rm ei}$, $\varrho_{\rm ec}$, and $\varrho_{\rm vol}$, are all dual-compatible central axis selectors. For the sake of completeness we recall that:

- $\varrho_{\rm si}(K)$ is the axial symmetry center of the revolution cone inscribed in K, i.e., the revolution cone of largest aperture angle contained in K. Equivalently, $\varrho_{\rm si}(K)$ is the unique solution to the maximization problem

  $$r(K) := \max_{c \in K \cap \mathbb{S}_n} \operatorname{dist}[c, \operatorname{bd}(K)].$$

  The number $r(K)$ is usually called the inradius of K, but other names can also be found in the specialized literature.

- $\varrho_{\rm sc}(K)$ is the axial symmetry center of the revolution cone circumscribed around K. We know that

  $$\varrho_{\rm sc}(K) = \varrho_{\rm si}(K^*). \tag{37}$$

- $\varrho_{\rm ei}(K)$ is the axial symmetry center of the ellipsoidal cone inscribed in K, whereas $\varrho_{\rm ec}(K)$ is the axial symmetry center of the ellipsoidal cone circumscribed around K; see Seeger and Torki [14] for general material concerning the theory of inscribed and circumscribed ellipsoidal cones. We know that

  $$\varrho_{\rm ec}(K) = \varrho_{\rm ei}(K^*). \tag{38}$$

- $\varrho_{\rm vol}(K)$ is the unique solution to the minimization problem

  $$\operatorname{Vol}(K^*) := \min_{c \in \mathbb{S}_n} \operatorname{vol}_n(\{y \in K^* : \langle c, y \rangle \leq 1\}).$$

  Geometrically speaking, the number $\operatorname{Vol}(K^*)$ is viewed as the volume of K^*. Such a definition of volume for a proper cone is consistent with (2). The definition of the vector $\varrho_{\rm vol}(K)$ may seem technical at first glance, but the reader can consult Seeger and Torki [12, 13] for motivational and computational issues concerning the theory of volumetric centers.

Proposition 6.2. *None of the axial symmetry indices listed in* (36) *is dual-invariant. However, for all $K \in \Pi_n$, we can write*

$$\Phi_{\rm sc}(K) = \Phi_{\rm si}(K^*), \tag{39}$$
$$\Phi_{\rm ec}(K) = \Phi_{\rm ei}(K^*). \tag{40}$$

Proof. The conjugacy relationship (39) is obtained by combining (37) and the fact that

$$f(\,\cdot\,, K^*) = f(\,\cdot\,, K) \quad \text{for all } \; K \in \Pi_n. \tag{41}$$

Similarly, formula (40) is obtained by combining (38) and (41). We claim that neither $\Phi_{\rm si}$ nor $\Phi_{\rm sc}$ is dual-invariant. In view of (39), we just need to find $P_0 \in \Pi_n$ such that $\Phi_{\rm si}(P_0) \neq \Phi_{\rm sc}(P_0)$. Consider a self-dual revolution cone R in $\mathbb{R}^n$ and a vector $b \in \mathbb{S}_n$ orthogonal to the axial symmetry center c_R. The half-revolution cone $P_0 = \{x \in R : \langle b, x\rangle \geq 0\}$ is an infra-dual proper cone, i.e., a proper cone such that $P_0 \subseteq P_0^*$. Furthermore,

$$\begin{aligned} \varrho_{\rm sc}(P_0) = c_R \;\; &\in \;\; {\rm bd}(P_0) = {\rm bd}(P_0 \cap P_0^*), \\ \varrho_{\rm si}(P_0) \;\; &\in \;\; {\rm int}(P_0) = {\rm int}(P_0 \cap P_0^*). \end{aligned}$$

By using (30) we get $\Phi_{\rm sc}(P_0) = 0$, whereas (29) implies that $\Phi_{\rm si}(P_0)$ is positive. We now claim that neither $\Phi_{\rm ei}$ nor $\Phi_{\rm ec}$ is dual-invariant. In view of (40), we just need to find $Q_0 \in \Pi_n$ such that $\Phi_{\rm ei}(Q_0) \neq \Phi_{\rm ec}(Q_0)$. To avoid a long theoretical discussion on how to construct such a proper cone Q_0, we simply resort to a numerical experiment in dimension $n = 3$. Let Q_0 be the polyhedral cone in $\mathbb{R}^3$ generated by the columns of the matrix

$$A = \begin{bmatrix} -1 & 1 & 1 & -1 \\ -2 & -1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

The dual cone Q_0^* is polyhedral and generated by the columns of the matrix

$$B = \begin{bmatrix} -1/3 & -1 & 0 & 1 \\ 2/3 & 0 & -1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

A matter of numerical computation yields

$$\begin{aligned} \varrho_{\rm ec}(Q_0) &= (-0.0561,\, -0.0638,\, 0.9964)^\top \\ \varrho_{\rm ei}(Q_0) &= (0.0079,\, -0.0079,\, 0.9955)^\top \\ \varrho_{\rm vol}(Q_0) &= (-0.0189,\, -0.0828,\, 0.9964)^\top \\ \varrho_{\rm vol}(Q_0^*) &= (-0.0298,\, -0.0776,\, 0.9965)^\top. \end{aligned}$$

Since Q_0 is a polyhedral proper cone, evaluating $f(\,\cdot\,, Q_0)$ at these vectors offers no difficulty. As pointed out in [15, Proposition 3.6], the function $f(\,\cdot\,, Q_0)$ admits an easily computable formula expressed in terms of the columns of A and B. By using such a formula we obtain

$$\begin{aligned} \Phi_{\rm ec}(Q_0) &= f(\varrho_{\rm ec}(Q_0), Q_0) &= 0.7009 \\ \Phi_{\rm ei}(Q_0) &= f(\varrho_{\rm ei}(Q_0), Q_0) &= 0.5119 \\ \Phi_{\rm vol}(Q_0) &= f(\varrho_{\rm vol}(Q_0), Q_0) &= 0.7385 \\ \Phi_{\rm vol}(Q_0^*) &= f(\varrho_{\rm vol}(Q_0^*), Q_0) &= 0.7332. \end{aligned}$$

This completes the proof of the proposition. □

The axial symmetry indices listed in (36) are all different, but some of them coincide when restricted to special subclasses of proper cones. For instance, $\Phi_{\rm ec}$, $\Phi_{\rm ei}$, and $\Phi_{\rm vol}$, coincide on the subclass of simplicial cones. Indeed, [14, Proposition 5.4] asserts that $\varrho_{\rm ec}(K) = \varrho_{\rm ei}(K) = \varrho_{\rm vol}(K)$ whenever K is simplicial.

7 Use of volume ratios

Immediately after the Minkowski coefficient (20), the second most popular way to measure the degree of central symmetry of a convex body C is by using the Kövner-Besicovitch coefficient

$$\Pi^{\uparrow}_{\rm vr}(C) := \sup_{\substack{D\in\mathcal{C}_n \\ D\subseteq C}} \frac{{\rm vol}_n(D)}{{\rm vol}_n(C)}$$

or, alternatively, the Estermann coefficient

$$\Pi^{\downarrow}_{\rm vr}(C) := \sup_{\substack{D\in\mathcal{C}_n \\ D\supseteq C}} \frac{{\rm vol}_n(C)}{{\rm vol}_n(D)}\,,$$

where $\mathcal{C}_n$ denotes the collection of centrally symmetric convex bodies in $\mathbb{R}^n$. The analysis of such coefficients has been the subject of a number of papers, among which we mention the classical works by Grünbaum [6] and Stein [16]. In a similar spirit, we may consider any of the following volume ratios

$$\Phi^{\uparrow}_{\rm vr}(K) := \frac{\upsilon^{\uparrow}(K)}{{\rm Vol}(K)}\,, \quad \Phi^{\downarrow}_{\rm vr}(K) := \frac{{\rm Vol}(K)}{\upsilon^{\downarrow}(K)}$$

as measure of axial symmetry of a proper cone K. Here,

$$\upsilon^{\uparrow}(K) := \sup_{\substack{Q\in\mathcal{A}_n \\ Q\subseteq K}} {\rm Vol}(Q) \tag{42}$$

is an optimization problem that consists in finding an axially symmetric proper cone of largest volume contained in K, whereas

$$\upsilon^{\downarrow}(K) := \inf_{\substack{Q\in\mathcal{A}_n \\ Q\supseteq K}} {\rm Vol}(Q) \tag{43}$$

amounts to finding an axially symmetric proper cone of smallest volume enclosing K. The volume function ${\rm Vol} : \Pi_n \to \mathbb{R}$ is known to be continuous, cf. [12, Theorem 5.7]. To prove that $\upsilon^{\uparrow}$ and $\upsilon^{\downarrow}$ are continuous functions on Π_n is far from being a trivial matter. For such an endeavor, we need first to understand well how the term ${\rm Vol}(K)$ behaves when the argument $K \in \Pi_n$ is almost non-proper, i.e., nearly flat or nearly unpointed. The next technical lemma contributes to clarify this issue. In what follows, the symbol $\theta_{\rm inn}(K)$ stands for the inner angle of K, i.e., the half-aperture angle of the revolution cone inscribed in K. Similarly, $\theta_{\rm out}(K)$ stands for the outer angle of K, i.e., the half-aperture angle of the revolution cone circumscribed around K. As shown in [7], we have the general identities

$$r(K) = \sin\left[\theta_{\rm inn}(K)\right], \quad r(K^*) = \cos\left[\theta_{\rm out}(K)\right].$$

Hence, the inner angle serves to measure the degree of solidity of the cone, whereas the outer angle serves to measure the degree of pointedness; see [5, 8] for general material on indices of solidity and pointedness.

Lemma 7.1 (Compensation inequalities). *There are positive constants a_n and b_n such that*

$$a_n \tan\left[\theta_{\rm inn}(K)\right] \tan^{n-2}[\theta_{\rm out}(K)] \;\geq\; {\rm Vol}(K), \tag{44}$$

$$b_n \tan\left[\theta_{\rm out}(K)\right] \tan^{n-2}[\theta_{\rm inn}(K)] \;\leq\; {\rm Vol}(K) \tag{45}$$

for all $K \in \Pi_n$.

Proof. Let $E^{\uparrow}(K)$ and $E^{\downarrow}(K)$ denote respectively the ellipsoidal cone inscribed in K and the ellipsoidal cone circumscribed around K. Proposition 6.2 in Seeger and Torki [14] ensures the existence of positive constants p_n and q_n such that

$$p_n \mathrm{Vol}(E^{\downarrow}(K)) \;\leq\; \mathrm{Vol}(K) \;\leq\; q_n \mathrm{Vol}(E^{\uparrow}(K)) \tag{46}$$

for all $K \in \Pi_n$. As explained in [14, Section 7], the volume of an ellipsoidal cone Q in $\mathbb{R}^n$ can be computed by using the explicit formula

$$\mathrm{Vol}(Q) = \mathrm{Vol}(\mathbb{L}_n)\, \Pi_{k=1}^{n-1} \gamma_k(Q),$$

where $\mathbb{L}_n$ is the n-dimensional Lorentz cone and the $\gamma_k(Q)$'s are the semiaxes lengths of the ellipsoid

$$\mathrm{T}(Q) = \{x \in Q : \langle c_Q, x\rangle = 1\}.$$

The smallest and the largest semiaxes lengths are given respectively by

$$\gamma_{\min}(Q) = \tan\left[\theta_{\mathrm{inn}}(Q)\right], \quad \gamma_{\max}(Q) = \tan\left[\theta_{\mathrm{out}}(Q)\right].$$

We obtain in this way

$$\mathrm{Vol}(\mathbb{L}_n) \tan\left[\theta_{\mathrm{inn}}(Q)\right] \tan^{n-2}[\theta_{\mathrm{out}}(Q)] \;\geq\; \mathrm{Vol}(Q), \tag{47}$$

$$\mathrm{Vol}(\mathbb{L}_n) \tan\left[\theta_{\mathrm{out}}(Q)\right] \tan^{n-2}[\theta_{\mathrm{inn}}(Q)] \;\leq\; \mathrm{Vol}(Q). \tag{48}$$

The second inequality in (46), combined with the inequality (47) evaluated at $Q = E^{\uparrow}(K)$, shows that (44) holds with $a_n = q_n \mathrm{Vol}(\mathbb{L}_n)$. Similarly, the first inequality in (46), combined with the inequality (48) evaluated at $Q = E^{\downarrow}(K)$, shows that (45) holds with $b_n = p_n \mathrm{Vol}(\mathbb{L}_n)$. □

Corollary 7.1. *Let $\{K_\nu\}_{\nu\in\mathbb{N}}$ be a sequence in Π_n. Then*

(a) $\lim_{\nu\to\infty} \theta_{\mathrm{inn}}(K_\nu) = 0$ *and* $\sup_{\nu\in\mathbb{N}} \theta_{\mathrm{out}}(K_\nu) < \pi/2$ *imply* $\lim_{\nu\to\infty} \mathrm{Vol}(K_\nu) = 0$.

(b) $\lim_{\nu\to\infty} \theta_{\mathrm{out}}(K_\nu) = \pi/2$ *and* $\inf_{\nu\in\mathbb{N}} \theta_{\mathrm{inn}}(K_\nu) > 0$ *imply* $\lim_{\nu\to\infty} \mathrm{Vol}(K_\nu) = \infty$.

Proof. Part (a) follows from (44), whereas part (b) follows from (45). □

The compensation inequalities have such a name because they fully determine the limiting behavior of $\mathrm{Vol}(K_\nu)$ when a vanishing degree of pointedness (respectively, solidity) is suitable compensated by a control on the degree of solidity (respectively, pointedness). We now come back to the main stream of the presentation.

Proposition 7.1. *For all $K \in \Pi_n$, the supremum in* (42) *and the infimum in* (43) *are both attained. Furthermore,*

(a) The function $\upsilon^{\uparrow} : \Pi_n \to \mathbb{R}$ is continuous and admits the characterization

$$\upsilon^{\uparrow}(K) = \max_{c\in\mathbb{S}_n\cap\,\mathrm{int}(K)} \mathrm{Vol}(K \cap R_{\vec{c}}(K))\,. \tag{49}$$

(b) Similarly, the function $\upsilon^{\downarrow} : \Pi_n \to \mathbb{R}$ is continuous and

$$\upsilon^{\downarrow}(K) = \min_{c\in\mathbb{S}_n\cap\,\mathrm{int}(K^*)} \mathrm{Vol}(K + R_{\vec{c}}(K))\,.$$

Proof. We concentrate on the analysis of the upward case $\upsilon^{\uparrow}$, the downward case $\upsilon^{\downarrow}$ can be treated along the same lines. We prove formula (49) and the attainment of the supremum in (42). If c is a feasible solution to (49), then the closed convex cone $K^c := K \cap R_{\vec{c}}(K)$ is proper and symmetric with respect to the line $\vec{c}$. In particular, K^c is a feasible solution to (42). Conversely, if Q is a feasible solution to (42), then the axial symmetry center c_Q is a feasible solution to (49) and $Q \subseteq K \cap R_{\vec{c}_Q}(K)$. This proves formula (49), except for the attainment of the maximum. Let $\{c_\nu\}_{\nu\in\mathbb{N}}$ be a maximizing sequence for (49). Without loss of generality we may assume that the limit $c_\infty := \lim_{\nu\to\infty} c_\nu$ exists, in which case it belongs to $\mathbb{S}_n \cap K$. We claim that

$$c_\infty \in \operatorname{int}(K). \tag{50}$$

The proof of (50) relies on Corollary 7.1(a). Since K is pointed, we have

$$\sup_{\nu\in\mathbb{N}} \theta_{\rm out}(K^{c_\nu}) \;\leq\; \theta_{\rm out}(K) \;<\; \pi/2.$$

Concerning the limiting behavior of $\theta_{\rm inn}(K^{c_\nu})$, observe that

$$\begin{aligned} c_\infty \in \operatorname{bd}(K) \quad &\Rightarrow \quad \lim_{\nu\to\infty} \operatorname{dist}[c_\nu, \operatorname{bd}(K)] = 0 \\ &\Rightarrow \quad \lim_{\nu\to\infty} r(K^{c_\nu}) = 0 \\ &\Rightarrow \quad \lim_{\nu\to\infty} \theta_{\rm inn}(K^{c_\nu}) = 0, \end{aligned}$$

where the second implication is due to the fact that

$$r(K^c) \;=\; \operatorname{dist}\left[c, \operatorname{bd}(K^c)\right] \;=\; \operatorname{dist}\left[c, \operatorname{bd}(K)\right]$$

for all $c \in \mathbb{S}_n \cap \operatorname{int}(K)$. In view of Corollary 7.1(a) and the above discussion, the case $c_\infty \in \operatorname{bd}(K)$ must be ruled out. Indeed, such a case leads to

$$\upsilon^{\uparrow}(K) = \lim_{\nu\to\infty} \operatorname{Vol}(K^{c_\nu}) = 0,$$

which contradicts the positive-valuedness of $\upsilon^{\uparrow}$. Now, since the claim (50) is true, we get

$$\lim_{\nu\to\infty} \delta(K^{c_\nu}, K^{c_\infty}) = 0$$

and, a posteriori,

$$\operatorname{Vol}(K^{c_\infty}) \;=\; \lim_{\nu\to\infty} \operatorname{Vol}(K^{c_\nu}) \;=\; \upsilon^{\uparrow}(K).$$

Hence, c_∞ is a solution to (49) and K^{c_∞} is a solution to (42). Next, we prove that $\upsilon^{\uparrow}$ is upper-semicontinuous at $K_\infty \in \Pi_n$. Let $\{K_\nu\}_{\nu\in\mathbb{N}}$ be a sequence in Π_n converging to K_∞ and such that

$$\lim_{\nu\to\infty} \upsilon^{\uparrow}(K_\nu) = \limsup_{K\to K_\infty} \upsilon^{\uparrow}(K).$$

For each ν, pick $c_\nu \in \mathbb{S}_n \cap \operatorname{int}(K_\nu)$ such that $\operatorname{Vol}(Q_\nu) = \upsilon^{\uparrow}(K_\nu)$, where

$$Q_\nu := K_\nu \cap R_{\vec{c}_\nu}(K_\nu).$$

Taking a subsequence if necessary, we may assume that the limit $c_\infty := \lim_{\nu\to\infty} c_\nu$ exists, in which case it belongs to $\mathbb{S}_n \cap K_\infty$. We prove that

$$c_\infty \in \operatorname{int}(K_\infty) \tag{51}$$

by applying Corollary 7.1(a) to the sequence $\{Q_\nu\}_{\nu\in\mathbb{N}}$. Since $\theta_{\rm out}(Q_\nu) \leq \theta_{\rm out}(K_\nu)$ and

$$\lim_{\nu\to\infty} \theta_{\rm out}(K_\nu) \;=\; \theta_{\rm out}(K_\infty) \;<\; \pi/2,$$

we get $\sup_{\nu\in\mathbb{N}} \theta_{\rm out}(Q_\nu) < \pi/2$. On the other hand,

$$\begin{aligned} c_\infty \in {\rm bd}(K_\infty) \quad &\Rightarrow \quad \lim_{\nu\to\infty} {\rm dist}[c_\nu, {\rm bd}(K_\nu)] = 0 \\ &\Rightarrow \quad \lim_{\nu\to\infty} \theta_{\rm inn}(K^{c_\nu}) = 0, \end{aligned}$$

where the first implication follows from a general convergence result for boundaries of proper cones, cf. [12, Lemma 5.6]. So, the case $c_\infty \in {\rm bd}(K_\infty)$ must be ruled out because it leads to

$$\limsup_{K\to K_\infty} \upsilon^\uparrow(K) = \lim_{\nu\to\infty} {\rm Vol}(Q_\nu) = 0,$$

which is a contradiction. Now, since the condition (51) is true, by relying on the theory of Painlevé-Kuratowski limits, we can check that $\{Q_\nu\}_{\nu\in\mathbb{N}}$ converges to

$$Q_\infty := K_\infty \cap R_{\vec{c}_\infty}(K_\infty). \tag{52}$$

Hence,

$$\limsup_{K\to K_\infty} \upsilon^\uparrow(K) = \lim_{\nu\to\infty} {\rm Vol}(Q_\nu) \;=\; \lim_{\nu\to\infty} {\rm Vol}(Q_\infty) \;\leq\; \upsilon^\uparrow(K_\infty),$$

which is the desired conclusion. Finally, we prove that $\upsilon^\uparrow$ is lower-semicontinuous at $K_\infty \in \Pi_n$. Let $\{K_\nu\}_{\nu\in\mathbb{N}}$ be a sequence in Π_n converging to K_∞ and such that

$$\lim_{\nu\to\infty} \upsilon^\uparrow(K_\nu) = \liminf_{K\to K_\infty} \upsilon^\uparrow(K).$$

Pick $c_\infty \in \mathbb{S}_n \cap {\rm int}(K_\infty)$ such that ${\rm Vol}(Q_\infty) = \upsilon^\uparrow(K_\infty)$, where Q_∞ is as in (52). In view of [10, Proposition 4.15], we have $c_\infty \in {\rm int}(K_\nu)$ for all ν large enough. If we define

$$Q_\nu := K_\nu \cap R_{\vec{c}_\infty}(K_\nu),$$

then $\lim_{\nu\to\infty} \delta(Q_\nu, Q_\infty) = 0$ and

$$\upsilon^\uparrow(K_\infty) \;=\; {\rm Vol}(Q_\infty) \;=\; \lim_{\nu\to\infty} {\rm Vol}(Q_\nu) \;\leq\; \lim_{\nu\to\infty} \upsilon^\uparrow(K_\nu) \;=\; \liminf_{K\to K_\infty} \upsilon^\uparrow(K),$$

which is the desired conclusion. □

Theorem 7.2. *$\Phi^\uparrow_{\rm vr}$ and $\Phi^\downarrow_{\rm vr}$ are axial symmetry indices on Π_n.*

Proof. $\Phi^\uparrow_{\rm vr}$ is positive-valued and continuous, being a quotient of two positive-valued continuous functions. Note that $\Phi^\uparrow_{\rm vr}$ satisfies the axioms A_1 and A_2, because $\upsilon^\uparrow(K) \leq {\rm Vol}(K)$ for all $K \in \Pi_n$, with equality if and only if K is axially symmetric. On the other hand,

$${\rm Vol}(U(K)) = {\rm Vol}(K)$$

for all $K \in \Pi_n$ and $U \in \mathbb{O}(n)$. This yields a similar identity for the function $\upsilon^\uparrow$ and, therefore,

$$\Phi^\uparrow_{\rm vr}(U(K)) = \frac{\upsilon^\uparrow(U(K))}{{\rm Vol}(U(K))} = \frac{\upsilon^\uparrow(K)}{{\rm Vol}(K)} = \Phi^\uparrow_{\rm vr}(K)\,.$$

In short, $\Phi^\uparrow_{\rm vr}$ satisfies the axiom A_3. The case of $\Phi^\downarrow_{\rm vr}$ is treated in a similar way. □

8 Miscellaneous results

8.1 Axial symmetrization and deformation lemma

Suppose that we are given a proper cone K in $\mathbb{R}^n$ that is not axially symmetric. The deformation lemma stated in a moment explains how to construct a continuous deformation that converts K into an axially symmetric proper cone. We start by writing a technical result that has an interest by its own.

Lemma 8.1. *Let $K \in \Pi_n$ and $c \in \mathbb{S}_n \cap \mathrm{int}(K^*)$. Then*

(a) For $\alpha, \beta \in \mathbb{R}$, not both equal to zero, the following set is a proper cone:

$$P_{\alpha,\beta} := R(\vec{c}, \alpha)(K) + R(\vec{c}, \beta)(K). \tag{53}$$

(b) For a nonzero $\beta \in \mathbb{R}$, the proper cone

$$Q_\beta := R(\vec{c}, -\beta)(K) + R(\vec{c}, \beta)(K) \tag{54}$$

is symmetric with respect to the line $\vec{c}$.

Proof. Generalized reflection matrices obey to the product rule

$$R(\vec{c}, \alpha)R(\vec{c}, \beta) = R(\vec{c}, -\alpha\beta) \tag{55}$$

for all $\alpha, \beta \in \mathbb{R}$. Suppose for instance that $\beta \neq 0$. In such a case, $R(\vec{c}, \beta)$ is invertible and

$$[R(\vec{c}, \beta)]^{-1} = R(\vec{c}, (1/\beta)).$$

The convex cone $P_{\alpha,\beta}$ has nonempty interior because the portion $R(\vec{c}, \beta)(K)$ has nonempty interior. In order to prove that $P_{\alpha,\beta}$ is closed, it suffices to check that

$$u, v \in K, \;\; R(\vec{c}, \alpha)u + R(\vec{c}, \beta)v = 0 \quad \Rightarrow \quad u = 0, \; v = 0. \tag{56}$$

Let u and v be as in the left-hand side of the implication (56). If $u = 0$, then also $v = 0$ and we are done. Suppose that $u \neq 0$. We must arrive to a contradiction. We have

$$v \;=\; -[R(\vec{c}, \beta)]^{-1} R(\vec{c}, \alpha)u \;=\; -R\left(\vec{c}, (1/\beta)\right) R(\vec{c}, \alpha)u \;=\; -R(\vec{c}, -(\alpha/\beta))u,$$

i.e.,

$$v \;=\; ((\alpha/\beta) - 1)\, \langle c, u\rangle c - (\alpha/\beta)\, u.$$

By taking the inner product with respect to $\langle c, u\rangle^{-1} c$, we get $\langle c, u\rangle^{-1}\langle c, v\rangle = -1$, contradicting the fact that $\langle c, u\rangle$ and $\langle c, v\rangle$ are nonnegative. Hence, (56) holds and $P_{\alpha,\beta}$ is closed. The implication (56) serves also to prove that $P_{\alpha,\beta}$ is pointed. The details are omitted. Part (b) is proved by writing the product rule (55) with $\alpha = 1$. We get

$$R_{\vec{c}}\, R(\vec{c}, \beta) = R(\vec{c}, -\beta), \;\; R_{\vec{c}}\, R(\vec{c}, -\beta) = R(\vec{c}, \beta),$$

and, therefore, $R_{\vec{c}}\,(Q_\beta) = Q_\beta$. □

Lemma 8.2 (Deformation lemma)**.** *Let $K \in \Pi_n$ and $c \in \mathbb{S}_n \cap \mathrm{int}(K^*)$. Then there exists a continuous function $\Upsilon : [0,1] \to \Pi_n$ such that $\Upsilon(0) = K$ and $\Upsilon(1)$ is symmetric with respect to $\vec{c}$.*

Proof. By taking $\alpha = -1$ in (53), we get a proper cone

$$\Upsilon(t) := K + R(\vec{c}, 2t-1)(K)$$

for each $t \in [0,1]$. By proceeding as in [15, Theorem 5.3], we can check that $\Upsilon : [0,1] \to \Pi_n$ is continuous. On the other hand,

$$\Upsilon(0) = K + R(\vec{c}, -1)(K) = K + K = K$$

and $\Upsilon(1) = K + R_{\vec{c}}(K)$ is symmetric with respect to $\vec{c}$. □

The continuous deformation map Υ in Lemma 8.2 is not unique and depends of course on the initial cone K and the choice of c. The next result is a direct consequence of Lemma 8.2.

Proposition 8.1. *For any axial symmetry index Φ on Π_n, the set*

$$\mathrm{Im}(\Phi) := \{\Phi(K) : K \in \Pi_n\}$$

is an interval.

Proof. We prove that $\mathrm{Im}(\Phi)$ is equal to $[\mu(\Phi), 1]$ if Φ attains its infimum, and equal to $]\mu(\Phi), 1]$ if Φ does not attain its infimum. Suppose first that Φ attains its infimum at some $K_0 \in \Pi_n$. Take any $c_0 \in \mathbb{S}_n \cap \mathrm{int}(K_0^*)$ and consider the function $h : [0,1] \to \mathbb{R}$ be given by

$$h(t) := \Phi\left(K_0 + R(\vec{c}_0, 2t-1)(K_0)\right).$$

Note that h is continuous and

$$\begin{aligned} h(0) &= \Phi\left(K_0\right) \;=\; \mu(\Phi), \\ h(1) &= \Phi\left(K_0 + R_{\vec{c}_0}\left(K_0\right)\right) = 1. \end{aligned}$$

By applying the Intermediate Value Theorem we get $\mathrm{Im}(\Phi) = [\mu(\Phi), 1]$. Suppose now that Φ does not attain its infimum. Let $\{K_\nu\}_{\nu \in \mathbb{N}}$ be a sequence in Π_n such that $\{\Phi(K_\nu)\}_{\nu \in \mathbb{N}}$ decreases to the infimal value $\mu(\Phi)$. By applying the previous argument to each K_ν, we get

$$\mathrm{Im}(\Phi) \;=\; \cup_{\nu \in \mathbb{N}} [\Phi(K_\nu), 1] \;=\;]\mu(\Phi), 1].$$

This completes the proof of the proposition. □

8.2 Comparing Φ_{met}, Φ_{rg}, and $\Phi_\diamond$

The proposition below displays some inequalities that relate the axial symmetry indices Φ_{met}, Φ_{rg}, and $\Phi_\diamond$.

Proposition 8.2. *Let $K \in \Pi_n$. Then*

$$\begin{aligned} \Phi_{\mathrm{met}}(K) &\leq (1/2)(1 + \Phi_{\mathrm{rg}}(K)), && (57) \\ \Phi_\diamond(K) &\leq \Phi_{\mathrm{rg}}(K), && (58) \\ \Phi_\diamond(K) &\leq \Phi_{\mathrm{met}}(K). && (59) \end{aligned}$$

Proof. In view of Theorem 4.1 and Proposition 2.2, $N_2\Phi_{\rm rg} = (1/2)(1+\Phi_{\rm rg})$ is a nonexpansive axial symmetry index. Hence, (57) is taken care by Theorem 3.1 (b). We now prove (58). Consider the auxiliary function $\sigma : \mathbb{S}_n \times \Pi_n \to \mathbb{R}$ given by

$$\sigma(c, K) := \min_{x\in K\cap \mathbb{S}_n} \langle c, x\rangle.$$

We claim that

$$1 - \Phi_{\rm rg}(K) \;\leq\; {\rm rg}(\varrho_\diamond(K), K) \;\leq\; \left[1 - \sigma^2(\varrho_\diamond(K), K)\right]^{1/2} [1 - \Phi_\diamond(K)] \tag{60}$$

for all $K \in \Pi_n$. The first inequality in (60) is obvious, so we concentrate on the second inequality. Let $c = \varrho_\diamond(K)$ and $\beta = \Phi_\diamond(K)$. In particular, $R(\vec{c}, \beta)(K) \subseteq K$ and, for all $x \in K \cap \mathbb{S}_n$, we get

$$\begin{aligned} {\rm dist}(R_{\vec{c}}\,x, K) &\leq \|R_{\vec{c}}\,x - R(\vec{c}, \beta)x\| \\ &= (1-\beta)\,\|\langle c, x\rangle c - x\| \\ &= (1-\beta)\left[1 - \langle c, x\rangle^2\right]^{1/2}. \end{aligned}$$

It suffices now to pass to the supremum with $x \in K \cap \mathbb{S}_n$. Since $\varrho_\diamond(K) \in {\rm int}(K^*)$, the term

$$\sigma(\varrho_\diamond(K), K) = {\rm dist}[\varrho_\diamond(K), {\rm bd}(K^*)]$$

belongs to the open interval $]0, 1[$. Hence, so does the factor in front of $1 - \Phi_\diamond(K)$. This shows (58). Finally, we take care of (59). Let c and β be as before, and Q_β be given by (54). The proper cone Q_β is symmetric with respect to $\vec{c}$ and

$$R(\vec{c}, \beta)(K) \subseteq K, \;\; R(\vec{c}, -\beta)(K) \subseteq K, \;\; Q_\beta \subseteq K.$$

Hence

$$\begin{aligned} {\rm dist}(K, \mathcal{A}_n) &\leq \delta(K, Q_\beta) \;=\; \delta(R_{\vec{c}}(K), Q_\beta) \;=\; \max_{x\in K\cap \mathbb{S}_n} {\rm dist}(R_{\vec{c}}\,x, Q_\beta) \\ &\leq \max_{x\in K\cap \mathbb{S}_n} {\rm dist}\left[R_{\vec{c}}\,x, R(\vec{c}, \beta)(K)\right] \;\leq\; \max_{x\in K\cap \mathbb{S}_n} \|R_{\vec{c}}\,x - R(\vec{c}, \beta)x\|. \end{aligned}$$

In this way we obtain

$${\rm dist}(K, \mathcal{A}_n) \leq \left[1 - \sigma^2(\varrho_\diamond(K), K)\right]^{1/2} [1 - \Phi_\diamond(K)]$$

and, a posteriori, the inequality (59). □

8.3 On the infimal value of an axial symmetry index

The maximal value of an axial symmetry index is obviously 1 and it is attained at any axially symmetric proper cone. By several reasons it is important to have an estimate of the infimal value

$$\mu(\Phi) := \inf_{K\in \Pi_n} \Phi(K) \tag{61}$$

and to know whether the solution set

$${\rm argmin}_{\Pi_n} \Phi \;:=\; \{K \in \Pi_n : \Phi(K) = \mu(\Phi)\}$$

is nonempty. Recall that the expression (61) has been used already in the statement of Proposition 2.1 and in the proof of Proposition 8.1. An axial symmetry index $\Phi : \Pi_n \to \mathbb{R}$ is called normalized if $\mu(\Phi) = 0$. If an axial symmetry index Φ is normalized, then so is the conjugate axial symmetry index Φ^*. In fact, we have the following general conjugacy result, whose proof is easy and omitted.

Proposition 8.3. *Let Φ be an axial symmetry index on Π_n. Then $\mu(\Phi^*) = \mu(\Phi)$ and*

$$\mathrm{argmin}_{\Pi_n} \Phi^* = \{K^* : K \in \mathrm{argmin}_{\Pi_n} \Phi\}.$$

In the next proposition we quickly review which axial symmetry indices are normalized and which are not. The notation $\mathcal{S}_n$ stands for the set of simplicial cones in $\mathbb{R}^n$.

Proposition 8.4. *We have:*

(a) $\Phi_{\rm sc}$ *and* $\Phi_{\rm si}$ *are normalized and attain their infimum. Furthermore,*

$$\begin{aligned} \mathrm{argmin}_{\Pi_n} \Phi_{\rm sc} &= \{K \in \Pi_n : \varrho_{\rm sc}(K) \in \mathrm{bd}(K)\}, \\ \mathrm{argmin}_{\Pi_n} \Phi_{\rm si} &= \{K \in \Pi_n : \varrho_{\rm si}(K) \in \mathrm{bd}(K^*)\}. \end{aligned}$$

(b) $\Phi_\diamond$, $\Phi_{\rm ei}$, $\Phi_{\rm ec}$, *and* $\Phi_{\rm vol}$, *are not normalized, but they attain their infimum. Furthermore,*

$$\begin{aligned} &\mu(\Phi_\diamond) = \mu(\Phi_{\rm ei}) = \mu(\Phi_{\rm ec}) = \mu(\Phi_{\rm vol}) = (n-1)^{-1}, \\ &\mathrm{argmin}_{\Pi_n} \Phi_\diamond = \mathrm{argmin}_{\Pi_n} \Phi_{\rm ei} = \mathrm{argmin}_{\Pi_n} \Phi_{\rm ec} = \mathcal{S}_n \subseteq \mathrm{argmin}_{\Pi_n} \Phi_{\rm vol}. \end{aligned}$$

(c) $\Phi_{\rm met}$, $\Phi_{\rm rg}$, $\Phi_{\rm vr}^{\uparrow}$, *and* $\Phi_{\rm vr}^{\downarrow}$, *are not normalized.*

Proof. Part (a). In view of Propositions 6.2 and 8.3, we just need to analyze $\Phi_{\rm sc}$. For all $K \in \Pi_n$, $\varrho_{\rm sc}(K)$ belongs to $K \cap \mathrm{int}(K^*)$. Hence, from (29)-(30) we see that $\Phi_{\rm sc}(K) = 0$ if and only if $\varrho_{\rm sc}(K)$ is on the boundary of K. *Part* (b). As shown in the proof of [15, Theorem 4.3], the equalities

$$\mu(\Phi) = (n-1)^{-1}, \quad \mathrm{argmin}_{\Pi_n} \Phi = \mathcal{S}_n$$

hold true for $\Phi = \Phi_\diamond$ and for $\Phi = \Phi_{\rm ei}$. In view of Propositions 6.2 and 8.3, these equalities are also true for $\Phi = \Phi_{\rm ec}$. So, we just need to examine the case of $\Phi_{\rm vol}$. As mentioned in Grünbaum [6], for any convex body C in $\mathbb{R}^n$, we can write $\mathrm{sym}(\mathrm{cm}(C), C) \geq 1/n$, where

$$\mathrm{cm}(C) := \frac{1}{\mathrm{vol}_n(C)} \int_C u du$$

is the center of mass of C. As shown in Seeger and Torki [13], the volumetric center $c := \varrho_{\rm vol}(K)$ of a proper cone K satisfies the fixed point condition $c = \mathrm{cm}(K^* \cap c^\dagger)$, where

$$c^\dagger := \{x \in \mathbb{R}^n : \langle c, x \rangle = 1\}.$$

Since $K^* \cap c^\dagger$ is a convex body in the $(n-1)$-dimensional affine space $c^\dagger$, we get

$$\begin{aligned} \Phi_\diamond(K) \geq \Phi_{\rm vol}(K) &= f(c, K) = \mathrm{sym}(c, K^* \cap c^\dagger) \\ &= \mathrm{sym}(\mathrm{cm}(K^* \cap c^\dagger), K^* \cap c^\dagger) \\ &\geq (n-1)^{-1}, \end{aligned}$$

where the second equality is due to [15, Proposition 3.2]. This shows that $\mu(\Phi_{\rm vol}) = (n-1)^{-1}$ and that $\Phi_{\rm vol}$ attains its infimum, for instance, at any simplicial cone. *Part* (c). That $\Phi_{\rm met}$ and $\Phi_{\rm rg}$ are not normalized follows from Proposition 8.2. For all $K \in \Pi_n$, we have

$$\mathrm{Vol}[E^\uparrow(K)] \leq \upsilon^\uparrow(K) \leq \mathrm{Vol}(K) \leq \upsilon^\downarrow(K) \leq \mathrm{Vol}[E^\downarrow(K)].$$

Hence,

$$\begin{aligned} \mu(\Phi_{\rm vr}^{\uparrow}) &\geq \inf_{K\in\Pi_n} \frac{\mathrm{Vol}[E^{\uparrow}(K)]}{\mathrm{Vol}(K)} \geq 1/q_n, \\ \mu(\Phi_{\rm vr}^{\downarrow}) &\geq \inf_{K\in\Pi_n} \frac{\mathrm{Vol}(K)}{\mathrm{Vol}[E^{\downarrow}(K)]} \geq p_n, \end{aligned}$$

where the positive constants p_n and q_n are as in (46). This shows that $\Phi_{\rm vr}^{\uparrow}$ and $\Phi_{\rm vr}^{\downarrow}$ are not normalized. □

It is possible to identify exactly all the minimizers of $\Phi_{\rm vol}$, but this task is space consuming and requires to introduce a highly technical class of proper cones. Hence, we skip this minor issue.

9 By way of application

We show next an interesting application of the axial symmetry indices $\Phi_{\rm ei}$ and $\Phi_{\rm ec}$. Consider the problem of evaluating the quality of the approximation of a proper cone K by means of its inscribed and circumscribed ellipsoidal cones:

$$E^{\uparrow}(K) \subseteq K \subseteq E^{\downarrow}(K).$$

We concentrate on the analysis of the inner approximation of K by its inscribed ellipsoidal cone, the outer approximation scheme can be treated in a similar way. We open a parenthesis and say some words on the classical inner approximation of a convex body C by means of its inscribed ellipsoid $e^{\uparrow}(C)$. The set $e^{\uparrow}(C)$ is sometimes called the John ellipsoid associated to C. Let $\mathcal{B}_n$ denote the set of convex bodies in $\mathbb{R}^n$. As mentioned in Section 7, the symbol $\mathcal{C}_n$ stands for the subset of centrally symmetric convex bodies in $\mathbb{R}^n$. There is a reach literature dealing with the uniform upper estimates

$$C \subseteq n\, e^{\uparrow}(C) \quad \text{for all } C \in \mathcal{B}_n, \tag{62}$$

$$C \subseteq \sqrt{n}\, e^{\uparrow}(C) \quad \text{for all } C \in \mathcal{C}_n. \tag{63}$$

As it is well known, the homothecy factor n is the smallest possible in (62), whereas the homothecy factor $\sqrt{n}$ is the smallest possible in (63). A natural question to ask in this context is the following one: which is the best homothecy factor when C is to be taken from a collection of convex bodies with prescribed degree of central symmetry? Blekherman [2] answered to the above question by using as measure of central symmetry of a convex body C the expression

$$\psi_{\rm John}(C) := \mathrm{sym}(\mathrm{j}(C), C),$$

where $\mathrm{sym}(\,\cdot, C)$ is the symmetry function introduced in (21) and $\mathrm{j}(C)$ is the John center of C, i.e., the center of the ellipsoid inscribed in C.

Theorem 9.1 (Blekherman, 2004). *Let C be a convex body in $\mathbb{R}^n$. Then*

$$C \subseteq \left[\frac{n}{\psi_{\rm John}(C)}\right]^{1/2} e^{\uparrow}(C). \tag{64}$$

For a proof of (64), we may consult also Belloni and Freund [2, Theorem 9] and Brandenberg and König [4, Theorem 7.1]. Note that (62) and (63) can both be seen as a particular instance of the upper estimate (64). We now establish a conic version of the Blekherman inequality (64). To start with, we need to clarify which is the meaning of an homothet of a proper cone. In fact, we just need to consider the homothecy operation for axially symmetric proper cones.

Definition 9.1. *Let $Q \in \Pi_n$ be axially symmetric and β be a positive real. Then the set*

$$\beta \odot Q := \mathbb{R}_+ \left(\beta \left(\mathrm{T}(Q) - c_Q\right) + c_Q\right) \tag{65}$$

is called the conic β-homothet of Q.

Geometrically speaking, the set $\mathrm{T}(Q) - c_Q$ is the transversal section of Q translated to the origin. The real β in (65) is called conic homothecy factor. We mention in passing that (65) is well defined also for a negative β. The operation $\odot$ has similar properties as the usual homothecy operation for origin-symmetric convex bodies. For the sake of completeness, we mention the following basic properties.

Lemma 9.1. *Let $Q \in \Pi_n$ be axially symmetric and β be a positive real. Then*

(a) $\beta \odot Q$ *is an axially symmetric proper cone and has* c_Q *as axial symmetry center.*

(b) $(\beta \odot Q)^* = (1/\beta) \odot Q^*$.

(c) $\mathrm{Vol}(\beta \odot Q) = \beta^{n-1}\mathrm{Vol}(Q)$.

(d) $\beta \odot [U(Q)] = U(\beta \odot Q)$ *for all* $U \in \mathbb{O}(n)$.

Proof. Note that $C = \mathrm{T}(Q) - c_Q$ is an origin-symmetric convex body in a linear subspace of dimension $n-1$, namely, in the hyperplane

$$c_Q^\perp := \{x \in \mathbb{R}^n : \langle c_Q, x\rangle = 0\}.$$

The homothet βC is understood in the usual sense, i.e., as a multiplication of the convex body C by the positive real β. All the properties listed in the lemma can be derived from analogous properties for origin-symmetric convex bodies. For saving space, we omit writing down the details. □

Lemma 9.2. *Let $Q \in \Pi_n$ be axially symmetric. Then*

(a) $\beta \odot Q$ *is increasing with respect to set inclusion as function of* $\beta > 0$.

(b) $\beta_1 \odot (\beta_2 \odot Q) = (\beta_1\beta_2) \odot Q$ *for all* $\beta_1, \beta_2 > 0$.

Proof. The proof is easy and therefore omitted. □

Without further ado, we state:

Theorem 9.2. *For all $K \in \Pi_n$ we have*

$$K \subseteq \left[\frac{n-1}{\Phi_{\mathrm{ei}}(K)}\right]^{1/2} \odot E^{\uparrow}(K). \tag{66}$$

Proof. By applying on K an orthogonal transformation if necessary, we may assume that $\varrho_{\rm ei}(K)$ is equal to e_n, the last canonical vector of $\mathbb{R}^n$. In such a case, we can write

$$K = {\rm fit}(M) := \{t(u,1) : t \geq 0, u \in M\},$$

where M is a convex body in $\mathbb{R}^{n-1}$. The set ${\rm fit}(M)$ is called the proper cone fitted by the convex body M. From the theory of inscribed ellipsoidal cones, cf. [14, Corollary 3.10], we know that

$$E^{\uparrow}(K) = {\rm fit}[e^{\uparrow}(M)]$$

and that $e^{\uparrow}(M)$ is centered at the origin, i.e., ${\rm j}(M) = 0$. By applying Theorem 9.1 to the convex body M, we get

$$M \subseteq \beta e^{\uparrow}(M) \quad \text{with} \quad \beta = \left[\frac{n-1}{\psi_{\rm John}(M)}\right]^{1/2}.$$

A posteriori,

$$K = {\rm fit}(M) \subseteq {\rm fit}(\beta e^{\uparrow}(M)) = \beta \odot {\rm fit}(e^{\uparrow}(M)) = \beta \odot E^{\uparrow}(K).$$

Finally, we observe that

$$\psi_{\rm John}(M) = {\rm sym}(0, M) = f(e_n, K) = \Phi_{\rm ei}(K),$$

where the second equality is obtained by applying [15, Proposition 3.2]. □

The quality of the upper estimate (66) is best (respectively, worst) when the proper cone K is axially symmetric (respectively, simplicial). For these extreme cases we get

$$K \subseteq \sqrt{n-1} \odot E^{\uparrow}(K) \quad \text{for all } K \in \mathcal{A}_n, \tag{67}$$

$$K \subseteq (n-1) \odot E^{\uparrow}(K) \quad \text{for all } K \in \mathcal{S}_n, \tag{68}$$

respectively. The conic homothecy factor $\sqrt{n-1}$ is the smallest possible in (67), whereas the conic homothecy factor $n-1$ is the smallest possible in (68). The next corollary is the lower counterpart of Theorem 9.2.

Corollary 9.3. *For all* $K \in \Pi_n$ *we have*

$$\left[\frac{\Phi_{\rm ec}(K)}{n-1}\right]^{1/2} \odot E^{\downarrow}(K) \subseteq K. \tag{69}$$

Proof. For passing from the upper bound (66) to the lower bound (69), we rely on the duality formula $\left[E^{\downarrow}(K)\right]^* = E^{\uparrow}(K^*)$ established in [14], together with the conjugacy relation (40) and the duality formula stated in Lemma 9.1(b). □

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